Acta Appl Math (2017) 147:187–195
DOI 10.1007/s10440-016-0075-0

Global Solutions to a Nonlocal Fisher-KPP Type Problem

Shen Bian[1]

Received: 5 December 2015 / Accepted: 11 September 2016 / Published online: 21 September 2016

Abstract We consider a nonlocal Fisher-KPP reaction-diffusion model arising from population dynamics, consisting of a certain type reaction term $u^\alpha(1-\int_\Omega u^\beta dx)$, where Ω is a bounded domain in $\mathbb{R}^n (n \geq 1)$. The energy method is applied to prove the global existence of the solutions and the results show that the long time behavior of solutions heavily depends on the choice of α, β. More precisely, for $1 \leq \alpha < 1+(1-2/p)\beta$, where p is the exponent from the Sobolev inequality, the problem has a unique global solution. Particularly, in the case of $n \geq 3$ and $\beta = 1$, $\alpha < 1+2/n$ is the known Fujita exponent (Fujita in J. Fac. Sci., Univ. Tokyo, Sect. 1A, Math. 13:109–124, 1966). Comparing to Fujita equation (Fujita in J. Fac. Sci., Univ. Tokyo, Sect. 1A, Math. 13:109–124, 1966), this paper will give an opposite result to our nonlocal problem.

Keywords Fisher-KPP equation · Reaction-diffusion · Global existence

1 Introduction

In this paper, we study the following nonlocal initial boundary value problem,

$$u_t - \Delta u = u^\alpha\left(1 - \int_\Omega u^\beta(x,t)dx\right), \quad x \in \Omega, t > 0, \tag{1a}$$

$$\nabla u \cdot \nu = 0, \quad x \in \partial\Omega, \tag{1b}$$

$$u(x,0) = u_0(x) \geq 0, \quad x \in \Omega, \tag{1c}$$

where u is the density of population, Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 1$, $\alpha, \beta \geq 1$ and ν is the outer unit normal vector on $\partial\Omega$. Without loss of generality, throughout this paper we assume $|\Omega| = 1$ (otherwise, rescale the problem by $|\Omega|$).

Partially supported by National Science Foundation of China (Grant No. 11501025) and the Fundamental Research Funds for the Central Universities (Grant No. ZY1528).

S. Bian
bianshen66@163.com

[1] Beijing University of Chemical Technology, 100029, Beijing, China

This kind of model is developed to describe the population dynamics [6, 9] with the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u), \tag{2}$$

where u is the population density, $\frac{\partial^2 u}{\partial x^2}$ describes the random displacement of the individuals of the population, the function $F(u)$ is considered as the rate of the reproduction of the population. Its usual form is the local version

$$F(u) = u^\alpha(1-u) - \gamma u, \tag{3}$$

the reaction term consists of the reproduction term which is represented by u to a power u^α and $(1-u)$ which stands for the local consumption of available resources, the last term $-\gamma u$ is the mortality of the population.

The nonlocal version is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^\alpha\left(1 - \int_{-\infty}^{\infty} \phi(x-y)u(y,t)dy\right) - \gamma u,$$

where $\int_{-\infty}^{\infty} \phi(y)dy = 1$. $\phi(x-y)$ represents the probability density function that describes the distribution of individuals around their average positions. Noting that if ϕ is a Dirac δ function, the nonlocal problem reduces to the local version (3).

In this paper, we will study the problem with nonlocal version reaction term. Nonlocal type reaction terms can describe also Darwinian evolution of a structured population density or the behaviors of cancer cells with therapy [5, 9]. There are some already known results on the reaction-diffusion equation with a nonlocal term. In [1], the authors considered the equation with reaction term

$$F(t,u) = f(t,u) + \int_\Omega g(t,u)dx, \quad t > 0, \tag{4}$$

where $f = e^u$ and $g = ke^u$ $(k > 0)$, for which the above problem represents an ignition model for a compressible reactive gas, and they proved the finite time blow-up of solutions.

Later, a power-like nonlinearity was investigated by Wang and Wang [11], i.e.

$$F(t,u) = \int_\Omega u^p(t,y)dy - ku^q(t,x), \quad t > 0,$$

with $p, q > 1$, and they proved that solutions blow up in finite time for some large initial data.

Moreover, [5] studied the closest models to the ones we are focusing in this work

$$F(u) = u^p - \frac{1}{|\Omega|}\int_\Omega u^p(t,y)dy,$$

this typical structure has mass conservation, and the authors showed that if $p > n/(n-2)$, the solutions will blow up in finite time with some large initial data and exist globally for some small initial data, while for $1 < p < n/(n-2)$, the solution exists globally for any initial value.

Recently, in [2], we considered the case $\beta = 1$ in (1a)–(1c), firstly the decay estimates of mass $\int_\Omega u(t,x)dx$ was presented and then using the decay properties we proved the global existence of solutions.

Compared to [5] and [2], there is no mass conservation or mass decay in our work, thus we need to explore other conditions for global existence of solutions. Moreover, if $\Omega = \mathbb{R}^n$ and $(1 - \int_{\mathbb{R}^n} u^\beta dx)$ remains positive, (1a)–(1c) has similar structure to Fujita equation [7] for which the problem has no global solution for $1 \le \alpha < 1 + 2/n$ $(n \ge 3)$. Therefore, we guess that our problem in bounded domain might also have no global solution for $1 \le \alpha < 1+2/n$, $\beta = 1$. However, we will give an opposite result in the nonlocal case.

Our main result can summarised as follows

Theorem 1 *Let $n \ge 1$. Assume u_0 is nonnegative and $u_0 \in L^k(\Omega)$ for any $1 < k < \infty$. If α satisfies*

$$1 \le \alpha < 1 + (1 - 2/p)\beta,$$

where $\beta \ge 1$ and p satisfies

$$p = \begin{cases} \frac{2n}{n-2}, & n \ge 3, \\ 2 < p < \infty, & n = 2, \\ \infty, & n = 1, \end{cases} \tag{5}$$

then problem (1a)–(1c) has a unique nonnegative classical solution.

This paper is mainly devoted to the proof of Theorem 1.

2 Global Existence of the Classical Solution

This part mainly focuses on the global existence of the classical solution to (1a)–(1c). We will use the following ODE inequality [3, 4] through this section.

Lemma 2 *Assume $y(t) \ge 0$ is a C^1 function for $t > 0$ satisfying*

$$y'(t) \le \eta - \gamma y(t)^a$$

for $a > 1$, $\eta > 0$, $\gamma > 0$. Then $y(t)$ has the following hyper-contractive property

$$y(t) \le (\eta/\gamma)^{1/a} + \left[\frac{1}{\gamma(a-1)t}\right]^{\frac{1}{a-1}} \quad \text{for any } t > 0. \tag{6}$$

Furthermore, if $y(0)$ is bounded, then

$$y(t) \le \max\big(y(0), (\eta/\gamma)^{1/a}\big). \tag{7}$$

The proof of global existence depends on *a priori* estimates in the following Proposition 3 and then we will use the compactness arguments to make the proof rigorous.

Proposition 3 *Let $n \ge 1$, p is defined as in (5). Assume $u_0 \in L^k(\Omega)$ for any $1 < k < \infty$. If α satisfies*

$$1 \le \alpha < 1 + (1 - 2/p)\beta,$$

where $\beta \geq 1$, *then any nonnegative solution of* (1*a*)–(1*c*) *satisfies that for any* $1 < k < \infty$ *and any* $t > 0$

$$\left\|u(\cdot,t)\right\|^k_{L^k(\Omega)} \leq C\big(k, \|u_0\|_{L^\beta(\Omega)}\big) + C\big(k, \|u_0\|_{L^\beta(\Omega)}\big)\, t^{-\frac{k-\beta}{\beta(1-2/p)}}, \tag{8}$$

and for any $0 < T < \infty$

$$\nabla u^{\frac{k}{2}} \in L^2\big(0,T;L^2(\Omega)\big). \tag{9}$$

Proof of Proposition 3 The proof will be given step by step. We firstly give the estimates on the boundedness of $\int_\Omega u^\beta dx$, then using the boundedness of L^β norm, we will prove that for all $k > \beta$, the L^k norm of the solution is bounded in time.

Step 1 (A priori estimates). Using ku^{k-1} $(k > 1)$ as a test function for (1a)–(1c) and integrating it by parts

$$\frac{d}{dt}\int_\Omega u^k dx + \frac{4(k-1)}{k}\int_\Omega \big|\nabla u^{\frac{k}{2}}\big|^2 dx + k\int_\Omega u^\beta dx \int_\Omega u^{k+\alpha-1}dx = k\int_\Omega u^{k+\alpha-1}dx. \tag{10}$$

Choosing $1 < k' < k+\alpha-1$, combining Hölder's inequality and the Sobolev embedding theorem one has

$$\begin{aligned}
&\int_\Omega u^{k+\alpha-1}dx \\
&\quad= \int_\Omega u^{\lambda\frac{k}{2}\frac{2(k+\alpha-1)}{k}} u^{(1-\lambda)\frac{k}{2}\frac{2(k+\alpha-1)}{k}}dx \\
&\quad\leq \big\|u^{\frac{k}{2}}\big\|^{\lambda\frac{2(k+\alpha-1)}{k}}_{L^p(\Omega)} \big\|u^{\frac{k}{2}}\big\|^{(1-\lambda)\frac{2(k+\alpha-1)}{k}}_{L^{\frac{2k'}{k}}(\Omega)} \\
&\quad\leq C(k)\Big(\big\|\nabla u^{\frac{k}{2}}\big\|^{\lambda}_{L^2(\Omega)}\big\|u^{\frac{k}{2}}\big\|^{1-\lambda}_{L^{\frac{2k'}{k}}(\Omega)} + \big\|u^{\frac{k}{2}}\big\|^{\lambda}_{L^2(\Omega)}\big\|u^{\frac{k}{2}}\big\|^{1-\lambda}_{L^{\frac{2k'}{k}}(\Omega)}\Big)^{\frac{2(k+\alpha-1)}{k}} \\
&\quad\leq C(k)\Big(\big\|\nabla u^{\frac{k}{2}}\big\|^{\frac{2\lambda(k+\alpha-1)}{k}}_{L^2(\Omega)}\big\|u^{\frac{k}{2}}\big\|^{\frac{2(1-\lambda)(k+\alpha-1)}{k}}_{L^{\frac{2k'}{k}}(\Omega)} + \big\|u^{\frac{k}{2}}\big\|^{\frac{2\lambda(k+\alpha-1)}{k}}_{L^2(\Omega)}\big\|u^{\frac{k}{2}}\big\|^{\frac{2(1-\lambda)(k+\alpha-1)}{k}}_{L^{\frac{2k'}{k}}(\Omega)}\Big),
\end{aligned} \tag{11}$$

where λ is the exponent from Hölder's inequality, i.e.

$$\lambda = \frac{\frac{k}{2k'} - \frac{k}{2(k+\alpha-1)}}{\frac{k}{2k'} - \frac{1}{p}} \in (0,1) \tag{12}$$

and p satisfies

$$\begin{cases} p = \frac{2n}{n-2}, & n \geq 3, \\ \frac{2(k+\alpha-1)}{k} < p < \infty, & n = 2, \\ p = \infty, & n = 1. \end{cases} \tag{13}$$

Now we will divide the analysis into three cases $n \geq 3$, $n = 2$ and $n = 1$.

For $n \geq 3$, $p = \frac{2n}{n-2}$ and then

$$\lambda = \frac{\frac{kn}{2k'} - \frac{kn}{2(k+\alpha-1)}}{\frac{kn}{2k'} + 1 - \frac{n}{2}} \in (0, 1) \tag{14}$$

with $k > \max\{\frac{(n-2)(\alpha-1)}{2}, 1\}$. Taking $k' > \frac{(\alpha-1)n}{2}$, simple computations arrive at

$$\frac{2\lambda(k+\alpha-1)}{k} = \frac{\frac{kn}{k'} + \frac{(\alpha-1)n}{k'} - n}{\frac{kn}{2k'} + 1 - \frac{n}{2}} < 2.$$

To sum up, for $k' > \max\{\frac{(\alpha-1)n}{2}, 1\}$, thanks to Young's inequality, from (11) one has

$$\begin{aligned} \int_\Omega u^{k+\alpha-1} dx \leq \frac{k-1}{k^2} \|\nabla u^{\frac{k}{2}}\|^2_{L^2(\Omega)} + C(k) \|u^{\frac{k}{2}}\|_{L^{\frac{2k'}{k}}(\Omega)}^{(1-\lambda)\frac{2(k+\alpha-1)}{k}\frac{1}{1-\frac{\lambda(k+\alpha-1)}{k}}} \\ + C(k) \|u^{\frac{k}{2}}\|_{L^2(\Omega)}^{\frac{2\lambda(k+\alpha-1)}{k}} \|u^{\frac{k}{2}}\|_{L^{\frac{2k'}{k}}(\Omega)}^{\frac{2(1-\lambda)(k+\alpha-1)}{k}}. \end{aligned} \tag{15}$$

Letting

$$r = (1-\lambda)\frac{2(k+\alpha-1)}{k} \frac{1}{1 - \frac{\lambda(k+\alpha-1)}{k}}, \tag{16}$$

together (10) with (15) yields

$$\begin{aligned} \frac{d}{dt} \int_\Omega u^k dx + k \int_\Omega u^\beta dx \int_\Omega u^{k+\alpha-1} dx + \frac{3(k-1)}{k} \|\nabla u^{\frac{k}{2}}\|^2_{L^2(\Omega)} \\ \leq C(k) \|u\|_{L^{k'}(\Omega)}^{\frac{kr}{2}} + C(k) \|u\|_{L^k(\Omega)}^{\lambda(k+\alpha-1)} \|u\|_{L^{k'}(\Omega)}^{(1-\lambda)(k+\alpha-1)}. \end{aligned} \tag{17}$$

On the other hand, taking k, k' such that

$$\beta < k' < k + \alpha - 1,$$

using Hölder's inequality we have

$$\|u\|_{L^{k'}(\Omega)} \leq \|u\|^\theta_{L^{k+\alpha-1}(\Omega)} \|u\|^{1-\theta}_{L^\beta(\Omega)}, \tag{18}$$

where

$$\theta = \frac{\frac{1}{\beta} - \frac{1}{k'}}{\frac{1}{\beta} - \frac{1}{k+\alpha-1}} \in (0, 1). \tag{19}$$

Hence

$$\|u\|_{L^{k'}(\Omega)}^{\frac{kr}{2}} \le \|u\|_{L^{k+\alpha-1}(\Omega)}^{\frac{kr\theta}{2}} \|u\|_{L^{\beta}(\Omega)}^{(1-\theta)\frac{kr}{2}}$$
$$\le \left(\|u\|_{L^{k+\alpha-1}(\Omega)}^{k+\alpha-1} \|u\|_{L^{\beta}(\Omega)}^{\beta}\right)^{\frac{kr\theta}{2(k+\alpha-1)}} \|u\|_{L^{\beta}(\Omega)}^{(1-\theta)\frac{kr}{2}-\frac{kr\theta\beta}{2(k+\alpha-1)}}. \tag{20}$$

Here we can choose k' such that

$$\left(1-\theta-\frac{\theta\beta}{k+\alpha-1}\right) \ge 0, \tag{21}$$

it equals to

$$\frac{k+\alpha-1+\beta}{k'} \le 2. \tag{22}$$

In addition, recalling the definition of r and θ, some computations deduce that

$$\frac{kr\theta}{2(k+\alpha-1)} < 1 \tag{23}$$

is equivalent to

$$(1-\lambda)(k+\alpha-1)\left(\frac{1}{\beta}-\frac{1}{k'}\right) < \left(1-\frac{\lambda(k+\alpha-1)}{k}\right)\left(\frac{k+\alpha-1}{\beta}-1\right)$$
$$\Leftrightarrow \quad \frac{1}{k+\alpha-1}+\frac{\lambda-1}{k'} < \frac{\lambda}{k}\left(1-\frac{\alpha-1}{\beta}\right), \tag{24}$$

plugging the definition of λ into (24) we have

$$1 \le \alpha < 1+\frac{2\beta}{n}. \tag{25}$$

Besides, when $n=2$, $\frac{2(k+\alpha-1)}{k} < p < \infty$, some computations yield that

$$1 \le \alpha < 1+\left(1-\frac{2}{p}\right)\beta, \tag{26}$$

follows (23). When $n=1$, $p=\infty$, $1 \le \alpha < 1+\beta$ also establishes (23).

Now we can take

$$k' = \frac{k+\alpha-1+\beta}{2} > \max\left\{1, \frac{(\alpha-1)n}{2}\right\},$$

so that $\theta = \frac{\frac{1}{\beta}-\frac{2}{k+\alpha-1+\beta}}{\frac{1}{\beta}-\frac{1}{k+\alpha-1}}$ and

$$\left(1-\theta-\frac{\theta\beta}{k+\alpha-1}\right) = 0. \tag{27}$$

Thus, from (20), using Young's inequality one obtains

$$\|u\|_{L^{k'}(\Omega)}^{\frac{kr}{2}} \le \frac{k}{4}\|u\|_{L^{k+\alpha-1}(\Omega)}^{k+\alpha-1}\|u\|_{L^{\beta}(\Omega)}^{\beta} + C(k). \tag{28}$$

recalling (18), together with (17) we have

$$\frac{d}{dt}\int_\Omega u^k dx + k\int_\Omega u^\beta dx \int_\Omega u^{k+\alpha-1}dx + \frac{3(k-1)}{k}\|\nabla u^{\frac{k}{2}}\|^2_{L^2(\Omega)}$$
$$\le \frac{k}{4}\|u\|^{k+\alpha-1}_{L^{k+\alpha-1}(\Omega)}\|u\|^{\beta}_{L^\beta(\Omega)} + C(k)$$
$$+ C(k)\|u\|^{\lambda(k+\alpha-1)}_{L^k(\Omega)}\left(\|u\|^{\theta}_{L^{k+\alpha-1}(\Omega)}\|u\|^{1-\theta}_{L^\beta(\Omega)}\right)^{(1-\lambda)(k+\alpha-1)}. \tag{29}$$

Besides, Hölder inequality yields that

$$\|u^\beta\cdot 1\|_{L^1(\Omega)} \le \|u^\beta\|_{L^{\frac{k+\alpha-1}{\beta}}(\Omega)}\|1\|_{L^{\frac{k+\alpha-1}{k+\alpha-1-\beta}}(\Omega)},$$

hence we have

$$\left(\int_\Omega u^\beta dx\right)^{\frac{k+\alpha-1}{\beta}} \le C(|\Omega|)\int_\Omega u^{k+\alpha-1}dx, \tag{30}$$

multiplying $\int_\Omega u^\beta dx$ to both sides one obtains

$$C(|\Omega|)\left(\int_\Omega u^\beta dx\right)^{1+\frac{k+\alpha-1}{\beta}} \le \int_\Omega u^\beta dx\int_\Omega u^{k+\alpha-1}dx. \tag{31}$$

Here $C(|\Omega|)$ is a constant depending on $|\Omega|$. Plugging (31) into (29) one has

$$\frac{d}{dt}\int_\Omega u^k dx + \frac{k}{2}\left(\int_\Omega u^\beta dx\right)^{\frac{k+\alpha-1+\beta}{\beta}} + \frac{k}{2}\int_\Omega u^\beta dx\int_\Omega u^{k+\alpha-1}dx + \frac{3(k-1)}{k}\|\nabla u^{\frac{k}{2}}\|^2_{L^2(\Omega)}$$
$$\le \frac{k}{4}\|u\|^{k+\alpha-1}_{L^{k+\alpha-1}(\Omega)}\|u\|^{\beta}_{L^\beta(\Omega)} + C(k)$$
$$+ C(k)\|u\|^{\lambda(k+\alpha-1)}_{L^k(\Omega)}\left(\|u\|^{\theta}_{L^{k+\alpha-1}(\Omega)}\|u\|^{1-\theta}_{L^\beta(\Omega)}\right)^{(1-\lambda)(k+\alpha-1)}. \tag{32}$$

Thus

$$\frac{d}{dt}\int_\Omega u^k dx + \frac{k}{4}\left(\int_\Omega u^\beta dx\right)^{\frac{k+\alpha-1+\beta}{\beta}} + \frac{k}{4}\int_\Omega u^\beta dx\int_\Omega u^{k+\alpha-1}dx + \frac{3(k-1)}{k}\|\nabla u^{\frac{k}{2}}\|^2_{L^2(\Omega)}$$
$$\le C(k) + C(k)\|u\|^{\lambda(k+\alpha-1)}_{L^k(\Omega)}\left(\|u\|^{\theta}_{L^{k+\alpha-1}(\Omega)}\|u\|^{1-\theta}_{L^\beta(\Omega)}\right)^{(1-\lambda)(k+\alpha-1)}. \tag{33}$$

Now we derive the L^β estimates. Taking

$$k=\beta$$

in (33), recalling (27) and using Young's inequality we obtain that the second term of the right side of (33)

$$C(\beta)\|u\|^{\lambda(\beta+\alpha-1)}_{L^\beta(\Omega)}\left(\|u\|^{\theta}_{L^{\beta+\alpha-1}(\Omega)}\|u\|^{1-\theta}_{L^\beta(\Omega)}\right)^{(1-\lambda)(\beta+\alpha-1)}$$
$$= C(\beta)\|u\|^{\lambda(\beta+\alpha-1)}_{L^\beta(\Omega)}\left(\|u\|^{\beta+\alpha-1}_{L^{\beta+\alpha-1}(\Omega)}\|u\|^{\beta}_{L^\beta(\Omega)}\right)^{\theta(1-\lambda)}$$

$$\leq \frac{\beta}{8}\|u\|_{L^{\beta+\alpha-1}(\Omega)}^{\beta+\alpha-1}\|u\|_{L^\beta(\Omega)}^{\beta} + C(\beta)\|u\|_{L^\beta(\Omega)}^{\lambda(\beta+\alpha-1)\frac{1}{1-\theta(1-\lambda)}}$$

$$\leq \frac{\beta}{8}\|u\|_{L^{\beta+\alpha-1}(\Omega)}^{\beta+\alpha-1}\|u\|_{L^\beta(\Omega)}^{\beta} + C(\beta) + \frac{\beta}{8}\|u\|_{L^\beta(\Omega)}^{\beta+\alpha-1+\beta}. \tag{34}$$

Hence from (33) and (34) we conclude that

$$\frac{d}{dt}\int_\Omega u^\beta dx + C_0(\beta)\left(\int_\Omega u^\beta dx\right)^{1+\frac{\beta+\alpha-1}{\beta}} + C_1(\beta)\int_\Omega \left|\nabla u^{\frac{\beta}{2}}\right|^2 dx \leq C_2(\beta), \tag{35}$$

then the result follows Lemma 2 that

$$\int_\Omega u^\beta dx \leq \max\left\{\|u_0\|_{L^\beta(\Omega)}^{\beta}, \left(\frac{C_2(\beta)}{C_0(\beta)}\right)^{\frac{1}{1+\frac{\beta+\alpha-1}{\beta}}}\right\}. \tag{36}$$

Step 2 (L^k estimates for $k > \beta$). Furthermore, for $k > \beta$, from (33) together with

$$\|u\|_{L^k(\Omega)} \leq \|u\|_{L^{k+\alpha-1}(\Omega)}^{\eta}\|u\|_{L^\beta(\Omega)}^{1-\eta}, \tag{37}$$

where

$$\eta = \frac{\frac{1}{\beta}-\frac{1}{k}}{\frac{1}{\beta}-\frac{1}{k+\alpha-1}} \in (0,1), \tag{38}$$

it concludes that

$$\frac{d}{dt}\int_\Omega u^k dx + C(k)\left(\int_\Omega u^\beta dx\right)^{1+\frac{\beta+\alpha-1}{\beta}} + C_0(k)\int_\Omega \left|\nabla u^{\frac{k}{2}}\right|^2 dx \leq C(k). \tag{39}$$

On the other hand, using Sobolev inequality and Young's inequality we observing the fact that

$$\int_\Omega u^k dx \leq C(k)\Big(\|\nabla u^{\frac{k}{2}}\|_{L^2(\Omega)}^{\lambda_0}\|u^{\frac{k}{2}}\|_{L^{\frac{2\beta}{k}}(\Omega)}^{1-\lambda_0} + \|u^{\frac{k}{2}}\|_{L^2(\Omega)}^{\lambda_0}\|u^{\frac{k}{2}}\|_{L^{\frac{2\beta}{k}}(\Omega)}^{1-\lambda_0}\Big)^2$$

$$\leq C(k)\Big(\|\nabla u^{\frac{k}{2}}\|_{L^2(\Omega)}^{2\lambda_0}\|u^{\frac{k}{2}}\|_{L^{\frac{2\beta}{k}}(\Omega)}^{2(1-\lambda_0)} + \|u^{\frac{k}{2}}\|_{L^2(\Omega)}^{2\lambda_0}\|u^{\frac{k}{2}}\|_{L^{\frac{2\beta}{k}}(\Omega)}^{2(1-\lambda_0)}\Big)$$

$$\leq C\big(k, \|u_0\|_{L^\beta(\Omega)}\big)\|\nabla u^{\frac{k}{2}}\|_{L^2(\Omega)}^{2\lambda_0} + \frac{1}{2}\|u\|_{L^k(\Omega)}^{k} + C\big(k, \|u_0\|_{L^\beta(\Omega)}\big), \tag{40}$$

where

$$\lambda_0 = \frac{\frac{kn}{2\beta}-\frac{n}{2}}{\frac{kn}{2\beta}-\frac{n}{p}} < 1. \tag{41}$$

Then

$$\left(\int_\Omega u^k dx\right)^{\frac{1}{\lambda_0}} \leq C_0(k)\|\nabla u^{\frac{k}{2}}\|_{L^2(\Omega)}^2 + C\big(k, \|u_0\|_{L^\beta(\Omega)}\big). \tag{42}$$

Substituting (42) into (39) arrives at

$$\frac{d}{dt}\int_{\Omega} u^k dx + C\big(k, \|u_0\|_{L^\beta(\Omega)}\big)\left(\int_{\Omega} u^k dx\right)^{\frac{1}{\lambda_0}} \leq C\big(k, \|u_0\|_{L^\beta(\Omega)}\big). \tag{43}$$

Consequently, it follows from Lemma 2 that for any $k > \beta$ and any $t > 0$

$$\int_{\Omega} u^k dx \leq C\big(k, \|u_0\|_{L^\beta(\Omega)}\big) + C\big(k, \|u_0\|_{L^\beta(\Omega)}\big)\, t^{-\frac{k-\beta}{\beta(1-2/p)}}. \tag{44}$$

In addition, integrating (35) and (39) from 0 to T in time, then for any $T > 0$ and $k > 1$

$$\int_{\Omega} u^k(T)dx + \int_0^T \int_{\Omega} \left|\nabla u^{\frac{k}{2}}\right| dxdt \leq \int_{\Omega} u_0^k dx + C\big(\|u_0\|_{L^\beta(\Omega)}, k\big)T.$$

Hence it follows the conclusion (9). This completes the proof. □

Now we can use the compactness arguments to complete the proof of Theorem 1. The proof is standard and here we give the key steps. Firstly, taking $k = 2$ and $k = 2\alpha$ in (8) and (9) we obtain $\|u\|_{L^2(0,T;L^2(\Omega))}$ and $\|u_t\|_{L^2(0,T;H^{-1}(\Omega))}$ are bounded for any $0 < T < \infty$, then by Aubin-Lions lemma [8] we have the strong compactness of u in $L^2(0, T; L^2(\Omega))$. Therefore, standard compactness arguments deduce the global existence of weak solutions (in the sense of distribution). In the second, from Proposition 3, the reaction term $u^\alpha(1 - \int_\Omega u^\beta dx) \in L^k(0, T; L^k(\Omega))$ for any $k > 1$ and $0 < T < \infty$, then from classical parabolic theory, the weak solution is strong solution in $W_k^{2,1}(0, T; \Omega)$. By virtue of Sobolev embedding we can bootstrap it to get global existence of classical solution. Moreover, since $u^\alpha(1 - \int_\Omega u^\beta dx)$ is bounded from above and below, then using comparison principle we can get the uniqueness of the classical solution [10]. Everything together we show that (1a)–(1c) has a unique global solution. This closes the proof of Theorem 1.

References

1. Bebernes, J.W., Bressan, A.: Thermal behaviour for a confined reactive gas. J. Differ. Equ. **44**, 118–133 (1982)
2. Bian, S., Chen, L., Latos, E.: Global existence and asymptotic behavior of solutions to a nonlocal Fisher-KPP type problem arXiv:1508.00063 (2015)
3. Bian, S., Liu, J.-G.: Dynamic and steady states for multi-dimensional Keller-Segel model with diffusion exponent $m > 0$. Commun. Math. Phys. **323**, 1017–1070 (2013)
4. Bian, S., Liu, J.-G., Zou, C.: Ultra-contractivity for Keller-Segel model with diffusion exponent $m > 1 - 2/d$. Kinet. Relat. Models **7**(1), 9–28 (2014)
5. Hu, B., Yin, H.M.: Semilinear parabolic equations with prescribed energy. Rend. Circ. Mat. Palermo **44**, 479–505 (1995)
6. Fisher, R.A.: The wave of advance of advantageous genes. Annu. Eugen. **7**, 355–369 (1937)
7. Fujita, H.: On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^\alpha$. J. Fac. Sci., Univ. Tokyo, Sect. 1A, Math. **13**, 109–124 (1966)
8. Jacques, S.: Compact sets in the space $L^p(0, T; B)$. Ann. Mat. Pura Appl. **146**(1), 65–96 (1986)
9. Kolmogorov, A., Petrovsky, I., Piskunov, N.: Investigation of the equation of diffusion combined with increasing of the substance and its application to a biology problem. Bull. Moscow State Univ. Ser. A, Math. Mech. **1**(6), 1–25 (1937)
10. Lieberman, G.M.: Second Order Parabolic Partial Differential Equations. World Scientific, Singapore (1996)
11. Wang, M., Wang, Y.: Properties of positive solutions for non-local reaction-diffusion problems. Math. Methods Appl. Sci. **19**, 1141–1156 (1996)