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Counting Central Configurations at the Bifurcation Points

Ya-Lun Tsai[1]

Abstract Enumeration problems for the central configurations of the Newtonian n body problem are hard for $n > 3$ in $\mathbb{R}^2$ and $n > 4$ in $\mathbb{R}^3$. These are problems in finding the numbers of classes of central configurations for all the masses in a parameter space of positive dimensions. Many results are obtained generically. That is, rigorous proofs of the counting problems only exists for parameters not at the bifurcation points. For the bifurcation points, only numerical evidences are provided due to the complexity of the problems.

In this paper, we propose an algorithm that rigorously proves results on counting central configurations for all masses in one dimensional parameter spaces. Especially, we provide an approach to find all bifurcation points and count real roots at those points, known only implicitly. A spatial restricted $(4 + 1)$-body problem and a planar $(1 + 3)$-body problem are successfully applied by our method. All results except for the equal masses for the restricted $(4 + 1)$-body problem are new and the results for the planar $(1 + 3)$-body problem are new at the bifurcation points.

Keywords Central configurations · Bifurcations · Parametric polynomial systems · Symbolic computations

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1 Introduction

Central configurations play an important role in the study of the Newtonian n-body problem. Given a initial position of a central configuration in $\mathbb{R}^3$ with zero initial velocity, the n particles accelerate toward the center of mass in such a way that the configuration collapses homothetically. If we have initial positions of a central configuration in $\mathbb{R}^2$ and proper initial velocities, each particle will traverse an elliptical orbit around the center of mass as in the

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Kepler problem and the configuration remains similar to the initial configuration throughout the motion, varying only in size.

Finding all central configurations is challenging in the field of celestial mechanics. They are completely known only for $n = 2, 3$ and for configurations in $\mathbb{R}^1$ and $\mathbb{R}^{n-1}$ for all n. For $n = 2$, every configuration of two bodies is central. Counting up to rotations and translations, we obtain one central configuration. For $n = 3$, there are five central configurations. Three of them are collinear configurations found by Euler and the others are the equilateral triangle discovered by Lagrange. Considering n bodies in $\mathbb{R}^1$, there are $\frac{n!}{2}$ central configurations [18]. On the other hand, the only central configuration in $\mathbb{R}^{n-1}$ is when the n bodies form a regular simplex [23].

For $n > 3$, only limited results are known about central configurations in $\mathbb{R}^\ell$, where $1 < \ell < n - 1$. For $n = 4$, Hampton and Moeckel [16] proved the finiteness of central configurations in $\mathbb{R}^2$ for any choices of four positive masses. For $n = 5$ with generic choices of masses, the finiteness of central configurations in $\mathbb{R}^2$ was proved by Albouy and Kaloshin [2], and the finiteness of central configurations in $\mathbb{R}^3$ was proved by Hampton and Jensen [15]. For $n > 5$, we do not even know if the number of central configurations is finite. Smale's 6-th problem in the list of his 18 mathematical problems for the 21 century [24] asked if the number of central configurations in $\mathbb{R}^2$ is finite for all n. It is still open for $n > 5$.

Even in the cases of 4 bodies in $\mathbb{R}^2$ and 5 bodies in $\mathbb{R}^3$, we do not have exact counts for the numbers of central configurations for all systems where finiteness are proved. Restricted cases where some of the masses are zero and/or some of the non zero masses are equal are among the few successful examples. There are two kinds of restricted cases for some of the masses being zero. One is when only one of the masses equals to zero, called the restricted $(n + 1)$-body problem. Another is when all but one of the masses approach zero and considering the limiting central configurations. This is called the $(1 + n)$-body problem.

The restricted $(2 + 1)$-body problem is well studied and also has five central configurations as in the 3-body problem. The number of central configurations of the restricted $(3 + 1)$-body problem in $\mathbb{R}^2$ is shown numerically to be 8, 9 or 10 [1, 20, 22]. The results are recently proved in [3, 4]. To our understanding, the paper in [4] is among the few dealing with the counting problems where masses are also on the bifurcation curve. They prove rigorously that, on the bifurcation curve, there are nine central configurations.

In this paper, we generalize the case of equal masses of the restricted $(4 + 1)$-body problem in $\mathbb{R}^3$ and consider two special cases, where one mass parameter is involved. We use different approach to count the numbers of central configurations and obtain 3, 4, 5 and 2, 3, 4 for the zero mass on different lines of symmetry. In particular, the numbers of such central configurations are four and three at the bifurcation points.

The cases of the $(1 + n)$-body problems are first considered by Maxwell when studying the rings of Saturn. For such problems, we have one dominant mass, assuming 1, and n small masses, $\mu_i \varepsilon$. Let $\varepsilon \to 0$ and consider the limiting central configurations. Such central configurations have the body of the dominant mass at the center of a circle in $\mathbb{R}^2$ which passes through the bodies of infinitesimal masses [9]. There are two central configurations of the $(1 + 2)$-body problem, one is collinear and the other one is the equilateral triangle [10].

The $(1 + 3)$-body problem is studied in [5, 6]. In [5], the special cases of $\mu_1 = \mu_2$ are considered. There numerical evidences shows the number of central configurations varies from 5 to 7. In [6], the general cases are studied and generic results of 5 or 7 central configurations are proved, while cases on the bifurcation curve are still in lack of rigorous proofs. It turns out that rigorous counts at two bifurcation points in the case of $\mu_1 = \mu_2$ are missing. In this paper, we use our method to show that, in the special cases of $\mu_1 = \mu_2$, the number

of central configurations are indeed $5, 6$, or 7. In particular, at two bifurcation points, the numbers of central configurations are five and six.

In this paper, some computations involve Groebner basis and resultants, which are exact symbolic computations. We use Mathematica 10, to perform such computations. We also use Mathematica to count roots of integral polynomials of one variable by its implemented command "CountRoots" that always give accurate answers as explained in [8]. Therefore, all of our arguments in this paper are rigorous. In [26], we provide a link for Mathematica notebooks containing results of our computations.

2 Two Restricted Problems and Main Results

The Newtonian n-body problem studies the dynamics of n particles with masses $m_i > 0$ and positions $q_i \in \mathbb{R}^d$, moving according to the Newton's laws of motion:

$$m_j \ddot{q}_j = \sum_{i \neq j} \frac{m_i m_j (q_i - q_j)}{\|q_i - q_j\|^3}, \quad 1 \leq j \leq n. \tag{1}$$

Definition 1 A configuration $(q_1, \ldots, q_n) \in \mathbb{R}^{dn} \setminus \triangle$ is central if there exists $\lambda < 0$ such that

$$\lambda(q_j - c) = \sum_{i \neq j} \frac{m_i (q_i - q_j)}{\|q_i - q_j\|^3}, \quad 1 \leq j \leq n, \tag{2}$$

where $c = \frac{1}{M}(m_1 q_1 + \cdots + m_n q_n)$, $M = m_1 + \cdots + m_n$, and $\triangle = \{q_i = q_j, i \neq j\}$.

2.1 A Restricted $(4 + 1)$-Body Problem in $\mathbb{R}^3$

Let five particles have masses $m_1, \ldots, m_4 > 0$ and $m_5 = 0$. The four particles with positive masses form a central configuration by themselves according to (2). Considering the central configuration in $\mathbb{R}^3$, we obtain a regular tetrahedron [23]. Fixing $q_1, \ldots, q_4$ in $\mathbb{R}^3$ such that the length of the sides of the regular tetrahedron is 1 and assuming the total mass $M = 1$, without lost of generality, we obtain $\lambda = -1$ in (2). The equation in (2) for the zero mass become $\partial_x G = \partial_y G = \partial_z G = 0$, where

$$G = \frac{1}{2}\|q_5 - c\|^2 + \frac{m_1}{\|q_5 - q_1\|} + \frac{m_2}{\|q_5 - q_2\|} + \frac{m_3}{\|q_5 - q_3\|} + \frac{m_4}{\|q_5 - q_4\|}.$$

By the identity,

$$\|q_5 - c\|^2 - \sum_{i=1}^{4} m_i \|q_5 - q_i\|^2 = \|c\|^2 - \sum_{i=1}^{4} m_i \|q_i\|^2,$$

we change the coordinate of q_5 to $r_i = \|q_5 - q_i\|$ for $i = 1, \ldots, 4$. So, we obtain

$$G = \frac{1}{2}\left(m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + m_4 r_4^2\right) + \frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_3}{r_3} + \frac{m_4}{r_4} + C,$$

for some constant C depending on c. With new coordinates, we impose a restriction for the mutual distances of five points in $\mathbb{R}^3$ given by the Cayley–Menger determinant below [19].

Therefore, we have the equation below for r_1, r_2, r_3, r_4.

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & r_1^2 \\ 1 & 1 & 0 & 1 & 1 & r_2^2 \\ 1 & 1 & 1 & 0 & 1 & r_3^2 \\ 1 & 1 & 1 & 1 & 0 & r_4^2 \\ 1 & r_1^2 & r_2^2 & r_3^2 & r_4^2 & 0 \end{vmatrix} = F = 0.$$

Now we find the critical points of G restricted to $F = 0$. By the Lagrange multiplier technique, we obtain the following equations for a multiplier ω.

$$\partial_{r_i} G = \omega \partial_{r_i} F, \quad \forall i = 1, 2, 3, 4,$$
$$F = 0. \tag{3}$$

Eliminating ω and clearing the denominators, we obtain polynomial systems from (3). For the case of equal masses, we have the following results [17, 21].

Proposition 1 *When $m_1 = m_2 = m_3 = m_4$, there are 5 central configurations for the zero mass locating at the line with $r_i = r_j$, and $r_k = r_\ell$, for distinct $i < j, k < \ell$ in $\{1, 2, 3, 4\}$. There are 4 central configurations for the zero mass locating at the line with $r_i = r_j = r_k$ for $i < j < k$ in $\{1, 2, 3, 4\}$. If the zero mass is not on any of the 7 lines of symmetry, there are no central configurations.*

To generalize the problem, we consider two special cases. The first case is when $m_1 = m_3 = k$, $m_2 = m_4 = \frac{1}{2} - k$. In this case, we consider $r_1 = r_3 = x$ and $r_2 = r_4 = y$ for simplicity. The second case is when $m_1 = m_2 = m_3 = k$, $m_4 = 1 - 3k$. In this case, we consider $r_1 = r_2 = r_3 = x$ and $r_4 = y$.

For the first case, we have the following system.

$$\begin{cases} f_1 = (1 - 2k)x^3 - (2 - 4k)x^5 + (2 - 4k)x^3y^2 - 2ky^3 \\ \qquad - 4kx^2y^3 - (1 - 4k)x^3y^3 + 2x^5y^3 + 4ky^5 - 2x^3y^5, \\ f_2 = 3 - 4x^2 + 4x^4 - 4y^2 - 8x^2y^2 + 4y^4. \end{cases} \tag{4}$$

Theorem 1 *The system (4) has 3, 4, and 5 positive zeros in $(0, \alpha) \cup (\frac{1}{2} - \alpha, \frac{1}{2})$, at $k = \alpha, \frac{1}{2} - \alpha$, and in $(\alpha, \frac{1}{2} - \alpha)$, respectively, where α is approximately 0.226696. Therefore, in the restricted $(4 + 1)$-body problem in $\mathbb{R}^3$, there are 3, 4, or 5 central configurations for the zero mass on the line of symmetry shown in Fig. 1 when $m_1 = m_3$, $m_2 = m_4$.*
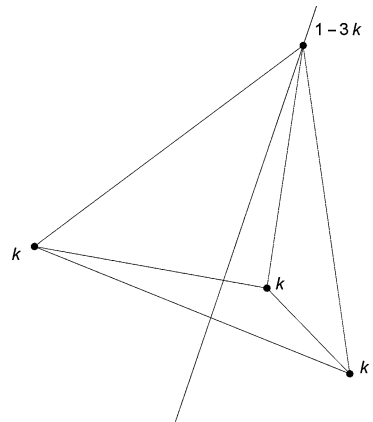
For the second case, we have the following system.

$$\begin{cases} f_3 = (1 - 3k)x^3 - (1 - 3k)x^5 + (1 - 3k)x^3y^2 - ky^3 \\ \qquad - 3kx^2y^3 - (1 - 4k)x^3y^3 + x^5y^3 + 3ky^5 - x^3y^5, \\ f_4 = 3 - 6x^2 + 3x^4 - 2y^2 - 6x^2y^2 + 3y^4, \\ f_5 = -1 + (-1 + x^2)z. \end{cases} \tag{5}$$

Fig. 1 $m_1 = m_3, m_2 = m_4$
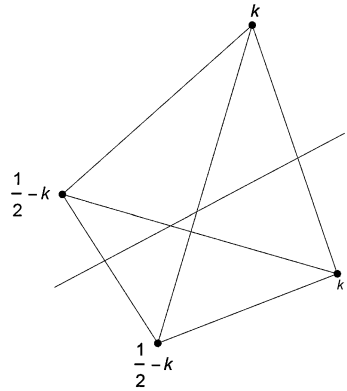
Fig. 2 $m_1 = m_2 = m_3$

Note that in system (5), we impose an extra variable z and the equation f_5 to remove the singularities when $x = 1$, where the zero mass particle collides with the particle of mass $1 - 3k$.

Theorem 2 *The system* (5) *has* 2, 3, *and* 4 *zeros with positive* x, y *coordinates in* $(0, \beta)$, *at* $k = \beta$, *and in* $(\beta, \frac{1}{3})$, *respectively, where* β *is approximately* 0.246659. *Therefore, in the restricted* $(4 + 1)$-*body problem in* $\mathbb{R}^3$, *there are* 2, 3, *or* 4 *central configurations for the zero mass on the line of symmetry shown in Fig.* 2 *when* $m_1 = m_2 = m_3$.

Remark 1 In Theorem 1, the number of central configurations 5 is obtained as in Proposition 1 by letting $k = \frac{1}{4}$. Similarly, in Theorem 2, the number of central configurations 4 is obtained as in Proposition 1 by letting $k = \frac{1}{4}$.

2.2 The $(1+3)$-Body Problem in $\mathbb{R}^2$

Now, let $q_i = (x_i, y_i) \in \mathbb{R}^2$ and $r_{i,j} = \|q_i - q_j\|$ be the distance between particles i and j. Let $\triangle_{i,j,k}$ denote the oriented area of the triangle with vertices q_i, q_j, and q_k. That is,

$$\triangle_{i,j,k} = \frac{1}{2} \begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix}.$$

We have the following characterization for non collinear central configurations [14].

Proposition 2 *Let $(q_1, \dots, q_n) \in \mathbb{R}^{2n} \setminus \triangle$ be a non collinear planar configuration. Then* (2) *are satisfied if and only if*

$$\sum_{k=1, k\neq i,j}^{n} m_k \triangle_{i,j,k} \left(\frac{1}{r_{ik}^3} - \frac{1}{r_{jk}^3} \right) = 0, \quad 1 \le i < j \le n.$$

Let $(q_1(\epsilon), q_2(\epsilon), q_3(\epsilon), q_4(\epsilon))$ be a central configuration of the planar 4-body problem, with $m_1 = \mu_1 \epsilon, m_2 = \mu_2 \epsilon, m_3 = \mu_3 \epsilon, m_4 = 1$. If the limit of each $q_i(\epsilon)$ exists as ϵ approaches zero such that none of the two limiting positions collide, we call the limiting configuration a non-collision central configuration of the planar $(1+3)$-body problem. We have the following well known results [9].

Proposition 3 *All the central configurations of the planar $(1+n)$-body problem lie on a circle centered at the particle with positive mass.*

To find the non-collision central configurations of the planar $(1+3)$-body problem, we can apply Proposition 2 and let ϵ approaches zero. By Proposition 3 we can then use, without lost of generality,

$$q_1 = \left(\frac{1-r_3^2}{1+r_3^2}, \frac{2r_3}{1+r_3^2} \right), \qquad q_2 = \left(\frac{1-r_4^2}{1+r_4^2}, \frac{2r_4}{1+r_4^2} \right), \qquad q_3 = (-1, 0), \qquad q_4 = (0, 0)$$

in the limiting equations. Here we consider $\mu_1 = \mu_2 = 1$ and $\mu_3 = k > 0$. Also, we impose two more variables r_1, r_2 such that $\frac{1}{r_1^2} = 1 + r_3^2$, and $\frac{1}{r_2^2} = 1 + r_4^2$. Then, we obtain the following system with four equations, four variables, and one parameter.

$$\begin{cases} f_6 = r_1 r_3^2 r_4 - 8 r_1 r_2^3 r_3^2 r_4 - 2 r_1 r_3 r_4^2 + 16 r_1 r_2^3 r_3 r_4^2 + r_1 r_4^3 - 8 r_1 r_2^3 r_4^3 \\ \quad + k(1 - 8 r_1^3 r_2^3 r_3^3 + r_3 r_4 + 24 r_1^3 r_2^3 r_3^2 r_4 - 8 r_1^3 r_2^3 r_3^4 r_4 - 24 r_1^3 r_2^3 r_3 r_4^2 \\ \quad + 24 r_1^3 r_2^3 r_3^3 r_4^2 + 8 r_1^3 r_2^3 r_4^3 - 24 r_1^3 r_2^3 r_3^2 r_4^3 + 8 r_1^3 r_2^3 r_3 r_4^4), \\ f_7 = r_2 r_3 - 8 r_1^3 r_2 r_3 + r_1 r_4 - 8 r_1 r_2^3 r_4, \\ f_8 = 1 - r_1^2 - r_1^2 r_3^2, \\ f_9 = 1 - r_2^2 - r_2^2 r_4^2. \end{cases} \quad (6)$$

The geometric meanings of r_3, r_4 are the y-coordinates of the points of intersection between the y-axis and the line segments connecting q_3 and q_1, q_2, respectively. Denote the

Fig. 3 The $1 + 3$ body problem
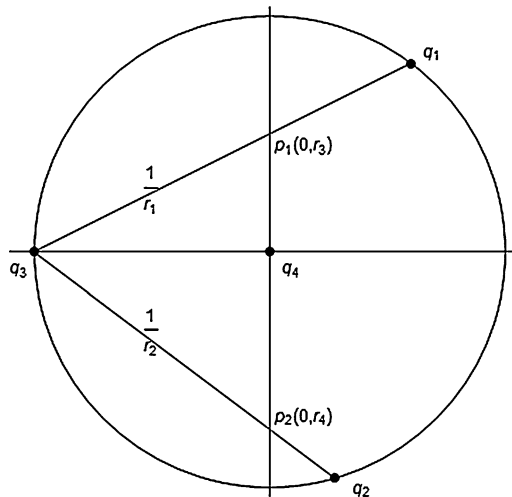
intersection points by p_1 and p_2. Then r_1 and r_2 are the inverse of the lengths of the hypotenuse of the triangles $p_1 q_3 q_4$ and $p_2 q_3 q_4$, respectively. Figure 3 shows the geometry. The goal here is to count the number of common zeros of the system (6) with $0 < r_1, r_2 < 1$ and $r_3 > r_4$ without lost of generality. (Note that it is easy to see there are no common zeros for $r_1 = 1$ or $r_2 = 1$ if $0 < k$.)

Theorem 3 *The system* (6) *has* 5, 6, *and* 7 *zeros with* $0 < r_1, r_2 < 1$ *and* $r_3 > r_4$ *in* $(0, a) \cup [b, \infty)$, *at* $s = a$, *and in* (a, b), *respectively, where* a, b *are approximately* 0.89616 *and* 1.42385. *Therefore, for the planar* $(1+3)$-*body problem with two equal infinitesimal masses, there are* 5, 6, *or* 7 *central configurations.*

Remark 2 In [6], the number of central configurations 5 or 7 on $(0, \infty) \setminus \{a, b\}$ are proved rigorously. As in [5], only numerical evidences are given for the numbers of central configurations at a and b.

3 Counting Algorithm

In this section, we present our method of counting real common zeros of a parametric polynomial system $\mathcal{F} = \{f_1, \ldots, f_n\}$, where $f_i \in \mathbb{Q}[k][x_1, \ldots, x_n]$. Here $\mathbb{Q}[k]$ is the domain of the coefficients.

We define the *specialization* at a point $r \in \mathbb{R}$. It is a ring homomorphism $\varphi^r : \mathbb{Q}[k][x_1, \ldots, x_n] \to \mathbb{R}[x_1, \ldots, x_n]$ such that $\varphi^a(f)$ is the real polynomial after substituting the parameter with the value r in f. We denote $\varphi^r(f) = f^r$ and $\varphi^r(\mathcal{F}) = \{f^r | f \in \mathcal{F}\} = \mathcal{F}^r$.

The enumeration problem of the system $\mathcal{F}$ is to count the number of common zeros in $\mathcal{X}$ of $\mathcal{F}^r$ for all $r \in \mathcal{P}$. Here $\mathcal{X} = \{x \in \mathbb{R}^n | p_1(x) > 0, \ldots, p_\ell(x) > 0\}$, where p_j are polynomials with integral coefficients and $\mathcal{P}$ is an open interval in $\mathbb{R}$, called the *parameter space*. (For example, if $n = 2$, $p_1 = x_1$, $p_2 = x_2$, and $\mathcal{P} = (0, \infty)$, we are to count the common zeros of the system in the first quadrant for all positive parameters.) To do so, we proceed as the following.

1. View $\mathcal{F}$ in $\mathbb{Q}[x_1, \ldots, x_n, k]$ and compute the Groebner basis $\mathcal{G}$ of $\mathcal{F}$ with a *block order*.
2. View $\mathcal{G}$ in $\mathbb{Q}(k)[x_1, \ldots, x_n]$, where $\mathbb{Q}(k)$ is the field of rational polynomials in k, and compute the Hermite matrices $H_j := H(\mathcal{F}, P_j)$ for all P_j, where P_j is any product of a subset from $\{p_1, \ldots, p_\ell\}$. There are 2^ℓ of them including of the product of no polynomials $P_0 := 1$.
3. From H_0, we obtain all points $a \in \mathcal{P}$ where the number of real zeros of $\mathcal{F}$ may change. Those points are zeros of a polynomial g, called the *bifurcation polynomial*, which is the numerator of H_0.
4. Call the zeros a_i of g the *bifurcation points* and pick a rational sample point $a \in (a_i, a_{i+1})$. Use Hermite root counting theorem to count the number of zeros in $\mathcal{X}$ by computing signatures of $H(\mathcal{F}^a, P_j)$ for all j.

    The number of zeros in $\mathcal{X}$ is a constant in (a_i, a_{i+1}) if $p_j \neq 0$ for all j at the common zeros. Therefore, generic results are obtained. Next, we focus on parameters at the bifurcation points a_i's. Let $H_j^i := H(\mathcal{F}^{a_i}, P_j)$ and d be the dimension of H_j.
5. For each a_i and H_j, obtain the rank $r_{i,j}$ of H_j^i from the principal minors of H_j. For each H_j, denote the leading principal minors of order t by $D_{j,t}$. It turns out $D_{j,t}^{a_i} = 0$ for all $t > r_{i,j}$, if $r_{i,j} < d$.
6. For each a_i, we pick an interval $(a_{i\ell}, a_{ir})$ containing it such that, for all j and $t \leq r_{i,j}$, if $D_{j,t}$ is not identically equal to zero, then there is no zeros in $(a_{i\ell}, a_{ir})$ and there is only one zero, a_i, of $D_{j,t}$ for all $t > r_{i,j}$, if $r_{i,j} < d$.
7. For each a_i and H_j, compute the number of sign changes $v_{i,j}$ in the list $\{1, D_{j,1}^{b_i}, \ldots, D_{j,r_{i,j}}^{b_i}\}$, where b_i can be either a_{li} or a_{ri}.
8. Obtain the signatures of H_j^i from $v_{i,j}$ and $r_{i,j}$ by the Jacobian theorem and compute the number of common zeros in $\mathcal{X}$ of $\mathcal{F}^{a_i}$ again by Hermite root counting theorem.

Remark 3 By Hermite root counting theorem, we not only count the common zeros of $\mathcal{F}^{a_i}$ in $\mathcal{X} = \{x \in \mathbb{R}^n \mid p_1(x) > 0, \ldots, p_\ell(x) > 0\}$. We also obtain the numbers of common zeros in any of the region $\{(-1)^{m_1} p_1 > 0, \ldots, (-1)^{m_\ell} p_\ell > 0\}$ among the 2^ℓ "quadrants".

In the following two subsections, we present tools we used. Moreover, we focus on showing that our counting method is rigorous.

3.1 Steps from (1) to (4): on Finding the Bifurcation Polynomial

Given any polynomial ring over $\mathbb{Q}$, a finite set of polynomials in it, a total ordering on monomials, we can symbolically manipulate the given set of polynomials and output the *Groebner basis*, another finite set of polynomials, which has many good properties [11]. They generate the same ideal and hence share the same set of common zeros. For certain orders on monomials, the Groebner base are much easier to solve than the original system.

Consider the quotient algebra A over the filed $\mathbb{C}$ of the ideal generated by the polynomials, a Groebner basis can verify if the polynomial system has finitely many complex common zeros by showing the dimension of A as a vector space over $\mathbb{C}$ is finite. In this case, the dimension gives an upper bound to the number of zeros. Moreover, we can obtain a basis of A and an algorithm to express any element in A with this basis. Therefore, it becomes possible to compute the *Hermite matrix* that is used to count the number of real roots.

Definition 2 Let $\mathcal{F} = \{f_1, \ldots, f_n\} \subset \mathbb{Q}[x_1, \ldots, x_n]$, $I = \langle \mathcal{F} \rangle$, $p \in \mathbb{Q}[x_1, \ldots, x_n]$ and $A = \mathbb{R}[x_1, \ldots, x_n]/I$ with a basis $\{b_1, \ldots, b_m\}$, the Hermite matrix, denoted by $H(\mathcal{F}, p)$, is the

$m \times m$ real symmetric matrix with entries $Trace(L(pb_i b_j))$, where $L(pb_i b_j)$ is the linear map on A defined by the left multiplication with $pb_i b_j$.

Hermite root counting method is given in the following proposition [7]. In [26], we provide a Mathematica notebook containing the implementation of our efficient algorithm outputting $H(\mathcal{F}, p)$ given $\mathcal{F}$, p and a Groebner basis of $\mathcal{F}$.

Proposition 4 *Matrix rank of $H(\mathcal{F}, p)$ equals to the number of complex roots of $\mathcal{F}$ with $p \neq 0$. Signature of $H(\mathcal{F}, p)$ equals to the number of real roots with $p > 0$ minus the number of real roots with $p < 0$.*

If we are to count real roots in $\mathcal{X} = \{x \in \mathbb{R}^n | \ p_1(x) > 0, \ldots, p_\ell(x) > 0\}$, where p_j are polynomials, we need to compute 2^ℓ Hermite matrices $H(\mathcal{F}, P_j)$ for all P_j, where P_j is any product of a subset from $\{p_1, \ldots, p_\ell\}$, including of the product of elements in the empty set $P_0 = 1$. Then solving a linear system of 2^ℓ variables, we obtain the numbers of real roots in all of the 2^ℓ regions $\{ (-1)^{m_1} p_1 > 0, \ldots, (-1)^{m_\ell} p_\ell > 0\}$.

In our situations, parameters are involved. Extra cares should be taken into consideration. Given $\mathcal{F} = \{f_1, \ldots, f_n\} \subset \mathbb{Q}[k][x_1, \ldots, x_n]$, a polynomial system with coefficients in the domain $\mathbb{Q}[k]$. The first goal is to compute a set of polynomials $\mathcal{G}$ in $\mathbb{Q}[k][x_1, \ldots, x_n]$ such that the *specialization* at any point $r \in \mathbb{R}$, $\mathcal{G}^r$ is a Groebner basis for $\mathcal{F}^r$ in $\mathbb{R}[x_1, \ldots, x_n]$. Here, we adopt the following proposition to achieve this goal for almost all $r \in \mathbb{R}$ [13].

Proposition 5 *View $\mathcal{F} \subset \mathbb{Q}[x_1, \ldots, x_n, k]$, consider any total ordering where any monomial involving one of the x_i's is greater than all monomials in k, called a block order, and compute a Groebner basis $\mathcal{G} = \{g_1, \ldots, g_s\}$ of $\mathcal{F}$ with this order. Using the total ordering on the variables $x_1, \ldots, x_n$ of the block order, we find the leading terms $\{h_1, \ldots, h_s\} \subset \mathbb{Q}[k]$ of $\mathcal{G}$ in $\mathbb{Q}[k][x_1, \ldots, x_n]$. If $h_i^r \neq 0$ for all i, then $\mathcal{G}^r$ is a Groebner basis of $\mathcal{F}^r$.*

Next, we will compute $H(\mathcal{F}^r, p)$ for all such r where $h_i^r \neq 0$ for all i. We use the following results whose proofs can be found in [25].

Proposition 6 *View $\mathcal{G}$ obtained in Proposition 5 as a set in $\mathbb{Q}(k)[x_1, \ldots, x_n]$ and consider $\mathcal{F}$ in $\mathbb{Q}(k)[x_1, \ldots, x_n]$. Then $\mathcal{G}$ is a Groebner basis of $\mathcal{F}$ in $\mathbb{Q}(k)[x_1, \ldots, x_n]$. Hence, we can compute the hermite matrix $H(\mathcal{F}, p)$ on the algebra A over the field $\mathbb{Q}(k)$. $H(\mathcal{F}, p)$ is a symmetric matrix with entries in $\mathbb{Q}(k)$, whose denominators can only contain factors h_i's in Proposition 5. We have the specialization $\varphi^r(H(\mathcal{F}, p)) = H(\mathcal{F}^r, p)$ for all r where $h_i^r \neq 0$ for all i.*

Then, we compute the *bifurcation polynomial g* of the system $\mathcal{F}$. Theoretically, the bifurcation polynomial is defined as the polynomial containing the projection of the common zeros of $\{\mathcal{F}, J\}$ in $\mathbb{C}[x_1, \ldots, x_n, k]$, where J is the Jacobian determinant of $\mathcal{F}$, into the parameter space $\mathcal{P}$ in $\mathbb{R}$. Those points of projection in the parameter space are called the *bifurcation points*. One approach to compute g is to compute a Groebner basis of $\{\mathcal{F}, J\}$ with a certain order to eliminate the variables $x_1, \ldots, x_n$. We will not use this method because the computations of such Groebner basis usually require much more computer memories then we have. Instead, we use the following method [25].

Proposition 7 *Consider the hermite matrix $H(\mathcal{F}, p)$ in $\mathbb{Q}(k)$ obtained in Proposition 6 and $p = 1$. If the determinant of $H(\mathcal{F}, 1)$ in $\mathbb{Q}(k)$ is not identically equal to zero. Then the*

zero set of the numerator g contains the bifurcation points. Therefore, we call such g the bifurcation polynomial.

3.2 Steps from (5) to (8): on Counting at the Bifurcation Points

Assume the bifurcation polynomial g has real zeros $\{a_1, \ldots, a_w\}$ in the parameter space $\mathcal{P}$. We will also use the Hermite root counting method to count the number of common zeros of $\mathcal{F}^{a_i}$ in $\mathcal{X}$. Since a_i's are only know implicitly, we can not compute $\varphi^{a_i}(H(\mathcal{F}, p))$ nor $H(\mathcal{F}^{a_i}, p)$ directly to obtain the signatures. However, as long as a_i is not a zero for any of the h_j's in Proposition 5, we have $\varphi^{a_i}(H(\mathcal{F}, p)) = H(\mathcal{F}^{a_i}, p)$ by Proposition 6. In fact, $\varphi^r(H(\mathcal{F}, p)) = H(\mathcal{F}^r, p)$ in a small closed interval $[a_{\ell i}, a_{r i}]$ containing a_i. We will pick such interval that is small enough for us to obtain the signatures of $\varphi^{a_i}(H(\mathcal{F}, p))$. The critical tool is the Jacobian theorem in the following [12].

Proposition 8 *Let H be a $d \times d$ real symmetric matrix and D_i for $i = 1, \ldots, d$ be the leading principal minors of H of order i. That is, D_i is the determinant of the sub matrix H^i of H, where $H^i_{s,t} = H_{s,t}$ for $1 \leq s, t \leq i$. Assume H has rank r.*

1. *If $D_i \neq 0$ for all $i \leq r$, then the signature of H is $r - 2v$, where v is the number of variation of sign in the sequence $1, D_1, \ldots, D_r$.*
2. *If in the sequence $1, D_1, \ldots, D_r \neq 0$, there are zeros but not three in succession, then the signature of H can be determined by $r - 2v$ omitting the zero D_k if $D_{k-1}D_{k+1} \neq 0$ and, in the cases of $D_k = D_{k+1} = 0$, setting the number of variation in the sequence $D_{k-1}, D_k, D_{k+1}, D_{k+2}$ to be 1 if $D_{k+2}D_{k-1} < 0$ and to be 2 if $D_{k+2}D_{k-1} > 0$.*
3. *If there are three consecutive zeros in $D_1, \ldots, D_{r-1}$, then the signs of the non zero D_k's do not determined the signature.*
4. *If $D_r = 0$, the signs of the non zero D_k's do not determined the signature.*

According to the Jacobian theorem, we only consider cases when $H(\mathcal{F}, p)$ does not have more than two consecutive zero polynomials in the sequence of its leading principal minors. Also, in applying the Jacobian theorem, it is critical to find the rank first. We list the following facts about ranks.

Definition 3 Let H be a $d \times d$ real matrix. A minor of order $i \leq d$ is the determinant of the $i \times i$ sub matrix M of H, where $M_{s,t} = H_{s,t}$ for $s, t \in \{1, \ldots, d\}$. The largest among the orders of the non-zero minors generated by H is the rank.

From the definition, it is easy to see the following facts. We omit the proofs.

Proposition 9 *Let H be a $d \times d$ real symmetric matrix and D_i for $i = 1, \ldots, d$ be the leading principal minors of H of order i. Let M^{d-1} be the matrix of order $d - 1$, where $M^{d-1}_{s,t} = H_{s,t}$ for $s, t \in \{1, \ldots, d-2, d\}$. Let M_{d-1} denote the determinant of M^{d-1}. We have the following.*

1. *If $D_d \neq 0$, then the rank of H is d.*
2. *If $D_d = 0$, $D_{d-1} \neq 0$, then the rank of H is $d - 1$.*
3. *If $D_d = D_{d-1} = M_{d-1} = 0$, $D_{d-2} \neq 0$, then the rank of H is $d - 2$.*

Now, we return to the goal of computing the signature of $\varphi^{a_i}(H(\mathcal{F}, p))$, where a_i a zero of the bifurcation polynomial g and is not a zero for any of the h_j's in Proposition 5.

Using Proposition 9, we obtain the ranks for the Hermite matrices considered in this paper involving the parameter k. It remains to compute the numbers of variation of sign in the sequences of leading principal minors.

Suppose the rank is r, $D_r^{a_i} \neq 0$, and there are no three or more consecutive zero polynomials in $D_1, \ldots, D_{r-1}$. A small interval $[a_{i\ell}, a_{ir}]$ containing a_i can be found easily satisfying that there is no zeros for the non zero D_k's for all $k = 1, \ldots, r-1$ in $[a_{i\ell}, a_{ir}]$. Then the sign of the non zero $D_k^{a_i}$ agrees with that of $D_k^{a_{i\ell}}$ or $D_k^{a_{ir}}$.

3.3 Challenging Parts and Other Remarks

Remark 4 To apply our approach, $\mathcal{F}$ need to meet some requirements.

1. In step (1), the Groebner basis $\mathcal{G}$ in $\mathbb{Q}[x_1, \ldots, x_n, k]$ should be computable.
2. In step (3), determinant of $H(\mathcal{F}, 1)$ in $\mathbb{Q}(k)$ is not identically equal to zero.
3. For steps (5) to (8), the H_j's having entries $\mathbb{Q}(k)$ and defined in the step (2) can not have three or more consecutive zero polynomials in the sequences of their leading principal minors up to the order of rank. And the leading principal minors of the order of rank are not zero polynomials.
4. For steps (5) to (8), a_i, a zero of the bifurcation polynomial g, is not a zero for any of the h_j's in Proposition 5.

   In fact, any zero of a h_j in the parameter space should be handle differently, for in Proposition 7, which is proved by Proposition 6 in [25], did not consider such points. It is even possible to obtain bifurcation points from them.

Remark 5 The orders on monomials can be determined by a matrix with row $\mathbf{w_i}$'s. Given two monomials $\mathbf{x_1}, \mathbf{x_2}$ with exponent vectors α and β, we have $\mathbf{x_1} > \mathbf{x_2}$ if $\mathbf{w_1} \cdot \alpha > \mathbf{w_1} \cdot \beta$. If $\mathbf{w_1} \cdot \alpha = \mathbf{w_1} \cdot \beta$, we compare $\mathbf{w_2} \cdot \alpha$ with $\mathbf{w_2} \cdot \beta$. Repeat the procedure until we find which one is larger. In this paper, we use the following matrix for variables $x_1, \ldots, x_n, k$.

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

This is a block order. Its restriction on x_j's is the order determined by the upper left $n \times n$ block called the graded reverse lexicographical order (or grevlex order). According to [11], for some operations, the grevlex ordering is the most efficient for computation. Therefore, we use this order as one of the block.

Remark 6 In step (6), we use the implemented command "CountRoots" of Mathematica 10 to count roots of integral polynomials of one variable. This command is based on [8]. In the introduction of this paper, the authors commented that "Unlike numerical methods the algorithm will always terminate with correct results."

Remark 7 In our work, we simply use the command "Det" of Mathematica to compute leading principal minors in one variable. For system (6), the order of H is 104 and we need

to compute 104×8 leading principal minors. It took us almost three months to have them all computed. The results are provided in [26].

4 The $(4+1)$-Body Problem

4.1 Counting Zeros for System (4)

The following results give a whole picture of the numbers of zeros in each quadrant when the parameters are in $(0, \frac{1}{2})$. So, Theorem 1 is proved.

Proposition 10 *For system* (4), *we obtain the bifurcation polynomial* g^4 *in the* Appendix. *There are 4 zeros of* g_4 *in* $(0, \frac{1}{2})$. *The neighbour points in the step* (6) *of our method are given in the Table* 1. *The ranks and sign variations of the neighbour and the bifurcation points are given in Table* 2. *Finally, the numbers of real roots in four quadrants for those points are given in Table* 3.

Proof We have $\mathcal{F} = \{f_1, f_2\} \subset \mathbb{Q}[x, y, k]$. Using the block order in Remark 5, we obtain $\mathcal{G} = \{g_1, g_2, g_3, g_4\}$ with leading coefficients $\{4, 4, 4, 16\} \subset \mathbb{Q}[k]$. Using $\mathcal{G}$, we find a basis $\{1, y, y^2, y^3, y^4, y^5, y^6, y^7, y^8, x, xy, xy^2, xy^3, xy^4, xy^5, xy^6, x^2, x^2y, x^2y^2, x^2y^3, x^2y^4, x^2y^5, x^2y^6, x^3, x^3y, x^3y^2, x^3y^3, x^3y^4\}$ for the algebra $A = \mathbb{Q}(k)[x, y]/\langle \mathcal{F} \rangle$. So, the dimension of A is 28 and there are at most 28 complex zeros for all k.

Computing four Hermite matrices, $H_0 = H(\mathcal{F}, 1)$, $H_1 = H(\mathcal{F}, x)$, $H_2 = H(\mathcal{F}, y)$, $H_3 = H(\mathcal{F}, xy)$ with $\mathcal{G}$, we find the determinant of $H(\mathcal{F}, 1)$ is not identically equal to zero. So, we obtain the bifurcation polynomial g^4 from the numerator. Using "CountRoots", we isolate four real roots a_1, a_2, a_3, a_4 of g^4 in $(0, \frac{1}{2})$.

Computing 28×4 leading principals minors, we find a_1, a_2, a_3, a_4 are also roots of the numerator of the determinants of H_1, H_2, H_3. And, all the leading principals minors of order 27 do not have a_1, a_2, a_3, a_4 as their zeros. Therefore, by Proposition 9, H_j^i has rank $r_{i,j} = 27$ for all i, j.

Choosing neighbour points of a_i's as in Table 1, we use "CountRoots" to verify the requirements in the step (6) in Sect. 3. Therefore, the numbers $v_{i,j}$ of variation of signs in the step (7) are obtained in Table 2.

Finally, using Table 2 to obtain signatures, and solving linear systems in four variables, we obtain Table 3. For example, in Table 2 at $k = a_1$, $r_{1,j} = 27$, $v_{1,0} = 10$, $v_{1,1} = 13$, $v_{1,2} = 12$, $v_{1,3} = 13$, signatures, $r_{1,j} - 2v_{1,j}$, are 7, 1, 3, 1.

Let s_1, s_2, s_3, s_4 denote the number of real roots (x, y) with the signs $(+, +)$, $(+, -)$, $(-, +)$, $(-, -)$, respectively. We get

$$7 = s_1 + s_2 + s_3 + s_4,$$

$$1 = s_1 + s_2 - s_3 - s_4,$$

$$3 = s_1 - s_2 + s_3 - s_4,$$

$$1 = s_1 - s_2 - s_3 + s_4.$$

Therefore, we obtain $(s_1, s_2, s_3, s_4) = (3, 1, 2, 1)$ as in Table 3. Note that, for all k, $x = 0$ or $y = 0$ are not zeros. Therefore, the number of roots at each quadrants is a constant in each (a_i, a_{i+1}) interval. So, Table 3 give the numbers of roots in each quadrant of the system (4) for all parameters in $(0, \frac{1}{2})$. $\qquad\square$

Table 1 Neighbors of the zeros of g^4 in $(0, \frac{1}{2})$

i	$a_{i\ell}$	a_{ir}
1	2.0712182	2.0712183
2	2.2669649	2.2669650
3	2.7330350	2.7330351
4	2.9287817	2.9287818

Table 2 Ranks and sign variations for the system (4)

r, v	$+, +$	$+, -$	$-, +$	$-, -$
$a_{1\ell}$	28, 10	28, 14	28, 12	28, 14
a_1	27, 10	27, 13	27, 12	27, 13
a_{1r}	28, 11	28, 13	28, 13	28, 13
$a_{2\ell}$	28, 11	28, 13	28, 13	28, 13
a_2	27, 10	27, 12	27, 12	27, 12
a_{2r}	28, 10	28, 12	28, 12	28, 12
$a_{3\ell}$	28, 10	28, 12	28, 12	28, 12
a_3	27, 10	27, 12	27, 12	27, 12
a_{3r}	28, 11	28, 13	28, 13	28, 13
$a_{4\ell}$	28, 11	28, 13	28, 13	28, 13
a_4	27, 10	27, 12	27, 13	27, 13
a_{4r}	28, 10	28, 12	28, 14	28, 14

Table 3 Numbers of zeros of the system (4)

k	$+, +$	$+, -$	$-, +$	$-, -$
$a_{1\ell}$	3	1	3	1
a_1	3	1	2	1
a_{1r}	3	1	1	1
$a_{2\ell}$	3	1	1	1
a_2	4	1	1	1
a_{2r}	5	1	1	1
$a_{3\ell}$	5	1	1	1
a_3	4	1	1	1
a_{3r}	3	1	1	1
$a_{4\ell}$	3	1	1	1
a_4	3	2	1	1
a_{4r}	3	3	1	1

4.2 Counting Zeros for System (5)

The following results give a whole picture of the numbers of zeros when the parameters are in $(0, \frac{1}{3})$. So, Theorem 2 is proved.

Proposition 11 *For system* (5), *we obtain the bifurcation polynomial g^5 in the Appendix. There are 3 zeros of g^5 in $(0, \frac{1}{3})$. The neighbour points in the step* (6) *of our method are given in the Table* 4. *The ranks and sign variations of the neighbour and the bifurcation*

Table 4 Neighbors of the zeros of g^5 in $(0, \frac{1}{3})$

i	$a_{i\ell}$	a_{ir}
1	2.3551745	2.3551746
2	2.4665893	2.4665894
3	2.4860582	2.4860583

Table 5 Ranks and sign variations for the system (5)

r, v	$+, +$	$+, -$	$-, +$	$-, -$
$a_{1\ell}$	24, 10	24, 12	24, 10	24, 12
a_1	23, 9	23, 11	23, 10	23, 12
a_{1r}	24, 9	24, 11	24, 11	24, 13
$a_{2\ell}$	24, 9	24, 11	24, 11	24, 13
a_2	23, 8	23, 10	23, 10	23, 12
a_{2r}	24, 8	24, 10	24, 10	24, 12
$a_{3\ell}$	24, 8	24, 10	24, 10	24, 12
a_3	23, 7	23, 10	23, 10	23, 11
a_{3r}	24, 7	24, 11	24, 11	24, 11

Table 6 Numbers of zeros of the system (5)

k	$+, +$	$+, -$	$-, +$	$-, -$
$a_{1\ell}$	2	0	2	0
a_1	2	1	2	0
a_{1r}	2	2	2	0
$a_{2\ell}$	2	2	2	0
a_2	3	2	2	0
a_{2r}	4	2	2	0
$a_{3\ell}$	4	2	2	0
a_3	4	2	2	1
a_{3r}	4	2	2	2

points are given in Table 5. Finally, the numbers of real roots in four quadrants for those points are given in Table 6.

Proof Now $\mathcal{F} = \{f_3, f_4, f_5\} \subset \mathbb{Q}[x, y, z, k]$. Using the block order in Remark 5, we obtain $\mathcal{G} = \{g_1, \ldots, g_{14}\}$, where the non constant leading coefficients in $\mathbb{Q}[k]$ are

$$
\begin{cases}
h_2 = 9k^2(-1 + 3k)(-6193005856 + 34523341224k - 88164313884k^2 \\
\qquad + 179027640450k^3 - 296194349727k^4 + 292634913915k^5 \\
\qquad - 135093148569k^6 + 22052144295k^7), \\
h_5 = c_5 k, \\
h_9 = c_9(-1 + 3k)^3(-485 - 162k + 243k^2), \\
h_{10} = c_{10}(-1 + 3k)^2, \\
h_{11} = c_{11}(-1 + 3k),
\end{cases}
$$

for some integers c_5, c_9, c_{10}, c_{11}. There is no zeros in $(0, \frac{1}{3})$ for all h_j's above.

Using $\mathcal{G}$, we find a basis $\{1, z, z^2, z^3, z^4, y, yz, yz^2, yz^3, y^2, y^2z, y^2z^2, y^3, x, xz, xz^2, xz^3,$ $xy, xyz, xyz^2, xy^2, x^2, x^2y, x^3\}$ for the algebra $A = \mathbb{Q}(k)[x, y, z]/\langle \mathcal{F} \rangle$. So, the dimension of A is 24 and there are at most 24 complex zeros for all k.

Computing four Hermite matrices, $H_0 = H(\mathcal{F}, 1)$, $H_1 = H(\mathcal{F}, x)$, $H_2 = H(\mathcal{F}, y)$, $H_3 = H(\mathcal{F}, xy)$ with $\mathcal{G}$, we find the determinant of $H(\mathcal{F}, 1)$ is not identically equal to zero. So, we obtain the bifurcation polynomial g^5 from the numerator. Using "CountRoots", we isolate three real roots a_1, a_2, a_3 of g^5 in $(0, \frac{1}{3})$.

Computing 24×4 leading principals minors, we find a_1, a_2, a_3 are also roots of the numerator of the determinants of H_1, H_2, H_3. And, all the leading principals minors of order 23 do not have a_1, a_2, a_3 as their zeros. Therefore, by Proposition 9, H_j^i has rank $r_{i,j} = 23$ for all i, j.

Choosing neighbour points of a_i's as in Table 4, we use "CountRoots" to verify the requirements in the step (6) in Sect. 3. Therefore, the numbers $v_{i,j}$ of variation of signs in the step (7) are obtained in Table 5.

Finally, using Table 5 to obtain signatures, and solving linear systems in four variables, we obtain Table 6. Note also, for all k, $x = 0$ or $y = 0$ are not zeros. Therefore, the number of roots at each quadrants is a constant in each (a_i, a_{i+1}) interval. So, Table 6 give the numbers of roots in each quadrant of the system (5) for all parameters in $(0, \frac{1}{3})$. □

5 The (1 + 3)-Body Problem

5.1 Counting Zeros for System (6)

The following results give a whole picture of the numbers of zeros for positive parameters. So, Theorem 3 is proved.

Proposition 12 *For system* (6), *we obtain the bifurcation polynomial g^6 in the Appendix. There are 11 zeros of g^6 in $(0, \infty)$. For a_2 to a_{10}, our method is applied. The neighbour points in the step* (6) *are given in the Table 7. The ranks and sign variations of the neighbour and the bifurcation points are given in Table 8. Finally, the numbers of real roots in the eight orthants for those points are given in Table 9.*

For a_1, it is a zero for some h_j in as remarked in Sect. 3.3. Our method does not apply. We will show it is not a bifurcation point. That is, the number of real zeros is a constant in the neighbourhood of this point, which is given from the results for $a_{2\ell}$.

For $a_{11} = 10$, it is again a zero for some h_j. We can just substitute k with 10 to obtain a system in $\mathbb{Q}$. We can use Hermite root counting theorem directly for this system to obtain the numbers of zeros. It is also recorded in Table 9. Same approached is applied to the sample point 100 on the right hand side of a_{11}.

Note there is a extra bifurcation point that is not obtain form g^6 in our method. That is when $k = 4$. This point of bifurcation explains why the numbers of zeros are not the same for $k = a_{9,r}$ and $k = a_{10\ell}$. It is again a zero for some h_j in as remarked in Sect. 3.3. We can again substitute k with 4 to obtain a system in $\mathbb{Q}$ and count roots directly. The results are also recorded in Table 9.

Proof Now $\mathcal{F} = \{f_6, f_7, f_8, f_9\} \subset \mathbb{Q}[r_1, r_2, r_3, r_4, k]$. Using the block order in Remark 5, we obtain $\mathcal{G} = \{g_1, \ldots, g_{59}\}$. Non constant factors in the leading coefficients that contain

positive zeros are $k - 4, k - 10, -312 + 1896k + 530k^2 + 588k^3 + 101k^4 + 31k^5 + k^6$. They have zeros 4, 10, and approximately 0.156489. These three points will be considered separately.

Using $\mathcal{G}$, we find a basis for the algebra $A = \mathbb{Q}(k)[r_1, r_2, r_3, r_4]/\langle\mathcal{F}\rangle$. They are $\{1, r_4, r_4^2, r_4^3, r_4^4, r_4^5, r_4^6, r_3, r_3r_4, r_3r_4^2, r_3r_4^3, r_3r_4^4, r_3r_4^5, r_3^2, r_3^2r_4, r_3^2r_4^2, r_3^2r_4^3, r_3^3, r_3^3r_4, r_3^3r_4^2, r_3^4,$ $r_3^4r_4, r_3^5, r_2, r_2r_4, r_2r_4^2, r_2r_4^3, r_2r_4^4, r_2r_4^5, r_2r_3, r_2r_3r_4, r_2r_3r_4^2, r_2r_3r_4^3, r_2r_3^2, r_2r_3^2r_4, r_2r_3^2r_4^2, r_2r_3^3,$ $r_2r_3^3r_4, r_2r_3^2, r_2^2r_4, r_2^2r_3, r_2^2r_3r_4, r_2^2r_3^2, r_2^2r_3^2r_4, r_2^2r_3^3, r_2^3, r_2^3r_4, r_2^3r_3, r_2^3r_3r_4, r_2^3r_3^2, r_2^4, r_2^4r_4, r_2^4r_3, r_2^5,$ $r_1, r_1r_4, r_1r_4^2, r_1r_4^3, r_1r_4^4, r_1r_3, r_1r_3r_4, r_1r_3r_4^2, r_1r_3r_4^3, r_1r_3^2, r_1r_3^2r_4, r_1r_3^2r_4^2, r_1r_2, r_1r_2r_4, r_1r_2r_4^2,$ $r_1r_2r_3, r_1r_2r_3r_4, r_1r_2r_3r_4^2, r_1r_2r_3^2, r_1r_2r_3^2r_4, r_1r_2^2, r_1r_2^2r_4, r_1r_2^2r_3, r_1r_2^2r_3r_4, r_1r_2^2r_3^2, r_1r_2^3, r_1r_2^3r_4,$ $r_1^2, r_1^2r_4, r_1^2r_4^2, r_1^2r_4^3, r_1^2r_3, r_1^2r_3r_4, r_1^2r_3r_4^2, r_1^2r_2, r_1^2r_2r_4, r_1^2r_2r_4^2, r_1^2r_2r_3, r_1^2r_2r_3r_4, r_1^2r_2^2, r_1^2r_2^2r_4,$ $r_1^2r_2^2r_3, r_1^2r_2^3, r_1^3, r_1^3r_4, r_1^3r_4^2, r_1^3r_3, r_1^3r_3r_4, r_1^3r_2, r_1^3r_2r_4\}$. Therefore, the dimension of A is 104 and there are at most 104 complex zeros for all $0 < k \neq 0.156489\ldots, 4, 10$.

For this system, we are interest in common zeros with $0 < r_1, r_2 < 1$ and $r_3 > r_4$. Therefore, we need to compute 2^3 Hermite matrices, $H_j = H(\mathcal{F}, P_j)$'s, where P_j is a product of any subset from $\{p_1 = \frac{1}{4} - (r_1 - \frac{1}{2})^2, p_2 = \frac{1}{4} - (r_2 - \frac{1}{2})^2, p_3 = r_3 - r_4\}$, including of $P_0 = 1$.

We find the determinant of H_0 is not identically equal to zero. So, we obtain the bifurcation polynomial g^6 from the numerator. Using "CountRoots", we isolate eleven real roots $a_1, \ldots, a_{11}$ of g^6 in $(0, \frac{1}{3})$. Here, $a_1 = 0.156489\ldots$ and $a_{11} = 10$ are the two zeros of some leading terms of $\mathcal{G}$ mentioned above. So, we use our method in Sect. 3 to count zeros only at $a_2, \ldots, a_{10}$.

We separate zeros into three groups. The first group is $\{a_4, a_8, a_{10}\}$. They are zeros of g^6 of multiplicity 1. The second group is $\{a_2, a_3, a_5, a_6\}$. They are zeros of multiplicity 2. The third group is $\{a_7, a_9\}$. They are zeros of multiplicity 3.

We find all a_i's are zeros of the determinants of H_j for all j. Let $D_{j,w}$ be the leading principal minors of H_j of order w and $M_{j,103}$ be the principal minor of H_j of order 103 as the definition of M_{d-1} in Proposition 9 form H.

For all a_i's in the first group, $D_{j,103}^i \neq 0$ for all j. Therefore, by Proposition 9, ranks $r_{i,j} = 103$ for $i = 4, 8, 10$ and all j. For all a_i's in the second and third groups, $D_{j,103}^i = M_{j,103}^i = 0$ for all j, and $D_{j,102}^i \neq 0$ for all j. Therefore, by Proposition 9, ranks $r_{i,j} = 102$ for $i = 2, 3, 5, 6, 7, 9$ and all j.

Choosing neighbour points of a_i's as in Table 7, we use "CountRoots" to verify the requirements in the step (6) in Sect. 3. Therefore, the numbers $v_{i,j}$ of variation of signs in the step (7) are obtained in Table 8. Finally, using Table 8 to obtain signatures, and solving linear systems in eight variables, we obtain Table 9. Note also, for all $k > 0$, zeros of p_1, p_2 or p_3 are not common zeros for $\mathcal{F}$. The number of common zeros at each orthant is a constant in each (a_i, a_{i+1}) interval for all $i = 1, \ldots, 10$.

The number of zeros at 4 and 10 and a sample point $100 > 10$ are obtained by applying Hermite root counting theorem for systems in $\mathcal{Q}[r_1, r_2, r_3, r_4]$. For the point $a_1 = 0.156489\ldots$, which is a zero of $-312 + 1896k + 530k^2 + 588k^3 + 101k^4 + 31k^5 + k^6$, we will show that is it not a real bifurcation point. Therefore, the number of zeros at that point and in the interval $(0, a_1]$ agree with that at $a_{2\ell}$.

Denote the resultant of f, g with respect to the variable x by $Res(f, g, x)$. Here are our computations. Project common zeros of $\mathcal{F} \subset \mathbb{Q}[r_1, r_2, r_3, r_4, k]$ into the r_1, r_3, r_4 space by computing $f_{10} = Res(f_6, f_9, r_2)$, $f_{11} = Res(f_7, f_9, r_2)$, $f_{12} = Res(f_8, f_9, r_2)^{\frac{1}{2}}$. Project common zeros of $\{f_{10}, f_{11}, f_{12}\} \subset \mathbb{Q}[r_1, r_3, r_4, k]$ into the r_3, r_4 space by computing $f_{13} = Res(f_{10}, f_{12}, r_1)$, $f_{14}f_{15}^2 = Res(f_{11}, f_{12}, r_1)$.

Project common zeros of $\{f_{13}, f_{14}\} \subset \mathbb{Q}[r_3, r_4, k]$ into the r_4 space by computing $Res(f_{13}, f_{14}, r_3) = c(1 + r_4^2)f_{16}f_{17}f_{18}f_{19}$. Also, project common zeros of $\{f_{13}, f_{15}\} \subset \mathbb{Q}[r_3, r_4, k]$ into the r_4 space by computing $Res(f_{13}, f_{15}, r_3) = f_{20}f_{21}$.

Table 7 Neighbors of 10 zeros of g^6 in $(0, \infty)$

i	$a_{i\ell}$	a_{ir}
1	0.15648944799874668896600194	0.15648944799874668896600195
2	0.20556926599905728622766624	0.20556926599905728622766625
3	0.67910805392204853182022205	0.67910805392204853182022206
4	0.89616163995321403519808056	0.89616163995321403519808057
5	0.99392026403213025507536460	0.99392026403213025507536470
6	1.00728250440625135445741400	1.00728250440625135445741500
7	1.06064742584296185594855900	1.06064742584296185594855960
8	1.25063752336357110125996600	1.25063752336357110125996700
9	1.42385134217617455964492600	1.42385134217617455964492700
10	4.00001429220011941250039210564680	4.00001429220011941250039210564681

Table 8 Ranks and sign variations for the system (6)

r, v	$+,+,+$	$+,+,-$	$+,-,+$	$+,-,-$	$-,+,+$	$-,+,-$	$-,-,+$	$-,-,-$
$a_{2\ell}$	104, 43	104, 48	104, 48	104, 52	104, 51	104, 51	104, 51	104, 52
a_2	102, 42	102, 47	102, 47	102, 51	102, 50	102, 50	102, 50	102, 51
a_{2r}	104, 43	104, 48	104, 48	104, 52	104, 51	104, 51	104, 51	104, 52
$a_{3\ell}$	104, 43	104, 48	104, 48	104, 52	104, 51	104, 51	104, 51	104, 52
a_3	102, 43	102, 47	102, 47	102, 52	102, 49	102, 50	102, 50	102, 50
a_{3r}	104, 45	104, 48	104, 48	104, 54	104, 49	104, 51	104, 51	104, 50
$a_{4\ell}$	104, 45	104, 48	104, 48	104, 54	104, 49	104, 51	104, 51	104, 50
a_4	103, 44	103, 47	103, 47	103, 53	103, 48	103, 50	103, 50	103, 49
a_{4r}	104, 44	104, 47	104, 47	104, 53	104, 48	104, 50	104, 50	104, 49
$a_{5\ell}$	104, 44	104, 47	104, 47	104, 53	104, 48	104, 50	104, 50	104, 49
a_5	102, 42	102, 46	102, 46	102, 53	102, 48	102, 49	102, 49	102, 47
a_{5r}	104, 42	104, 47	104, 47	104, 55	104, 50	104, 50	104, 50	104, 47
$a_{6\ell}$	104, 42	104, 47	104, 47	104, 55	104, 50	104, 50	104, 50	104, 47
a_6	102, 42	102, 46	102, 46	102, 53	102, 48	102, 49	102, 49	102, 47
a_{6r}	104, 44	104, 47	104, 47	104, 53	104, 48	104, 50	104, 50	104, 49
$a_{7\ell}$	104, 44	104, 47	104, 47	104, 53	104, 48	104, 50	104, 50	104, 49
a_7	102, 44	102, 47	102, 47	102, 51	102, 48	102, 48	102, 48	102, 47
a_{7r}	104, 45	104, 48	104, 48	104, 52	104, 49	104, 49	104, 49	104, 48
$a_{8\ell}$	104, 45	104, 48	104, 48	104, 52	104, 49	104, 49	104, 49	104, 48
a_8	103, 44	103, 48	103, 48	103, 51	103, 48	103, 49	103, 49	103, 47
a_{8r}	104, 44	104, 49	104, 49	104, 51	104, 48	104, 50	104, 50	104, 47
$a_{9\ell}$	104, 44	104, 49	104, 49	104, 51	104, 48	104, 50	104, 50	104, 47
a_9	102, 44	102, 49	102, 49	102, 51	102, 48	102, 50	102, 50	102, 47
a_{9r}	104, 45	104, 50	104, 50	104, 52	104, 49	104, 51	104, 51	104, 48
$a_{10\ell}$	104, 44	104, 50	104, 50	104, 52	104, 48	104, 52	104, 52	104, 48
a_{10}	103, 44	103, 50	103, 50	103, 51	103, 48	103, 51	103, 51	103, 47
a_{10r}	104, 45	104, 51	104, 51	104, 51	104, 49	104, 51	104, 51	104, 47

Table 9 Numbers of zeros of the system (6)

k	$+,+,+$	$+,+,-$	$+,-,+$	$+,-,-$	$-,+,+$	$-,+,-$	$-,-,+$	$-,-,-$
$a_{2\ell}$	5	4	2	2	2	2	0	1
a_2	5	4	2	2	2	2	0	1
a_{2r}	5	4	2	2	2	2	0	1
$a_{3\ell}$	5	4	2	2	2	2	0	1
a_3	5	4	1	2	1	2	0	1
a_{3r}	5	4	0	2	0	2	0	1
$a_{4\ell}$	5	4	0	2	0	2	0	1
a_4	6	4	0	2	0	2	0	1
a_{4r}	7	4	0	2	0	2	0	1
$a_{5\ell}$	7	4	0	2	0	2	0	1
a_5	7	4	0	3	0	3	0	1
a_{5r}	7	4	0	4	0	4	0	1
$a_{6\ell}$	7	4	0	4	0	4	0	1
a_6	7	4	0	3	0	3	0	1
a_{6r}	7	4	0	2	0	2	0	1
$a_{7\ell}$	7	4	0	2	0	2	0	1
a_7	7	2	0	2	0	2	0	1
a_{7r}	7	2	0	2	0	2	0	1
$a_{8\ell}$	7	2	0	2	0	2	0	1
a_8	7	2	0	2	0	2	1	1
a_{8r}	7	2	0	2	0	2	2	1
$a_{9\ell}$	7	2	0	2	0	2	2	1
a_9	5	2	0	2	0	2	2	1
a_{9r}	5	2	0	2	0	2	2	1
4	5	2	0	2	0	2	2	1
$a_{10\ell}$	5	3	0	2	0	2	3	1
a_{10}	5	2	0	2	0	2	3	1
a_{10r}	5	1	0	2	0	2	3	1
10	5	1	0	2	0	2	3	1
100	5	1	0	2	0	2	3	1

If there is a k such that the Jacobian of $\mathcal{F}$ with respect to r_1, r_2, r_3, r_4 is zero at the common zero, than the projection onto the r_4 space must be a zero of multiplicity greater than one [25]. Therefore, at least one of the polynomial from $f_{16}, \ldots, f_{21}$ has such zero of multiplicity greater than one. Let $f_{i+6} = Res(f_i, \frac{df_i}{dr_4}, r_4)$, for $i = 16, \ldots, 21$.

Finally, we find that $Res(f_i, -312 + 1896k + 530k^2 + 588k^3 + 101k^4 + 31k^5 + k^6) \neq 0$, for all $i = 22, \ldots, 27$. So, any zero of $-312 + 1896k + 530k^2 + 588k^3 + 101k^4 + 31k^5 + k^6$ is not a bifurcation point. $\qquad\square$

Appendix

$$g^4 = 14348907k^8(-1+2k)^8(-20913127289445694459314442378351149 + 1211062836562186314175259515862678452 8k$$
$$-32183304368752284073180110476338691059 2k^2 + 528281604545215476186461089077328360051 2k^3$$
$$-62321870992149340241017528750838818320 64k^4 + 599525084805566810091138728319212884859904k^5$$
$$-50895684347085605453914613409821946500 38272k^6 + 361667441346580307103886196840628156172124 16k^7$$
$$-17368227171369142650898298880299822911 38600096k^8 + 19457100670540162246175089377074126532929126 4k^9$$
$$+43612564991843996059962035884537104632024 92416k^{10} - 364216377490034671583732006725528903097159516 16k^{11}$$
$$+124882780920217517416458592876205800627856474 112k^{12} + 296860576236707349921588671957722017714777708 68k^{13}$$
$$-2466456064067386975579005444474891589338089309 08k^{14} + 1338102205179984624345234638589611724312686886912 0k^{15}$$
$$-413746700228927015503499041158440688909016052531 20k^{16} + 8075332391791940624124440710950611871018545420697 6k^{17}$$
$$-83736644329390615981846331514643026939784938389504 k^{18} - 30007914497106340313051523250403681924098155347968 k^{19}$$
$$+272731526235208997199716801978606751657411295051776 k^{20} - 47489330868599930451356683179118661258718550478028 8k^{21}$$
$$+37793679149679004153153928748410193651784800324812 8k^{22} + 4572557181049194512938076201830817803163042788147 2k^{23}$$
$$-461651770005669715931369926596333817388529501601792 k^{24} + 531482425572495898790826588549220379127061961244672 k^{25}$$
$$-303978834654331499546743054391553714615507518226432 k^{26} + 69287711932141399434591769259304063291764502429696 k^{27}$$
$$+27556767991099841456412079571314810360028677013504k^{28} - 36931239344498259899134718751866345172519622803456k^{29}$$
$$+2557005718206264653188963454500672667662357941452 8k^{30} - 131712195131130415035853222965081157436220039495 68k^{31}$$
$$+4161404052216203236353315247932243607000751013888k^{32} - 187806263065237241933283988669334605412687675392k^{33}$$
$$-5198801802142649034940000681731598530877388226 56k^{34} + 26944283317176411487590506455970022800969485516 8k^{35}$$
$$-6843319870189804470915796185666374210833874944 0k^{36} + 64595892702916054719089467499830359004733767 68k^{37}$$
$$+24912590316663409254044317757389920432354754 56k^{38} - 9757587067569828101457829512389784716181501 0400k^{39}$$
$$+9757587067569828101457829512389784716181501 5040k^{40}).$$

$$g^5 = 10003018326377130933368111046328 32k^{14}(-6193005856 + 34523341224k - 881643138884k^2 + 179027640450k^3 - 296194349727k^4 + 292634913915k^5$$
$$-135093148569k^6 + 22052144295k^7)^2(7290000000000 - 437400000000000k + 10671484725000000k^2 - 9112390285500000 0k^3 + 643286524024321875k^4$$
$$-2270895961839958125 0k^5 + 500801355667529080875k^6 - 619785003215347611390 0k^7 + 54882543612603736731564k^8 - 42114633128714335603 2660k^9$$
$$+295454655742870047159658k^{10} - 1764863078414956041955708k^{11} + 8377196992060985910807125 8k^{12} - 3069754512728107874138041 80k^{13}$$
$$+8573912082515194140551097 52k^{14} - 1794120238816087989509616300k^{15} + 272619139498337506639544923 9k^{16} - 283857497147970556688777401 8k^{17}$$
$$+1728344797213312641067462323k^{18} - 37350374248580151836867904k^{19} - 13000314295221162948561071 52k^{20} + 198344186395145053346360937 6k^{21}$$
$$-2053166141818806972470118720k^{22} + 15312808661649450265512 89856k^{23} - 694191759631599704931041472k^{24} + 158306825224330412352 7104k^{25}$$
$$+2865252156973234610693875 20k^{26} - 258835251578200089068123136k^{27} + 13351410874366972726686105 6k^{28} - 40315836246102348631824384k^{29}$$
$$+52809328743617147281274 88k^{30}).$$

$$g^6 = (-10+k)^2(-312 + 1896k + 530k^2 + 588k^3 + 101k^4 + 31k^5 + k^6)^2(2963399638413312 + 23142740033323008k + 72951703384555008k^2$$
$$+111198530932125696k^3 + 57853164006793584k^4 - 74362983171264576k^5 - 15043284900478624 0k^6 - 90543746418355752k^7 + 13327925036628816k^8$$
$$+44887125905050176k^9 + 20891973423983679k^{10})^3(-8438743501575408 + 2678966190976320k + 345472406710968960k^2 + 40029645576123828 0k^3$$
$$-1138465254406519512 0k^4 + 12123699015371647968k^5 + 119990316536124348321k^6 - 15950084552134832073 6k^7 - 177185206222332959386 8k^8$$
$$+9249906747903520991616k^9 - 2560802346451818104726 0k^{10} + 5319268506123072931584 0k^{11} - 87060482984162666161152k^{12}$$
$$+10502946756022690220008 32k^{13} - 9033026688160114998006 72k^{14} + 544412526846470436275 2k^{15} - 359782824755109048422 4k^{16}$$
$$-50207178662075239133184k^{17} + 4510139049847521493401 6k^{18} + 28988560858374445547 52k^{19} - 11491835434562035187712k^{20}$$
$$+544400220773632720896k^{22})(130826199714027687546065564006018101914573913451569359994533161147196600096890290176$$
$$+6379334690817350097484339882960120779072556541638429744495331286415681833295983673344k$$
$$+143343436290907434190236021879697731112605548862658775303662260908808778086044764471296k^2$$
$$+19259714686001006400785297263923199816866840295826013421046484539673476699445338623180 80k^3$$
$$+16429902038782606583966068303176355643061275650052560033878736641831321770977372649505024k^4$$
$$+79818441780782785298622655114118418614152880689308033436557543804575226170277636842717184k^5$$
$$+11861556659920223788540462451206599915743797275844984652670697944025985808933496087680 00k^6$$
$$-33264258777173494048907661148544188474912014861300004392474168258124949363371872282439454 72k^7$$
$$-29810408973203405650504903643812092463921474518251682329413306945917906711732246082125588848k^8$$

$$-13776667519643318965247053853298542340870348149167233004847884741901790019401990113658345062 4k^{9}$$

$$-22113217086871592283745006253267216677870131610205289015498062638412065374479106206894902905 6k^{10}$$

$$+18076636684768433797872474486547042812230723649065074678539709017346335840371944833834494115 84k^{11}$$

$$+17210516363703688109410764981931187566113319520547741849634865036725432133642018859904797599 616k^{12}$$

$$+76453250500788007077602612456964514915210305030229523300641479285800042780910399312512265073 92k^{13}$$

$$+17182607494519759743947794434595588687869644732777420174273292559135855063604957372250627146 816k^{14}$$

$$-18779979921972946213048034501990575709832952998835401120669151837852790469024856014907073415 0912k^{15}$$

$$-35069355407552015160616287656713989793348923976302284747316456668027913051630826285793163062 0928k^{16}$$

$$-16685981007862080222261942014908814524889083182754418387932967366610605254027552236687716057 516160k^{17}$$

$$-46741539140815536453664256883776051478980280491609165478123328651577225431704752392429668464 103648k^{18}$$

$$-61791521559184706472527367904713974509060889517035425682666785763021907402125508683791314157 012224k^{19}$$

$$+13124960185895542801362683045669053531888369853452320789005957351159703116814171185594191788 454000k^{20}$$

$$+10685445607100468981114431811630387296567216296805055664534395698870091001296043841218515886 03022464k^{21}$$

$$+34630799924144609882743543089886733342230039010967435823053872438540269060178782281773279786 1546441k^{22}$$

$$+64198221134348792502823023155211082994342982584659798104257374981070217477256298287696571335 421600k^{23}$$

$$+19795199039398038934381723536257222852266709118749215401409866283633219083176037401484454910 21677176k^{24}$$

$$-34683376930038393458953245095561876552348902973219361538733201261128603393761859204472584541 895138972k^{25}$$

$$-15408710460780750814627966603304383823356070135928971219015357864230638057019372379062266060 6190028364k^{26}$$

$$-42745524318263845067214016926020719172392224103405554855001671153546652659687731982412920190 9624857024k^{27}$$

$$-92308968232339289166604707414095845551710805238289830570619420534548192125773402564602310185 5683462418k^{28}$$

$$-17089729939437323888036342062320343955578946431577332836001995206272429277564317811873917877 9929839696k^{29}$$

$$-30119902058529231109453082608913568677081473890867705276920749555033091309271941703730669702 26410019726k^{30}$$

$$-56789344873446022792279633680625177215463536086344457446569877438197497862763875508900613362 79696102364k^{31}$$

$$-11815279314924989958518720012276790070346076895949444644113895087051704250275013490466032537 01891281152k^{32}$$

$$-24801774380754131156491823684222603099127511055198039944478179594317357994308496365195196441 449729521496k^{33}$$

$$-47168261827276459284542617976633789007784920708407136843055560592776881329950703073730765279 36197929299k^{34}$$

$$-75093411932431688760387739140737443441255559997177462428517787642799225677995702550689867686 64668904 4176k^{35}$$

$$-90735490170765155437366044777137870223414013922975351374908498666176348567238357739348175822 928477777300k^{36}$$

$$-57849476055832003614944342766728571184345788180763119952952913029256133284610070847957480209 57580476 2128k^{37}$$

$$+70880080173819969063467196699223343380987603661719313659638637345083086195282278990831134960 2215941535 95k^{38}$$

$$+32940662045591092045568625242700345497093064145482729470846204464274424435173769987709738873 26855958 48392k^{39}$$

$$+70262020332676098538659340835133357938191901122960499248139896350040243341234847214443029427 26842203139792k^{40}$$

$$+10993551105163268737848202196475878144894611575920586506842551062767815859158915085399397512 5283452283 8396k^{41}$$

$$+13602677653109012484064935633654795304445154560761855105706533271361520838495728303984406377 93330806071530k^{42}$$

$$+13126790385811208388123992055626317487513111210939036813537311670350878530849808087515446511 21280365610480k^{43}$$

$$+85642520600260211836951145941324154370234844237796250455950361231657545170649791841780264439 2316745079506k^{44}$$

$$+36202229725222209897870509764907554989756506518674102892227680007033063428322967526133340650 47564720416k^{45}$$

$$-94683910397411337879700064870354530357142853055541027993936853926426307648982172564242998943 6746737392464k^{46}$$

$$-18022335674146725169665508054722808662223808699267311600459834985272662516691075887198383989 80164400794052k^{47}$$

$$-22679699797231433188675232168960521515503885752559750204120375029200198556888479255234452132 87333679308040k^{48}$$

$$-22188017127940437013937530650269156913176660445533621939962058427089622287065037655424779611 45186583426768k^{49}$$

$$-17096614730643097590569947530230212532965806458089053634868948527846330503374126406231217428 0290326456929k^{50}$$

$$-93678125640449875817308252897482895359891262152599209119034808887666625869333760819090941046 9912078373792k^{51}$$

$$-14442574492849686660268340300390562292005525784959378434948320107575064054028614271981297989 57525867575656k^{52}$$

$$+47097206797101762126765626245184422139579132641964703642183533550159612693817057627147179257 6057499620064k^{53}$$

$$+81602915363411431270542407261307754102805145622126043073498549236788774278930173146599841915 074331962671k^{54}$$

$$+90139824186299873021656620613610547038376212617982565345208067234886733315670606251964711313 651808966 8240k^{55}$$

$$+80399792318593069249664702154455061903076973615444672092242778126622983115436143765621840539 4908377769368k^{56}$$

$$+61869999345474756018315321239587536365398906910530318157267300513350180338881751130520126198 1862448965788k^{57}$$

$$+ 4224520784646955554745540307423727998303337074690143414053390172322884975044440111727445619250877425144496 k^{58}$$
$$+ 2596113141688840067882778477696043715845360419467566867901890450712014551125572368361009986985979734316 8k^{59}$$
$$+ 1447115221364174305454615036061098804986343713387359596123842117532692407812965045231474019983885319859 06k^{60}$$
$$+ 734791690501306461514537258781889961428629558174497102193863668422800271593168362019037389017126091120 k^{61}$$
$$+ 340555092816547453491834387349755720632605645819423785150542378506393684816163655803863343527843134670 02k^{62}$$
$$+ 144133290567027843478760126500857287732295049939028235063242249264235113439143668088851835120470031202 8k^{63}$$
$$+ 55659281394499739074597715806801310702174698892001473951352092132297224416378819767397745357819898379 36k^{64}$$
$$+ 19573251917971685521710335748260199969929951345195198032607252504247209288045941134854936712202888548 24k^{65}$$
$$+ 6249168376150360603342686029370541304367116152535943482188543264819886938027458918939722820303875640 27k^{66}$$
$$+ 1803839694586279612029893832656877436549916227493193351573555118535867178300269321352090421419201880 0k^{67}$$
$$+ 46817949116451900797748670248298961704611767473584313537835991741717254695275449730282109053029547852 k^{68}$$
$$+ 10849875318994151603032797323948467554024140287730197036749234364809086774742896450246169431629221552 k^{69}$$
$$+ 2225221508950651459592654266022542551525792485857827235132999395189877944485800821244395576606621885 k^{70}$$
$$+ 399333836316577577906961899511206835549858229188976348726535631026857736989583546786091985945900680 k^{71}$$
$$+ 61796521467123963355105984078485972861372383293338624242889260861139163023969503929397554019685408 k^{72}$$
$$+ 8089022711223645600529464708861676042153179708190779666179924655790669763003713977485687587578 92k^{73}$$
$$+ 872542331281041329790157302889074619752457775630885310148043768798640392163509372031654114995970 k^{74}$$
$$+ 74745328349994581611779939844485006064910130668357818169305194856380178617038980552481143084848 k^{75}$$
$$+ 481135022720708804765766113503474507560415130957588904696870550482754272851481328650429437825 4k^{76}$$
$$+ 21294227565741154420247327595008296597666022046976788027752226258585440011983925841546405578 88k^{77}$$
$$+ 558102291224288417144610808458714792425717866659349236147446638931617230156687379944309454 96k^{78}$$
$$+ 7633562313898569523355203622559753390265018723694588749256635763949096709673638936520250 0k^{79}$$
$$+ 1550152988085496822992578379085646470777530030706561536010310454161996471356618527375000 k^{80}$$
$$+ 691243125044510274005912103124481176332055353520718216999811338315381410100700000000000 0k^{81}$$
$$+ 1190887660945238808095380017180843932807535881669966446858733623752809140437597656 25k^{82})^2.$$

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