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Global Dirichlet Heat Kernel Estimates for Symmetric Lévy Processes in Half-Space

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Abstract In this paper, we derive explicit sharp two-sided estimates for the Dirichlet heat kernels of a large class of symmetric (but not necessarily rotationally symmetric) Lévy processes on half spaces for all $t > 0$. These Lévy processes may or may not have Gaussian component. When Lévy density is comparable to a decreasing function with damping exponent β, our estimate is explicit in terms of the distance to the boundary, the Lévy exponent and the damping exponent β of Lévy density.

Keywords Dirichlet heat kernel · Transition density · Survival probability · Exit time · Lévy system · Lévy process · Symmetric Lévy process

Mathematics Subject Classification (2000) Primary 60J35 · 47G20 · 60J75 · Secondary 47D07

1 Introduction

Classical Dirichlet heat kernel is the fundamental solution of the heat equation in an open set with zero boundary values. Except for a few special cases, explicit form of the Dirichlet heat kernel is impossible to obtain. Thus the best thing we can hope for is to establish sharp two-sided estimates of Dirichlet heat kernels. See [21] for upper bound estimates and [28] for the lower bound estimate for Dirichlet heat kernels of diffusions in bounded $C^{1,1}$ domains.

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The generator of a discontinuous Lévy process is an integro-differential operator and so it is a non-local operator. Dirichlet heat kernels (if they exist) of the generators of discontinuous Lévy processes on an open set D are the transition densities of such Lévy processes killed upon leaving D. Due to this connection, obtaining sharp estimates on Dirichlet heat kernels is a fundamental problem both in probability theory and in analysis.

Before [10], sharp two-sided estimates for the Dirichlet heat kernel of any non-local operator in open sets are unknown. Jointly with R. Song, in [10] for the fractional Laplacian $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ with zero exterior condition, we succeeded in establishing sharp two-sided estimates in any $C^{1,1}$ open set D and over any finite time interval (see [3] for an extension to non-smooth open sets). When D is bounded, one can easily deduce large time heat kernel estimates from short time estimates by a spectral analysis. The approach developed in [10] provides a road map for establishing sharp two-sided heat kernel estimates of other discontinuous processes in open subsets of $\mathbb{R}^d$ (see [2, 3, 6, 11, 12, 14–16, 23]). In [13, 14, 20], sharp two-sided estimates for the Dirichlet heat kernels $p_D(t,x,y)$ of $\Delta^{\alpha/2}$ and of $m-(m^{2/\alpha}-\Delta)^{\alpha/2}$ are obtained for all $t>0$ in two classes of unbounded open sets: half-space-like $C^{1,1}$ open sets and exterior open sets. Since the estimates in [13, 14, 20] hold for all $t>0$, they are called global Dirichlet heat kernel estimates. An important question in this direction is for how general discontinuous Lévy processes one can prove sharp two-sided global Dirichlet heat kernel estimates in unbounded open subsets of $\mathbb{R}^d$.

We conjectured in [13, (1.9)] that, when D is a half space-like $C^{1,1}$ open set, the following two-sided estimates hold for a large class of rotationally symmetric Lévy process X whose Lévy exponent of X is $\Psi(|\xi|)$: there are constants $c_1, c_2, c_3 \geq 1$ such that for every $(t,x,y) \in (0,\infty)\times D\times D$,

$$\begin{aligned}
&\frac{1}{c_1}\left(\frac{1}{\sqrt{t\Psi(1/\delta_D(x))}}\wedge 1\right)\left(\frac{1}{\sqrt{t\Psi(1/\delta_D(y))}}\wedge 1\right)p\big(t,c_2(y-x)\big)\\
&\quad\leq p_D(t,x,y)\\
&\quad\leq c_1\left(\frac{1}{\sqrt{t\Psi(1/\delta_D(x))}}\wedge 1\right)\left(\frac{1}{\sqrt{t\Psi(1/\delta_D(y))}}\wedge 1\right)p\big(t,c_3(y-x)\big)
\end{aligned}\tag{1.1}$$

where $p(t,x)$ is the transition density of X. In this paper, we use ":=" as a way of definition. For $a,b\in\mathbb{R}$, $a\wedge b:=\min\{a,b\}$ and $a\vee b:=\max\{a,b\}$.

Recently, the above conjecture is confirmed in [6, Theorem 5.8] for rotationally symmetric unimodal Lévy process whose Lévy exponent $x\to\Psi(|x|)$ satisfies the following upper and lower scaling properties: there are constants $0<\beta_1<\beta_2<2$ and $C>1$ so that

$$c^{-1}\left(\frac{R}{r}\right)^{\beta_1}\leq\frac{\Psi(R)}{\Psi(r)}\leq c\left(\frac{R}{r}\right)^{\beta_2}\quad\text{for any } R\geq r>0. \tag{1.2}$$

Condition (1.2) implies that the Lévy process is purely discontinuous, and by [4, Corollary 23], its Lévy intensity kernel $x\to j(|x|)$ satisfies

$$\frac{c^{-1}}{|x|^d\Phi(|x|)}\leq j(|x|)\leq\frac{c}{|x|^d\Phi(|x|)}\quad\text{for all } x\neq 0, \tag{1.3}$$

where $\Phi(r):=\max_{|x|\leq r}1/\Psi(1/|x|)$. It is easy to see that $\Phi(r)$ is comparable to $1/\Psi(1/r)$. It follows from (1.2) that the same two-sided estimates hold for Φ in place of Ψ. Thus condition (1.2) excludes damped Lévy processes such as relativistic stable processes. We remark here that under condition (1.2), it follows as a special case from [18] that the transition density $p(t,x)$ of the rotationally symmetric unimodal Lévy process has the following two-sided estimates:

$$c^{-1}\left(\Phi^{-1}(t)\wedge\frac{t}{|x|^d\Phi(|x|)}\right)\le p(t,x)\le c\left(\Phi^{-1}(t)\wedge\frac{t}{|x|^d\Phi(|x|)}\right)$$
$$\text{for all } t>0 \text{ and } x\in\mathbb{R}^d. \tag{1.4}$$

In this paper, we mainly focus on estimate (1.1) when D is a half space and we prove that (1.1) holds for a large class of symmetric Lévy processes which may not be isotropic and may have damped Lévy kernel. Moreover, our symmetric Lévy processes may or may not have Gaussian component. Once the global Dirichlet heat kernel estimates for upper half space and short time heat kernel estimates on $C^{1,1}$ open sets are obtained, one can then use the "push inward" method introduced in [20] to extend the results to half-space-like $C^{1,1}$ open sets. See Remark 7.2. For recent results on short time Dirichlet heat kernel estimates for symmetric Lévy processes in $C^{1,1}$ open sets, we refer the reader to [15, 16]. Note that, for all symmetric Lévy process in $\mathbb{R}$, except compound Poisson processes, the survival probability $\mathbb{P}_x(\zeta>t)$ of its subprocess in the half line $(0,\infty)$ is comparable to

$$1\wedge\frac{\max_{0\le y\le 1/x}1/\sqrt{\Psi(y)}}{\sqrt{t}},$$

where $x\to\Psi(|x|)$ is its characteristic exponent (see [26, Theorem 4.6] and [5, Proposition 2.6]). This fact, which is used several times in this paper, is essential in our approach.

We now give more details on the main results of this paper. In this paper, $d\ge 1$ and $X=(X_t,\mathbb{P}_x)_{t\ge 0,x\in\mathbb{R}^d}$ is a symmetric discontinuous Lévy process (but possibly with Gaussian component) on $\mathbb{R}^d$ with Lévy exponent $\Psi(\xi)$ and Lévy density J where $\mathbb{P}_x(X_0=x)=1$. That is, X is a right continuous symmetric process having independent stationary increments with

$$\mathbb{E}_x\left[e^{i\xi\cdot(X_t-X_0)}\right]=e^{-t\Psi(\xi)}\quad\text{for every } x\in\mathbb{R}^d \text{ and } \xi\in\mathbb{R}^d. \tag{1.5}$$

Throughout this paper, we assume that X is not a compounded Poisson process, which corresponds exactly to the case that Ψ is unbounded. It is known that

$$\Psi(\xi)=\sum_{i,j=1}^d a_{ij}\xi_i\xi_j+\int_{\mathbb{R}^d}\big(1-\cos(\xi\cdot y)\big)J(y)dy\quad\text{for } \xi=(\xi_1,\dots,\xi_d)\in\mathbb{R}^d,$$

where $A=(a_{ij})$ is a constant, symmetric, non-negative definite matrix and J is a symmetric non-negative function on $\mathbb{R}^d\setminus\{0\}$ with $\int_{\mathbb{R}^d}(1\wedge|z|^2)J(z)dz<\infty$.

When

$$\int_{\mathbb{R}^d}\exp\big(-t\Psi(\xi)\big)d\xi<\infty\quad\text{for } t>0, \tag{\textbf{ExpL}}$$

the transition density $p(t,x,y)=p(t,y-x)$ of X exists as a bounded continuous function for each fixed $t>0$, and it is given by

$$p(t,x):=(2\pi)^{-d}\int_{\mathbb{R}^d}e^{-i\xi\cdot x}e^{-t\Psi(\xi)}d\xi,\quad t>0.$$

Moreover,

$$p(t,x) \le (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t\Psi(\xi)} d\xi = p(t,0) < \infty. \tag{1.6}$$

Clearly, condition (**ExpL**) holds if $\inf_{|\xi|=1} \sum_{i,j=1}^d a_{ij}\xi_i\xi_j > 0$. Conversely, suppose that, for every $t > 0$, X_t has a probability density function $p(t,x)$ under $\mathbb{P}_0$ and that $x \mapsto p(t,x)$ is L^2-integrable. Then (1.5) can be rewritten as

$$\int_{\mathbb{R}^d} p(t,x) e^{ix\cdot\xi} dx = e^{-t\Psi(\xi)}$$

and so, by the Plancherel theorem, $e^{-t\Psi(\xi)} \in L^2(\mathbb{R}^d)$ for every $t > 0$; that is, (**ExpL**) holds. Hence condition (**ExpL**) is equivalent to the existence of transition density function $p(t,x)$ of X that is $L^2(\mathbb{R}^d)$-integrable in x for every $t > 0$. If we assume that X has a transition density function $p(t,x)$ that is continuous in x, then, since $\int_{\mathbb{R}^d} p(t,x)^2 dx = p(2t,0) < \infty$, (**ExpL**) holds. See [25, Proposition 4.1] for additional discussion on condition (**ExpL**).

Let

$$\Psi^*(r) := \sup_{|z|\le r} \Psi(z) \tag{1.7}$$

and use Φ to denote the non-decreasing function

$$\Phi(r) = \frac{1}{\Psi^*(1/r)} \quad \text{for } r > 0. \tag{1.8}$$

Note that since $\Psi(z)$ is a continuous unbounded function on $\mathbb{R}^d$, Ψ^* is a non-decreasing continuous function on $[0,\infty)$ with $\Psi^*(0) = 0$ and $\lim_{r\to\infty} \Psi^*(r) = \infty$. Consequently, Φ is a non-decreasing continuous function on $[0,\infty)$ with $\Phi(0) = 0$ and $\lim_{r\to\infty} \Phi(r) = \infty$. The right continuous inverse function of Φ will be denoted by the usual notation $\Phi^{-1}(r)$; that is,

$$\Phi^{-1}(t) = \inf\{s > 0 : \Phi(s) > t\}.$$

Note that $0 < \Phi^{-1}(t) < \infty$ for every $t > 0$ and $\lim_{t\to 0+} \Phi^{-1}(t) = 0$. Define for $r > 0$,

$$\Psi_1^*(r) := \sup_{s\in(-r,r)} \Psi\big((0,\dots,0,s)\big).$$

We consider the following condition: there exists a constant $c \ge 1$ such that

$$\Psi^*(r) \le c\Psi_1^*(r) \quad \text{for all } r > 0. \tag{\textbf{Comp}}$$

Condition (**Comp**) is a mild assumption that is satisfied by a large class of symmetric Lévy processes, see Lemma 2.9. Under assumptions (**ExpL**) and (**Comp**), we derive in Lemma 2.10 a useful upper bound estimate for Dirichlet heat kernels.

In general, the explicit estimates of the transition density $p(t,y)$ in $\mathbb{R}^d$ depend heavily on the corresponding Lévy measure and Gaussian component (see [8, 18]). On the other hand, scale-invariant parabolic Harnack inequality holds with the explicit scaling in terms of Lévy exponent for a large class of symmetric Lévy processes (see [18, Theorem 4.12], [8, Theorem 4.11] and our Theorem 5.3). Motivated by this, we first develop a rather general version of Dirichlet heat kernel upper bound estimate in Proposition 3.6 under the assumption that parabolic Harnack inequality (**PHI(Φ)**) and (**UJS**) hold. See Sect. 3 for the definition of (**PHI(Φ)**). We say (**UJS**) holds if there exists a positive constant c such that for every $y \in \mathbb{R}^d$,

$$J(y) \le \frac{c}{r^d} \int_{B(0,r)} J(y-z)dz \quad \text{whenever } r \le |y|/2. \tag{UJS}$$

Note that (**UJS**) is very mild assumption in our setting. In fact, (**UJS**) always holds if $J(x) \asymp j(|x|)$ for some non-increasing function j (see [7, page 1070]). Moreover, if J is continuous on $\mathbb{R}^d \setminus \{0\}$, then (**PHI($\Phi$)**) implies (**UJS**). In fact, using (3.1) below instead of [7, (2.10)], this follows from the proof of [7, Proposition 4.1]. We also show in Theorem 3.2 that (**PHI(Φ)**) implies (**ExpL**).

Assume in addition that for every $t > 0$, $x \to p(t,x)$ is weakly radially decreasing in the following sense: there exist constants $c > 0$ and $C_1, C_2 > 0$ such that

$$p(t,x) \le c p(C_1 t, C_2 y) \quad \text{for } t \in (0,\infty) \text{ and } |x| \ge |y| > 0. \tag{HKC}$$

We remark here that the same assumption with $C_1 = 1$ for small t was made in [15]. Then our Dirichlet heat kernel upper bound estimate obtained in Propositions 3.6 yields the desired upper bound estimate in (1.1). Moreover, we show that this assumption on $p(t,x)$ (see Sect. 4 below) and the upper bound of $p_D(t,x)$ imply a very useful lower bound of $p_D(t,x)$; see Theorem 4.5.

Jointly with T. Kumagai, in [8, 9, 18] we have established two-sided sharp heat kernel estimates for a large class of symmetric Markov processes. In Sects. 5–7, we assume the jumping kernels of our Lévy process satisfy the assumptions of [8, 9, 18], that is, conditions (**UJS**), (5.4) and (5.5) of this paper. Then all the aforementioned conditions (**ExpL**), (**Comp**), (**PHI(Φ)**), (**HKC**) are satisfied. Using the two-sided heat kernel estimates for symmetric Markov processes on $\mathbb{R}^d$ from [8, 9, 18] (see Theorem 5.3) and our lower bound estimates for Dirichlet heat kernels in Theorem 4.5, we obtain two-sided global Dirichlet heat kernel estimates (7.4), essentially prove the conjecture (1.1) for such symmetric Lévy processes and for $D = \mathbb{H}$. See Remark 7.2(i) for details. Furthermore, our estimates are explicit in terms of the distance to the boundary, the Lévy exponent and the damping exponent β of Lévy density; see Theorem 7.1.

In this paper, we use the following notations. For any two positive functions f and g, $f \asymp g$ means that there is a positive constant $c \ge 1$ so that $c^{-1}g \le f \le cg$ on their common domain of definition. For any open set V, we denote by $\delta_V(x)$ the distance of a point x to the boundary of V, i.e., $\delta_V(x) = \text{dist}(x, \partial V)$. We sometimes write point $z = (z_1, \dots, z_d) \in \mathbb{R}^d$ as $(\widetilde{z}, z_d)$ with $\widetilde{z} \in \mathbb{R}^{d-1}$. We denote $\mathbb{H} := \{x = (\widetilde{x}, x_d) \in \mathbb{R}^d : x_d > 0\}$ the upper half space. For a set W in $\mathbb{R}^d$, $\overline{W}$ and $|W|$ denotes the closure and the Lebesgue measure of W in $\mathbb{R}^d$, respectively. Throughout the rest of this paper, the positive constants $a_0, a_1, M_1, C_i, i = 0, 1, 2, \dots$, can be regarded as fixed. In the statements of results and the proofs, the constants $c_i = c_i(a, b, c, \dots), i = 0, 1, 2, \dots$, denote generic constants depending on $a, b, c, \dots$, whose exact values are unimportant. They start anew in each statement and each proof. The dependence of the constants on the dimension $d \ge 1$ may not be mentioned explicitly.

2 Setup and Preliminary Estimates

Let X be a symmetric Lévy process on $\mathbb{R}^d$ with Lévy exponent $\Psi(z)$ and Lévy density $J(z)$. Recall the definition of the non-decreasing functions $\Psi^*(r)$ and $\Phi(r)$ from (1.7) and (1.8), respectively. We emphasize that the Lévy process X does not need to be rotationally symmetric. The following is known and true for any negative definite function (see [22, Lemma 1]).

Lemma 2.1 *For every $t > 0$ and $\lambda \geq 1$,*

$$1 \leq \frac{\Phi(\lambda t)}{\Phi(t)} \leq 2\big(1 + \lambda^2\big).$$

For an open set D, denote by $\tau_D := \inf\{t > 0 : X_t \notin D\}$ the first exit time of D.

Theorem 2.2 *There exists a constant $c = c(d) > 0$ such that*

$$\mathbb{P}_0(|X_t| > r) \leq ct/\Phi(r) \quad \text{for } (t,r) \in (0,\infty) \times (0,\infty). \tag{2.1}$$

Consequently, there exists $\varepsilon_1 = \varepsilon_1(d) > 0$ such that for all $r > 0$,

$$\mathbb{P}_0\big(\tau_{B(0,r/2)} > \varepsilon_1 \Phi(r)\big) \geq 1/2. \tag{2.2}$$

Proof (2.1) is a consequence of [27, (3.2)] and [22, Corollary 1].

Since the Lévy process X is conservative, (2.1) implies by [1, Lemma 3.8] that for every $t, r > 0$,

$$\mathbb{P}_0(\tau_{B(0,2r)} \leq t) = \mathbb{P}_0\Big(\sup_{s \leq t} |X_s| > 2r\Big) \leq 2c_2 t/\Phi(r).$$

Thus $\mathbb{P}_0(\tau_{B(0,r/2)} \leq \varepsilon_1 \Phi(r)) \leq 2c_2\varepsilon_1\Phi(r)/\Phi(r/2)$, which by Lemma 2.1 is no larger than $20c_2\varepsilon_1$. Taking $\varepsilon_1 = 1/(40c_2)$ proves the theorem. □

Recall that J is the Lévy density of X, which gives rise to a Lévy system for X describing the jumps of X. For any $x \in \mathbb{R}^d$, stopping time S (with respect to the filtration of X), and nonnegative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s, y, y) = 0$ for all $y \in \mathbb{R}^d$ and $s \geq 0$ we have

$$\mathbb{E}_x\Big[\sum_{s \leq S} f(s, X_{s-}, X_s)\Big] = \mathbb{E}_x\Big[\int_0^S \Big(\int_{\mathbb{R}^d} f(s, X_s, y) J(X_s - y) dy\Big) ds\Big] \tag{2.3}$$

(e.g., see [18, Appendix A] and the proof of [17, Lemma 4.7]).

The following is a special case of [22, Corollary 1], whose upper bound will be used in the next lemma.

Lemma 2.3 *For every $r > 0$,*

$$\frac{1}{2\Phi(r)} \leq \frac{\|A\|}{r^2} + \int_{\mathbb{R}^d} J(z)\Big(1 \wedge \frac{|z|^2}{r^2}\Big) dz \leq \frac{8(1+2d)}{\Phi(r)}$$

where

$$\|A\| := \sup_{|\xi| \leq 1} \sum_{i,j=1}^d a_{i,j}\xi_i\xi_j.$$

Using Lemma 2.3, the proof of the next lemma is rather routine (see [24, Lemma 4.10]). In fact, this lemma is proved in [22, Lemma 3 and Corollary 1] for $a = 1/2$. The proof for general a is similar. Thus we skip the proof.

Lemma 2.4 *For every $a \in (0, 1)$, there exists $c = c(a) > 0$ so that for any $r > 0$ and any open set U with $U \subset B(0, r)$,*

$$\mathbb{P}_x\big(X_{\tau_U} \in B(0,r)^c\big) \le \frac{c}{\Phi(r)}\mathbb{E}_x[\tau_U], \quad x \in U \cap B(0, ar).$$

Note that for d-th coordinate X_t^d of $X_t = (X_t^1, \dots, X_t^d)$ is a Lévy process with

$$\mathbb{E}_x\big[e^{i\eta(X_t^d - X_0^d)}\big] = \mathbb{E}_{(\widetilde{0},x)}\big[e^{i(\widetilde{0},\eta)\cdot(X_t - X_0)}\big] = e^{-t\Psi((\widetilde{0},\eta))} \quad \text{for every } x \in \mathbb{R} \text{ and } \eta \in \mathbb{R}.$$

That is, X_t^d is a 1-dimensional symmetric Lévy process with Lévy exponent $\Psi_1(\eta) := \Psi((\widetilde{0}, \eta))$. Throughout this paper we let $\Psi_1^*(r) := \sup_{z\in(-r,r)} \Psi_1(z)$ and use Φ_1 to denote the increasing function

$$\Phi_1(r) = \frac{1}{\Psi_1^*(r^{-1})}, \quad r > 0.$$

Clearly

$$\Psi_1^*(r) \le \Psi^*(r) \quad \text{and} \quad \Phi(r) \le \Phi_1(r).$$

Since $\tau_{\mathbb{H}} := \inf\{t > 0 : X_t^d > 0\}$, by [5, Proposition 2.6] (see also [26, Theorems 3.1 and 4.6]) all symmetric Lévy processes, except compound Poisson processes, enjoy the following estimates of the survival probability on $\mathbb{H}$.

Lemma 2.5 *Suppose that Ψ_1 is unbounded, then there exists an absolute constant $C > 0$ such that*

$$C^{-1}\left(\sqrt{\frac{\Phi_1(\delta_{\mathbb{H}}(x))}{t}} \wedge 1\right) \le \mathbb{P}_x(\tau_{\mathbb{H}} > t) \le C\left(\sqrt{\frac{\Phi_1(\delta_{\mathbb{H}}(x))}{t}} \wedge 1\right).$$

Let

$$\tau_r^1 := \inf\big\{t > 0 : X_t^d \notin (0, r)\big\}$$

Combining [5, Lemma 2.3 and Proposition 2.4] we have

Lemma 2.6 *Suppose that Ψ_1 is unbounded, then there exists an absolute constant $c > 0$ such that for any $r \in (0, \infty)$ and*

$$\mathbb{E}_{(\widetilde{0},x)}\big[\tau_r^1\big] \le c\Phi_1(r)^{1/2}\Phi_1\big(\delta_{(0,r)}(x)\big)^{1/2} \quad \text{for } x \in (0, r).$$

Recall that, when (**ExpL**) holds, the transition density $p(t, x, y) := p(t, x - y)$ of X exists as a bounded continuous function. In this case, for an open set D we define

$$p_D(t, x, y) := p(t, x, y) - \mathbb{E}_x\big[\, p(t - \tau_D, X_{\tau_D}, y) : \tau_D < t\big] \quad \text{for } t > 0, x, y \in D \tag{2.4}$$

Using the strong Markov property of X, it is easy to verify that $p_D(t, x, y)$ is the transition density for X^D, the subprocess of X killed upon leaving an open set D.

Lemma 2.7 *Suppose* (**ExpL**) *holds. Then for every* $(t,x,y)\in(0,\infty)\times\mathbb{H}\times\mathbb{H}$,

$$p_{\mathbb{H}}(t,x,y)\le 3C^2p(t/3,0)\left(\sqrt{\frac{\Phi_1(\delta_{\mathbb{H}}(x))}{t}}\wedge 1\right)\left(\sqrt{\frac{\Phi_1(\delta_{\mathbb{H}}(y))}{t}}\wedge 1\right)$$

where C is the constant in Lemma 2.5.

Proof Since by (1.6)

$$\sup_{z,w\in\mathbb{H}} p_{\mathbb{H}}(t/3,z,w)\le \sup_{z\in\mathbb{H}} p(t/3,z)=p(t/3,0),$$

using the semigroup property and symmetry we have

$$\begin{aligned}p_{\mathbb{H}}(t,x,y)&=\int_{\mathbb{H}}\int_{\mathbb{H}} p_{\mathbb{H}}(t/3,x,z)p_{\mathbb{H}}(t/3,z,w)p_{\mathbb{H}}(t/3,w,y)dzdw\\&\le p(t/3,0)\mathbb{P}_x(\tau_{\mathbb{H}}>t/3)\mathbb{P}_y(\tau_{\mathbb{H}}>t/3).\end{aligned}$$

Now the lemma follows from Lemma 2.5. □

Using (2.3), the proof of next lemma is the same as the one in [16, Lemma 3.1] so it is omitted.

Lemma 2.8 *Suppose* (**ExpL**) *holds. Suppose that U_1,U_3,E are open subsets of $\mathbb{R}^d$, with $U_1,U_3\subset E$ and $\mathrm{dist}(U_1,U_3)>0$. Let $U_2:=E\setminus(U_1\cup U_3)$. If $x\in U_1$ and $y\in U_3$, then for every $t>0$ we have*

$$\begin{aligned}p_E(t,x,y)&\le \mathbb{P}_x(X_{\tau_{U_1}}\in U_2)\cdot\sup_{s<t,z\in U_2} p_E(s,z,y)\\&\quad+\int_0^t \mathbb{P}_x(\tau_{U_1}>s)\mathbb{P}_y(\tau_E>t-s)ds\cdot\sup_{u\in U_1,z\in U_3} J(u-z).\end{aligned}\tag{2.5}$$

Recall condition (**Comp**) from Introduction. The next lemma says that it is a mild assumption.

Lemma 2.9 *Suppose there are a non-negative function j on $(0,\infty)$ and $a\ge 0$ $c_i\ge 1$, $i=1$, 2, such that*

$$c_1^{-1}a|y|^2\le\sum_{i,j=1}^d a_{i,j}y_iy_j\le c_1a|y|^2\quad\textit{and}\quad c_1^{-1}j\big(|y|/c_2\big)\le J(y)\le c_1 j\big(c_2|y|\big)$$
$$\textit{for all } y\in\mathbb{R}^d,\tag{2.6}$$

Then $c^{-1}\Psi_1^*(r)\le\Psi^*(r)\le c\Psi_1^*(r)$, and so (**Comp**) *holds.*

Proof Let

$$\phi\big(|\xi|\big)=a|\xi|^2+\int_{\mathbb{R}^d}\big(1-\cos(\xi\cdot y)\big)j\big(|y|\big)dy.$$

By a change of variables, (2.6) implies that

$$\Psi(\xi) \le c_1\left(a|\xi|^2 + \int_{\mathbb{R}^d}\left(1-\cos(\xi\cdot y)\right)j\left(c_2|y|\right)dy\right)$$
$$\le c_3\left(a\left|c_2^{-1}\xi\right|^2 + \int_{\mathbb{R}^d}\left(1-\cos\left(c_2^{-1}\xi\cdot z\right)\right)j\left(|z|\right)dz\right) = c_3\phi\left(|\xi|/c_2\right)$$

and

$$\Psi(\xi) \ge c_1^{-1}\left(a|\xi|^2 + \int_{\mathbb{R}^d}\left(1-\cos(\xi\cdot y)\right)j\left(|y|/c_2\right)dy\right) \ge c_5\phi\left(c_2|\xi|\right)$$

Thus by Lemma 2.1, which holds for any negative definite function, $\Psi^*(r) \asymp \sup_{s\le r}\phi(s) \asymp \Psi_1^*(r)$ for all $r>0$. □

Using Lemma 2.8, we can obtain the following upper bound of $p_{\mathbb{H}}(t,x,y)$.

Lemma 2.10 *Suppose* (**ExpL**) *and* (**Comp**) *hold. For each* $a>0$, *there exists a constant* $c=c(a,\Psi)>0$ *such that for every* $(t,x,y)\in(0,\infty)\times\mathbb{H}\times\mathbb{H}$ *with* $a\Phi^{-1}(t)\le|x-y|$,

$$p_{\mathbb{H}}(t,x,y) \le c\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}}\wedge 1\right)\Bigg(\sup_{(s,z):s\le t,\,\frac{|x-y|}{2}\le|z-y|\le\frac{3|x-y|}{2}} p_{\mathbb{H}}(s,z,y)$$
$$+\left(\sqrt{t\Phi\left(\delta_{\mathbb{H}}(y)\right)}\wedge t\right)\sup_{w:|w|\ge\frac{|x-y|}{3}} J(w)\Bigg). \tag{2.7}$$

Proof If $\delta_{\mathbb{H}}(x) > a\Phi^{-1}(t)/(24)$, by Lemma 2.1

$$\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \ge \sqrt{\frac{\Phi(a\Phi^{-1}(t)/(24))}{\Phi(\Phi^{-1}(t))}} \ge \sqrt{\frac{1}{2}\frac{a^2}{a^2+(24)^2}}.$$

Thus (2.7) is clear.

We now assume $\delta_{\mathbb{H}}(x)\le a\Phi^{-1}(t)/(24)\le|x-y|/(24)$ and let $x_0=(\widetilde{x},0)$, $U_1:=B(x_0,a\Phi^{-1}(t)/(12))\cap\mathbb{H}$, $U_3:=\{z\in\mathbb{H}:|z-x|>|x-y|/2\}$ and $U_2:=\mathbb{H}\setminus(U_1\cup U_3)$. Recall that X_t^d is the d-th coordinate process of X with Lévy exponent $\Psi_1(\eta)=\Psi((\widetilde{0},\eta))$. Clearly,

$$\tau_{U_1} \le \inf\left\{t>0: X_t^d\notin\left(0,a\Phi^{-1}(t)/12\right)\right\} =: \tau_1^d.$$

Applying Lemma 2.6 on the interval $(0,a\Phi^{-1}(t)/12)$ and assumption (**Comp**), and noting Lemma 2.1, we have

$$\mathbb{E}_x[\tau_{U_1}] \le \mathbb{E}^{X^d}_{\delta_{\mathbb{H}}(x)}\left[\tau_1^d\right] \le c_1\sqrt{t\Phi\left(\delta_{\mathbb{H}}(x)\right)}. \tag{2.8}$$

Since $|z-x|>2^{-1}|x-y|\ge a2^{-1}\Phi^{-1}(t)$ for $z\in U_3$, we have for $u\in U_1$ and $z\in U_3$,

$$|u-z|\ge|z-x|-|x_0-x|-|x_0-u|\ge\frac{1}{2}|x-y|-6^{-1}a\Phi^{-1}(t)\ge\frac{1}{3}|x-y|.$$

Thus, $U_1\cap U_3=\emptyset$ and,

$$\sup_{u\in U_1,z\in U_3} J(u-z) \le \sup_{(u,z):|u-z|\ge\frac{1}{3}|x-y|} J(u-z) = \sup_{w:|w|\ge\frac{1}{3}|x-y|} J(w). \tag{2.9}$$

Since for $z \in U_2$

$$\frac{3}{2}|x-y| \geq |x-y|+|x-z| \geq |z-y| \geq |x-y|-|x-z| \geq \frac{|x-y|}{2} \geq a2^{-1}\Phi^{-1}(t),$$

we have

$$\sup_{s\leq t, z\in U_2} p_{\mathbb{H}}(s,z,y) \leq \sup_{s\leq t, \frac{|x-y|}{2}\leq |z-y| \leq \frac{3|x-y|}{2}} p_{\mathbb{H}}(s,z,y). \tag{2.10}$$

Moreover, by Lemma 2.5 and (**Comp**)

$$\begin{aligned}
&\int_0^t \mathbb{P}_x(\tau_{U_1}>s)\mathbb{P}_y(\tau_{\mathbb{H}}>t-s)ds \\
&\quad \leq \int_0^t \mathbb{P}_x(\tau_{\mathbb{H}}>s)\mathbb{P}_y(\tau_{\mathbb{H}}>t-s)ds \\
&\quad \leq c_3\int_0^t \sqrt{\frac{\Phi_1(\delta_{\mathbb{H}}(y))}{s}}\left(\sqrt{\frac{\Phi_1(\delta_{\mathbb{H}}(y))}{t-s}}\wedge 1\right)ds \\
&\quad \leq c_4\sqrt{\Phi\big(\delta_{\mathbb{H}}(x)\big)}\left(\sqrt{\Phi\big(\delta_{\mathbb{H}}(y)\big)}\wedge\sqrt{t}\right)\int_0^t \frac{1}{\sqrt{s(t-s)}}ds \\
&\quad = c_5\sqrt{\Phi\big(\delta_{\mathbb{H}}(x)\big)}\left(\sqrt{\Phi\big(\delta_{\mathbb{H}}(y)\big)}\wedge\sqrt{t}\right).
\end{aligned}$$

Applying this and (2.5), (2.8), (2.9) and (2.10), we obtain,

$$\begin{aligned}
p_{\mathbb{H}}(t,x,y) &\leq c_6\int_0^t \mathbb{P}_x(\tau_{U_1}>s)\mathbb{P}_y(\tau_{\mathbb{H}}>t-s)ds \sup_{w:|w|\geq\frac{1}{3}|x-y|} J(w) \\
&\quad + c_6\mathbb{P}_x(X_{\tau_{U_1}}\in U_2)\sup_{s\leq t, z\in U_2} p(s,z,y) \\
&\leq c_7\sqrt{\Phi\big(\delta_{\mathbb{H}}(x)\big)}\left(\sqrt{\Phi\big(\delta_{\mathbb{H}}(y)\big)}\wedge\sqrt{t}\right)\sup_{w:|w|\geq\frac{1}{3}|x-y|} J(w) \\
&\quad + c_6\mathbb{P}_x(X_{\tau_{U_1}}\in U_2)\sup_{s\leq t, \frac{|x-y|}{2}\leq|z-y|\leq\frac{3|x-y|}{2}} p(s,z,y).
\end{aligned}$$

Finally, applying Lemmas 2.4 and 2.1 and then (2.8), we have

$$\begin{aligned}
\mathbb{P}_x(X_{\tau_{U_1}}\in U_2) &\leq \mathbb{P}_x\left(X_{\tau_{U_1}}\in B\big(x_0, a\Phi^{-1}(t)/(12)\big)^c\right) \\
&\leq \frac{c_8}{t}\mathbb{E}_x[\tau_{U_1}] \leq c_9 t^{-1/2}\sqrt{\Phi\big(\delta_{\mathbb{H}}(x)\big)}.
\end{aligned}$$

Thus we have proved (2.7). □

Example 2.11 Let $d=1$, and

$$j(y) = |y|^{-(1+\alpha)}\left(1+\sum_{n=1}^{\infty} n\mathbf{1}_{[n,n+2^{-n}]}\big(|y|\big)\right),$$

or

$$j(y) = |y|^{-(1+\alpha)} + \sum_{n=1}^{\infty} n \mathbf{1}_{[n,n+2^{-n}]}\big(|y|\big).$$

In either case, the Lévy exponent $\Psi(\xi)$ for symmetric Lévy process having $j(y)$ as its Lévy intensity is comparable to $|\xi|^{\alpha}$. So conditions (**ExpL**) and (**Comp**) are satisfied. Consequently results in this section are valid for this Lévy process. However, in either case, j does not satisfy (**UJS**) at points n when n is large (nor (5.5) below).

3 Consequences of Parabolic Harnack Inequality

Let $Z_s := (V_s, X_s)$ be the space-time process of X, where $V_s = V_0 - s$. The law of the space-time process $s \mapsto Z_s$ starting from (t, x) will be denoted as $\mathbb{P}^{(t,x)}$.

Definition 3.1 A non-negative Borel measurable function $h(t, x)$ on $[0, \infty) \times \mathbb{R}^d$ is said to be *parabolic* (or *caloric*) on $(a, b] \times B(x_0, r)$ if for every relatively compact open subset U of $(a, b] \times B(x_0, r)$, $h(t, x) = \mathbb{E}_{(t,x)}[h(Z_{\tau_U^Z})]$ for every $(t, x) \in U \cap ([0, \infty) \times \mathbb{R}^d)$, where $\tau_U^Z := \inf\{s > 0 : Z_s \notin U\}$.

It follows from the strong Markov property of X and (2.4), $(t, x) \mapsto p_D(t, x, y)$ is parabolic on $(0, \infty) \times D$ for every $y \in D$.

Throughout this section, we assume the following (scale-invariant) parabolic Harnack inequality (**PHI(Φ)**) holds for X: For every $\delta \in (0, 1)$, there exists $c = c(d, \delta) > 0$ such that for every $x_0 \in \mathbb{R}^d$, $t_0 \geq 0$, $R > 0$ and every non-negative function u on $[0, \infty) \times \mathbb{R}^d$ that is parabolic on $(t_0, t_0 + 4\delta\Phi(R)] \times B(x_0, R)$,

$$\sup_{(t_1,y_1)\in Q_-} u(t_1, y_1) \leq c \inf_{(t_2,y_2)\in Q_+} u(t_2, y_2), \qquad \textbf{(PHI(}\boldsymbol{\Phi}\textbf{))}$$

where $Q_- = (t_0 + \delta\Phi(R), t_0 + 2\delta\Phi(R)] \times B(x_0, R/2)$ and $Q_+ = [t_0 + 3\delta\Phi(R), t_0 + 4\delta\Phi(R)] \times B(x_0, R/2)$.

Theorem 3.2 *Suppose that* (**PHI(Φ)**) *holds. Then* (**ExpL**) *holds and so the Lévy process X has a bounded continuous density function $p(t, x)$. Moreover, there is a constant $c > 0$ so that*

$$p(t, x) \leq c\big(\Phi^{-1}(t)\big)^{-d} \quad \textit{for every } t > 0 \textit{ and } x \in \mathbb{R}^d. \tag{3.1}$$

Proof Let $f \geq 0$ be an arbitrary bounded L^1-integrable function on $\mathbb{R}^d$. Clearly $u(t, x) := P_t f(x) = \mathbb{E}_x[f(X_t)]$ is a non-negative parabolic function on $(0, \infty) \times \mathbb{R}^d$. Thus by (**PHI($\Phi$)**) and the symmetry of the semigroup $\{P_t; t > 0\}$, for every $x_0 \in \mathbb{R}^d$ and $t > 0$,

$$\begin{aligned} P_t f(x_0) &\leq c_1 \inf_{z\in B(x_0,\Phi^{-1}(t))} P_{3t} f(z) \leq c_2\big(\Phi^{-1}(t)\big)^{-d} \int_{B(x_0,\Phi^{-1}(t))} P_{3t} f(z)dz \\ &= c_2\big(\Phi^{-1}(t)\big)^{-d} \int_{\mathbb{R}^d} P_{3t}\mathbf{1}_{B(x_0,\Phi^{-1}(t))}(z) f(z)dz \leq c_2\big(\Phi^{-1}(t)\big)^{-d} \int_{\mathbb{R}^d} f(z)dz. \end{aligned}$$

This implies that X has a transition density function $p(t, x)$ and $p(t, x) \leq c_2(\Phi^{-1}(t))^{-d}$ a.e. on $\mathbb{R}^d$. Consequently, as mentioned earlier in the Introduction, (**ExpL**) holds by the

Plancherel theorem, which in turn implies that $p(t,x)$ is bounded and jointly continuous and so (3.1) holds. □

Under the assumptions (**PHI(Φ)**) and (**UJS**), we can derive an interior lower bound for $p_D(t,x,y)$ for all $t>0$; see Propositions 3.4 and 3.5. Similar bound for $t\le T$ was obtained in [16] for subordinate Brownian motions with Gaussian component. In this section, we use the convention that $\delta_D(\cdot)\equiv\infty$ when $D=\mathbb{R}^d$.

The next lemma holds for every symmetric Lévy process and it follows from [27, (3.2)] and [22, Corollary 1].

Lemma 3.3 *For any positive constants a,b, there exists $c=c(a,b,\Psi)>0$ such that for all $z\in\mathbb{R}^d$ and $t>0$,*

$$\inf_{y\in B(z,a\Phi^{-1}(t)/2)}\mathbb{P}_y(\tau_{B(z,a\Phi^{-1}(t))}>bt)\ge c.$$

For the next two results, D is an arbitrary nonempty open set.

Proposition 3.4 *Suppose* (**PHI(Φ)**) *holds. Let $a>0$ be a constant. There exists $c=c(a)>0$ such that*

$$p_D(t,x,y)\ge c\big(\Phi^{-1}(t)\big)^{-d} \tag{3.2}$$

for every $(t,x,y)\in(0,\infty)\times D\times D$ with $\delta_D(x)\wedge\delta_D(y)\ge a\Phi^{-1}(t)\ge 4|x-y|$.

Proof We fix $(t,x,y)\in(0,\infty)\times D\times D$ satisfying $\delta_D(x)\wedge\delta_D(y)\ge a\Phi^{-1}(t)\ge 4|x-y|$. Note that $|x-y|\le a\Phi^{-1}(t)/4$ and that

$$B\big(x,a\Phi^{-1}(t)/4\big)\subset B\big(y,a\Phi^{-1}(t)/2\big)\subset B\big(y,a\Phi^{-1}(t)\big)\subset D.$$

So by the symmetry of p_D, (**PHI(Φ)**), Theorem 3.2, and Lemma 2.1, there exists $c_1=c_1(a)>0$ such that

$$c_1p_D(t/2,x,w)\le p_D(t,x,y)\quad\text{for every } w\in B\big(x,a\Phi^{-1}(t)/4\big).$$

This together with Lemma 3.3 yields that

$$\begin{aligned}p_D(t,x,y)&\ge\frac{c_1}{|B(x,a\Phi^{-1}(t)/4)|}\int_{B(x,a\Phi^{-1}(t)/4)}p_D(t/2,x,w)dw\\&\ge c_2\big(\Phi^{-1}(t)\big)^{-d}\int_{B(x,a\Phi^{-1}(t)/4)}p_{B(x,a\Phi^{-1}(t)/4)}(t/2,x,w)dw\\&=c_2\big(\Phi^{-1}(t)\big)^{-d}\mathbb{P}_x(\tau_{B(x,a\Phi^{-1}(t)/4)}>t/2)\ge c_3\big(\Phi^{-1}(t)\big)^{-d},\end{aligned}$$

where $c_i>0$ for $i=2,3$. □

Recall the condition (**UJS**) from the Introduction.

Proposition 3.5 *Suppose* (**PHI(Φ)**) *and* (**UJS**) *hold. For every $a>0$, there exists a constant $c=c(a)>0$ such that $p_D(t,x,y)\ge ctJ(x-y)$ for every $(t,x,y)\in(0,\infty)\times D\times D$ with $\delta_D(x)\wedge\delta_D(y)\ge a\Phi^{-1}(t)$ and $a\Phi^{-1}(t)\le 4|x-y|$.*

Proof By Lemma 3.3, starting at $z \in B(y,(12)^{-1}a\Phi^{-1}(t))$, with probability at least $c_1 = c_1(a) > 0$ the process X does not move more than $(18)^{-1}a\Phi^{-1}(t)$ by time t. Thus, using the strong Markov property and the Lévy system in (2.3), we obtain

$$\begin{aligned}
&\mathbb{P}_x\big(X_t^D \in B\big(y, 6^{-1}a\Phi^{-1}(t)\big)\big)\\
&\geq c_1\mathbb{P}_x\big(X^D_{t\wedge\tau_{B(x,(18)^{-1}a\Phi^{-1}(t))}} \in B\big(y,(12)^{-1}a\Phi^{-1}(t)\big) \text{ and}\\
&\qquad t\wedge\tau_{B(x,(18)^{-1}a\Phi^{-1}(t))} \text{ is a jumping time}\big)\\
&= c_1\mathbb{E}_x\left[\int_0^{t\wedge\tau_{B(x,(18)^{-1}a\Phi^{-1}(t))}}\int_{B(y,(12)^{-1}a\Phi^{-1}(t))} J(X_s-u)duds\right]. \qquad (3.3)
\end{aligned}$$

By (**UJS**), we obtain

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^{t\wedge\tau_{B(x,(18)^{-1}a\Phi^{-1}(t))}}\int_{B(y,(12)^{-1}a\Phi^{-1}(t))} J(X_s-u)duds\right]\\
&= \mathbb{E}_x\left[\int_0^t\int_{B(y,(12)^{-1}a\Phi^{-1}(t))} J\big(X_s^{B(x,(18)^{-1}a\Phi^{-1}(t))}-u\big)duds\right]\\
&\geq c_2\Phi^{-1}(t)^d\int_0^t \mathbb{E}_x\big[J\big(X_s^{B(x,(18)^{-1}a\Phi^{-1}(t))}-y\big)\big]ds\\
&\geq c_2\Phi^{-1}(t)^d\int_{t/2}^t\int_{B(x,(72)^{-1}a\Phi^{-1}(t/2))} J(w-y)\\
&\qquad\times p_{B(x,(18)^{-1}a\Phi^{-1}(t))}(s,x,w)dwds. \qquad (3.4)
\end{aligned}$$

Since, for $t/2 < s < t$ and $w \in B(x,(72)^{-1}a\Phi^{-1}(t/2))$

$$\delta_{B(x,(18)^{-1}a\Phi^{-1}(t))}(w) \geq (18)^{-1}a\Phi^{-1}(t) - (72)^{-1}a\Phi^{-1}(t/2) \geq 2^{-1}(18)^{-1}a\Phi^{-1}(s)$$

and

$$|x-y| < (72)^{-1}a\Phi^{-1}(t/2) \leq 4^{-1}(18)^{-1}a\Phi^{-1}(s),$$

we have by Theorem 3.2 and Lemma 3.4 that for $t/2 < s < t$ and $w \in B(x,(72)^{-1}a \times \Phi^{-1}(t/2))$,

$$p_{B(x,(18)^{-1}a\Phi^{-1}(t))}(s,x,w) \geq c_3\big(\Phi^{-1}(s)\big)^{-d} \geq c_3\big(\Phi^{-1}(t)\big)^{-d}. \qquad (3.5)$$

Combining (3.3), (3.4) with (3.5) and applying (**UJS**) again, we get

$$\begin{aligned}
&\mathbb{P}_x\big(X_t^D \in B\big(y,6^{-1}a\Phi^{-1}(t)\big)\big)\\
&\geq c_4 t\int_{B(x,(72)^{-1}a\Phi^{-1}(t/2))} J(w-y)dw\\
&\geq c_5 t\big(\Phi^{-1}(t/2)\big)^d J(x-y) \geq c_6 t\big(\Phi^{-1}(t)\big)^d J(x-y). \qquad (3.6)
\end{aligned}$$

In the last inequality we have used Lemma 2.1. The proposition now follows from the Chapman-Kolmogorov equation along with (3.3), (3.4) and Proposition 3.4. Indeed,

$$\begin{aligned} p_D(t,x,y) &= \int_D p_D(t/2,x,z)p_D(t/2,z,y)dz \\ &\geq \int_{B(y,a\Phi^{-1}(t/2)/6)} p_D(t/2,x,z)p_D(t/2,z,y)dz \\ &\geq c_7\big(\Phi^{-1}(t/2)\big)^{-d}\mathbb{P}_x\big(X^D_{t/2}\in B\big(y,a\Phi^{-1}(t/2)/6\big)\big) \\ &\geq c_6c_7 t J(x-y). \end{aligned}$$ □

We now apply Lemma 2.10 to get the following heat kernel upper bound.

Proposition 3.6 *Suppose* (**Comp**), (**PHI(Φ)**) *and* (**UJS**) *hold. Then there exists a constant* $c>0$ *such that for every* $(t,x,y)\in(0,\infty)\times\mathbb{H}\times\mathbb{H}$

$$p_{\mathbb{H}}(t,x,y)\leq c\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}}\wedge 1\right)\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}}\wedge 1\right)\sup_{|w|\geq|x-y|/6}p(t,w).$$

Proof By Lemma 2.7 and Theorem 3.2,

$$p_{\mathbb{H}}(t,x,y)\leq c_1\big(\Phi^{-1}(t)\big)^{-d}\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}}\wedge 1\right)\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}}\wedge 1\right).$$

If $\Phi^{-1}(t)\geq|x-y|$, by Proposition 3.4, $p(t,x-y)\geq c_2(\Phi^{-1}(t))^{-d}$. Thus

$$p_{\mathbb{H}}(t,x,y)\leq c_3\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}}\wedge 1\right)\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}}\wedge 1\right)p(t,x-y). \tag{3.7}$$

We extend the definition of $p(t,w)$ by setting $p(t,w)=0$ for $t<0$ and $w\in\mathbb{R}^d$. For each fixed $x,y\in\mathbb{R}^d$ and $t>0$ with $|x-y|>8r$, one can easily check that $(s,w)\mapsto p(s,w-y)$ is a parabolic function in $(-\infty,\infty)\times B(x,2r)$. Suppose $\Phi^{-1}(t)\leq|x-y|$ and let (s,z) with $s\leq t$ and $\frac{|x-y|}{2}\leq|z-y|\leq\frac{3|x-y|}{2}$. By (**PHI($\Phi$)**), there is a constant $c_4\geq 1$ so that for every $t>0$,

$$\sup_{s\leq t}p(s,z-y)\leq c_4 p(t,z-y).$$

Hence we have

$$\begin{aligned} \sup_{s\leq t,\,\frac{|x-y|}{2}\leq|z-y|\leq\frac{3|x-y|}{2}} p(s,z-y) &\leq c_4 \sup_{\frac{|x-y|}{2}\leq|z-y|\leq\frac{3|x-y|}{2}} p(t,z-y) \\ &= c_4\sup_{\frac{|x-y|}{2}\leq|z|\leq\frac{3|x-y|}{2}}p(t,z). \end{aligned}\tag{3.8}$$

Using this and Lemma 2.10 and Proposition 3.5, we have for every $(t,x,y)\in(0,\infty)\times\mathbb{H}\times\mathbb{H}$ with $\Phi^{-1}(t)\leq|x-y|$,

$$\begin{aligned} &p_{\mathbb{H}}(t,x,y) \\ &\leq c_5\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}}\wedge 1\right)\Big(\sup_{\frac{|x-y|}{2}\leq|z|\leq\frac{3|x-y|}{2}}p(t,z)+\Big(\sqrt{t\Phi\big(\delta_{\mathbb{H}}(y)\big)}\wedge t\Big)\sup_{|w|\geq\frac{|x-y|}{3}}J(w)\Big) \end{aligned}$$

$$\le c_6\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}}\wedge 1\right)\Big(\sup_{|z|\ge|x-y|/2} p(t,z)+\sup_{|w|\ge|x-y|/3} p(t,w)\Big)$$
$$\le 2c_6\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}}\wedge 1\right)\sup_{|w|\ge|x-y|/3} p(t,w).$$

In view of (3.7), the last inequality holds in fact holds for all $(t,x,y)\in(0,\infty)\times\mathbb{H}\times\mathbb{H}$. Thus we have by an analogy of (3.8) that for every $(t,x,y)\in(0,\infty)\times\mathbb{H}\times\mathbb{H}$ with $|x-y|\ge\Phi^{-1}(t)$,

$$\sup_{s\le t,\,\frac{|x-y|}{2}\le|z-y|\le\frac{3|x-y|}{2}} p_{\mathbb{H}}(s,z,y)\le c_7\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}}\wedge 1\right)\sup_{|z-y|\ge|x-y|/2}\ \sup_{|w|\ge|z-y|/3} p(t,w)$$
$$\le c_8\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}}\wedge 1\right)\sup_{|w|\ge|x-y|/6} p(t,w). \tag{3.9}$$

Therefore by Lemma 2.10, Proposition 3.5 and (3.9),

$$\begin{aligned}
&p_{\mathbb{H}}(t,x,y)\\
&\le c_9\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}}\wedge 1\right)\Bigg(\sup_{s\le t,\,\frac{|x-y|}{2}\le|z-y|\le\frac{3|x-y|}{2}} p_{\mathbb{H}}(s,z,y)\\
&\quad+\left(\sqrt{t\Phi\big(\delta_{\mathbb{H}}(y)\big)}\wedge t\right)\sup_{|w|\ge\frac{|x-y|}{3}} J(w)\Bigg)\\
&\le c_{10}\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}}\wedge 1\right)\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}}\wedge 1\right)\\
&\quad\times\left(\sup_{|w|\ge|x-y|/6} p(t,w)+\sup_{|w|\ge|x-y|/3} p(t,w)\right)\\
&\le 2c_{10}\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}}\wedge 1\right)\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}}\wedge 1\right)\sup_{|w|\ge|x-y|/6} p(t,w).
\end{aligned}$$

□

4 Condition (HKC) and Its Consequence

Under the condition (**HKC**), clearly we have the following by Proposition 3.6.

Theorem 4.1 *Suppose that conditions* (**Comp**), (**PHI(Φ)**), (**HKC**), *and* (**UJS**) *hold. Then there exists a constant $C_3>0$ such that for every $(t,x,y)\in(0,\infty)\times\mathbb{H}\times\mathbb{H}$*

$$p_{\mathbb{H}}(t,x,y)\le C_3\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}}\wedge 1\right)\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}}\wedge 1\right)p\big(C_1t,6^{-1}C_2(x-y)\big).$$

We consider the following condition.

$$\lim_{M\to\infty}\sup_{r>0}\frac{\Psi^*(r)}{\Psi^*(Mr)}=0. \tag{4.1}$$

It is equivalent to

$$\lim_{M\to\infty}\sup_{t>0}\frac{t}{\Phi(M\Phi^{-1}(t))}=\lim_{M\to\infty}\sup_{t>0}t\Psi^*\big(M^{-1}\Psi^{*-1}(1/t)\big)=0. \tag{4.2}$$

The following gives a sufficient condition for (4.1).

Proposition 4.2 *Suppose that X has a transition density function $p(t,x)$ that is continuous at $x=0$ for every $t>0$ and $p(t,0)\le c(\Phi^{-1}(t))^{-d}<\infty$ for all $t>0$. Then condition (4.1) holds. In particular,* (**PHI(Φ)**) *implies that the condition (4.1) holds.*

Proof Since X has a transition density function $p(t,x)$ that is continuous at $x=0$ for every $t>0$, $\int_{\mathbb{R}^d}p(t/2,x)^2dx=p(t,0)<\infty$. It follows then $e^{-t\Psi(\xi)}$ is integrable on $\mathbb{R}^d$ and so

$$p(t,x)=(2\pi)^{-d}\int_{\mathbb{R}^d}e^{-ix\cdot\xi}e^{-t\Psi(\xi)}d\xi.$$

In particular, $\int_{\mathbb{R}^d}e^{-t\Psi(\xi)}d\xi=(2\pi)^dp(t,0)\le c_1(\Phi^{-1}(t))^{-d}$ for every $t>0$. In other words,

$$\int_{\mathbb{R}^d}e^{-\Psi(\xi)/r}d\xi\le c_1\big(\Phi^{-1}(1/r)\big)^{-d}=c_1\big(\big(\Psi^*\big)^{-1}(r)\big)^d,\quad r>0.$$

Thus, for all $R,r>0$ we have

$$e^{-\Psi^*(R)/r}\big|B(0,R)\big|\le\int_{B(0,R)}e^{-\Psi(\xi)/r}d\xi\le c_1\big(\big(\Psi^*\big)^{-1}(r)\big)^d. \tag{4.3}$$

Note that $\Psi^*(r)$ is a non-decreasing continuous function on $[0,\infty)$ with $\Psi^*(0)=0$ and $\lim_{r\to\infty}\Psi^*(r)=\infty$. Thus for every $r>0$ and $\lambda>1$, there is $R>0$ so that $\Psi^*(R)=\lambda r$. Hence we have from (4.3) that $e^{-\lambda}((\Psi^*)^{-1}(\lambda r))^d\le c_2((\Psi^*)^{-1}(r))^d$, and so

$$\frac{(\Psi^*)^{-1}(\lambda r)}{(\Psi^*)^{-1}(r)}\le\big(c_2e^{\lambda}\big)^{1/d}. \tag{4.4}$$

For $M>1$, let $\lambda=\lambda(M)=\log(M^d/c_2)$ so that $(c_2e^\lambda)^{1/d}=M$. Then by (4.4) with $s=(\Psi^*)^{-1}(r)$ we have $Ms\ge(\Psi^*)^{-1}(\lambda r)$. In other words, $\Psi^*(Ms)\ge\lambda r=\log(M^d/c_2)\Psi^*(s)$. Therefore

$$\sup_{s>0}\frac{\Psi^*(s)}{\Psi^*(Ms)}\le\frac{1}{\log(M^d/c_2)},$$

which goes to zero as $M\to\infty$. The last assertion of the theorem follows directly from Theorem 3.2. □

Lemma 4.3 *Suppose that (4.1) holds. Then for each fixed $c>0$ the function*

$$H_c(M):=c^{-d}\sup_{t>0}\mathbb{P}\big(|X_t|>cM\Phi^{-1}(t)\big)$$

vanishes at ∞; that is, $\lim_{M\to\infty}H_c(M)=0$.

Proof By Theorem 2.2, we have

$$\sup_{t>0} \mathbb{P}\big(|X_t| > cM\Phi^{-1}(t)\big) \le c_1 \sup_{t>0} \frac{t}{\Phi(cM\Phi^{-1}(t))},$$

which goes to zero as $M \to \infty$ by (4.2). □

For the remainder of this section, we assume that conditions (**Comp**), (**PHI(Φ)**), (**HKC**) and (**UJS**) hold, and discuss some lower bound estimates of $p_{\mathbb{H}}(t, x, y)$ under these conditions. We first note that by (**Comp**) and Lemma 2.5, there exists $C_0 > 0$ such that

$$C_0^{-1}\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1\right) \le \mathbb{P}_x(\tau_{\mathbb{H}} > t) \le C_0\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1\right). \tag{4.5}$$

We denotes by e_d the unit vector in the positive direction of the x_d-axis in $\mathbb{R}^d$.

Lemma 4.4 *There exist $a_1 > 0$ and $M_1 > 4a_1$ such that for every $x \in \mathbb{H}$ and $t > 0$ we have*

$$\int_{\{u\in\mathbb{H}\cap B(\xi_x(t), M_1\Phi^{-1}(t)):\Phi(\delta_{\mathbb{H}}(u))>a_1t\}} p_{\mathbb{H}}(t, x, u)du \ge 4^{-1}C_0^{-1}\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1\right)$$

where $\xi_x(t) := x + a_1\Phi^{-1}(t)e_d$ and C_0 is the constant in (4.5).

Proof By Theorem 4.1 and a change of variable, for every $t > 0$ and $x \in \mathbb{H}$,

$$\begin{aligned}
&\int_{\{u\in\mathbb{H}:\Phi(\delta_{\mathbb{H}}(u))\le at\}} p_{\mathbb{H}}(t, x, u)du \\
&\le C_3\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1\right)\int_{\{u\in\mathbb{H}:\Phi(\delta_{\mathbb{H}}(u))\le at\}}\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(u))}{t}} \wedge 1\right) \\
&\quad \times p\big(C_1t, 6^{-1}C_2(x-u)\big)du \\
&\le C_3\sqrt{a}\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1\right)\int_{\{u\in\mathbb{H}:\Phi(\delta_{\mathbb{H}}(u))\le at\}} p\big(C_1t, 6^{-1}C_2(x-u)\big)du \\
&\le C_3\sqrt{a}\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1\right)\int_{\mathbb{R}^d} p\big(C_1t, 6^{-1}C_2(x-u)\big)du \\
&= C_3(6/C_2)^d\sqrt{a}\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1\right)\int_{\mathbb{R}^d} p(C_1t, w)dw \\
&= C_3(6/C_2)^d\sqrt{a}\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1\right).
\end{aligned} \tag{4.6}$$

Choose $a_1 > 0$ small so that $C_3(6/C_2)^d\sqrt{a_1} \le (8C_0)^{-1}$ where C_0 is the constant in (4.5).

For $x \in \mathbb{H}$, we let $\xi_x(t) := x + a_1\Phi^{-1}(t)e_d$. For every $t > 0$, $M \ge 2a_1$ and $u \in \mathbb{H} \cap B(\xi_x(t), M\Phi^{-1}(t))^c$, we have

$$|x-u| \geq \left|\xi_x(t)-u\right| - \left|x-\xi_x(t)\right| \geq \left|\xi_x(t)-u\right| - a_1\Phi^{-1}(t)$$
$$\geq \left(1-\frac{a_1}{M}\right)\left|\xi_x(t)-u\right| \geq \frac{1}{2}\left|\xi_x(t)-u\right|$$

Thus using Theorem 4.1 and condition (**HKC**), by a change of variable we have for every $t>0$ and $M \geq 2a_1$,

$$\begin{aligned}
&\int_{\mathbb{H}\cap B(\xi_x(t),M\Phi^{-1}(t))^c} p_{\mathbb{H}}(t,x,u)du \\
&\leq C_3\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}}\wedge 1\right)\int_{\mathbb{H}\cap B(\xi_x(t),M\Phi^{-1}(t))^c}\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(u))}{t}}\wedge 1\right) \\
&\quad\times p\big(C_1t, 6^{-1}C_2(x-u)\big)du \\
&\leq C_3\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}}\wedge 1\right)\int_{\mathbb{H}\cap B(\xi_x(t),M\Phi^{-1}(t))^c} p\big(C_1^2t, (12)^{-1}C_2^2\big(\xi_x(t)-u\big)\big)du \\
&\leq C_3\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}}\wedge 1\right)\int_{B(0,M\Phi^{-1}(t))^c} p\big(C_1^2t, (12)^{-1}C_2^2u\big)du \\
&= C_3\big((12)^{-1}C_2^2\big)^d\left(\int_{B(0,(12)^{-1}C_2^2M\Phi^{-1}(t))^c} p\big(C_1^2t, v\big)dv\right)\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}}\wedge 1\right) \\
&\leq C_3H_{(12)^{-1}C_2^2}(M)\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}}\wedge 1\right).
\end{aligned} \tag{4.7}$$

By Lemma 4.3, and Proposition 4.2, we can choose $M_1 > 4a_1$ large so that $C_3H_{(12)^{-1}C_2^2}(M_1) < 8^{-1}\cdot C_0^{-1}$. Then by (4.5), (4.6), (4.7) and our choice of a_1 and M_1, we conclude that

$$\begin{aligned}
&\int_{\{u\in\mathbb{H}\cap B(\xi_x(t),M_1\Phi^{-1}(t)):\Phi(\delta_{\mathbb{H}}(u))>a_1t\}} p_{\mathbb{H}}(t,x,u)du \\
&= \int_{\mathbb{H}} p_{\mathbb{H}}(t,x,u)du - \int_{\mathbb{H}\cap B(\xi_x(t),M_1\Phi^{-1}(t))^c} p_{\mathbb{H}}(t,x,u)du \\
&\quad - \int_{\{u\in\mathbb{H}:\Phi(\delta_{\mathbb{H}}(u))\leq a_1t\}} p_{\mathbb{H}}(t,x,u)du \\
&\geq 4^{-1}\cdot C_0^{-1}\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}}\wedge 1\right).
\end{aligned}$$

□

For $x\in\mathbb{H}$ and $t>0$, let $\xi_x(t) := x + a_1\Phi^{-1}(t)e_d$ and define

$$\mathcal{B}(x,t) := \left\{z\in\mathbb{H}\cap B\big(\xi_x(t), M_1\Phi^{-1}(t)\big) : \Phi\big(\delta_{\mathbb{H}}(z)\big) > a_1t\right\}. \tag{4.8}$$

Theorem 4.5 *There exist $c_1, c_2 > 0$ such that for all $(t,x,y)\in(0,\infty)\times\mathbb{H}\times\mathbb{H}$,*

$$\begin{aligned}
&p_{\mathbb{H}}(t,x,y) \\
&\geq c_1\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}}\wedge 1\right)\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}}\wedge 1\right)\left(\inf_{(u,v)\in\mathcal{B}(x,t)\times\mathcal{B}(y,t)} p_{\mathbb{H}}(t/3,u,v)\right)
\end{aligned} \tag{4.9}$$

$$\geq c_2 \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1 \right) \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1 \right) \times$$
$$\times \begin{cases} \inf_{\substack{(u,v):2M_1\Phi^{-1}(t)\leq|u-v|\leq 3|x-y|/2 \\ \Phi(\delta_{\mathbb{H}}(u))\wedge\Phi(\delta_{\mathbb{H}}(v))>a_1 t}} p_{\mathbb{H}}(t/3,u,v) & \text{if } |x-y| > 4M_1\Phi^{-1}(t), \\ (\Phi^{-1}(t))^{-d} & \text{if } |x-y| \leq 4M_1\Phi^{-1}(t). \end{cases} \tag{4.10}$$

Proof By Chapman-Kolmogorov equation,

$$\begin{aligned} p_{\mathbb{H}}(t,x,y) &\geq \int_{\mathcal{B}(y,t)} \int_{\mathcal{B}(x,t)} p_{\mathbb{H}}(t/3,x,u) p_{\mathbb{H}}(t/3,u,v) p_{\mathbb{H}}(t/3,v,y) du dv \\ &\geq \left(\inf_{(u,v)\in\mathcal{B}(x,t)\times\mathcal{B}(y,t)} p_{\mathbb{H}}(t/3,u,v) \right) \int_{\mathcal{B}(y,t)} \int_{\mathcal{B}(x,t)} \\ &\quad \times p_{\mathbb{H}}(t/3,x,u) p_{\mathbb{H}}(t/3,v,y) du dv. \end{aligned}$$

Thus (4.9) follows from Lemma 4.4.

Observe that for $(u,v) \in \mathcal{B}(x,t) \times \mathcal{B}(y,t)$,

$$\left|\xi_x(t) - \xi_y(t)\right| = |x-y|, \qquad \delta_{\mathbb{H}}(u) \wedge \delta_{\mathbb{H}}(v) \geq a_1\Phi^{-1}(t), \tag{4.11}$$

and

$$\begin{aligned} |x-y| - 2M_1\Phi^{-1}(t) \leq |u-v| &\leq |x-y| + \left|u - \xi_x(t)\right| + \left|v - \xi_y(t)\right| \\ &\leq |x-y| + 2M_1\Phi^{-1}(t). \end{aligned} \tag{4.12}$$

When $|x-y| > 4M_1\Phi^{-1}(t)$, we have by (4.12) that for $(u,v) \in \mathcal{B}(x,t) \times \mathcal{B}(y,t)$,

$$|x-y|/2 \leq |u-v| \leq 3|x-y|/2$$

and so $2M_1\Phi^{-1}(t) \leq |u-v|$. Thus, for $|x-y| > 4M_1\Phi^{-1}(t)$,

$$\inf_{(u,v)\in\mathcal{B}(x,t)\times\mathcal{B}(y,t)} p_{\mathbb{H}}(t/3,u,v) \geq \inf_{\substack{(u,v):2M_1\Phi^{-1}(t)\leq|u-v|\leq 3|x-y|/2 \\ \Phi(\delta_{\mathbb{H}}(u))\wedge\Phi(\delta_{\mathbb{H}}(v))>a_1 t}} p_{\mathbb{H}}(t/3,u,v). \tag{4.13}$$

When $|x-y| \leq 4M_1\Phi^{-1}(t)$, by (4.12) $|u-v| \leq 6M_1\Phi^{-1}(t)$ for $(u,v) \in \mathcal{B}(x,t) \times \mathcal{B}(y,t)$. Thus using (4.11) and (**PHI(Φ)**) (at most $2 + 12[M_1/a_1]$ times) and Lemma 2.1 and Proposition 3.4, we get

$$\begin{aligned} p_{\mathbb{H}}(t/3,u,v) \geq c_1 p_{\mathbb{H}}(t/6,u,u) \geq c_2 \left(\Phi^{-1}(t)\right)^{-d} \\ \text{for every } (u,v) \in \mathcal{B}(x,t) \times \mathcal{B}(y,t). \end{aligned} \tag{4.14}$$

(4.10) now follows from (4.9), (4.13) and (4.14). □

5 Heat Kernel Upper Bound Estimates in Half Spaces

In this section, we consider a large class of symmetric Lévy processes with concrete condition on the Lévy densities. Under these conditions, we can check that conditions (**Comp**), (**HKC**) and (**PHI(Φ)**) all hold. Thus we can apply Proposition 3.6 and Theorem 4.5 to establish sharp two-sided estimates of the transition density of such Lévy processes in half spaces.

Suppose that ψ_1 is an increasing function on $[0, \infty)$ with $\psi_1(r) = 1$ for $0 < r \le 1$ and there are constants $a_2 \ge a_2 > 0$, $\gamma_2 \ge \gamma_1 > 0$ and $\beta \in [0, \infty]$ so that

$$a_1 e^{\gamma_1 r^\beta} \le \psi_1(r) \le a_2 e^{\gamma_2 r^\beta} \quad \text{for every } 1 < r < \infty. \tag{5.1}$$

Suppose that ϕ_1 is a strictly increasing function on $[0, \infty)$ with $\phi_1(0) = 0$, $\phi_1(1) = 1$ and there exist constants $0 < a_3 < a_4$ and $0 < \beta_1 \le \beta_2 < 2$ so that

$$a_3 \left(\frac{R}{r}\right)^{\beta_1} \le \frac{\phi_1(R)}{\phi_1(r)} \le a_4 \left(\frac{R}{r}\right)^{\beta_2} \quad \text{for every } 0 < r < R < \infty. \tag{5.2}$$

Since $0 < \beta_1 \le \beta_2 < 2$, (5.2) implies that

$$\int_0^r \frac{s}{\phi_1(s)} ds \asymp \frac{r^2}{\phi_1(r)}, \qquad \int_r^\infty \frac{1}{s\phi_1(s)} ds \asymp \frac{1}{\phi_1(r)} \quad \text{for every } r > 0. \tag{5.3}$$

Throughout the remainder of this paper, we assume that (**UJS**) holds and that there are constants $\gamma \ge 1$, κ_1, κ_2 and $a_0 \ge 0$ such that

$$\gamma^{-1} a_0 |\xi|^2 \le \sum_{i,j=1}^d a_{i,j} \xi_i \xi_j \le \gamma a_0 |\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^d, \tag{5.4}$$

and

$$\gamma^{-1} \frac{1}{|x|^d \phi_1(|x|) \psi_1(\kappa_2 |x|)} \le J(x) \le \gamma \frac{1}{|x|^d \phi_1(|x|) \psi_1(\kappa_1 |x|)} \quad \text{for } x \in \mathbb{R}^d. \tag{5.5}$$

Note that (**UJS**) holds if $\kappa_1 = \kappa_2$ in (5.5).

Recall Φ is the function defined in (1.8). The next lemma gives explicit relation between Φ and ϕ_1.

Lemma 5.1 *When* $\beta = 0$,

$$\Phi(r) \asymp \begin{cases} \phi_1(r) \mathbf{1}_{\{a_0 = 0\}} + r^2 \mathbf{1}_{\{a_0 > 0\}} & \text{for } r \in [0, 1], \\ \phi_1(r) & \text{for } r \ge 1; \end{cases} \tag{5.6}$$

while for $\beta \in (0, \infty]$,

$$\Phi(r) \asymp \begin{cases} \phi_1(r) \mathbf{1}_{\{a_0 = 0\}} + r^2 \mathbf{1}_{\{a_0 > 0\}} & \text{for } r \in [0, 1], \\ r^2 & \text{for } r \ge 1. \end{cases} \tag{5.7}$$

Proof By Lemma 2.3 and (5.4),

$$\frac{1}{\Phi(r)} \asymp \frac{a_0}{r^2} + \int_{\mathbb{R}^d} \left(1 \wedge \frac{|z|^2}{r^2}\right) J(z) dz$$

Thus, by (5.5) and (5.1)

$$c_0^{-1}\left(\frac{a_0}{r^2}+r^{-2}\int_0^r \frac{s}{\phi_1(s)}e^{-\kappa_2\gamma_2 s^\beta}ds+\int_r^\infty \frac{1}{s\phi_1(s)}e^{-\kappa_2\gamma_2 s^\beta}ds\right)$$
$$\le \frac{1}{\Phi(r)}\le c_0\left(\frac{a_0}{r^2}+r^{-2}\int_0^r \frac{s}{\phi_1(s)}e^{-\kappa_1\gamma_1 s^\beta}ds+\int_r^\infty \frac{1}{s\phi_1(s)}e^{-\kappa_1\gamma_1 s^\beta}ds\right). \quad (5.8)$$

When $\beta=0$, it follows from (5.3) and (5.8) that

$$\frac{1}{\Phi(r)}\asymp \frac{a_0}{r^2}+r^{-2}\int_0^r \frac{s}{\phi_1(s)}ds+\int_r^\infty \frac{1}{s\phi_1(s)}ds\asymp \frac{a_0}{r^2}+\frac{1}{\phi_1(r)} \quad \text{for } r>0. \quad (5.9)$$

Note that taking $R=1$ and $r=1$ in (5.2), we have

$$\phi_1(r)\ge a_4^{-1}r^{\beta_2}\ge a_4^{-1}r^2 \quad \text{for } r\in[0,1] \quad \text{and}$$
$$\phi_1(R)\le a_4R^{\beta_2}\le a_4R^2 \quad \text{for } R\ge 1. \quad (5.10)$$

This together with (5.9) establishes (5.6).

When $r\ge 1$ and $\beta>0$,

$$\int_r^\infty s^{-\beta_1-1}e^{-\kappa_1\gamma_1 s^\beta}ds\le c_1\int_r^\infty s^{-3}ds\le c_1r^{-2}/2.$$

Thus by (5.2), for $\beta>0$ and $r\ge 1$,

$$r^{-2}\int_0^r \frac{s}{\phi_1(s)}e^{-\kappa_1\gamma_1 s^\beta}ds+\int_r^\infty \frac{1}{s\phi_1(s)}e^{-\kappa_1\gamma_1 s^\beta}ds$$
$$\le r^{-2}\int_0^\infty \frac{s}{\phi_1(s)}e^{-\kappa_1\gamma_1 s^\beta}ds+\int_r^\infty \frac{1}{s\phi_1(s)}e^{-\kappa_1\gamma_1 s^\beta}ds\le c_2r^{-2},$$

while

$$r^{-2}\int_0^r \frac{s}{\phi_1(s)}e^{-\kappa_2\gamma_2 s^\beta}ds+\int_r^\infty \frac{1}{s\phi_1(s)}e^{-\kappa_2\gamma_2 s^\beta}ds\ge r^{-2}\int_0^1 \frac{s}{\phi_1(s)}e^{-\kappa_2\gamma_2 s^\beta}ds\ge c_3r^{-2}.$$

By (5.3), for $r\le 1$ and $\beta>0$,

$$r^{-2}\int_0^r \frac{s}{\phi_1(s)}e^{-\kappa_1\gamma_1 s^\beta}ds+\int_r^\infty \frac{1}{s\phi_1(s)}e^{-\kappa_1\gamma_1 s^\beta}ds$$
$$\le r^{-2}\int_0^r \frac{1}{s\phi_1(s)}ds+\int_r^1 \frac{1}{s\phi_1(s)}ds+\int_1^\infty e^{-\kappa_1\gamma_1 s^\beta}ds$$
$$\le \frac{c_4}{\phi_1(r)}+c_4\le \frac{c_5}{\phi_1(r)}$$

and

$$r^{-2}\int_0^r \frac{s}{\phi_1(s)}e^{-\kappa_2\gamma_2 s^\beta}ds+\int_r^\infty \frac{1}{s\phi_1(s)}e^{-\kappa_2\gamma_2 s^\beta}ds\ge e^{-\kappa_2\gamma_2}r^{-2}\int_0^r \frac{s}{\phi_1(s)}ds\ge \frac{c_6}{\phi_1(r)}.$$

These combined with (5.8) and (5.10) immediately yield (5.7). □

As an immediate consequence of Lemmas 2.9 and 5.1, we have the following.

Corollary 5.2 *The conditions* (4.1), (**ExpL**) *and* (**Comp**) *hold.*

Since we have assumed (**UJS**), (5.4) and (5.5), our Lévy process X belongs to a subclass of the processes considered in [8, 9, 18, 19]. Therefore $p(t,x,y)$ is Hölder continuous on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ and for every open set D, transition density $p_D(t,x,y)$ for the killed process X^D is Hölder continuous on $(0,\infty)\times D\times D$. Define

$$p^c(t,r)=t^{-d/2}\exp\bigl(-r^2/t\bigr). \tag{5.11}$$

Recall that a_0 is the ellipticity constant in (5.4). For each $a, T>0$, we define a function $h_{a,T}(t,r)$ on $(t,r)\in(0,T]\times[0,\infty)$ as

$$h_{a,T}(t,r):=\begin{cases} a_0p^c(t,ar)+(\Phi^{-1}(t)^{-d}\wedge(tj(ar)) & \text{if } \beta\in[0,1] \text{ or } r\in[0,1],\\ t\exp(-a(r(\log\frac{Tr}{t})^{(\beta-1)/\beta}\wedge r^\beta)) & \text{if } \beta\in(1,\infty) \text{ with } r\ge 1,\\ (t/(Tr))^{ar} & \text{if } \beta=\infty \text{ with } r\ge 1; \end{cases} \tag{5.12}$$

and, for each $a, T>0$, define a function $k_{a,T}(t,r)$ on $(t,r)\in[T,\infty)\times[0,\infty)$ as

$$k_{a,T}(t,r):=\begin{cases} \Phi^{-1}(t)^{-d}\wedge[(a_0p^c(t,ar))+tj(ar)] & \text{if } \beta=0,\\ t^{-d/2}\exp(-a(r^\beta\wedge\frac{r^2T}{t})) & \text{if } \beta\in(0,1],\\ t^{-d/2}\exp(-ar((1+\log^+\frac{rT}{t})^{(\beta-1)/\beta}\wedge\frac{rT}{t})) & \text{if } \beta\in(1,\infty),\\ t^{-d/2}\exp(-ar((1+\log^+\frac{rT}{t})\wedge\frac{r^2T}{t})) & \text{if } \beta=\infty. \end{cases} \tag{5.13}$$

Note that $r\to h_{a,T}(t,r)$ and $r\to k_{a,T}(t,r)$ are decreasing.

Theorem 5.3 *The parabolic Harnack inequality* (**PHI(Φ)**) *holds. Moreover, for each positive constant T, there are positive constants c_i, $i=1,\dots 6$, which depend on the ellipticity constant a_0 of* (5.4), *such that*

$$c_2^{-1}h_{c_1,T}\bigl(t,|x|\bigr)\le p(t,x)\le c_2h_{c_3,T}\bigl(t,|x|\bigr)\quad \textit{for every } (t,x)\in(0,T]\times\mathbb{R}^d,$$

and

$$c_4^{-1}k_{c_5,T}\bigl(t,|x|\bigr)\le p(t,x)\le c_4k_{c_6,T}\bigl(t,|x|\bigr)\quad \textit{for every } (t,x)\in[T,\infty)\times\mathbb{R}^d.$$

In particular, the condition (**HKC**) *holds.*

The above two-sided estimates on $p(t,x,y)$ follow from [18, Theorem 1.2] and [8, Theorems 1.2 and 1.4] when $a_0=0$, and from [9, 19] when $a_0>0$. Note that even though in [18, Theorem 1.2] and [8, Theorems 1.2 and 1.4] two-sided estimates for $p(t,x,y)$ are stated separately for the cases $0<t\le 1$ and $t>1$, the constant 1 does not play any special role. In fact, for example for $T<1$ one can easily check

$$c_3^{-1}h_{c_2,1}(t,r)\le h_{c_1,T}(t,r)\le c_3h_{c_2,1}(t,r)\quad \text{on } t<T,$$

and the two-sided estimates for $p(t,x)$ hold for the cases $0<t\le T$ and $t>T$, and can be stated in the above way.

Remark 5.4 We remark here that in [8, Theorems 1.2(2.b)], the $|\log \frac{|x-y|}{t}|$ term should replaced by $1+\log^+ \frac{|x-y|}{t}$. In the proof of [8, Theorems 1.2(2.b)], the case that $|x-y| \asymp t$ when $\beta \in (1,\infty)$ missed to be considered. Once taking into account of that missed case, One can see from [8] that (5.13) is the correct form. See the statement and the proof of Proposition 6.7 below for the lower bound.

We now present the main result of this section.

Theorem 5.5 *There exist* $c_1, c_2 > 0$ *such that for all* $(t,x,y) \in (0,\infty) \times \mathbb{H} \times \mathbb{H}$,

$$p_{\mathbb{H}}(t,x,y) \le c_1 \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1\right)\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1\right) \times \begin{cases} h_{c_2,1}(t,|x-y|/6) & \text{if } t \in (0,1), \\ k_{c_2,1}(t,|x-y|/6) & \text{if } t \in [1,\infty). \end{cases}$$

Proof Since $r \to h_{a,T}(t,r)$ and $r \to k_{a,T}(t,r)$ are decreasing, by Theorem 5.3

$$\sup_{w:|w|\ge \frac{|x-y|}{6}} p(t,w) \le c_1 \begin{cases} \sup_{w:\frac{|x-y|}{6}\le|w|} h_{c_2,1}(t,|w|) & \text{if } t \in (0,1), \\ \sup_{w:\frac{|x-y|}{6}\le|w|} k_{c_2,1}(t,|w|) & \text{if } t \in [1,\infty), \end{cases}$$

$$\le c_1 \begin{cases} h_{c_2,1}(t,|x-y|/6) & \text{if } t \in (0,1), \\ k_{c_2,1}(t,|x-y|/6) & \text{if } t \in [1,\infty). \end{cases}$$

This together with Proposition 3.6 proves the theorem. □

6 Interior Lower Bound Estimates

In this section, we derive following preliminary lower bound estimates on $p_{\mathbb{H}}(t,x,y)$. Recall that we have assumed (**UJS**), (5.4) and (5.5).

Theorem 6.1 *Let* a, T *be positive constants. There exist* $c = c(a,\beta_1,\beta_2,\beta,T) > 0$ *and* $C_4 = C_4(a,\beta_1,\beta_2,\beta,T) > 0$ *such that*

$$p_{\mathbb{H}}(t,x,y) \ge c \begin{cases} h_{C_4,T}(t,|x-y|) & \text{if } t \in (0,T), \\ k_{C_4,T}(t,|x-y|) & \text{if } t \in [T,\infty), \end{cases}$$

for every $(t,x,y) \in (0,\infty) \times \mathbb{H} \times \mathbb{H}$ *with* $\delta_{\mathbb{H}}(x) \wedge \delta_{\mathbb{H}}(y) \ge a\Phi^{-1}(t)$.

We will prove this theorem through several propositions. The following proposition follows immediately from Propositions 3.4 and 3.5, Lemma 5.1 and condition (5.5).

Proposition 6.2 *Let* D *be an open subset of* $\mathbb{R}^d$. *For every* $a > 0$, *there exists a constant* $c = c(a) > 0$ *so that*

$$p_D(t,x,y) \ge c\left(\left(\Phi^{-1}(t)\right)^{-d} \wedge t\, j\left(|x-y|\right)\right)$$

for every $(t,x,y) \in (0,\infty) \times D \times D$ *with* $\delta_D(x) \wedge \delta_D(y) \ge a\Phi^{-1}(t)$.

Proposition 6.2 yields the interior lower bound for $p_D(t,x,y)$ and $p(t,x,y)$ for the case $\beta=0$ and $a_0=0$. Proposition 6.2 also yield the interior lower bound for $p_D(t,x,y)$ and $p(t,x,y)$ for the case $\beta\in(0,1]$, $t\le T$ and $a_0=0$. As a direct consequence of Proposition 3.5, we have

Corollary 6.3 *Suppose $\beta\in(0,\infty)$. For every $a,T,C_*>0$, there exist $c_1,c_2>0$ so that*

$$p_{\mathbb{H}}(t,x,y)\ge c_1te^{-c_2|x-y|^\beta}$$

for every $(t,x,y)\in[T,\infty)\times\mathbb{H}\times\mathbb{H}$ with $\delta_{\mathbb{H}}(x)\wedge\delta_{\mathbb{H}}(y)\ge a\Phi^{-1}(t)$ and $|x-y|\ge C_*\Phi^{-1}(t)$. In particular, when $0<\beta\le 1$, for every $a,T,C_*>0$, there exist $c_1,c_2>0$ such that

$$\begin{aligned}&p_{\mathbb{H}}(t,x,y)\ge c_1te^{-c_2|x-y|^\beta}\quad \textit{when } |x-y|^{2-\beta}\ge t/C_*,\\&\delta_{\mathbb{H}}(x)\wedge\delta_{\mathbb{H}}(y)\ge a\Phi^{-1}(t)\ \textit{and}\ t\ge T.\end{aligned}$$

The last assertion in Corollary 6.3 holds because $\Phi^{-1}(t)\asymp t^{1/2}$ for $t\ge T$ (by Lemma 5.1), and for $t\ge T$ and x,y with $|x-y|^{2-\beta}\ge t/C_*$, one has $|x-y|^2\ge ct$ where $c=(T/C_*)^{1/(2-\beta)}C_*^{-1}$.

A standard chaining argument give the following Gaussian lower bound. The proof is similar to the one of [8, Theorem 5.4].

Proposition 6.4 *Suppose $\beta\in(0,\infty]$. For every $C_*,a,T>0$, there exist constants $c_1,c_2>0$ such that*

$$p_{\mathbb{H}}(t,x,y)\ge c_1t^{-d/2}\exp\left(-\frac{c_2|x-y|^2}{t}\right)$$

for every $(t,x,y)\in[T,\infty)\times\mathbb{H}\times\mathbb{H}$ with $\delta_{\mathbb{H}}(x)\wedge\delta_{\mathbb{H}}(y)\ge a\Phi^{-1}(t)$ and $C_*|x-y|\le t/T$.

Proof By considering t/T instead of t, without loss of generality we assume $T=1$. Fix a constant $C_*>0$ and let $R:=|x-y|$. When $t\ge 1\ge R$, by Proposition 6.2 and (5.7), $p_D(t,x,y)\ge c_1\Phi^{-1}(t)^{-d}\ge c_2t^{-d/2}$. When $t\ge R^2\ge 1$, note that $R^2\le c_3\Phi(R)$ for some $c_3>0$. Thus in view of Lemma 2.1, by applying the parabolic Harnack inequality (**PHI(Φ)**) at most $3(1+16a^{-2})c_3$ times, we have from Proposition 3.4 that $p_D(t,x,y)\ge c_4\Phi^{-1}(t)^{-d}\ge c_5t^{-d/2}$. Hence we only need to consider the case $1\vee(C_*R)\le t\le R^2$ (so $C_*\le 1$), which we now assume. By (5.7), there exist a constant $c_0\in(0,1)$ such that

$$c_0^{-1}\sqrt{s}\ge\Phi^{-1}(s)\ge c_0\sqrt{s}\quad \text{for every } s\ge 2^{-1}(C_*)^2.$$

Thus $\delta_{\mathbb{H}}(x)\wedge\delta_{\mathbb{H}}(y)\ge ac_0\sqrt{t}$.

Let n be the smallest positive integer so that $t/n\ge(R/n)^2$. Then

$$1\le R^2/t\le n<1+R^2/t\le 2R^2/t\quad \text{and}\quad 2(R/n)^2\ge t/n\ge(R/n)^2. \tag{6.1}$$

Since $t\ge C_*R$, by (6.1)

$$\frac{t}{n}\ge\frac{t}{1+R^2/t}=\frac{t^2}{t+R^2}\ge 2^{-1}\left(\frac{t}{R}\right)^2\ge 2^{-1}(C_*)^2. \tag{6.2}$$

Let $x = x_0, x_1, \ldots, x_n = y$ be the points equally spaced on the line segment connecting x to y so that $|x_i - x_{i+1}| = R/n$ for $i = 0, \ldots, n-1$. Set $B_i := B(x_i, 2^{-1}ac_0 R/n)$. Since $t/n \geq (R/n)^2$ (by (6.1)) and $t/n \geq 2^{-1}(C_*)^2$ (by (6.2)), we have for every $(y_i, y_{i+1}) \in B_i \times B_{i+1}$,

$$\delta_{\mathbb{H}}(y_i) \wedge \delta_{\mathbb{H}}(y_{i+1}) \geq ac_0\sqrt{t} - 2^{-1}ac_0 R/n \geq 2^{-1}ac_0\sqrt{t/n} \geq 2^{-1}ac_0^2\Phi^{-1}(t/n)$$

and

$$4|y_i - y_{i+1}| \leq 4(1 + ac_0)R/n \leq 4(1 + ac_0)\sqrt{t/n} \leq 4\big(c_0^{-1} + a\big)\Phi^{-1}(t/n).$$

By Proposition 3.4 and applying (**PHI(Φ)**) at most N times, where N depends only on a, Φ and C_* (or by Proposition 6.2), we have

$$p_{\mathbb{H}}(t/n, y_i, y_{i+1}) \geq c_6(t/n)^{-d/2} \quad \text{for every } (y_i, y_{i+1}) \in B_i \times B_{i+1}. \tag{6.3}$$

Using (6.3) and then (6.1), we have

$$\begin{aligned} p_{\mathbb{H}}(t, x, y) &\geq \int_{B_1} \cdots \int_{B_{n-1}} p_{\mathbb{H}}(t/n, x, y_1) \ldots p_{\mathbb{H}}(t/n, y_{n-1}, y) dy_1 \ldots dy_{n-1} \\ &\geq c_6(t/n)^{-d/2} \prod_{i=1}^{n-1} \big(c_7(t/n)^{-d/2}(R/n)^d\big) \geq c_6(t/n)^{-d/2}\big(c_7 2^{-d/2}\big)^{n-1} \\ &\geq c_8(t/n)^{-d/2}\exp(-c_9 n) \geq c_{10} t^{-d/2}\exp\left(-\frac{c_{11}|x-y|^2}{t}\right). \end{aligned} \tag{6.4}$$

□

Proposition 6.5 *Suppose $a_0, a > 0$. There are positive constants c_1 and c_2 so that*

$$p_{\mathbb{H}}(t, x, y) \geq c_1\Phi^{-1}(t)^{-d} \wedge \left(t^{-d/2}\exp\left(-\frac{c_2|x-y|^2}{t}\right) + tj\big(|x-y|\big)\right)$$

for every $(t, x, y) \in (0, \infty) \times \mathbb{H} \times \mathbb{H}$ with $\delta_{\mathbb{H}}(x) \wedge \delta_{\mathbb{H}}(y) \geq a\Phi^{-1}(t)$ if either $\beta \in [0, 1]$ or $|x - y| \leq 1$.

Proof We first consider following five cases: (1) $t \geq 1$ and $|x - y| \leq 1$ when $\beta \in [0, \infty]$, (2) $t \geq 1$ and $x, y \in \mathbb{R}^d$ when $\beta = 0$, (3) $|x - y|^{2-\beta} \geq t \geq 1$ when $\beta \in (0, 1]$, (4) $t \leq 1$ and $|x - y| \geq 1$ when $\beta \in (0, 1]$ (5) $|x - y|^2 \leq t \leq 1$ when $\beta \in [0, \infty]$.

Using the condition (5.5), we see that for these five cases it holds that for every $c_1 > 0$ there is $c_2 > 0$ such that

$$t^{-d/2}\exp\left(-\frac{c_1|x-y|^2}{t}\right) \leq c_2 tj\big(|x-y|\big). \tag{6.5}$$

Hence by Propositions 6.2 and 6.4 and (5.6)–(5.7), it suffices to consider the case when $t \leq |x - y|^2 \leq 1$, which we will assume for the remainder of the proof.

By (5.6)–(5.7), there is a constant $c_1 \in (0, 1/2)$ so that $c_1 r^2 \leq \Phi(r) \leq r^2/c_1$. Set $R = |x - y|$. Let n to be the smallest integer so that $t/n \geq c_1^{-1}(R/n)^2$. Observe that $\frac{R^2}{c_1 t} \leq n \leq \frac{2R^2}{c_1 t}$. Let $x_0 = x, x_1, \ldots, x_n = y$ be the evenly spaced points on be the line segment connecting x

to y so that $|x_i - x_{i+1}| = R/n$. Let $B_i = B_i(x_i, aR/4n)$. Then for every we have for every $(y_i, y_{i+1}) \in B_i \times B_{i+1}$,

$$\delta_{\mathbb{H}}(y_i) \wedge \delta_{\mathbb{H}}(y_{i+1}) \geq a\sqrt{c_1 t} - aR/2n \geq a\sqrt{c_1 t}/2 \geq 2^{-1} a c_1 \Phi^{-1}(t)$$

and

$$4|y_i - y_{i+1}| \leq 4(1+a)R/n \leq 8\sqrt{c_1 t/n} \leq 8\Phi^{-1}(t/n).$$

By Proposition 3.4 and applying (**PHI(Φ)**) at most finite many times (or by Proposition 6.2), we have

$$p_{\mathbb{H}}(t/n, y_i, y_{i+1}) \geq c_3 (t/n)^{-d/2} \quad \text{for every } (y_i, y_{i+1}) \in B_i \times B_{i+1}. \tag{6.6}$$

Using (6.6) and then (6.1), by the same argument in (6.4) we have

$$p_{\mathbb{H}}(t, x, y) \geq c_4 t^{-d/2} \exp\left(-\frac{c_5 |x-y|^2}{t}\right).$$

This together with Proposition 6.2 gives the desired lower bound interior estimate for $t \leq |x-y|^2 \leq 1$. This completes the proof of the proposition. □

Proposition 6.2, Corollary 6.3 and Propositions 6.4–6.5 give the desired interior lower bound stated in Theorem 6.1 for $p_{\mathbb{H}}(t, x, y)$ when $\beta \in [0, 1]$. We now consider the case $\beta = \infty$ and the case $\beta \in (1, \infty)$ separately.

Proposition 6.6 *Suppose that $T, a > 0$ and $\beta = \infty$. Then, there exist constants $c_i = c_i(a, T) > 0$, $i = 1, 2$, such that for any x, y in $\mathbb{H}$ with $\delta_{\mathbb{H}}(x) \wedge \delta_{\mathbb{H}}(y) \geq a\Phi^{-1}(t)$, we have*

$$p_{\mathbb{H}}(t, x, y) \geq c_1 \left(\frac{t}{T|x-y|}\right)^{c_2|x-y|} \quad \textit{when } |x-y| \geq 1 \vee (t/T). \tag{6.7}$$

Proof By considering t/T instead of t, without loss of generality we assume $T = 1$. We let $R_1 := |x-y| \geq 1$. We define k as the integer satisfying $(4 \leq) 4R_1 \leq k < 4R_1 + 1 < 5R_1$ and $r_t := 2^{-1} a \Phi^{-1}(t)$. Let $x = x_0, x_1, \ldots, x_k = y$ be the points equally spaced on the line segment connecting x to y so that $|x_i - x_{i+1}| = R_1/k$ for $i = 0, \ldots, k-1$ and $B_i := B(x_i, r_{t/k})$, with $i = 0, 1, 2, \ldots, k$. Then, $\delta_{\mathbb{H}}(x_i) > 2r_t$ and $B_i \subset B(x_i, r_t) \subset B(x_i, 2r_t) \subset \mathbb{H}$, with $i = 0, 1, 2, \ldots, k$.

Since $4(1 \vee t) \leq 4R_1 \leq k$, we have $r_{t/k} \leq a/8$ and, for each $y_i \in B_i$

$$\begin{aligned} |y_i - y_{i+1}| &\leq |y_i - x_i| + |x_i - x_{i+1}| + |x_{i+1} - y_{i+1}| \\ &\leq \frac{a\Phi^{-1}(1/4)}{2} + \frac{R_1}{k} + \frac{a\Phi^{-1}(1/4)}{2} \leq a\Phi^{-1}(1/4) + 1/4. \end{aligned} \tag{6.8}$$

Moreover, $\delta_{\mathbb{H}}(y_i) \geq \delta_{\mathbb{H}}(x_i) - |y_i - x_i| > r_t > r_{t/k}$.

Thus, by Proposition 6.2 and (6.8) and using the fact $t/k \leq R_1/k \leq 1/4$, there are constants $c_i = c_i(a) > 0$, $i = 1, 2$, such that for $(y_i, y_{i+1}) \in B_i \times B_{i+1}$ we have

$$\begin{aligned} p_{\mathbb{H}}(t/k, y_i, y_{i+1}) &\geq c_1 \left(\phi_1^{-1}(t/k)^{-d} \wedge \frac{t/k}{|y_i - y_{i+1}|^d \phi_1(|y_i - y_{i+1}|)}\right) \\ &\geq c_2 \big(\phi_1^{-1}(t/k)^{-d} \wedge t/k\big) = c_2 \big(1 \wedge (t/k)\big) = c_2 t/k. \end{aligned} \tag{6.9}$$

Observe that $4R_1 \le k < 2(k-1) < 8R_1$, $\phi_1^{-1}(t/k) \ge a_3^{1/\beta_1}(t/k)^{1/\beta_1}$. Thus, from (6.9) we obtain

$$\begin{aligned} p_{\mathbb{H}}(t,x,y) &\ge \int_{B_1}\cdots\int_{B_{k-1}} p_{\mathbb{H}}(t/k,x,y_1)\ldots p_{\mathbb{H}}(t/k,y_{k-1},y)dy_{k-1}\ldots dy_1 \\ &\ge (c_2t/k)^k\prod_{i=1}^{k-1}|B_i| \ge (c_2t/k)^k c_3^{k-1}(t/k)^{d(k-1)/\beta_1} \\ &\ge c_4(c_5t/k)^{c_6k} \ge c_7(c_8t/R_1)^{c_9R_1} \ge c_{10}(t/R_1)^{c_{11}R_1}. \end{aligned}$$

□

Proposition 6.7 *Suppose that $T>0$, $a>0$ and $\beta\in(1,\infty)$. Then, there exist constants $c_i=c_i(a,\beta,T)>0$, $i=1,2$ such that for any x,y in $\mathbb{H}$ with $\delta_{\mathbb{H}}(x)\wedge\delta_{\mathbb{H}}(y)\ge a\Phi^{-1}(t)$ we have*

$$p_{\mathbb{H}}(t,x,y) \ge c_1 t\exp\left(-c_2\left(|x-y|\left(\log\frac{T|x-y|}{t}\right)^{\frac{\beta-1}{\beta}} \wedge \left(|x-y|\right)^{\beta}\right)\right)$$
$$\text{if } t\le T,\ |x-y|>1,$$

and

$$p_{\mathbb{H}}(t,x,y) \ge c_1 t^{-d/2}\exp\left(-c_2\left(|x-y|\left(1+\log^+\frac{T|x-y|}{t}\right)^{\frac{\beta-1}{\beta}}\right)\right)$$
$$\text{if } t>T, |x-y|>t/T.$$

Proof Without loss of generality we assume $T=1$. We fix $a>0$, and we let $R_1:=|x-y|$.

(i) If $1\le R_1\le 3$ and $t\le 1$, the proposition holds by virtue of Proposition 6.2.
(ii) If $R_1(\log(R_1/t))^{(\beta-1)/\beta}\ge(R_1)^\beta$ (when $t\le 1$), the proposition holds also by virtue of Proposition 6.2.
(iii) If $t>1$ and $3t\ge R_1\ge t$, the proposition holds by virtue of Proposition 6.4.
(iv) We now assume $(t,R_1)\in((0,1]\times(3,\infty))\cup((1,\infty)\times(3t,\infty))$ and $R_1(\log(R_1/t))^{(\beta-1)/\beta}<(R_1)^\beta$, which is equivalent to $R_1\exp\{-(R_1)^\beta\}<t$. Note that $R_1/t>3$.

Let $k\ge 2$ be a positive integer such that

$$1 < R_1\left(\log\frac{R_1}{t}\right)^{-1/\beta} \le k < R_1\left(\log\frac{R_1}{t}\right)^{-1/\beta}+1 < 2R_1\left(\log\frac{R_1}{t}\right)^{-1/\beta}. \tag{6.10}$$

We define $r_t:=(2^{-1}a\Phi^{-1}(t/R_1))\wedge((6)^{-1}(\log(R_1/t))^{1/\beta})$. Then, by (6.10) we have

$$\left(2^{-1}a\Phi^{-1}(t/R_1)\right)\wedge\frac{R_1}{6k} \le r_t \le \frac{1}{6}\left(\log\frac{R_1}{t}\right)^{1/\beta} < \frac{R_1}{3k}. \tag{6.11}$$

Let $x=x_0,x_1,\ldots,x_k=y$ be the points equally spaced on the line segment connecting x to y so that $|x_i-x_{i+1}|=R_1/k$ for $i=0,\ldots,k-1$ and $B_i:=B(x_i,r_t)$, with $i=0,1,2,\ldots,k$. Then, $\delta_{\mathbb{H}}(y_i)\ge 2^{-1}a\Phi^{-1}(t)>2^{-1}a\Phi^{-1}(t/k)$ for every $y_i\in B_i$. Note that from (6.11) we obtain

$$\frac{1}{3}\frac{R_1}{k} \le |x_i-x_{i+1}|-2r_t \le |y_i-y_{i+1}| \le |x_i-x_{i+1}|+2r_t \le \frac{5}{3}\frac{R_1}{k} \tag{6.12}$$

for every $(y_i, y_{i+1}) \in B_i \times B_{i+1}$. We also observe that, by (6.10)

$$\frac{t}{k} \le \frac{t}{R_1}\big(\log(R_1/t)\big)^{1/\beta} \le \sup_{s\ge 3} s^{-1}(\log s)^{1/\beta} < \infty$$

and

$$\frac{R_1}{2k} \ge \frac{1}{4}\big(\log(R_1/t)\big)^{1/\beta} \ge \frac{1}{4}(\log 3)^{1/\beta} > 0.$$

Thus, using Proposition 6.2 along with (6.10) and (6.12) we obtain

$$\begin{aligned} p_{\mathbb{H}}(t/k, y_i, y_{i+1}) &\ge c_1 \frac{t}{k} j\big(|y_i - y_{i+1}|\big) \ge c_2 \frac{t}{k}(R_1/k)^{-d-\beta_2} e^{-c_3(R_1/k)^\beta} \\ &\ge c_4 \frac{t}{R_1}\left(\frac{k}{2R_1}\right)^{d+\beta_2-1} e^{-c_3(R_1/k)^\beta} \\ &\ge c_4 \frac{t}{R_1}\left(\log\frac{R_1}{t}\right)^{-\frac{d+\beta_2-1}{\beta}}\left(\frac{t}{R_1}\right)^{c_3} \ge c_4\left(\frac{t}{R_1}\right)^{c_5}. \end{aligned} \tag{6.13}$$

Since the definition of r_t yields

$$r_t \ge c_6\Big((t/R_1)^{(\beta_2\wedge\beta)^{-1}} \wedge \big(\log(R_1/t)\big)^{1/\beta}\Big) \ge c_7 (t/R_1)^{(\beta_2\wedge\beta)^{-1}}$$

by using (6.10), (6.13) and the semigroup property we conclude that

$$\begin{aligned} &p_{\mathbb{H}}(t,x,y) \\ &\quad\ge \int_{B_1}\cdots\int_{B_{k-1}} p_{\mathbb{H}}(t/k, x, y_1)\cdots p_{\mathbb{H}}(t/k, y_{k-1}, y)dy_1\cdots dy_{k-1} \\ &\quad\ge c_4^k c_7^{k-1}\left(\frac{t}{R_1}\right)^{c_5 k + (\beta_2\wedge\beta)^{-1}(k-1)d} \\ &\quad\ge c_8 \exp\big(-c_9 k \log(R_1/t)\big) \\ &\quad\ge c_8 \exp\big(-c_9\big(R_1\log(R_1/t)^{-1/\beta}+1\big)\log(R_1/t)\big) \\ &\quad\ge c_8 \exp\big(-2c_9\big(R_1 \log(R_1/t)^{\frac{\beta-1}{\beta}}\big)\big) \\ &\quad\ge c_8 \begin{cases} t\exp(-2c_9(R_1\log(R_1/t)^{\frac{\beta-1}{\beta}})) & \text{if } (t,R_1)\in(0,1]\times(3,\infty) \\ t^{-d/2}\exp(-2c_9(R_1(1+\log^+(R_1/t)^{\frac{\beta-1}{\beta}}))) & \text{if } (t,R_1)\in(1,\infty)\times(3t,\infty). \end{cases} \end{aligned}$$ □

Propositions 6.6–6.7 together with Proposition 6.2 and Propositions 6.4–6.5 yield the interior lower bound estimates of Theorem 6.1 for $\beta \in (1, \infty]$.

Remark 6.8 Assume that D is an connected open set with the following property: there exist $\lambda_1 \in [1,\infty)$ and $\lambda_2 \in (0,1]$ such that for every $r \le 1$ and x, y in D with $\delta_D(x)\wedge\delta_D(y) \ge r$ there exists in D a length parameterized rectifiable curve l connecting x to y with the length $|l|$ of l less than or equal to $\lambda_1|x-y|$ and $\delta_D(l(u)) \ge \lambda_2 r$ for $u \in [0,|l|]$.

Under this assumption, we can also prove Theorem 6.1 on such D with minor modifications. We omit the details here; see [23, Sect. 3] for the case $t < T$ and $\phi(r) = r^\alpha$.

7 Two-Sided Heat Kernel Estimates

In this section we prove the two-sided estimates of $p_{\mathbb{H}}(t,x,y)$ under conditions (5.4) and (5.5).

Theorem 7.1 *Suppose* (**UJS**), (5.4) *and* (5.5) *hold. There exist* $c_1, c_2, c_3 > 0$ *such that for all* $(t,x,y) \in (0,\infty)\times\mathbb{H}\times\mathbb{H}$,

$$p_{\mathbb{H}}(t,x,y) \le c_1\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}}\wedge 1\right)\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}}\wedge 1\right) \times \begin{cases} h_{c_2,1}(t,|x-y|/6) & \text{if } t\in(0,1),\\ k_{c_2,1}(t,|x-y|/6) & \text{if } t\in[1,\infty),\end{cases}$$

and

$$p_{\mathbb{H}}(t,x,y) \ge c_1^{-1}\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}}\wedge 1\right)\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}}\wedge 1\right) \times \begin{cases} h_{c_3,1}(t,3|x-y|/2) & \text{if } t\in(0,1),\\ k_{c_3,1}(t,3|x-y|/2) & \text{if } t\in[1,\infty).\end{cases}$$

Proof By Theorem 5.5, we only need to show the lower bound of $p_{\mathbb{H}}(t,x,y)$. For this, we will apply Theorem 4.5. Let C_4 be the constant C_4 in Theorem 6.1 with $T=1/3$.

Since $r\to h_{a,T}(t,r)$ and $r\to k_{a,T}(t,r)$ are decreasing, we have by Theorem 6.1 that for $|x-y|>4M_1\Phi^{-1}(t)$,

$$\begin{aligned}\inf_{\substack{(u,v):2M_1\Phi^{-1}(t)\le|u-v|\le 3|x-y|/2\\ \Phi(\delta_{\mathbb{H}}(u))\wedge\Phi(\delta_{\mathbb{H}}(v))>a_1 t}} p_{\mathbb{H}}(t/3,u,v) &\ge c_1\begin{cases} h_{C_4,1/3}(t/3,3|x-y|/2) & \text{if } t\in(0,1),\\ k_{C_4,1/3}(t/3,3|x-y|/2) & \text{if } t\in[1,\infty),\end{cases}\\ &\ge c_2\begin{cases} h_{c_3,1}(t,3|x-y|/2) & \text{if } t\in(0,1),\\ k_{c_3,1}(t,3|x-y|/2) & \text{if } t\in[1,\infty).\end{cases}\end{aligned} \tag{7.1}$$

When $6M_1\Phi^{-1}(t)\ge r$ and $t\ge 1$, by (5.7), we have $c_8M_1t^{1/2}\ge r$. Thus on $6M_1\Phi^{-1}(t)\ge r$ and $t\ge 1$

$$\begin{aligned}k_{C_4,1/3}(t/3,r) &\ge c_4\begin{cases}\Phi^{-1}(t)^{-d} & \text{if } \beta=0,\\ t^{-d/2}\exp(-C_4(r^\beta\wedge 3\frac{r^2}{t})) & \text{if } \beta\in(0,1],\\ t^{-d/2}\exp(-C_4((r(1+\log^+\frac{r}{t})^{\frac{\beta-1}{\beta}})\wedge 3\frac{r^2}{t})) & \text{if } \beta\in(1,\infty),\\ t^{-d/2}\exp(-C_4((r(1+\log^+\frac{r}{t}))\wedge 3\frac{r^2}{t})) & \text{if } \beta=\infty,\end{cases}\\ &\ge c_5\Phi^{-1}(t)^{-d}.\end{aligned} \tag{7.2}$$

So by (7.2) and Theorem 6.1, for $|x-y|\le 4M_1\Phi^{-1}(t)$,

$$\Phi^{-1}(t)\ge c_6\begin{cases} h_{c_7,1}(t,3|x-y|/2) & \text{if } t\in(0,1),\\ k_{c_7,1}(t,3|x-y|/2) & \text{if } t\in[1,\infty).\end{cases} \tag{7.3}$$

Combining (4.10), (7.1) and (7.3), we conclude that

$$p_{\mathbb{H}}(t,x,y) \geq c_8\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1\right)\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1\right) \times \begin{cases} h_{c_9,1}(t, 3|x-y|/2) & \text{if } t \in (0,1), \\ k_{c_9,1}(t, 3|x-y|/2) & \text{if } t \in [1,\infty). \end{cases} \qquad \square$$

Remark 7.2 (i) In view of Theorem 5.3, we can restate Theorem 7.1 as follows. There are positive constants c_i, $1 \leq i \leq 5$, so that

$$\begin{aligned} &\frac{1}{c_1}\left(\frac{1}{\sqrt{t\Psi(1/\delta_D(x))}} \wedge 1\right)\left(\frac{1}{\sqrt{t\Psi(1/\delta_D(y))}} \wedge 1\right) p\big(c_2 t, c_3(y-x)\big) \\ &\leq p_D(t,x,y) \\ &\leq c_1\left(\frac{1}{\sqrt{t\Psi(1/\delta_D(x))}} \wedge 1\right)\left(\frac{1}{\sqrt{t\Psi(1/\delta_D(y))}} \wedge 1\right) p\big(c_4 t, c_5(y-x)\big) \end{aligned} \tag{7.4}$$

where $p(t,x)$ is the transition density of X. This essentially confirms the conjecture (1.1) for this class of symmetric Lévy processes and for $D = \mathbb{H}$.

(ii) Recently sharp two-sided Dirichlet heat kernel estimates have been established in [15, 16] for a large class of symmetric Lévy processes in $C^{1,1}$ open sets for $t \leq 1$. The Lévy process considered in [15, 16] satisfy the conditions (5.4), (5.5) and (**UJS**) of this paper. Now assume X is a symmetric Lévy process considered [15, 16]. Then using the "push inward" method of [20] (see [13] for its use in relativistic stable processes case) and the short time heat kernel estimates in [15, 16], we can obtain global sharp two-sided Dirichlet heat kernel estimates on half-space-like $C^{1,1}$ open sets from the Dirichlet heat kernel estimates established in this paper on half-spaces. We leave the details to the interested reader.

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