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Global Existence and Finite Time Blow-up for a Reaction-Diffusion System with Three Components

Huiling Li[1] · **Yang Zhang**[2]

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Abstract This paper concerns global existence and finite time blow-up behavior of positive solutions for a nonlinear reaction-diffusion system with different diffusion coefficients. By use of algebraic matrix theory and modern analytical theory, we extend results of Wang (Z. Angew. Math. Phys. 51:160–167, 2000) to a more general system. Furthermore, we give a complete answer to the open problem which was brought forward in Wang (Z. Angew. Math. Phys. 51:160–167, 2000).

Keywords Global existence · Finite time blow-up · Structure of the matrix · Reaction-diffusion system · Three components · Different diffusion coefficients

Mathematics Subject Classification (2000) 35B40 · 35K55 · 35K61

1 Introduction and Main Results

In this paper, global existence and finite time blow-up behaviors of positive solutions for a nonlinear reaction-diffusion system are to be discussed:

$$\begin{cases} u_{it} = d_i \Delta u_i + u_1^{p_{i1}} u_2^{p_{i2}} u_3^{p_{i3}}, & x \in \Omega,\ t > 0, \\ u_i(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u_i(x,0) = u_{i0}(x), & x \in \Omega,\ i = 1,2,3, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, initial values $u_{i0}(x)$ ($1 \le i \le 3$) are non-negative and continuous functions which satisfy compatibility conditions.

H. Li
huilingli@seu.edu.cn

Y. Zhang
zhangyanghit830217@163.com

1 School of Mathematics, Southeast University, Nanjing 210096, P.R. China

2 Department of Mathematics, Harbin Engineering University, Harbin 150001, P.R. China

The exponents p_{ij} $(i, j = 1, 2, 3)$ are non-negative constants, and diffusion coefficients d_i are positive constants for all $1 \le i \le 3$.

System (1.1) is usually used as a model to describe heat propagation in a three-component combustible mixture (cf. [2]). In this case, u_1, u_2 and u_3 represent temperatures of the interacting components, and corresponding d_i $(1 \le i \le 3)$ are thermal conductivity, which are supposed constant.

Written system (1.1) as integral equations, by constructing bounded monotone iterative sequences it can be proved that system (1.1) has a local non-negative solution (cf. [15]). However, uniqueness does not hold (cf. [12]). The comparison principle holds, see Sect. 1 of this paper.

For system of two components in a bounded domain Ω of the form

$$\begin{cases} u_{it} = \Delta u_i + u_1^{p_{i1}} u_2^{p_{i2}}, & x \in \Omega,\ t > 0, \\ u_i(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u_i(x,0) = u_{i0}(x) \ge 0, & x \in \Omega,\ i = 1, 2, \end{cases} \tag{1.2}$$

Chen [4] in 1997 investigated special case: p_{12}, $p_{21} > 0$, $p_{12} > p_{22} - 1$ and $p_{21} > p_{11} - 1$, and critical exponents were proved. Later, Wang [22] considered a general case, and obtained significant results which cover that of [4].

Let λ be the first eigenvalue and $\varphi(x)$ the corresponding eigenfunction of the problem

$$-\Delta\varphi(x) = \lambda\varphi(x), \quad x \in \Omega; \qquad \varphi(x) = 0, \quad x \in \partial\Omega. \tag{1.3}$$

It is well known that $\lambda > 0$, $\varphi(x) > 0$ in Ω and $\partial\varphi/\partial\eta < 0$ on $\partial\Omega$, here η is the unit outward normal vector on $\partial\Omega$. Then main results of [22] are read as follows:

(i) If

$$p_{11} \le 1, p_{22} \le 1 \quad \text{and} \quad p_{12}p_{21} \le (1 - p_{11})(1 - p_{22}), \tag{1.4}$$

then all solutions of (1.2) exist globally.

(ii) If

$$p_{11} > 1,\ p_{12} > 0,\ p_{21} = 0,\ p_{22} = 1,\ \lambda < 1,\ p_{11} \le 1 + p_{12}(1 - \lambda)/\lambda, \tag{1.5}$$

or

$$p_{22} > 1,\ p_{21} > 0,\ p_{12} = 0,\ p_{11} = 1,\ \lambda < 1,\ p_{22} \le 1 + p_{21}(1 - \lambda)/\lambda. \tag{1.6}$$

Furthermore, if $p_{11} = 1 + p_{12}(1 - \lambda)/\lambda$ in (1.5) or $p_{22} = 1 + p_{21}(1 - \lambda)/\lambda$ in (1.6), we also assume $\lambda < 2/3$. Then, for any initial data $u_{i0}(x) \ge 0, \not\equiv 0$ $(i = 1, 2)$, solutions of (1.2) blow up in finite time.

(iii) If (1.4), (1.5) and (1.6) do not hold, then solutions of (1.2) exist globally for small initial data, and blow up in finite time for large initial data.

There are many other related works on global existence and blow-up in finite time of solutions to reaction-diffusion equations or systems with two components, see for example [1, 6–19, 23–26] and references therein.

In this paper, we are going to try to generalize results of Wang [22] to system with three components and different diffusion coefficients, and we mainly focus on conditions of global existence and finite time blow-up for positive solutions of system (1.1). Our main conclusions will be introduced and proved in Sect. 3, and from these it is not difficult to see that fine structure of the matrix P, which is a nonlinear function of the exponents p_{ij}

$(i, j = 1, 2, 3)$, is crucially important. This is one of very interesting features for our results. The tools we adopt are a combination of algebraic matrix and modern analytical theory. Here, it is noteworthy that we give a complete and final resolution to the open problem which was presented in [22], please see Remark 4 below.

Here we point out that Li and his collaborators [16, 17] have applied properties of M-matrix and a comparison principle to investigate global existence conditions for quasilinear parabolic systems. There are different properties between M-matrix and the matrix in this paper. Problem in this paper can be considered as a general version of [16] somewhat in the aspect of different diffusion coefficients. Problem and results of this paper differ from that of [17], and two of the most obvious differences are: the first is that for irreducible matrix, our blow-up condition is independent of the value of the first eigenvalue of $-\Delta$ on Ω with null Dirichlet boundary condition, and results of [17] do depend on it; the second is that results of [17] do not cover our results. All of these will be discussed in Sect. 4. The rest of the paper is organized as follows: in Sect. 2, we establish a comparison principle and some preliminaries on properties with respect to matrix. We then discuss the global existence and finite blow-up of solution in Sect. 3.

2 Preliminaries

In this section, we first prove a comparison principle and then some results related to matrix, all of which will play an important role in the proof of our main theorems.

Proposition 1 (*Comparison principle*) *Assume that f_i is a continuous, non-decreasing and non-negative function, and assume that continuous functions u_i and v_i satisfy $u_i, v_i > 0$ in $\Omega \times (0, T)$ and*

$$\begin{cases} u_{it} - d_i \Delta u_i - f_i(u_1, u_2, u_3) \geq 0, & (x,t) \in \Omega \times (0,T), \\ v_{it} - d_i \Delta v_i - f_i(v_1, v_2, v_3) \leq 0, & (x,t) \in \Omega \times (0,T), \\ u_i(x,t) > v_i(x,t) \geq 0, & (x,t) \in \partial\Omega \times (0,T), \\ u_i(x,0) = u_{i0}(x) > v_i(x,0) = v_{i0}(x) \geq 0, & x \in \bar{\Omega},\ i = 1,2,3. \end{cases}$$

Then $u_i(x,t) > v_i(x,t)$ for any $(x,t) \in \bar{\Omega} \times [0,T)$ $(i = 1,2,3)$.

Proof Let $w_i = u_i - v_i$, then w_i satisfies

$$\begin{cases} w_{it} - d_i \Delta w_i \geq f_i(u_1, u_2, u_3) - f_i(v_1, v_2, v_3), & (x,t) \in \Omega \times (0,T), \\ w_i(x,t) > 0, & (x,t) \in \partial\Omega \times (0,T), \\ w_i(x,0) > 0, & x \in \bar{\Omega},\ i = 1,2,3. \end{cases} \tag{2.1}$$

Set

$$t_i = \sup\big\{\, t \leq T \mid w_i(x,s) > 0 \text{ for all } (x,s) \in \bar{\Omega} \times (0,t)\big\}, \quad i = 1,2,3.$$

On the contrary we assume that the conclusion would be not true, and which joined with initial conditions of (2.1) and continuity of u_i and v_i implies

$$t_i > 0, \quad i = 1,2,3.$$

Put $t^* = \min_{1 \leq i \leq 3} t_i$, and without loss of generality we may think $t^* = t_1$. Note that w_1 is continuous. By the definition of t^* and initial-boundary conditions of (2.1), the existence of $x_1 \in \Omega$ can be obtained such that $w_1(x_1, t_1) = 0$.

On the other hand, monotonicity of f_1 and the definition of $t^* = t_1$ show that $w_i \geq 0$ $(i = 1, 2, 3)$ for all $(x, t) \in \bar{\Omega} \times [0, t_1]$ and w_1 satisfies

$$\begin{cases} w_{1t} - d_1 \Delta w_1 \geq f_1(u_1, u_2, u_3) - f_1(v_1, v_2, v_3) \geq 0, & (x, t) \in \Omega \times (0, t_1), \\ w_1(x, t) > 0, & (x, t) \in \partial\Omega \times (0, t_1), \\ w_1(x, 0) > 0, & x \in \bar{\Omega}. \end{cases}$$

As $w_1(x_1, t_1) = 0$ and $x_1 \in \Omega$, in view of the strong maximum principle for single equation it follows that $w_1(x, t) \equiv 0$ for every $(x, t) \in \bar{\Omega} \times [0, t_1)$, which is a contradiction. Therefore, we arrive at conclusions of Proposition 1. □

We next establish some results on matrix, and we begin with notation and definition of matrix.

Notation 1 $|F| = \det F$ *is the determinant of matrix* F.

Definition 1 Let $A = (a_{ij})_{n\times n}$ with $n \geq 2$. If there exists an array matrix F, such that

$$F^T A F = \begin{bmatrix} A_1 & \vdots & * \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & A_2 \end{bmatrix},$$

where A_1 is a r-th sub-matrix and A_2 is $(n - r)$-th sub-matrix with $1 \leq r \leq n - 1$, then A is called to be reducible. Otherwise, A is irreducible.

We write $M = (m_{ij})_{n\times n}$ with $m_{ij} \geq 0$ $(i, j = 1, \ldots, n)$ and $A = (a_{ij})_{n\times n} = I - M$. It is obvious that A is reducible if and only if M is reducible.

Lemma 1 (cf. [20]) *Suppose that* M *is a non-negative matrix. If* M *is irreducible, then* M *has a positive eigenvalue* λ_0 *which is the largest, i.e.* $|\mu| \leq \lambda_0$ *for any eigenvalue* μ *of* M, *and the corresponding eigenvector* $\alpha = (\alpha_1, \ldots, \alpha_n)^T$ *is positive, i.e.* $\alpha_i > 0$ $(1 \leq i \leq n)$.

Proposition 2 (cf. [25]) *Suppose that* M *is a non-negative matrix, and all the principal minor determinants of* $A = I - M$ *are non-negative. If* A *is irreducible, then there exists* $\alpha = (\alpha_1, \ldots, \alpha_n)^T$ *with* $\alpha_i > 0$ *for all* $i = 1, \ldots, n$, *such that* $A\alpha \geq \mathbf{0}$, *i.e.* $\alpha_i - \sum_{j=1}^n m_{ij}\alpha_j \geq 0$.

Proposition 3 (cf. [25]) *Let* M *be a non-negative matrix. Assume that all the lower-order principal minor determinants of* $A = I - M$ *are non-negative and* $|A| < 0$. *Then* A *is irreducible and there exists*

$$\alpha = (\alpha_1, \ldots, \alpha_n)^T$$

with $\alpha_i > 0$ $(1 \leq i \leq n)$, *such that* $A\alpha < \mathbf{0}$, *i.e.* $\alpha_i - \sum_{j=1}^n m_{ij}\alpha_j < 0$.

From the above results we have

Lemma 2 *Let* M *be a non-negative* 3-*th matrix. Suppose that all the lower-order principle minor determinants of* $A = I - M$ *are non-negative and* $|A| < 0$. *Then* A *is irreducible and* *for any positive constant* σ *there exists positive constants* ℓ_i $(1 \leq i \leq 3)$, *such that*

$$-\sigma = \ell_i - \sum_{j=1}^{3} m_{ij}\ell_j, \quad 1 \leq i \leq 3. \tag{2.2}$$

Proof From Proposition 3 we know that M is irreducible. Direct computation gives

$$\ell_1 = \frac{1}{|A|}\begin{vmatrix} -\sigma & -m_{12} & -m_{13} \\ -\sigma & 1-m_{22} & -m_{23} \\ -\sigma & -m_{32} & 1-m_{33} \end{vmatrix}$$

$$= -\frac{\sigma}{|A|}\left(\begin{vmatrix} 1-m_{22} & -m_{23} \\ -m_{32} & 1-m_{33} \end{vmatrix} + \begin{vmatrix} m_{12} & -m_{13} \\ m_{32} & 1-m_{33} \end{vmatrix} + \begin{vmatrix} -m_{12} & -m_{13} \\ 1-m_{22} & -m_{23} \end{vmatrix}\right) \geq 0,$$

here we use the fact that $|A| < 0$ and

$$\begin{vmatrix} 1-m_{22} & -m_{23} \\ -m_{32} & 1-m_{33} \end{vmatrix} \geq 0,$$

$$\begin{vmatrix} m_{12} & -m_{13} \\ m_{32} & 1-m_{33} \end{vmatrix} = m_{12}(1-m_{33}) + m_{13}m_{23} \geq 0,$$

$$\begin{vmatrix} -m_{12} & -m_{13} \\ 1-m_{22} & -m_{23} \end{vmatrix} = m_{12}m_{23} + m_{13}(1-m_{22}) \geq 0.$$

Hence, if $\ell_1 = 0$, then by the fact that $m_{ij} \geq 0$ and $1 - m_{ii} \geq 0$ $(i, j = 1, 2, 3)$ we find that

$$m_{13}m_{23} = 0.$$

Since we have proved that A is irreducible which guarantees that $m_{13}m_{23} \neq 0$. It is a contraction. Therefore, $\ell_1 > 0$.

Similarly, we can obtain that $\ell_i > 0$ for $i = 2, 3$, and the desired conclusion holds. □

Lemma 3 *Let* M *be a non-negative* 3*-th matrix. Suppose that* M *is irreducible and* λ_0 *is the largest eigenvalue of* M*. If* $\lambda_0 = 1$*, then all the principal minor determinants of* $A = I - M$ *are non-negative.*

Proof Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T$ be the eigenvector of M corresponding with λ_0, then by Lemma 1 we see that $\alpha_i > 0$ for all $1 \leq i \leq 3$, and

$$A\alpha = (1 - \lambda_0)\alpha = \mathbf{0}. \tag{2.3}$$

It follows that

$$|A| = 0.$$

Eq. (2.3) can be changed into

$$\alpha_i = \sum_{j=1}^{3} m_{ij}\alpha_j, \quad i = 1, 2, 3. \tag{2.4}$$

As $m_{31}m_{32} \neq 0$ follows by the fact that M is irreducible, due to $m_{ij} \geq 0$ and $\alpha_i > 0$ $(i, j = 1, 2, 3)$, from (2.4) we obtain $1 - m_{33} > 0$. Similarly, we can prove that $1 - m_{11} > 0$ and $1 - m_{22} > 0$.

Up to now, we have concluded that both $|A|$ itself and all the first order principal minor determinants of A are non-negative. Therefore, to accomplish the proof we only need to infer that all the second order principal minor determinants of A are non-negative. Conversely, assume that at least one of the second order principal minors of A is negative, and let's say without loss of generality that

$$\begin{vmatrix} 1 - m_{22} & -m_{23} \\ -m_{32} & 1 - m_{33} \end{vmatrix} < 0 .$$

Note that $1 > m_{ii} \geq 0$ $(1 \leq i \leq 3)$. It yields

$$\begin{aligned} |A| &= (1 - m_{11}) \begin{vmatrix} 1 - m_{22} & -m_{23} \\ -m_{32} & 1 - m_{33} \end{vmatrix} + m_{12} \begin{vmatrix} -m_{21} & -m_{23} \\ -m_{31} & 1 - m_{33} \end{vmatrix} - m_{13} \begin{vmatrix} -m_{21} & 1 - m_{22} \\ -m_{31} & -m_{32} \end{vmatrix} \\ &< -m_{12}m_{21}(1 - m_{33}) - m_{12}m_{23}m_{31} - m_{13}m_{21}m_{32} - m_{13}m_{31}(1 - m_{22}) \\ &\leq 0, \end{aligned}$$

which contradicts to $|A| = 0$. Therefore, all the assertions of Lemma 3 have concluded. □

Lemma 4 *Let M be a non-negative 3-th matrix. Suppose that M is irreducible and λ_0 is the largest eigenvalue of M. If $\lambda_0 \neq 1$, then for any constant $L > 0$, there exists positive constants ℓ_i such that*

$$\ell_i \geq \ell_1^{m_{i1}} \ell_2^{m_{i2}} \ell_3^{m_{i3}} (1 + L)^{m_{i1} + m_{i2} + m_{i3}}, \quad 1 \leq i \leq 3. \tag{2.5}$$

Proof (2.5) can be rewritten as

$$\ln \ell_i - \sum_{j=1}^{3} m_{ij} \ln \ell_j \geq (m_{i1} + m_{i2} + m_{i3}) \ln(1 + L), \quad i = 1, 2, 3. \tag{2.6}$$

Put

$$L^* = \max_{1 \leq i \leq 3} (m_{i1} + m_{i2} + m_{i3}),$$

then (2.6) holds provided that

$$\ln \ell_i - \sum_{j=1}^{3} m_{ij} \ln \ell_j \geq L^* \ln(1 + L). \tag{2.7}$$

From Lemma 1, we know that $\lambda_0 > 0$ and its corresponding eigenvector $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T > \mathbf{0}$, such that

$$(I - M)\alpha = (1 - \lambda_0)\alpha. \tag{2.8}$$

When $0 < \lambda_0 < 1$, it follows from (2.8) that

$$\alpha_i - \sum_{j=1}^{3} m_{ij}\alpha_j > \frac{(1 - \lambda_0)\alpha_i}{2} > 0, \quad i = 1, 2, 3.$$

Let

$$\ln \ell_i = K\alpha_i, \quad i = 1, 2, 3, \tag{2.9}$$

where the positive constant K is to be determined later. Then the above results in

$$\ln \ell_i - \sum_{j=1}^{3} m_{ij} \ln \ell_j > \frac{K(1-\lambda_0)\alpha_i}{2} > 0, \quad i = 1, 2, 3.$$

Consequently, if we choose positive constant K suitable large so that

$$K(1-\lambda_0) \min_{1\le i\le 3} \alpha_i \ge 2L^* \ln(1+L),$$

then (2.7) follows.

When $\lambda_0 > 1$, (2.8) implies

$$-\alpha_i - \sum_{j=1}^{3} m_{ij}(-\alpha_j) > \frac{(\lambda_0 - 1)\alpha_i}{2} > 0, \quad i = 1, 2, 3. \tag{2.10}$$

By letting

$$\ln \ell_i = -K\alpha_i, \quad i = 1, 2, 3 \tag{2.11}$$

with

$$K(\lambda_0 - 1) \min_{1\le i\le 3} \alpha_i \ge 2L^* \ln(1+L),$$

then from (2.10) it can be found that (2.7) holds.

Therefore, existence of positive constants ℓ_i $(1 \le i \le 3)$ is ensured by (2.9) and (2.11), which correspond to the case $\lambda_0 < 1$ and the case $\lambda_0 > 1$, respectively. □

Remark 1 Going through the proofs of Lemmas 2–4, we find that all results in these three lemmas hold for any r-th matrix with $r \ge 1$.

3 Main Theorems and Their Proofs

In this section, we will state our main results. Let constants $p_{ij} \ge 0$ $(i, j = 1, 2, 3)$ and be defined by (1.1). Denote

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}, \qquad P_1 = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ 0 & p_{22} & p_{23} \\ 0 & p_{32} & p_{33} \end{bmatrix}, \qquad P_2 = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ 0 & 0 & p_{33} \end{bmatrix}, \tag{3.1}$$

and let $A = I - P$.

We will illustrate our results on the basis of properties of the matrix P.

3.1 *P* Is Irreducible

When P is irreducible, conditions on global existence and blow-up in a finite time of solutions to problem (1.1) will be established as follows.

Theorem 1 *Assume P is irreducible. If all the principal minor determinants of A are nonnegative, then solutions of (1.1) exist globally.*

Proof With the help of Proposition 2, there exists $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T > \mathbf{0}$, such that $A\alpha \geq \mathbf{0}$. For $1 \leq i \leq 3$, write

$$\ell_i = \frac{\alpha_i}{\| \alpha \|}, \quad d = \max_{1 \leq i \leq 3} d_i, \quad i = 1, 2, 3.$$

It is easy to see that $0 < \ell_i < 1$ $(1 \leq i \leq 3)$.

Let $k(t) \in C^1([0, +\infty))$ be a positive function satisfying $k'(t) \geq 0$. It is well known that for positive constant $\theta = 1 + \sum_{1 \leq i \leq 3} (\max_{x \in \bar{\Omega}} u_{i0}(x))^{1/\ell_i}$, the unique nonnegative to the linear problem

$$\begin{cases} w_t = d\Delta w + k(t)w, & x \in \Omega,\ t > 0, \\ w(x,t) = \theta, & x \in \partial\Omega,\ t > 0, \\ w(x,0) = \theta, & x \in \Omega \end{cases} \tag{G}$$

exists globally. Furthermore, $w(x,t) \geq \theta > 1$ directly from the maximum principal. As θ is a sub-solution of the above problem, results of [19] (refer to Lemma 4.1, p. 199) assert that $w_t \geq 0$ for all $x \in \Omega$ and $t > 0$.

Let $w(x,t)$ be the unique nonnegative solution to problem (G) with $k(t) = k = \max_{1 \leq i \leq 3} d/(d_i \ell_i)$, and let

$$\bar{u}_i = w^{\ell_i}, \quad x \in \bar{\Omega},\ t > 0,\ i = 1, 2, 3,$$

it can be inferred that for $1 \leq i \leq 3$ and $w_t \geq 0$,

$$\begin{aligned} d_i \Delta \bar{u}_i + \bar{u}_1^{p_{i1}} \bar{u}_2^{p_{i2}} \bar{u}_3^{p_{i3}} &= d_i \ell_i w^{\ell_i - 1} \Delta w + d_i \ell_i (\ell_i - 1) w^{\ell_i - 2} |\nabla w|^2 + w^{p_{i1}\ell_1 + p_{i2}\ell_2 + p_{i3}\ell_3} \\ &\leq d_i \ell_i w^{\ell_i - 1} \Delta w + w^{\ell_i} = \frac{d_i \ell_i}{d} w^{\ell_i - 1} \left(d\Delta w + \frac{d}{d_i \ell_i} w \right) \\ &\leq \frac{d_i \ell_i}{d} w^{\ell_i - 1} (d\Delta w + kw) = \frac{d_i \ell_i}{d} w^{\ell_i - 1} w_t \\ &\leq \ell_i w^{\ell_i - 1} w_t = \bar{u}_{it}, \quad x \in \Omega,\ t > 0, \\ \bar{u}_i(x,t) \geq 1 > 0 &= u_i(x,t), \quad x \in \partial\Omega,\ t > 0, \\ \bar{u}_i(x,0) &> u_{i0}(x) \geq 0, \quad x \in \bar{\Omega}. \end{aligned}$$

Therefore, the comparison principle (cf. Proposition 1) asserts that $\bar{u}_i(x,t) > u_i(x,t)$ $(1 \leq i \leq 3)$ for all $x \in \bar{\Omega}$ and $t \geq 0$. As $(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ exists globally, (u_1, u_2, u_3) exists globally. We have completed the proof. □

Theorem 2 *Let P be irreducible. If assumptions of Theorem 1 do not hold, then all solutions of (1.1) are globally bounded for small initial data, and blow up in a finite time for large initial data.*

Proof of Theorem 2 (Global existence) In this part, we are going to deduce conditions to ensure global existence of solutions for problem (1.1). Let $\psi(x)$ be the uniqueness solution of

$$\begin{cases} -\Delta\psi = d = \max_{1\le i\le 3}\dfrac{1}{d_i}, & x \in \Omega, \\ \psi = 0, & x \in \partial\Omega. \end{cases} \tag{3.2}$$

Then for some constant $L > 0$,

$$0 \le \psi(x) \le L, \quad \forall x \in \bar{\Omega}. \tag{3.3}$$

Since P is irreducible, it follows from Lemma 1 that P has an eigenvalue $\lambda_0 > 0$ which is the largest, and the corresponding eigenvector $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T > \mathbf{0}$. In addition, Lemma 3 and assumptions of Theorem 2 assert $\lambda_0 \ne 1$. Then by Lemma 4 we know the existence of positive constants ℓ_i $(1 \le i \le 3)$, such that estimates (2.5) hold. For such fixed ℓ_i, take

$$\bar{u}_i(x,t) = \ell_i\big(1 + \psi(x)\big), \quad x \in \bar{\Omega},\ t > 0.$$

A series of computations and (2.5) give

$$\begin{aligned} \bar{u}_{it} - d_i\Delta\bar{u}_i - \bar{u}_i^{p_{i1}}\bar{u}_2^{p_{i2}}\bar{u}_3^{p_{i3}} &= dd_i\ell_i - \ell_1^{p_{i1}}\ell_2^{p_{i2}}\ell_3^{p_{i3}}\big(1 + \psi(x)\big)^{p_{i1}+p_{i2}+p_{i3}} \\ &\ge \ell_i - \ell_1^{p_{i1}}\ell_2^{p_{i2}}\ell_3^{p_{i3}}(1+L)^{p_{i1}+p_{i2}+p_{i3}} \ge 0, \quad x \in \Omega,\ t > 0, \\ \bar{u}_i(x,t) = \ell_i > 0 = u_i(x,t), &\quad x \in \partial\Omega,\ t > 0,\ i = 1,2,3. \end{aligned}$$

Therefore, if we choose initial data such that

$$u_{i0}(x) < \ell_i\big(1 + \psi(x)\big) = \bar{u}_i(x,0), \quad \forall x \in \bar{\Omega},\ i = 1,2,3,$$

then by Proposition 1, it can be deduced that for $1 \le i \le 3$,

$$u_i(x,t) < \bar{u}_i(x,t), \quad x \in \bar{\Omega},\ t \ge 0.$$

Consequently, all solutions (u_1, u_2, u_3) of (1.1) exist globally for small $u_{i0}(x)$ $(1 \le i \le 3)$. □

Before demonstrating blow-up results of Theorem 2, we introduce two lemmas, which will not just once be used in the coming discussion.

Lemma 5 *Assume that $D \subset \mathbb{R}^n$ is a bounded domain with smooth boundary ∂D. Let λ_D be the first eigenvalue of $-\Delta$ in D with homogeneous Dirichlet boundary condition and $\varphi_D(x)$ the corresponding eigenfunction satisfying $\max_{\bar{D}}\varphi_D(x) = 1$. Let k and σ be any positive constants. Suppose $w(x,t)$ is the unique non-negative solution to the problem*

$$\begin{cases} w_t - d\Delta w = k e^{\beta t}\varphi_D^b(x) w^{1+\sigma}, & x \in D,\ t > 0, \\ w(x,t) = 0, & x \in \partial D,\ t > 0, \\ w(x,0) = w_0(x) \ge 0, \not\equiv 0 & x \in D, \end{cases} \tag{3.4}$$

where β is a constant and constants $b \ge 0$ and $d > 0$. Then we have

(i) *If $\beta/(\sigma d\lambda_D) > 1$ or $\beta/(\sigma d\lambda_D) = 1 > -1 + b/\sigma$, then $w(x,t)$ blows up in a finite time T for any initial datum $w_0(x)$, and meanwhile, if $\beta/(\sigma d\lambda_D) < 1$ or $\beta/(\sigma d\lambda_D) = 1 \le -1 + b/\sigma$, such phenomenon appears only for $w(x,t)$ with $w_0(x)$ satisfying (3.6).*

(ii) *Assume the initial function* $w_0(x)$ *satisfies*

$$w_0(x)=0 \quad \text{on } \partial D \quad \text{and} \quad d\Delta w_0(x)+k\varphi_D^b(x)w_0^{1+\sigma}(x)\ge 0 \quad \text{for all } x\in D, \tag{3.5}$$

then $w_t(x,t)\ge 0$ *for every* $(x,t)\in D\times(0,T)$, *where* T *is the maximal existence time of* $w(x,t)$.

Remark 2 (1) To complete the upcoming proof of Lemma 6, just as stated in Remark 3, the assumption in above assertion (ii) is needed only for the case $d_1\ne d_2$.

(2) It should be emphasized here that there exist nontrivial nonnegative functions such that assumptions in above assertion (i) and assertion (ii) hold simultaneously. Indeed, by upper and sub-solution method, it can be deduced that for any given positive constant M, boundary problem

$$\begin{cases} -d\Delta v=k\varphi_D^b(x)v^{1+\sigma}(x), & x\in D,\\ v(x)=0, & x\in\partial D\end{cases}$$

has a nontrivial nonnegative solution $v(x)$ between 0 and $\varepsilon\varphi_D(x)$ with positive constant $\varepsilon\le(\lambda_D/k)^{1/\sigma}$. Therefore, when $\beta/(\sigma d\lambda_D)>1$ or when $\beta/(\sigma d\lambda_D)=1>-1+b/\sigma$, take $w_0(x)=v(x)$, then (3.5) holds, and when $\beta/(\sigma d\lambda_D)<1$ or when $\beta/(\sigma d\lambda_D)=1\le -1+b/\sigma$, take $w_0(x)=\ell v(x)$ with $\ell\ge 1$ sufficiently large such that (3.6) holds, then

$$d\Delta w_0(x)+k\varphi_D^b(x)w_0^{1+\sigma}(x)\ge \ell\big(d\Delta v(x)+k\varphi_D^b(x)v^{1+\sigma}(x)\big)=0, \quad x\in\Omega,$$

and (3.5) follows.

Proof of Lemma 5 Directly from the maximum principal it is not difficult to see that $w(x,t)\ge 0$ for all $(x,t)\in D\times[0,T)$. The assumption (3.5) illustrates that $w_0(x)$ is a sub-solution of problem (3.4). Consequently, we again apply Lemma 4.1 of [19] to obtain that $w_t\ge 0$ for all $(x,t)\in D\times(0,T)$.

We next verify assertion (i). Although some part was given in the proof of main theorem of [22], we will state it for the sake of completion. From definitions of λ_D and $\varphi_D(x)$, we obtain that $\lambda_D>0$, $\varphi_D(x)>0$ in D and $\partial\varphi_D/\partial\eta<0$ on ∂D (where η is the unit outward normal vector on ∂D).

(a) When $\beta/(\sigma d\lambda_D)>\max\{1,-1+b/\sigma\}$ or $\beta/(\sigma d\lambda_D)=1$ or $\beta/(\sigma d\lambda_D)<1$, there exists a positive constant θ such that

$$\beta/(\sigma d\lambda_D)>\theta>\max\{1,-1+b/\sigma\} \quad \text{when } \beta/(\sigma d\lambda_D)>\max\{1,-1+b/\sigma\},$$
$$\theta=\beta/(\sigma d\lambda_D) \quad \text{when } \beta/(\sigma d\lambda_D)=1>-1+b/\sigma,$$
$$\theta>\max\{1,-1+b/\sigma\} \quad \text{when } \beta/(\sigma d\lambda_D)=1\le -1+b/\sigma \text{ or } \beta/(\sigma d\lambda_D)<1.$$

For the above θ, multiplying (3.4) with $\varphi_D^\theta(x)$ and integrating the result, it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_D w\varphi_D^\theta \mathrm{d}x+\theta d\lambda_D\int_D w\varphi_D^\theta\mathrm{d}x$$
$$=d\theta(\theta-1)\int_D w|\nabla\varphi_D|^2\varphi_D^{\theta-2}\mathrm{d}x+k\mathrm{e}^{\beta t}\int_D w^{1+\sigma}\varphi_D^{b+\theta}\mathrm{d}x.$$

Since

$$\left(\int_D w\varphi_D^\theta \mathrm{d}x\right)^{1+\sigma} \le \left(\int_D \varphi_D^{\theta-b/\sigma}\mathrm{d}x\right)^\sigma \int_D w^{1+\sigma}\varphi_D^{b+\theta}\mathrm{d}x$$

and $\theta - b/\sigma > -1$ and $\partial\varphi_D/\partial\eta < 0$ on ∂D guarantee that

$$B := \int_D \varphi_D^{\theta-b/\sigma}\mathrm{d}x < +\infty,$$

above estimates and $\theta \ge 1$ result in

$$g'(t) \ge kB^{-\sigma}\mathrm{e}^{(\beta-\sigma d\theta\lambda_D)t}g^{1+\sigma}(t), \quad t>0; \qquad g(0)=\int_D w_0(x)\varphi_D^\theta(x)\mathrm{d}x,$$

where

$$g(t) = \mathrm{e}^{\theta d\lambda_D t}\int_D w(x,t)\varphi_D^\theta(x)\mathrm{d}x.$$

Integrating it leads to

$$g^{-\sigma}(t) \le \begin{cases} g^{-\sigma}(0) - \sigma k B^{-\sigma}t, & \text{if } \beta = \sigma\theta d\lambda_D, \\ g^{-\sigma}(0) + \dfrac{\sigma k B^{-\sigma}}{\beta - \sigma\theta d\lambda_D}(1-\mathrm{e}^{(\beta-\sigma\theta d\lambda_D)t}), & \text{if } \beta \ne \sigma\theta d\lambda_D. \end{cases}$$

If

$$\begin{aligned} g(0) = \int_D w_0(x)\varphi_D^\theta(x)\mathrm{d}x > 0 \quad &\text{when } \theta \le \beta/(\sigma d\lambda_D), \\ \int_D w_0(x)\varphi_D^\theta(x)\mathrm{d}x > B\left(\frac{\sigma\theta d\lambda_D - \beta}{k\sigma}\right)^{1/\sigma} \quad &\text{when } \theta > \beta/(\sigma d\lambda_D), \end{aligned} \tag{3.6}$$

then due to $\sigma > 0$, we find that

$$\lim_{t\to T}\int_D w(x,t)\varphi_D^\theta(x)\mathrm{d}x = +\infty$$

for some $0 < T < \infty$. Consequently, $w(x,t)$ blows up in the finite time T.

(b) When $1 < \beta/(\sigma d\lambda_D) \le -1 + b/\sigma$, we choose $D^* \subset\subset D$ such that the first eigenvalue λ_{D^*} of $-\Delta$ in D^* with homogeneous Dirichlet condition satisfies

$$\beta/(\sigma d\lambda_{D^*}) > 1. \tag{3.7}$$

Set $\delta = \min_{\bar{D}^*}\varphi_D(x)$, then $\delta > 0$. By the strong maximum principle for single equation, from (3.4) it follows that $w(x,t) > 0$ for all $x \in D$ and $0 < t < T$ with T maximal existence of w. Thus, by $b \ge 0$ we have

$$\begin{cases} w_t - d\Delta w \ge k\delta^b\mathrm{e}^{\beta t}w^{1+\sigma} =: k^*\mathrm{e}^{\beta t}w^{1+\sigma}, & x\in D^*,\ t>0, \\ w(x,t) > 0, & x \in \partial D^*,\ t>0, \\ w(x,0) = w_0(x) \ge 0, \not\equiv 0, & x \in D^*. \end{cases}$$

By virtue of comparison principle, we know nonnegative solution $v(x,t)$ to problem

$$\begin{cases} v_t - d\Delta v = k^* e^{\beta t} v^{1+\sigma}, & x \in D^*,\ t>0,\\ v(x,t)=0, & x\in \partial D^*,\ t>0,\\ v(x,0)=w_0(x)\ge 0,\ \not\equiv 0, & x\in D^* \end{cases} \tag{3.8}$$

possesses

$$v(x,t)\le w(x,t),\quad \forall (x,t)\in D^*\times[0,\ T).$$

On the other hand, problem (3.8) is just the problem (3.4) with $D=D^*$, $k=k^*$ and $b=0$. At this moment, (3.7) and $b=0$ show that $-1+b/\sigma=-1$ and

$$\beta/(\sigma d\lambda_{D^*})>1=\max\{1,\ -1+b/\sigma\}.$$

Thus, the above case (a) have been proven that $v(x,t)$ blows up in a finite time T^*. Therefore, $w(x,t)$ blows up in the finite time $T\le T^*$. The Lemma is concluded. □

Lemma 6 *Let D, λ_D and $\varphi_D(x)$ be defined as in Lemma 5. Assume constants $d_i,\mu_i>0$ and $q_{ij}\ge 0$ $(i,j=1,2,3)$ satisfy $q_{11}\le 1$, $q_{22}\le 1$ and $(1-q_{11})(1-q_{22})<q_{12}q_{21}$. Denote*

$$d=\min\{d_1,\ d_2\},\qquad \beta=\gamma\min\{q_{13},\ q_{23}\},$$

$$\sigma=\begin{cases}\dfrac{q_{12}q_{21}-(1-q_{11})(1-q_{22})}{1+q_{12}-q_{22}}, & \textit{if}\ \dfrac{1+q_{21}-q_{11}}{1+q_{12}-q_{22}}\ge 1,\\ \dfrac{q_{12}q_{21}-(1-q_{11})(1-q_{22})}{1+q_{21}-q_{11}}, & \textit{if}\ \dfrac{1+q_{21}-q_{11}}{1+q_{12}-q_{22}}<1.\end{cases} \tag{3.9}$$

Suppose non-negative functions $v_i(x,t)$ $(1\le i\le 2)$ satisfy

$$\begin{cases} v_{jt}-d_j\Delta v_j\ge \mu_j e^{q_{j3}\gamma t}\varphi_D^{q_{j3}} v_1^{q_{j1}} v_2^{q_{j2}}, & x\in D,\ t>0,\\ v_j(x,t)\ge 0, & x\in\partial D,\ t>0,\\ v_j(x,0)=v_{j0}(x)\ge 0,\ \not\equiv 0, & x\in D,\ j=1,2,\end{cases} \tag{3.10}$$

where γ is a arbitrary constant. Then, both $v_1(x,t)$ and $v_2(x,t)$ blow up in a finite time for any initial data $v_{i0}(x)$ $(1\le j\le 2)$ if $\beta/(\sigma d\lambda_D)>1$, and while, blow-up in a finite time happens only for suitable large initial data if $\beta/(\sigma d\lambda_D)\le 1$.

Proof As $q_{jj}\le 1$ $(1\le j\le 2)$ and $(1-q_{11})(1-q_{22})<q_{12}q_{21}$ imply that $q_{12}>0$ and $q_{21}>0$, for $\sigma>0$ defined by (3.9), we can take

$$\begin{cases}\ell_1=1,\quad \ell_2=\dfrac{1+q_{21}-q_{11}}{1+q_{12}-q_{22}}, & \text{if}\ \dfrac{1+q_{21}-q_{11}}{1+q_{12}-q_{22}}\ge 1,\\ \ell_2=1,\quad \ell_1=\dfrac{1+q_{12}-q_{22}}{1+q_{21}-q_{11}}, & \text{if}\ \dfrac{1+q_{21}-q_{11}}{1+q_{12}-q_{22}}<1.\end{cases}$$

It is obvious that

$$-\sigma=(1-q_{11})\ell_1-q_{12}\ell_2,\qquad -\sigma=-q_{21}\ell_1+(1-q_{22})\ell_2. \tag{3.11}$$

When $\beta/(\sigma d\lambda_D)>1$, we choose a sub-domain $D^*\subset\subset D$, such that the first eigenvalue λ_{D^*} of $-\Delta$ in D^* with homogeneous Dirichlet boundary condition satisfies

$$\beta/(\sigma d\lambda_D)>\beta/\big(\sigma d\lambda_D^*\big)>1. \tag{3.12}$$

Denote $\varphi_{D^*}(x)$ as the eigenfunction corresponding to λ_{D^*} with $\max_{\bar{D}^*}\varphi_{D^*}(x)=1$. When $\beta/(\sigma d\lambda_D)\le 1$, we still consider such domain D^*, and in this moment we see

$$\beta/(\sigma d\lambda_{D^*})<\beta/(\sigma d\lambda_D)\le 1. \tag{3.13}$$

From the definition of $\varphi_D(x)$ we know that there exists a constant δ such that $\varphi_D(x)\ge\delta$ for all $x\in\bar{D}^*$. By the strong maximum principle for single equation it can be inferred from (3.10) that $v_j(x,t)>0$ for all $x\in D$ and $t>0$. Thus,

$$v_j(x,t)>0,\quad x\in\partial D^*,\ t>0,\ j=1,2.$$

Consequently, (3.10) yields

$$\begin{cases} v_{jt}-d_j\Delta v_j\ge\mu_j\delta^{q_{j3}}\mathrm{e}^{q_{j3}\gamma t}v_1^{q_{j1}}v_2^{q_{j2}}, & x\in D^*,\ t>0,\\ v_j(x,t)>0, & x\in\partial D^*,\ t>0,\\ v_j(x,0)=v_{j0}(x)\ge 0,\not\equiv 0, & x\in D^*,\ j=1,2.\end{cases}$$

For $\sigma>0$, $\ell_1\ge 1$ and $\ell_2\ge 1$ given above, define

$$\underline{v}_j(x,t)=w^{\ell_j}(x,t),\quad x\in D^*,\ 0<t<T,\ 1\le j\le 2,$$

where $w(x,t)$ is the unique non-negative solution of (3.4) in D^* stead of D with such constants d, β, σ defined as above and $b=0$, $k=\min_{1\le j\le 2}\{\mu_j d\delta^{q_{j3}}/(d_j\ell_j)\}$. Lemma 5, (3.12) and (3.13) show that $w(x,t)$ blows up in a finite time T for any $w_0(x)\ge 0,\not\equiv 0$ when $\beta/(\sigma d\lambda_{D^*})>1$, and when $\beta/(\sigma d\lambda_{D^*})<1$, $w(x,t)$ blows up in a finite time T only for $w_0(x)\ge 0$ satisfying

$$\begin{cases} \displaystyle\int_{D^*} w_0(x)\varphi_{D^*}^{\theta}(x)\mathrm{d}x>B\left(\frac{\sigma\theta d\lambda_{D^*}-\beta}{k\sigma}\right)^{1/\sigma},\\ \theta>1,\quad B=\displaystyle\int_{D^*}\varphi_{D^*}^{\theta}(x)\mathrm{d}x<+\infty.\end{cases}$$

In remain part of this proof, we further suppose that $w_0(x)$ satisfies the condition (3.5) with d, k and b, and then Lemma 5 shows that $w_t\ge 0$ for every $(x,t)\in D^*\times(0,T)$.

By definitions of k, β, b, σ and ℓ_j, thanks to $w_t\ge 0$, we deduce that

$$\begin{aligned} &d_j\Delta\underline{v}_j+\mu_j\delta^{q_{j3}}\mathrm{e}^{q_{j3}\gamma t}\underline{v}_1^{q_{j1}}\underline{v}_2^{q_{j2}}\\ &\quad=d_j\ell_j w^{\ell_j-1}\Delta w+d_j\ell_j(\ell_j-1)w^{\ell_j-2}|\nabla w|^2+\mu_j\delta^{q_{j3}}\mathrm{e}^{q_{j3}\gamma t}w^{q_{j1}\ell_1+q_{j2}\ell_2}\\ &\quad\ge d_j\ell_j w^{\ell_j-1}\Delta w+\mu_j\delta^{q_{j3}}\mathrm{e}^{q_{j3}\gamma t}w^{\ell_j+\sigma}\\ &\quad\ge\frac{d_j\ell_j}{d}w^{\ell_j-1}\big(d\Delta w+k\mathrm{e}^{\beta t}w^{1+\sigma}\big)=\frac{d_j\ell_j}{d}w^{\ell_j-1}w_t\\ &\quad\ge\ell_j w^{\ell_j-1}w_t=\underline{v}_{jt},\quad (x,t)\in D^*\times(0,T),\ j=1,2.\end{aligned}$$

We make an extension of $w_0(x)$ to D by defining $w_0(x)=0$ in $\bar{D}\backslash\bar{D}^*$. Select

$$v_{j0}(x)>{w_0}^{\ell_j}(x),\quad x\in\bar{D},\ j=1,2,$$

then from the above and the comparison principle, we derive that

$$v_j(x,t) > \underline{v}_j(x,t), \quad (x,t) \in \bar{D}^* \times [0,T^*), \ j=1,2.$$

Therefore, (v_1, v_2) blows up in a finite time T^* with $T^* \le T$ for suitable initial data. □

Remark 3 By going through process of the above proof, it is not difficult to see that when $d_1 = d_2$ (which implies $d = d_1 = d_2$), we can remove this hypothesis that $w_0(x)$ satisfies the condition (3.5) with $b = 0$ and d, k defined as above, and conclusions in Lemma 6 still hold.

With the help of Lemma 5 and Lemma 6, we are able to finish the proof of Theorem 2. Let $\varphi(x)$ be as in (1.3). Further, from now on we may assume in this paper that $0 \le \varphi(x) \le 1$ for all $x \in \bar{\Omega}$ (this can be achieved by adjust the value of $\varphi(x)$ if necessary).

Proof of Theorem 2 (Blow-up results) We are going to demonstrate blow-up conditions for solutions to problem (1.1) now, and we will develop our discussion according to three cases.

Case 1. $|A| < 0$ *and all the lower-order principal minor determinants of* A *are non-negative.*

In Lemma 2 we take $M = P$, and then the existence of $\ell_i > 0$ $(1 \le i \le 3)$ satisfying (2.2) are obtained. Select $\sigma > 0$ so large that $\ell_i \ge 1$ $(1 \le i \le 3)$. Let $\Omega_1 \subset\subset \Omega$ be any sub-domain. For such fixed $\sigma > 0$ and ℓ_i, denote

$$d = \min_{1\le i\le 3} d_i, \quad \underline{u}_i(x,t) = w^{\ell_i}(x,t), \quad i = 1,2,3,$$

where $w(x,t)$ is the unique non-negative solution of (3.4) with $k = \min_{1\le i\le 3} d/(d_i\ell_i)$, $\beta = b = 0$ and $D = \Omega_1$. It has been known from Lemma 5 that $w(x,t)$ blows up in a finite time T and $w_t \ge 0$ in $\Omega_1 \times (0,T)$ for some suitable initial datum $w_0(x)$ satisfying all conditions in assertions (i) and (ii) of Lemma 5. Define $w_0(x) = 0$ in $\bar{\Omega}\backslash\bar{\Omega}_1$, and select

$$u_{i0}(x) \ge {w_0}^{\ell_i}(x) + \varphi(x), \quad x \in \bar{\Omega}, \ i = 1,2,3,$$

which implies

$$u_{i0}(x) > \underline{u}_i(x,0), \quad x \in \bar{\Omega}_1, \ i = 1,2,3. \tag{3.14}$$

By the maximum principle, we have $u_i(x,t) > 0$ $(1 \le i \le 3)$ for all $(x,t) \in \bar{\Omega}_1 \times [0,T^*)$, where T^* is the maximal existence time of (u_1,u_2,u_3). It follows that

$$u_i(x,t) > 0 = \underline{u}_i(x,t), \quad (x,t) \in \partial\Omega_1 \times \big(0,T^*\big), \ i = 1,2,3. \tag{3.15}$$

By definitions of k and d, from (2.2) and $w_t \ge 0$, we deduce that for $1 \le i \le 3$,

$$\begin{aligned} d_i\Delta\underline{u}_i + \underline{u}_1^{p_{i1}}\underline{u}_2^{p_{i2}}\underline{u}_3^{p_{i3}} &= d_i\ell_i w^{\ell_i-1}\Delta w + d_i\ell_i(\ell_i-1)w^{\ell_i-2}|\nabla w|^2 + w^{p_{i1}\ell_1+p_{i2}\ell_2+p_{i3}\ell_3} \\ &\ge \frac{d_i\ell_i}{d}w^{\ell_i-1}\left(d\Delta w + \frac{d}{d_i\ell_i}w^{p_{i1}\ell_1+p_{i2}\ell_2+p_{i3}\ell_3+1-\ell_i}\right) \\ &\ge \frac{d_i\ell_i}{d}w^{\ell_i-1}\big(d\Delta w + kw^{1+\sigma}\big) = \frac{d_i\ell_i}{d}w^{\ell_i-1}w_t \\ &\ge \ell_i w^{\ell_i-1}w_t = \underline{u}_{it}, \quad (x,t) \in \Omega_1 \times \big(0,T^*\big). \end{aligned} \tag{3.16}$$

Using the comparison principle once again, one has that, from (3.14)–(3.16),

$$u_i(x,t) > \underline{u}_i(x,t), \quad (x,t) \in \bar{\Omega}_1 \times [0, T^*),\ i = 1,2,3.$$

Therefore, (u_1, u_2, u_3) blows up in a finite time T^* with $T^* \le T$.

Case 2. $|A| \ge 0$, *all the first principal minor determinants of A are non-negative, and there is one of the second principal minors of A whose determinant is negative.*

Assume that $p_{ii} \le 1$ $(1 \le i \le 3)$, and we may think without loss of generality that $(1 - p_{22})(1 - p_{33}) < p_{23}p_{32}$. Take $u_{10}(x)$ large enough such that

$$u_{10}(x) \ge \varphi(x) \quad \text{for all } x \in \bar{\Omega}.$$

Remember that (u_1, u_2, u_3) is a non-negative solution of problem (1.1). It follows that

$$\begin{cases} u_{1t} - d_1\Delta u_1 \ge 0 = (\mathrm{e}^{-\lambda d_1 t}\varphi(x))_t - d_1\Delta(\mathrm{e}^{-\lambda d_1 t}\varphi(x)), & x \in \Omega,\ t > 0, \\ u(x,t) = 0 = \mathrm{e}^{-\lambda d_1 t}\varphi(x), & x \in \partial\Omega,\ t > 0, \end{cases}$$

and thus,

$$u_1(x,t) \ge \mathrm{e}^{-\lambda d_1 t}\varphi(x), \quad (x,t) \in \bar{\Omega} \times [0, T)$$

where T is maximal existence time of u_1. Consequently,

$$\begin{cases} u_{it} - d_i\Delta u_i \ge \mathrm{e}^{-\lambda d_1 p_{i1} t}\varphi^{p_{i1}} u_2^{p_{i2}} u_3^{p_{i3}}, & x \in \Omega,\ t > 0, \\ u_i(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u_i(x,0) = u_{i0}(x) \ge 0, & x \in \Omega,\ i = 2,3. \end{cases}$$

In Lemma 6, we put $D = \Omega$, $\gamma = -\lambda d_1$, $\mu_j = 1$, $d_j = d_i$, $q_{j3} = p_{i1}$, $q_{j1} = p_{i2}$ and $q_{j2} = p_{i3}$ for all $i = 2,3$ and $j = 1,2$. It is not difficult to see that conditions of Lemma 6 hold. Therefore, Lemma 6 shows that for some suitable large (u_{20}, u_{30}), the corresponding (u_2, u_3) blows up in a finite time T^* with $T^* \le T$, and so does (u_1, u_2, u_3).

Case 3. Both case 1 and the case 2 do not happen.

Without loss of generality we assume that $p_{11} > 1$. Let $u_{i0}(x) \ge \varphi(x)$ $(2 \le i \le 3)$ for all $x \in \bar{\Omega}$. Then as above, we have

$$u_i(x,t) \ge \mathrm{e}^{-\lambda d_i t}\varphi(x), \quad (x,t) \in \bar{\Omega} \times [0, \infty),\ i = 2,3,$$

which leads to

$$\begin{cases} u_{1t} - d_1\Delta u_1 \ge \exp\{-\lambda(d_2 p_{12} + d_3 p_{13})t\}\varphi^{p_{12}+p_{13}} u_1^{p_{11}}, & x \in \Omega,\ t > 0, \\ u_1(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u_1(x,0) = u_{10}(x) \ge 0, & x \in \Omega. \end{cases}$$

In Lemma 5, we choose $D = \Omega$, $d = d_1$, $k = 1$, $\beta = -\lambda(d_2 p_{12} + d_3 p_{13})$, $b = p_{12} + p_{13}$ and $\sigma = p_{11} - 1$, then the corresponding solution $w(x,t)$ of problem (3.4) blows up in a finite time T for suitable large $w_0(x)$ satisfying conditions in assertion (i) of Lemma 5. From $p_{11} > 1$ and the comparison principle for single equation, we achieve

$$u_1(x,t) \ge w(x,t), \quad \forall (x,t) \in \bar{\Omega} \times [0, T^*),$$

with some constant $T^* \le T$, provided that

$$u_{10}(x) \ge w_0(x) \quad \text{for all } x \in \Omega.$$

Hence, u_1 blows up in the finite time T^*, and it follows that (u_1, u_2, u_3) blows up in a finite time for suitable large initial data (u_{10}, u_{20}, u_{30}).

Therefore, we have finished all the proof of Theorem 2. □

3.2 *P* Is Reducible

In this subsection, we will consider the case that P is reducible, and we will find conditions for global existence and blow-up in finite time of solutions to problem (1.1).

For reducible matrix P, we may think without loss of generality that $P = P_1$ or $P = P_2$ with P_1 and P_2 defined by (3.1). We first investigate the case $P = P_1$.

3.2.1 When $P = P_1$

Note that $P = P_1$ implies $p_{21} = p_{31} = 0$. Hence, any solution (u_1, u_2, u_3) of problem (1.1) satisfies

$$\begin{cases} u_{it} - d_i \Delta u_i = u_2^{p_{i2}} u_3^{p_{i3}}, & x \in \Omega,\ t > 0, \\ u_i(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u_i(x,0) = u_{i0}(x) \geq 0, & x \in \Omega,\ i = 2, 3. \end{cases} \tag{3.17}$$

Theorem 3 *For $P = P_1$, if all the principal minor determinants of A are non-negative, then all solutions of* (1.1) *exist globally.*

Proof From assumptions of Theorem 3 it can be found that

$$p_{11} \leq 1, \quad p_{22} \leq 1, \quad p_{33} \leq 1, \quad (1 - p_{22})(1 - p_{33}) \geq p_{23} p_{32}.$$

Thus, with the aid of problem (G) we are able to follow ideas of [22] to obtain global existence of (u_2, u_3), and then from this and $p_{11} \leq 1$, it is not difficult to see that u_1 exists globally. In fact, when $p_{23} p_{32} = 0$, it is clear that (u_2, u_3) exists globally. When $p_{23} p_{32} \neq 0$, the condition $p_{22} \leq 1$, $p_{33} \leq 1$ and $(1 - p_{22})(1 - p_{33}) \geq p_{23} p_{32}$ can be rewritten as

$$\frac{1 - p_{22}}{p_{23}} \cdot \frac{1 - p_{33}}{p_{32}} \geq 1.$$

There exist $0 < \ell_i < 1$ for $2 \leq i \leq 3$, such that

$$(1 - p_{22})/p_{23} \geq \ell_3/\ell_2, \quad (1 - p_{33})/p_{32} \geq \ell_2/\ell_3.$$

In problem (G), set

$$d = \max_{2 \leq i \leq 3} d_i, \quad k(t) = k = \max_{2 \leq i \leq 3} d/(d_i \ell_i), \quad \theta = 1 + \sum_{2 \leq i \leq 3} \left(\max_{x \in \bar{\Omega}} u_{i0}(x) \right)^{1/\ell_i}.$$

As in the proof of Theorem 1, $w(x,t)$ exists globally, $w(x,t) \geq \theta > 1$ and $w_t \geq 0$ for all $x \in \Omega$ and $t > 0$. Moreover, by letting

$$\bar{u}_i = w^{\ell_i}, \quad x \in \bar{\Omega},\ t > 0,\ i = 2, 3,$$

and applying arguments similar as in the proof of Theorem 1, one can arrive at the conclusion

$$\bar{u}_i(x,t) > u_i(x,t), \quad \forall (x,t) \in \bar{\Omega} \times [0,\infty),\ i = 2,3.$$

It means that (u_2, u_3) exists globally. We complete this theorem. □

Theorem 4 *For $P = P_1$, if $p_{ii} \le 1$ $(1 \le i \le 3)$ and $(1 - p_{22})(1 - p_{33}) < p_{23}p_{32}$, then solution of (1.1) exists globally for small initial data, and while blows up in a finite time for large initial data.*

Proof Let (u_1, u_2, u_3) be any non-negative solution of problem (1.1), then (u_2, u_3) satisfies (3.17). By virtue of conditions in this theorem, directly comparing (3.17) and (3.10) gives (u_2, u_3) blows up in a finite time T for large $(u_{20}(x), u_{30}(x))$. Therefore, (u_1, u_2, u_3) blows up in a finite time T for large initial data.

In the following, we only need to show (u_1, u_2, u_3) exists globally for small initial data. Let $M = (p_{ij})$ and $A = I - M$ with $i, j = 2, 3$. Then by Proposition 3 we see that A is irreducible, and so, M is irreducible. Hence, Lemma 1 implies existence of the largest eigenvalue λ_0 with $\lambda_0 > 0$. By virtue of Remark 1 and Lemma 3, we know that $\lambda_0 \neq 1$. Therefore, thanks to Remark 1 and Lemma 4 we find that there exist positive constants ℓ_i, such that

$$\ell_i \ge \ell_2^{p_{i2}} \ell_3^{p_{i3}} (1+L)^{p_{i2}+p_{i3}}, \quad i = 2,3, \tag{3.18}$$

where the positive constant L is determined by (3.3). Let $\psi(x)$ be defined in (3.2) with $d = \max_{2 \le i \le 3} 1/d_i$. Similar to the discussion of global existence part in Theorem 2, with the help of estimates (3.18) it follows that u_2 and u_3 exist globally provided that

$$u_{i0}(x) < \ell_i\big(1 + \psi(x)\big), \quad \forall x \in \bar{\Omega},\ i = 2,3. \tag{3.19}$$

As $p_{11} \le 1$, u_1 exists globally following from global existence of u_2 and u_3. Therefore, (u_1, u_2, u_3) exist globally for such initial data satisfying (3.19). We conclude this theorem. □

Theorem 5 *Let $P = P_1$ and let λ be the first eigenvalue of problem (1.3). Then we have*

(A) *Assume $p_{32} = 0$, $p_{33} = 1$ and $p_{22} > 1$, then the following results hold:*

(A_1) *When $p_{23}(1 - d_3\lambda)/(d_2\lambda) > p_{22} - 1$ or $p_{23}(1 - d_3\lambda)/(d_2\lambda) = p_{22} - 1 > 1 + p_{23} - p_{22}$, all solutions of (1.1) blow up in a finite time for any $u_{i0}(x) \ge 0, \not\equiv 0$ $(1 \le i \le 3)$.*

(A_2) *When $p_{23}(1 - d_3\lambda)/(d_2\lambda) < p_{22} - 1$, solution of (1.1) blows up in a finite time for suitable large $u_{i0}(x)$ $(1 \le i \le 3)$, and exists globally with uniform bounds for suitable small $u_{i0}(x)$ $(1 \le i \le 3)$.*

(A_3) *When $p_{22} - 1 = p_{23}(1 - d_3\lambda)/(d_2\lambda) \le 1 + p_{23} - p_{22}$, solution of (1.1) blows up in a finite time for suitable large $u_{i0}(x)$ $(1 \le i \le 3)$, and exists globally for small $u_{i0}(x)$ $(1 \le i \le 3)$.*

(B) *Assume $p_{23} = 0$, $p_{22} = 1$ and $p_{33} > 1$, then the following results hold:*

(B_1) *When $p_{32}(1 - d_2\lambda)/(d_3\lambda) > p_{33} - 1$ or $p_{32}(1 - d_2\lambda)/(d_3\lambda) = p_{33} - 1 > 1 + p_{32} - p_{33}$, all solutions of (1.1) blow up in a finite time for any $u_{i0}(x) \ge 0, \not\equiv 0$ $(1 \le i \le 3)$.*

(B_2) *When $p_{32}(1 - d_2\lambda)/(d_3\lambda) < 1 + p_{32} - p_{33}$, solution of (1.1) blows up in a finite time for suitable large $u_{i0}(x)$ $(1 \le i \le 3)$, and exists globally with uniform bounds for suitable small $u_{i0}(x)$ $(1 \le i \le 3)$.*

(B_3) *When $p_{33} - 1 = p_{32}(1 - d_2\lambda)/(d_3\lambda) \le 1 + p_{32} - p_{33}$, solution of (1.1) blows up in a finite time for suitable large $u_{i0}(x)$ $(1 \le i \le 3)$, and exists globally for small $u_{i0}(x)$ $(1 \le i \le 3)$.*

(C) *Suppose both the above assumptions and the conditions of Theorem 3 and Theorem 4 do not hold, then solution of (1.1) blows up in a finite time for suitable large $u_{i0}(x)$ $(1 \le i \le 3)$, and exists globally for small $u_{i0}(x)$ $(1 \le i \le 3)$.*

Remark 4 It is not difficult to see that, when $d_2 = d_3 = 1$, conditions in (A_3) are equivalent to $2/3 \le \lambda < 1$ and $p_{22} = 1 + p_{23}(1-\lambda)/\lambda$, and the conditions in (B_3) are equivalent to $2/3 \le \lambda < 1$ and $p_{33} = 1 + p_{32}(1-\lambda)/\lambda$. Therefore, Theorem 5 answers the open problem in [22].

Proof of Theorem 5 Let (u_1, u_2, u_3) be any non-negative solution of problem (1.1), then $u_i(x,t) > 0$ $(1 \le i \le 3)$ for all $(x,t) \in \Omega \times (0,T)$ with the maximal existence time T for (u_1, u_2, u_3), and (u_2, u_3) satisfies (3.17). Note that if (u_2, u_3) blows up in a finite time, obviously, (u_1, u_2, u_3) blows up in a finite time, and if (u_2, u_3) exists globally, then from the first equation of (1.1) we know that $u_1(x,t)$ exits globally for small $u_{10}(x)$, and so does (u_1, u_2, u_3). Hence, in the following we only need to check whether (u_2, u_3) exists globally for all cases other than case (C_5) which will be discussed independently.

We first demonstrate case (A) and case (B). Under the help of assertion (i) in Lemma 5 and arguments of paper [22], we are going to prove case (A), and case (B) can be verified similarly.

Blow-up results in case (A). Notice that u_3 satisfies

$$\begin{cases} u_{3t} - d_3\Delta u_3 = u_3, & x \in \Omega,\ t > 0, \\ u_3(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u_3(x,0) = u_{30}(x) \ge 0, \not\equiv 0 & x \in \Omega. \end{cases}$$

By the maximum principle we can suppose without loss of generality that $u_{30}(x) \ge \varepsilon\varphi(x)$ for some $\varepsilon > 0$. Thus, $u_3(x,t) \ge \varepsilon \exp\{(1 - d_3\lambda)t\}\varphi(x)$, and

$$\begin{cases} u_{2t} - d_2\Delta u_2 \ge \varepsilon^{p_{23}} \exp\{p_{23}(1 - d_3\lambda)t\}\varphi^{p_{23}}(x)u_2^{p_{22}}, & x \in \Omega,\ t > 0, \\ u_2(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u_2(x,0) = u_{20}(x) \ge 0, \not\equiv 0, & x \in \Omega. \end{cases}$$

Put $D = \Omega$, $d = d_2$, $\lambda_D = \lambda$, $k = \varepsilon^{p_{23}}$, $\beta = p_{23}(1 - d_3\lambda)$, $b = p_{23}$ and $\sigma = p_{22} - 1$ in Lemma 5, then assertion (i) of Lemma 5 and the comparison principle for single equation show that $u_2(x,t)$ satisfies all blow-up results in (A).

Global existence results in case (A). From problem (G) and the problem satisfied by u_3, it is know that $u_3(x,t)$ exists globally for any nonnegative $u_{30}(x)$. Then, similar as analysis in the beginning of this proof, global existence of $u_2(x,t)$ for small $u_{20}(x)$ follows from (3.17). Therefore, (u_2, u_3) exists globally for small $u_{20}(x)$ and $u_{30}(x)$.

When $p_{23}(1 - d_3\lambda)/(d_2\lambda) < p_{22} - 1$, analogously as above it can be concluded that $u_3(x,t) \le \exp\{(1 - d_3\lambda)t\}\varphi(x)$ provided that $u_{30}(x) \le \varphi(x)$. By $p_{23} \ge 0$, it leads to

$$u_{2t} - d_2\Delta u_2 \le \exp\big\{p_{23}(1 - d_3\lambda)t\big\}u_2^{p_{22}}, \quad (x,t) \in \Omega \times (0,T).$$

As $p_{23}(1 - d_3\lambda) < (p_{22} - 1)d_2\lambda$, there must be a constant $0 < \theta < d_2\lambda$ with $p_{23}(1 - d_3\lambda) < (p_{22} - 1)\theta$. For such θ, by taking a positive constant $\varepsilon \le (d_2\lambda - \theta)^{1/(p_{22}-1)}$, a direct computation shows that $\bar{u}_2(x,t) = \varepsilon e^{-\theta t}\varphi(x)$ satisfies

$$\bar{u}_{2t} - d_2\Delta \bar{u}_2 \ge \exp\big\{p_{23}(1 - d_3\lambda)t\big\}\bar{u}_2^{p_{22}}, \quad x \in \Omega,\ t > 0.$$

From the comparison principle it follows that $u_2 \le \bar{u}_2$ if $u_{20}(x) \le \varepsilon\varphi(x)$ in Ω. The definitions of $\bar{u}_2$, $\varphi(x)$ and ε imply that $u_2(x,t) < (d_2\lambda)^{1/(p_{22}-1)}$.

The next thing left to do is to prove case (C). By going through all conditions with respect to p_{ij} in Theorem 3, Theorem 4 and case (A) and (B) in Theorem 5, conditions in case (C) with respect to p_{ij} are as follows: (C_1) $p_{22} > 1$, $p_{33} = 1$ and $p_{32} > 0$; (C_2) $p_{22} > 1$, $p_{33} \neq 1$; (C_3) $p_{33} > 1$, $p_{22} = 1$ and $p_{23} > 0$; (C_4) $p_{33} > 1$, $p_{22} < 1$; (C_5) $p_{11} > 1$, $p_{22} \leq 1$ and $p_{33} \leq 1$.

Global existence results in cases (C_1)–(C_4). Define

$$\bar{u}_i = \ell_i\big(1 + \psi(x)\big), \quad x \in \Omega,\ t > 0,\ \ i = 2, 3,$$

where positive constants ℓ_i are determined later, and function $\psi(x)$ is fixed by (3.2) which possesses property (3.3). Note that $P = P_1$ implies $p_{21} = p_{31} = 0$. A series of calculations derive

$$\bar{u}_{it} - d_i \Delta \bar{u}_i \geq \bar{u}_2^{p_{i2}} \bar{u}_3^{p_{i3}}, \quad x \in \Omega,\ t > 0,\ \ i = 2, 3,$$

provided that

$$\begin{cases} (1 - p_{22}) \ln \ell_2 \geq p_{23} \ln \ell_3 + (p_{22} + p_{23}) \ln(1 + L), \\ (1 - p_{33}) \ln \ell_3 \geq p_{32} \ln \ell_2 + (p_{32} + p_{33}) \ln(1 + L). \end{cases} \tag{3.20}$$

For case (C_1), we select constants $0 < \ell_i \leq 1$ $(2 \leq i \leq 3)$ satisfying

$$-\ln \ell_2 = \max\left\{\frac{p_{32} + p_{33}}{p_{32}},\ \frac{p_{22} + p_{23}}{p_{22} - 1}\right\} \ln(1 + L), \quad \ell_3 = 1,$$

then estimate (3.20) holds. For case (C_2), when $p_{33} > 1$, constants $0 < \ell_i \leq 1$ $(2 \leq i \leq 3)$ satisfying

$$-\ln \ell_2 = \frac{(p_{22} + p_{23}) \ln(1 + L)}{p_{22} - 1}, \qquad -\ln \ell_3 = \frac{(p_{32} + p_{33}) \ln(1 + L)}{p_{33} - 1}$$

make estimate (3.20) hold, and when $p_{33} < 1$, if we take $0 < \ell_2 \leq 1$ and $\ell_3 \geq 1$ such that

$$-\ln \ell_2 = \frac{[(1 - p_{33}) p_{22} + p_{23}(1 + p_{32})] \ln(1 + L)}{(p_{22} - 1)(1 - p_{33})}, \qquad \ln \ell_3 = \frac{(p_{32} + p_{33}) \ln(1 + L)}{1 - p_{33}},$$

estimate (3.20) follows. For case (C_3) and case (C_4), we can verify the validity of estimate (3.20) by applying similar arguments as in case (C_1) and case (C_2). Therefore, the comparison principle and estimate (3.20) show

$$\bar{u}_i(x, t) > u_i(x, t), \quad \forall x \in \Omega,\ t > 0,\ i = 2, 3$$

provided that $u_{i0}(x) < \ell_i(1 + \psi(x))$ $(2 \leq i \leq 3)$ in Ω for such ℓ_i fixed above. It implies that (u_2, u_3) exists globally.

Global existence results in cases (C_5). Note that $p_{22} \leq 1$ and $p_{33} \leq 1$ in this case. When $(1 - p_{22})(1 - p_{33}) \geq p_{23} p_{32}$, it has been shown in the proof Theorem 3 that (u_2, u_3) exists globally for any $u_{i0}(x) \geq 0, \not\equiv 0$ $(2 \leq i \leq 3)$, and when $(1 - p_{22})(1 - p_{33}) < p_{23} p_{32}$, it has been shown in the proof Theorem 4 that (u_2, u_3) exists globally for small $u_{i0}(x) \geq 0, \not\equiv 0$ $(2 \leq i \leq 3)$.

Blow-up results in case (C). Similar as in proof Theorem 2 (Blow-up results), from $u_{i0}(x) \geq \varphi(x)$ in Ω we deduce that

$$u_i(x, t) \geq \mathrm{e}^{-\lambda d_i t} \varphi(x), \quad (x, t) \in \bar{\Omega} \times [0, T),\ 1 \leq i \leq 3$$

with the maximal existence time T of (u_1, u_2, u_3). When $p_{11} > 1$, we obtain

$$\begin{cases} u_{1t} - d_1\Delta u_1 \geq \exp\{-\lambda(d_2 p_{12} + d_3 p_{13})t\}\varphi^{p_{12}+p_{13}} u^{p_{11}}, & x \in \Omega,\ 0 < t < T, \\ u_1(x,t) = 0, & x \in \partial\Omega,\ 0 < t < T, \\ u_{10}(x) \geq \varphi(x) \geq 0, \not\equiv 0, & x \in \Omega. \end{cases}$$

In view of assertion (i) of Lemma 5 and the comparison principle for single equation, we find that u_1 blows up in a finite time T for large $u_{10}(x)$, and so does (u_1, u_2, u_3). When $p_{22} > 1$ or $p_{33} > 1$, by applying similar arguments to u_2 or u_3, we also find (u_1, u_2, u_3) blows up in a finite time T for large $(u_{10}(x),\ u_{20}(x),\ u_{30}(x))$. Therefore, we conclude this theorem. □

3.2.2 *When $P = P_2$*

By proceeding our discussion as in Sect. 3.2.1, we see that behaviors of solution for problem (1.1) with $P = P_2$ and $p_{12}p_{21} = 0$ are similar to that for $P = P_1$. So, from now on we suppose, unless otherwise noted, that $p_{12}p_{21} \neq 0$. By $P = P_2$ it follows that solution (u_1, u_2, u_3) of problem (1.1) is bound to satisfy

$$\begin{cases} u_{3t} - d_3\Delta u_3 = u_3^{p_{33}}, & x \in \Omega,\ t > 0, \\ u_3(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u_3(x,0) = u_{30}(x) \geq 0, & x \in \Omega. \end{cases} \tag{3.21}$$

Theorem 6 *Let $P = P_2$ and $p_{33} \neq 1$. If all the principal minor determinants of A are non-negative, then all solutions of (1.1) exist globally.*

Proof Let (u_1, u_2, u_3) be any solution of problem (1.1). From the assumptions, $p_{33} < 1$ follows. Hence, by virtue of the comparison principle for a single equation we have that for any $u_{30}(x) \geq 0$,

$$u_3(x,t) < \big[(1 - p_{33})t + a\big]^{1/(1-p_{33})} =: C(t), \quad \forall x \in \bar{\Omega},\ t > 0, \tag{3.22}$$

where $a = 1 + \max_{\bar{\Omega}} u_{30}(x)$. Then from problem (1.1) we find that

$$\begin{cases} u_{it} \leq d_i\Delta u_i + (C(t))^{p_{i3}} u_1^{p_{i1}} u_2^{p_{i2}}, & x \in \Omega,\ t > 0, \\ u_i(x,t) = 0, & x \in \partial\Omega,\ t \in (0,T), \\ u_i(x,0) = u_{i0}(x) \geq 0, & x \in \Omega,\ i = 1, 2. \end{cases} \tag{3.23}$$

As all the principal minor determinants of A are non-negative and $P = P_2$, we have

$$p_{11} \leq 1,\ p_{22} \leq 1 \quad \text{and} \quad (1 - p_{11})(1 - p_{22}) \geq p_{12}p_{21}.$$

Define

$$\bar{u}_i = w^{\ell_i}, \quad x \in \bar{\Omega},\ t > 0,\ i = 1, 2,$$

where positive constants $0 < \ell_i < 1$ satisfy

$$(1 - p_{11})/p_{12} \geq \ell_2/\ell_1, \qquad (1 - p_{22})/p_{21} \geq \ell_1/\ell_2$$

when $p_{12}p_{21} \neq 0$ (the case $p_{12}p_{21} = 0$ is trivial), and $w(x,t)$ is the unique nonnegative solution to problem (G) with

$$d = \max_{1\le i\le 2} d_i, \qquad k(t) = \max_{1\le i\le 2} \frac{d}{d_i \ell_i}\big(C(t)\big)^{p_{i3}}, \qquad \theta = 1 + \sum_{1\le i\le 2}\Big(\max_{x\in\bar\Omega} u_{i0}(x)\Big)^{1/\ell_i}.$$

The remaining part is exactly the same as that in the proof of Theorem 3, and we will arrive at the conclusion: (u_1,u_2) exists globally. Therefore, by remembering that u_3 exists globally, (u_1,u_2,u_3) exists globally. □

Theorem 7 *Let $P = P_2$ and $p_{33} \neq 1$. If conditions of Theorem 6 do not hold, then solution of (1.1) exists globally for small initial data, and blows up in a finite time for large initial data.*

Proof Let (u_1,u_2,u_3) be any solution of problem (1.1). When $p_{33}\neq 1$, from (3.2) and (3.3), it can be deduced by the comparison principle that

$$u_3(x,t) < \ell_3\big(1+\psi(x)\big) \le \ell_3(1+L) =: C, \quad \forall x\in\Omega,\ t>0 \tag{3.24}$$

provided that

$$\ell_3 = (1+L)^{p_{33}/(1-p_{33})}, \quad u_{30}(x) < \ell_3\big(1+\psi(x)\big), \quad \forall x\in\Omega. \tag{3.25}$$

By $P=P_2$ it is easy to obtain

$$|A| = (1-p_{33})\begin{vmatrix} 1-p_{11} & -p_{12} \\ -p_{21} & 1-p_{22}\end{vmatrix}.$$

We are going to carry out our discussion according to two cases.

Case 1. $p_{33}<1$, $\max\{p_{11},\ p_{22}\}\le 1$ *and* $(1-p_{11})(1-p_{22}) < p_{12}p_{21}$. It follows that $|A|<0$. As above, $u_3(x,t)$ exists globally for any $u_{30}(x)$.

Blow-up part. we go all the analysis procedure of Case 2 in the proof of Theorem 2 (Blow-up results), and the desired conclusion in this theorem is derived. The only thing which need us do is to change the role of u_i and p_{ij} with each other.

Global existence part. Let $\psi(x)$ and L be as in (3.2) and (3.3), respectively. For $1\le i\le 2$, put

$$\bar u_i(x,t) = \ell_i\big(1+\psi(x)\big), \quad (x,t)\in\Omega\times[0,\infty),$$

where constants ℓ_i are to be fixed later. By computation, it is not difficult to see from (3.24) that

$$\bar u_{it} \ge d_i\Delta \bar u_i + \bar u_1^{p_{i1}}\bar u_2^{p_{i2}}C^{p_{i3}}, \quad x\in\Omega,\ t>0,\ i=1,2$$

hold, provided that

$$(1-p_{11})\ln\ell_1 - p_{12}\ln\ell_2 \ge p_{13}\ln C + (p_{11}+p_{12}+p_{13})\ln(1+L), \tag{3.26}$$

$$-p_{21}\ln\ell_1 + (1-p_{22})\ln\ell_2 \ge p_{23}\ln C + (p_{21}+p_{22}+p_{23})\ln(1+L). \tag{3.27}$$

As $p_{11}\le 1$, $p_{22}\le 1$ and $(1-p_{11})(1-p_{22}) < p_{12}p_{21}$, by directly solving we make sure the existence of positive constants ℓ_1 and ℓ_2 such that (3.26) and (3.27) hold. Obviously,

$$\bar u_i(x,t) = \ell_i > 0 = u_i(x,t), \quad \forall (x,t)\in\partial\Omega\times(0,\infty),\ 1\le i\le 2.$$

Therefore, if

$$u_{i0}(x) < \bar{u}_i(x,0) = \ell_i\big(1+\psi(x)\big), \quad \forall x \in \bar{\Omega},\ 1 \le i \le 2,$$

then Proposition 1 asserts that

$$u_i(x,t) < \bar{u}_i(x,t), \quad \forall (x,t) \in \bar{\Omega} \times [0,\infty),\ 1 \le i \le 2,$$

which joining with (3.24) asserts that (u_1, u_2, u_3) exists globally.

Case 2. At least one of the first order principal minor determinants of A is negative. In the following we divide our discussion into two cases.

(A) $p_{33} > 1$. By (3.21), it is well known to us all that $u_3(x,t)$ blows up in a finite time for suitable large $u_{30}(x)$, and from (3.24), $u_3(x,t)$ exists globally with an uniform bounds for all $u_{30}(x)$ satisfying (3.25). Therefore, solutions (u_1, u_2, u_3) of problem (1.1) blow up in a finite time for large $u_{i0}(x)$ $(1 \le i \le 3)$.

In the following we only need to analysis behaviors of (u_1, u_2, u_3) for small $u_{i0}(x)$ $(1 \le i \le 3)$. In view of (3.24), (3.23) holds with $C(t) = C$. From assumptions of this theorem and $p_{12}p_{21} \ne 0$, one of the following must happen:

(A_1) $\max\{p_{11},\ p_{22}\} \le 1$ and $(1-p_{11})(1-p_{22}) \ge p_{12}p_{21}$.

(A_2) $\max\{p_{11},\ p_{22}\} \le 1$ and $(1-p_{11})(1-p_{22}) < p_{12}p_{21}$.

(A_3) $\max\{p_{11},\ p_{22}\} > 1$ ($p_{12} \ge 0,\ p_{21} \ge 0$ and $p_{12}p_{21} \ne 0$ imply $p_{12} > 0$ and $p_{21} > 0$).

Therefore, following proofs of Theorem 3, Theorem 4 and Theorem 5, by Proposition 1 we know that solution (u_1, u_2) to problem (3.23) exists globally for small initial data, and so does (u_1, u_2, u_3).

(B) $p_{33} < 1$ and $\max\{p_{11},\ p_{22}\} > 1$. As $u_3(x,t)$ has global bounds for any $u_{30}(x)$ satisfying (3.25). Hence, blow-up of (u_1, u_2, u_3) depends on that of (u_1, u_2). As above, from (3.23) with $M(T) = C$ we conclude that solution (u_1, u_2) exists globally for small initial data, and thus, (u_1, u_2, u_3) exists globally for small $u_{i0}(x)$ $(1 \le i \le 3)$.

In the case of large $u_{i0}(x)$ $(1 \le i \le 3)$, we may think that $u_{i0}(x) \ge \varphi(x)$ $(i = 2, 3)$. In view of $\max\{p_{11},\ p_{22}\} > 1$, $p_{12} > 0$ and $p_{21} > 0$, we may suppose without loss of generality that $p_{11} > 1$. Then, we follow analysis exactly the same as Case 3 in the proof of Theorem 2 (Blow-up results), and in the final we will certainly arrive at the conclusion that $u_1(x,t)$ blows up in a finite time for suitable large $u_{10}(x)$. Therefore, (u_1, u_2, u_3) blows up in a finite time for large $u_{i0}(x)$ $(1 \le i \le 3)$. We complete this theorem. □

Theorem 8 *Let $P = P_2$ and $p_{33} = 1$. If all the principal minor determinants of A are nonnegative, then all solutions of* (1.1) *exist globally.*

Proof Let (u_1, u_2, u_3) be any solution of problem (1.1). From $p_{33} = 1$ and (3.21), by virtue of the comparison principle for a single equation we have that for any $u_{30}(x) \ge 0$,

$$u_3(x,t) < a\mathrm{e}^t =: C(t), \quad a = 1 + \max_{\bar{\Omega}} u_{30}(x),\ \forall x \in \bar{\Omega},\ t > 0,$$

then (3.23) follows. Going along with arguments in the proof of Theorem 6 we arrive at conclusions in this theorem. □

Theorem 9 *Suppose $P = P_2$ and $p_{33} = 1$. Let λ be the first eigenvalue of problem (1.3). Assume $p_{11} \le 1$, $p_{22} \le 1$ and $(1 - p_{11})(1 - p_{22}) < p_{12}p_{21}$. Define*

$$d = \min\{d_1, d_2\}, \qquad \beta = (1 - d_3\lambda)\min\{p_{13}, p_{23}\},$$

$$\sigma = \begin{cases} \dfrac{p_{12}p_{21} - (1 - p_{11})(1 - p_{22})}{1 + p_{12} - p_{22}}, & \text{if } \dfrac{1 + p_{21} - p_{11}}{1 + p_{12} - p_{22}} \ge 1, \\ \dfrac{p_{12}p_{21} - (1 - p_{11})(1 - p_{22})}{1 + p_{21} - p_{11}}, & \text{if } \dfrac{1 + p_{21} - p_{11}}{1 + p_{12} - p_{22}} < 1. \end{cases}$$

If $\beta/(\sigma d\lambda) > 1$, then all solutions of (1.1) blow up in a finite time for any $u_{i0}(x) \ge 0, \not\equiv 0$ $(1 \le i \le 3)$, and meanwhile, if $\beta/(\sigma d\lambda) \le 1$, then solution of (1.1) blows up in a finite time for suitable large $u_{i0}(x)$ $(1 \le i \le 3)$, and exists globally for small $u_{i0}(x)$ $(1 \le i \le 3)$. Especially, when $\sigma d\lambda \min\{1 + p_{12} - p_{22}, 1 + p_{21} - p_{11}\} > (1 - d_3\lambda)\max\{p_{13}(1 - p_{22}) + p_{12}p_{23}, p_{23}(1 - p_{11}) + p_{21}p_{13}\}$, then solution of (1.1) is global bounded for suitable small $u_{i0}(x)$ $(1 \le i \le 3)$.

Proof The proof is similar as that of Theorem 5, here we omit it. □

Theorem 10 *Assume $P = P_2$, $p_{33} = 1$ and $\max\{p_{11}, p_{22}\} > 1$. Then the following conclusions hold.*

(1) *If $p_{11} > 1$, then solution of (1.1) blows up in a finite time for any $u_{i0}(x) \ge 0, \not\equiv 0$ $(1 \le i \le 3)$ when $p_{13}(1 - d_3\lambda) - p_{12}d_2\lambda > (p_{11} - 1)d_1\lambda$ or when $p_{13}(1 - d_3\lambda) - p_{12}d_2\lambda = (p_{11} - 1)d_1\lambda > (1 + p_{12} + p_{13} - p_{11})d_1\lambda$; solution of (1.1) blows up in a finite time for suitable large $u_{10}(x)$ when $p_{13}(1 - d_3\lambda) - p_{12}d_2\lambda < (p_{11} - 1)d_1\lambda$ or when $p_{13}(1 - d_3\lambda) - p_{12}d_2\lambda = (p_{11} - 1)d_1\lambda \le (1 + p_{12} + p_{13} - p_{11})d_1\lambda$.*

(2) *If $p_{22} > 1$, then solution of (1.1) blow up in a finite time for any $u_{i0}(x) \ge 0, \not\equiv 0$ $(1 \le i \le 3)$ when $p_{23}(1 - d_3\lambda) - p_{21}d_1\lambda > (p_{22} - 1)d_2\lambda$ or $p_{23}(1 - d_3\lambda) - p_{21}d_1\lambda = (p_{22} - 1)d_2\lambda > (1 + p_{21} + p_{23} - p_{22})d_2\lambda$; solution of (1.1) blows up in a finite time for suitable large $u_{20}(x)$ when $p_{23}(1 - d_3\lambda) - p_{21}d_1\lambda < (p_{22} - 1)d_2\lambda$ or when $p_{23}(1 - d_3\lambda) - p_{21}d_1\lambda = (p_{22} - 1)d_2\lambda \le (1 + p_{21} + p_{23} - p_{22})d_2\lambda$.*

(3) *Suppose that conditions in the above* (1) *and* (2) *do not hold.* Case (a_1) *If $\lambda d_3 \ge 1$ and $p_{12}p_{21} > 0$, then solution of (1.1) exists globally with uniform bounds for suitable small $u_{i0}(x)$ $(1 \le i \le 3)$.* Case (a_2) *If $p_{12}p_{21} = 0$, then solution of (1.1) exists globally for small $u_{i0}(x)$ $(1 \le i \le 3)$.* Case (a_3) *If $\lambda d_3 < 1$ and $p_{12}p_{21} > 0$, we further assume that* $\min\{p_{11}, p_{22}\} \ge 1$, *$p_{13}(1 - d_3\lambda) - p_{12}d_2\lambda < (p_{11} - 1)d_1\lambda$ and $p_{23}(1 - d_3\lambda) - p_{21}d_1\lambda < (p_{22} - 1)d_2\lambda$, then solution of (1.1) exists globally for suitable small $u_{i0}(x)$ $(1 \le i \le 3)$.*

Proof Let (u_1, u_2, u_3) be any solution of problem (1.1), then $u_i(x, t) > 0$ for all $(x, t) \in \Omega \times (0, T)$, where $T > 0$ is the maximal existence time of (u_1, u_2, u_3).

Blow-up results. We need only deduce assertion (1), since assertion (2) can be obtained analogously. Applying arguments as in the proof of Theorem 5, we find that $u_{i0}(x) \ge \varepsilon\varphi(x)$ $(1 \le i \le 3)$ for some $\varepsilon > 0$. Thus,

$$u_3(x, t) \ge \varepsilon \exp\big\{(1 - \lambda d_3)t\big\}\varphi(x), \qquad u_2(x, t) \ge \varepsilon \exp\{-\lambda d_2 t\}\varphi(x), \quad \forall x \in \Omega,\ t > 0.$$

It follows that

$$\begin{cases} u_{1t} - d_1\Delta u_1 \ge \varepsilon^{p_{12}+p_{13}}\varphi^{p_{12}+p_{13}}\exp\{[-\lambda d_2 p_{12} + (1 - \lambda d_3)p_{13}]t\}u_1^{p_{11}}, & x \in \Omega,\ t > 0, \\ u_1(x, t) = 0, & x \in \partial\Omega,\ t > 0, \\ u_1(x, 0) = u_{10}(x), & x \in \Omega. \end{cases}$$

Therefore, when $p_{11} > 1$, Lemma 5 has proven that $u_1(x,t)$ satisfies requirements in assertion (1), and thus, assertion (1) are available for (u_1, u_2, u_3).

Global existence. Suppose $u_{30}(x) \le \varphi(x)$ in Ω, it follows that

$$u_3(x,t) \le \exp\big\{(1-\lambda d_3)t\big\}\varphi(x) \le \exp\big\{(1-d_3\lambda)t\big\}, \quad \forall x \in \Omega,\ t > 0. \tag{3.28}$$

Remember that $\max\{p_{11},\ p_{22}\} > 1$, and we may think without loss of generality that $p_{11} > 1$ in the following.

Case (a_1): $\lambda d_3 \ge 1$ and $p_{12}p_{21} > 0$. By (3.28) we have $u_3(x,t) \le 1$, and hence,

$$\begin{cases} u_{it} - d_i\Delta u_i \le u_1^{p_{i1}} u_2^{p_{i2}}, & x \in \Omega,\ t > 0,\\ u_i(x,t) = 0,\ x \in \partial\Omega;\ \ u_i(x,0) = u_{i0}(x), & x \in \Omega,\ 1 \le i \le 2. \end{cases}$$

Let

$$\bar u_1 = \ell\big(1 + \psi(x)\big), \qquad \bar u_2 = \big(1 + \psi(x)\big), \quad x \in \Omega,\ t > 0,$$

where positive constant $\ell < 1$ satisfies

$$-\ln \ell \ge \max\left\{\frac{(p_{11} + p_{12})\ln(1+L)}{p_{11} - 1},\ \frac{(p_{21} + p_{22})\ln(1+L)}{p_{21}}\right\},$$

and the positive constant L and the nonnegative function $\psi(x)$ are defined by (3.3) and (3.2), respectively. It is not difficult to check that

$$\begin{cases} \bar u_{it} - d_i\Delta \bar u_i \ge \bar u_1^{p_{i1}} \bar u_2^{p_{i2}}, & x \in \Omega,\ t > 0,\\ \bar u_1(x,t) = \ell > 0,\ \bar u_2(x,t) = 1 > 0, & x \in \partial\Omega,\ t > 0,\ 1 \le i \le 2. \end{cases}$$

By the comparison principle, we achieve

$$\bar u_i(x,t) > u_i(x,t), \quad \forall x \in \Omega,\ t > 0,\ 1 \le i \le 2,$$

provided that $u_{10}(x) < \ell(1 + \psi(x))$ and $u_{20}(x) < 1 + \psi(x)$ in Ω, which illustrates that u_1 and u_2 exist globally with uniform bounds. Recall that $u_3 \le 1$. Consequently, (u_1, u_2, u_3) has an uniform bounds for suitable small initial data.

Case (a_2): $p_{12}p_{21} = 0$. When $p_{12} = 0$, from the equation of u_1 and $p_{11} > 1$, by the comparison principle for the single equation we find that $u_1(x,t)$ exists globally for small $u_{10}(x)$. Then, by the equation of u_2 it follows that $u_2(x,t)$ exists globally for small $u_{20}(x)$. Hence, (u_1, u_2, u_3) exists globally for suitable small $(u_{10},\ u_{20},\ u_{30})$. When $p_{21} = 0$, global existence of (u_1, u_2, u_3) for suitable small $(u_{10},\ u_{20},\ u_{30})$ can be accomplished similarly.

Case (a_3): $\lambda d_3 < 1$ and $p_{12}p_{21} > 0$. From (3.28) it follows

$$\begin{cases} u_{it} - d_i\Delta u_i \le u_1^{p_{i1}} u_2^{p_{i2}} \exp\{p_{i3}(1-\lambda d_3)t\}, & x \in \Omega,\ t > 0,\\ u_i(x,t) = 0,\ x \in \partial\Omega,\ t > 0;\ \ u_i(x,0) = u_{i0}(x), & x \in \Omega,\ 1 \le i \le 2. \end{cases}$$

As $p_{23}(1-d_3\lambda) - p_{21}d_1\lambda < (p_{22}-1)d_2\lambda$ and $p_{13}(1-d_3\lambda) - p_{12}d_2\lambda < (p_{11}-1)d_1\lambda$, we can choose a domain $\Omega \subset\subset \Omega_1$ such that the first eigenvalue λ_1 to the problem (1.3) in Ω_1 satisfying

$$\begin{aligned} p_{13}(1-d_3\lambda) &< \big[(p_{11}-1)d_1 + p_{12}d_2\big]\lambda_1,\\ p_{23}(1-d_3\lambda) &< \big[(p_{22}-1)d_2 + p_{21}d_1\big]\lambda_1, \end{aligned} \tag{3.29}$$

and $\varphi_1(x)$ the corresponding function satisfying $\max_{\bar{\Omega}_1}\varphi_1(x)=1$. It is well known that $\varphi_1(x)>0$ in Ω_1, and there exists a constant $\varepsilon>0$, such that $\varphi_1(x)\geq\varepsilon$ for all $x\in\bar{\Omega}$. With the help of (3.29), there are positive constants $0<\theta_1<d_1\lambda_1$ and $0<\theta_2<d_2\lambda_1$, such that

$$(p_{11}-1)\theta_1+p_{12}\theta_2\geq p_{13}(1-d_3\lambda),\qquad p_{21}\theta_2+(p_{22}-1)\theta_2\geq p_{23}(1-d_3\lambda). \tag{3.30}$$

For such fixed such θ_i, take positive constant δ such that

$$\ln\delta=\min\left\{\frac{\ln(d_1\lambda_1-\theta_1)}{p_{11-1}},\ \frac{\ln(d_2\lambda_1-\theta_2)}{p_{21}}\right\}, \tag{3.31}$$

and define

$$\bar{u}_1=\delta\exp\{-\theta_1 t\}\varphi_1(x),\qquad \bar{u}_2=\exp\{-\theta_2 t\}\varphi_1(x),\quad x\in\Omega,\ t>0,$$

then direct computation combining with (3.30)–(3.31) gives that for $1\leq i\leq 2$,

$$\begin{cases}\bar{u}_{it}-d_i\Delta\bar{u}_i\geq\bar{u}_1^{p_{i1}}\bar{u}_2^{p_{i2}}\exp\{p_{i3}(1-\lambda d_3)t\}, & x\in\Omega,\ t>0,\\ \bar{u}_i(x,t)=\varepsilon>0, & x\in\partial\Omega,\ t>0.\end{cases}$$

Therefore,

$$\bar{u}_i(x,t)>u_i(x,t),\quad \forall x\in\Omega,\ t>0,\ 1\leq i\leq 2,$$

provided that $u_{10}(x)\leq\delta\varepsilon$ and $u_{20}(x)\leq\varepsilon$ in Ω. Consequently, (u_1,u_2,u_3) exists globally for suitable small initial data. We complete the proof. □

Remark 5 (1) When $d_1=d_2$, from the above proof one can find that there is no need to require $\min\{p_{11},\ p_{22}\}\geq 1$.

(2) All above results in this paper can be extended to problem (1.1) with m-components $(m\geq 4)$.

4 Further Discussion

In this section, we do some discussion about results between this paper and papers [16, 17]. In 2003, Li and his collaborator considered the homogeneous Dirichlet boundary value problem

$$u_{it}=c_iu_i^{\alpha_i}\left(\Delta u_i+\prod_{j=1}^{m}u_j^{p_{ij}}\right),\quad x\in\Omega,\ t>0 \tag{4.1}$$

and the homogeneous Dirichlet boundary value problem

$$u_{it}-\Delta u_i=\prod_{j=1}^{m}u_j^{p_{ij}},\quad x\in\Omega,\ t>0, \tag{4.2}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain, the constants $c_i>0$, $\alpha_i>0$, $p_{ij}\geq 0$ $(1\leq i,j\leq m)$. Under the assumption that $u_{i0}(x)$ $(1\leq i\leq m)$ are continuous, positive and bounded in Ω, for irreducible matrix $P=(p_{ij})$, their main results are read as follows:

Theorem B ([16, Theorem 1.1]) *If $I - P$ is an M-matrix, then all solutions of* (4.2) *are global, and if $I - P$ is not an M-matrix, then there exist both nontrivial global solutions and nonglobal solutions of* (4.2).

Theorem C ([17, Theorem 1.1]) (*i*) *When $I - P$ is a nonsingular M-matrix, all solutions of* (4.1) *exist globally* (*uniformly bounded in time*).

(*ii*) *When $I - P$ is not an M-matrix, there exist both nontrivial global solutions and nonglobal solutions of* (4.1).

(*iii*) *When $I - P$ is a singular M-matrix, assume that $\underline{\ell} = (\underline{\ell}_1, \dots, \underline{\ell}_m)$ is a solution of $(I - P)\ell = 0$ with $\underline{\ell}_i > 0$ $(1 \le i \le m)$ and $\min_{1\le i\le m} \underline{\ell}_i = 1$, and $\bar{\ell} = (\bar{\ell}_1, \dots, \bar{\ell}_m)$ is also a solution of $(I - P)\ell = 0$ with $\bar{\ell}_i > 0$ $(1 \le i \le m)$ and $\max_{1\le i\le m} \bar{\ell}_i = 1$. Let $\underline{\lambda} = \min_{1\le i\le m} 1/\underline{\ell}_i$ and $\bar{\lambda} = \max_{1\le i\le m} 1/\bar{\ell}_i$. Then there exist no global nontrivial solutions of* (4.1) *for $\lambda < \underline{\lambda}$, and all solutions of* (4.1) *exist globally* (*uniformly bounded in time*) *for $\lambda > \bar{\lambda}$, where λ is defined by* (1.3).

Elementary approaches adopted in [16, 17] and our paper are exactly the same (comparison principle and matrix theory). Properties of M-matrix are used in the former, and the latter cares only about whether all of the principal minor determinants are nonnegative. Just as stated in Remark 5(2), all results in our paper can be extended to m-system ($m \ge 4$). Papers [16, 17] only focus on irreducible matrix $P = (p_{ij})$, and we investigate both reducible matrix $P = (p_{ij})$ and irreducible matrix $P = (p_{ij})$. Authors of [16] assumed all diffusion coefficients are 1 or equal, and our methods are used for different coefficients. Results of paper [17] do not cover the case $\alpha_i = 0$ for all $1 \le i \le m$, which just corresponds to our problem (1.1).

Now we will directly compare results of three papers. Notice that the former two only discuss irreducible P which was concerned in Sect. 3.1 of our paper. Let $P = (p_{ij})_{3\times 3}$ and $A = I - P$ for convenience. We rearrange assumptions about A. By [3, p. 134, (A_1)], [3, p. 149, (A_1)] and [3, p. 156, (1) and (4)], we have

$$A \text{ is an } M\text{-matrix} \iff \text{all the principal minors of } A \text{ are nonnegative.} \tag{4.3}$$

$$A \text{ is a nonsingular } M\text{-matrix} \iff \text{all the principal minors of } A \text{ are positive,} \tag{4.4}$$

$$A \text{ is a singular, irreducible } M\text{-matrix} \Longrightarrow |A| = 0, \text{ and each principal submatrix of } A \text{ other than } A \text{ itself is a nonsingular } M\text{-matrix.} \tag{4.5}$$

Thanks to (4.3), by the definition it can be deduced that if A is not an M-matrix, then A must be

$$\text{at least one of principal minor determinants of } A \text{ is negative.} \tag{4.6}$$

Therefore, from above it is easily seen that all results in Theorem B (that is the main result and also the only result in [16]) are included in our Theorem 1 and Theorem 2 for the special case $d_i = 1$ $(1 \le i \le m)$. In other words, we generalize results of [16] for different diffusion coefficients.

Now it comes to paper [17] and this article. When A is an irreducible M-matrix, with the help of (4.3)–(4.5), one can find that such matrix A certainly meets requirements of our Theorem 1. In this time, conclusions of Theorem C(i) cohere with that of our Theorem 1, and conclusions of Theorem C(iii) differ with that of Theorem 1. However, some matrix A in our Theorem 1, does not satisfy conditions of Theorem C(i) and conditions of Theorem C(iii).

When A is an irreducible but not M-matrix, we find from (4.6) that A satisfies conditions of our Theorem 2, and conclusions in these two papers are the same.

In fact, such difference has taken place in their single equation, and we take the critical case as an example. For the problem

$$\begin{cases} u_t = u^\alpha(\Delta u + u), & x \in \Omega,\ t > 0, \\ u(x,t) = 0,\ x \in \partial\Omega,\ t > 0; & u(x,0) = u_0(x) > 0,\ x \in \Omega \end{cases} \tag{4.7}$$

with constant $\alpha > 0$, it is well known that whether solution exists globally or blows up in a finite time depends on the first eigenvalue λ of $-\Delta$ in Ω with null Dirichlet boundary condition (refer to [5, 7, 8, 14] for example; see also [17]). More precisely, there exist global nontrivial solutions for problem (4.7) if and only if $\lambda > 1$. On the other hand, [21, p. 8, Proposition 3] has proved that all solutions of problem

$$\begin{cases} u_t = \Delta u + u, & x \in \Omega,\ t > 0, \\ u(x,t) = 0,\ x \in \partial\Omega,\ t > 0; & u(x,0) = u_0(x) \ge 0, \not\equiv 0,\ x \in \Omega \end{cases} \tag{4.8}$$

are global. Hence, any solution of problem (4.8) exists globally no matter how much is the value of λ.

The above analysis clarifies that for problem (1.1) and problem (4.1), assumptions of theorems and the corresponding behaviors of solutions are different in some cases, which in return illustrate that the two problems are different and can not discuss them as the same one problem.

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