Acta Appl Math (2014) 132:139–149
DOI 10.1007/s10440-014-9912-1

On the Prandtl Boundary Layer Equations in Presence of Corner Singularities

M. Cannone · M.C. Lombardo · M. Sammartino

Received: 15 December 2013 / Accepted: 3 April 2014 / Published online: 17 May 2014

Abstract In this paper we prove the well-posedness of the Prandtl boundary layer equations on a periodic strip when the initial and the boundary data are not assigned to be compatible.

Keywords Boundary layer · Incompatible data · Analytic norm

1 Introduction

Prandtl's equations describe the behavior of an incompressible flow near a boundary in the zero viscosity limit. There is an extensive literature on boundary layer associated with incompressible flows and the related question of the behavior of the Navier-Stokes solutions in the vanishing viscosity limit (see for instance [2, 5, 6, 9, 12, 13, 16, 20, 25–28, 30–34, 37–39, 42, 45]). We shall not survey the literature here, but emphasize that the boundary layer problem associated with the NS equations is still open and that there is a need to develop tools and methods to tackle it. There are not many well-posedness results, and in fact recent results suggest that these equations could be ill-posed in certain Sobolev spaces [14, 19, 21, 23]. On the other hand it is known that, when the data are analytic with respect to the tangential variable, Prandtl's equations are well posed [7, 27, 29, 38]. However, so far, it has always been assumed that the initial and boundary data, are compatible.

It is well known that, if the initial and boundary data assigned for a PDE do not obey an infinite set of compatibility conditions, singularities will arise in the solution at the corners of

In honor of Professor Salvatore Rionero, on the occasion of his 80th birthday.

M. Cannone
Université Paris-Est, Laboratoire d'Analyse et de Mathématiques Appliquées, UMR, 8050 CNRS, 5, boulevard Descartes, Cité Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, France
e-mail: marco.cannone@univ-mlv.fr

M.C. Lombardo (✉) · M. Sammartino
Department of Mathematics, University of Palermo, Via Archirafi 34, 90123 Palermo, Italy
e-mail: mariacarmela.lombardo@math.unipa.it

M. Sammartino
e-mail: marco.sammartino@math.unipa.it

the time-space domain [40, 41]. For dissipative equations, these singularities are known to be short lived. The issue of compatibility between initial and boundary data has been therefore receiving considerable attention, other than for it has an intrinsic theoretical interest also because of its consequences on the numerical solutions of the corresponding PDEs [3, 10, 11, 15, 24].

In [8] it has been recently proved that the Prandtl equations on a 2D (or 3D) half space are well posed, even when initial and boundary data are incompatible, under the assumption of analyticity in the tangential variable. In this paper we shall consider a periodic domain in the x-variable and introduce different real analytic norms. This result generalizes previously proved theorems removing the compatibility hypothesis, and is also of interest for the two reasons that follow.

First, in almost all the cases where the Prandtl equations have been investigated numerically, an incompatibility between initial and boundary data was present [17, 18, 44]. It was left to the numerical scheme to handle the initial singularity but it was never proved that this initial singularity and the following high gradients (in the normal direction) did not interact with the tangential structure where viscosity is absent.

Second, it could be a step toward the analysis of the zero viscosity limit of Navier-Stokes solutions for not well prepared initial data. This is a problem of relevant interest and has been recently afforded for the linearized Navier-Stokes equations [22].

We shall be concerned with the Prandtl's equations written as:

$$\partial_t u^P + u^P \partial_x u^P + v^P \partial_Y u^P + \partial_x p^E = \partial_{YY} u^P, \quad (x, Y) \in [0, 2\pi] \times \mathbb{R}^+, \tag{1}$$

$$\partial_x u^P + \partial_Y v^P = 0, \tag{2}$$

$$u^P(x, Y = 0, t) = 0, \tag{3}$$

$$u^P(x, Y, t = 0) = u_0(x, Y), \tag{4}$$

$$u^P(x, Y \to \infty, t) = U(x, t), \tag{5}$$

$$v^P(x, Y = 0, t) = 0, \tag{6}$$

where the Euler matching datum $U(x, t)$ and the pressure $p^E = p^E(x, t)$ satisfy the Bernoulli law:

$$\partial_t U + U \partial_x U + \partial_x p^E|_{y=0} = 0. \tag{7}$$

All the data will be assumed to be periodic in x, i.e.:

$$u_0(0, Y) = u_0(2\pi, Y) \tag{8}$$

$$U(0, t) = U(2\pi, t) \tag{9}$$

but, in general, initial and boundary data are not compatible, i.e., in general:

$$u_0(x, Y = 0) \neq 0 = u^P(x, Y = 0, t = 0). \tag{10}$$

2 Function Spaces and Main Result

In this section we shall introduce the function spaces where the well-posedness Theorem will be proved.

Definition 1 The space $\mathcal{H}^\sigma$ is the space of the 2π periodic real functions $f(x)$ such that:

$$|f|_\sigma \equiv \sum_{k\in\mathbb{Z}} \left|\hat{f}(k)\right| e^{|k|\sigma} < \infty,$$

where $\hat{f}(k)$ are the Fourier coefficients of f.

Following [27], we introduce the Y-weight given by:

$$\rho(Y) = \langle Y\rangle^\alpha, \quad \text{where } \alpha > 1/2 \text{ and } \langle Y\rangle = \sqrt{1+Y^2}.$$

Definition 2 The space $\mathcal{H}^{\sigma,\alpha}$, with $\alpha > 0$, is the space of the real functions $f(x, Y)$, 2π periodic with respect to x, such that:

- $\rho(Y)\partial_Y^j f \in L^\infty(\mathbb{R}^+, \mathcal{H}^\sigma)$ when $j \le 2$;

the norm in $\mathcal{H}^{\sigma,\alpha}$ is defined as:

$$|f|_{\sigma,\alpha} \equiv \sum_{j\le 2} \sup_{Y\in\mathbb{R}^+} \rho(Y)\left|\partial_Y^j f(\cdot, Y)\right|_\sigma.$$

Definition 3 The space $\mathcal{H}^\sigma_{\beta,T}$,with $\beta > 0$, is the space of the functions $f(x,t)$, 2π periodic with respect to x, such that:

- $\partial_t^i f(x,t) \in \mathcal{H}^{\sigma-\beta t}$ $\forall 0 \le t \le T < \sigma/\beta$, where $0 \le i \le 1$.

Moreover:

$$|f|_{\sigma,\beta,T} \equiv \sum_{0\le i\le 1} \sup_{0\le t\le T} \left|\partial_t^i f(\cdot,t)\right|_{\sigma-\beta t} < \infty.$$

Definition 4 The space $\mathcal{H}^{\sigma,\alpha}_{\beta,T}$ is the space of the functions $f(x,Y,t)$, 2π periodic with respect to x, such that:

- $f, \partial_t f, \partial_Y^j f \in \mathcal{H}^{\sigma-\beta t,\alpha}$, $\forall 0 \le t \le T$, and $j \le 2$.

Moreover:

$$|f|_{\sigma,\alpha,\beta,T} \equiv \sum_{0\le j\le 2} \sup_{0\le t\le T} \left|\partial_Y^j f(\cdot,\cdot,t)\right|_{\sigma-\beta t,\alpha} + \sup_{0\le t\le T} \left|\partial_t f(\cdot,\cdot,t)\right|_{\sigma-\beta t,\alpha} < \infty.$$

Once we have introduced the functional spaces, we can state the main result of this paper:

Theorem 1 (Main result) *Suppose that the Euler datum U and the initial datum u_0 satisfy the following hypotheses*:

(i) *U satisfies the Bernoulli law Eq.* (7);
(ii) $U(x,t) \in \mathcal{H}^{\sigma_0}_{\beta_0,T_0}$;
(iii) $u_0(x,Y) - U(x,t=0) \in \mathcal{H}^{\sigma_0,\alpha}$.

Then there exist $0 < \sigma < \sigma_0$, $\beta > \beta_0 > 0$ and $0 < T < T_0$ such that Prandtl equations Eqs. (1)–(5) *admit, in* $[0,T]$, *a unique solution u^P. This solution can be written as*:

$$u^P(x,Y,t) = u^S + u^R, \tag{11}$$

where

1. u^S *is an initial layer corrector and has the following form*:

$$u^S = -2u_0(x, Y = 0)\,\mathrm{erfc}\left(\frac{Y}{2\sqrt{t}}\right).$$

2. u^R *is the solution of the Prandtl equation with compatible data and with a source term that keeps into account the interaction with* u^S; u^R *can be decomposed as follows*:

$$u^R(x, Y, t) = \tilde{u}(x, Y, t) + U$$

where $\tilde{u} \in \mathcal{H}^{\sigma,\alpha}_{\beta,T}$.

Moreover the solution $u^P(x, Y, t)$, *for* $t > 0$, *is analytic both in* x *and* Y.

We now make some comments on the hypotheses we have imposed to the data.

Hypothesis (i) simply means that the datum U is consistent with the Euler equations, and it is imposed on the Prandtl equations once the Euler solution is known. The physical meaning is that the Prandtl solutions has to match, outside the boundary layer, the inviscid Euler flow.

Hypothesis (ii) means that the Fourier spectrum of the Euler datum is exponentially decaying; this kind of analyticity requirement is typical for the Prandtl equations, see e.g. [7, 8, 27, 38], and it is needed to compensate for the lack of dissipation along the streamwise direction x. Without this requirement (or without the monotonicity hypothesis *à la* Oleinik) there are strong evidences that the Prandtl equations might be ill-posed, see the recent papers [19, 21].

Hypothesis (iii) has two different meanings: first, it is a requirement of analyticity in the streamwise variable x for the initial datum $u_0(x, Y)$ (see the comments above concerning the meaning of analyticity); second, it enforces the compatibility between the initial datum and the Euler datum requiring the polynomial decay of their difference when $Y \to \infty$. It would be certainly interesting to analyze the case where there is incompatibility between the Euler and the initial data: however, in this paper we shall be concerned only with the incompatibility between the boundary and the initial data.

To prove the above theorem, we shall use a Cauchy-Kowalewski-type (Fixed Point) Theorem in the abstract setup of weighted Banach spaces. The abstract form of the Cauchy-Kowalewski Theorem has attracted the attention of many authors, among which Treves [43], Nirenberg [35], Nishida [36], Caflisch [4], Asano [1], just to mention a few. Here we shall use a version of the Abstract Cauchy-Kowalewski Theorem (ACK) whose proof is given in [29].

Let $\{X_\rho : 0 < \rho \le \rho_0\}$ be a Banach scale with norms $|\ |_\rho$, such that $X_{\rho'} \subset X_{\rho''}$ and $|\ |_{\rho''} \le |\ |_{\rho'}$ when $\rho'' \le \rho' \le \rho_0$. In this Banach scale, for t in $[0, T]$, consider the equation:

$$u + F(t, u) = 0. \tag{12}$$

The formal statement of the ACK Theorem can be found in [29]. Here we just mention that the key point to ensure the existence of a solution to the problem (12) is the quasi-contractiveness of the operator F, namely:

$$\left|F\big(t, u^1\big) - F\big(t, u^2\big)\right|_{\rho'} \le C \int_0^t ds \left(\frac{|u^1 - u^2|_{\rho(s)}}{\rho(s) - \rho'} + \frac{|u^1 - u^2|_{\rho'}}{\sqrt{t - s}}\right) \tag{13}$$

when $0 < \rho' < \rho(s) < \rho_0$.

3 Construction of the Solution

To prove Theorem 1, we set:

$$u^P = u^S + u^R = u^S + u^D + w_1, \tag{14}$$

where we have further decomposed the regular part of the solution u^R as

$$u^R = u^D + w_1.$$

We shall now separately consider the different terms appearing on the right hand side of (14).

Singular Solution The singular solution $\mathbf{u}^S = (u^S, v^S)$ satisfies the following equations:

$$(\partial_t - \partial_{YY})u^S = 0 \tag{15}$$

$$u^S(x, Y = 0, t) = -u_0(x, Y = 0) \tag{16}$$

$$u^S(x, Y, t = 0) = 0 \tag{17}$$

and

$$v^S = -\int_0^Y \partial_x u^S\big(x, Y', t\big)\, dY'. \tag{18}$$

It absorbs the incompatibility between the initial and the boundary data. The explicit expression of u^S is given by:

$$u^S = -\frac{2u_0(x, Y = 0)}{\sqrt{\pi t}} \int_Y^\infty \exp\left(-\frac{s^2}{4t}\right) ds = -2u_0(x, Y = 0)\,\mathrm{erfc}\left(\frac{Y}{2\sqrt{t}}\right). \tag{19}$$

Notice that u^S displays a corner singularity in $(x, Y = 0, t = 0)$.

Regular Solution The regular solution $\mathbf{u}^D = (u^D, v^D)$ satisfies the following equations:

$$\partial_t u^D + u^D \partial_x u^D + v^D \partial_Y u^D + \partial_x p^E = \partial_{YY} u^D \tag{20}$$

$$\partial_x u^D + \partial_Y v^D = 0 \tag{21}$$

$$u^D(x, Y = 0, t) = u_0(x, Y = 0) \tag{22}$$

$$u^D(x, Y, t = 0) = u_0(x, Y) \tag{23}$$

$$u^D(x, Y \to \infty, t) = U(x, t) \tag{24}$$

Notice that u^D has initial data $u_0(x, Y)$ and boundary data which are compatible.

Following the same procedure adopted in [29], one can recast Eqs. (20)–(24) in a suitable form for the application of the Fixed Point Theorem, and verify that its hypothesis are satisfied. The only difference is the way one has to estimate the x-derivative of u^D appearing on the left hand side of (20). The desired Cauchy estimate is obtained by means of the following Proposition, where the norm of the derivative of a function in $\mathcal{H}^{\sigma''}$ can be bounded in terms of the norm of the function itself in a smaller Banach space. Namely one has:

Proposition 1 *Let $f \in \mathcal{H}^{\sigma''}$. If $\sigma' < \sigma''$ then*

$$|\partial_x f|_{\sigma'} \le \frac{|f|_{\sigma''}}{\sigma'' - \sigma'}. \tag{25}$$

Proof

$$\begin{aligned} |\partial_x f|_{\sigma'} &= \sum_{k\in\mathbb{Z}} |k| \big|\hat{f}(k)\big| e^{|k|\sigma'} \\ &\le \sum_{k\in\mathbb{Z}} |k| \big(\sigma'' - \sigma'\big) e^{-|k|(\sigma''-\sigma')} \frac{|\hat{f}(k)|}{\sigma''-\sigma'} e^{|k|\sigma''} \le c \frac{|f|_{\sigma''}}{\sigma''-\sigma'}, \end{aligned}$$

where the last inequality follows from the fact that, for $x > 0$, xe^{-x} is bounded by a constant. □

Analogous estimates can be proved in the spaces $\mathcal{H}^{\sigma}_{\beta,T}$, $\mathcal{H}^{\sigma,\alpha}$ and $\mathcal{H}^{\sigma,\alpha}_{\beta,T}$. Therefore one can prove the following

Theorem 2 *Suppose $U \in \mathcal{H}^{\sigma_0}_{\beta_0,T}$ and $u_0 - U \in \mathcal{H}^{\sigma_0,\alpha}$. Then there exist $0 < \sigma_1 < \sigma_0$, $\beta_1 > \beta_0 > 0$ and $0 < T_1 < T$ such that Eqs.* (20)–(24) *admit a unique mild solution u^D. This solution can be written as*:

$$u^D(x, Y, t) = u(x, Y, t) + U, \tag{26}$$

where $u \in \mathcal{H}^{\sigma_1,\alpha}_{\beta_1,T_1}$.

The Interaction Part It is straightforward to verify that the interaction part $\mathbf{w} = (w_1, w_2)$ satisfies the following equations [8]:

$$w_1 = F(t, w_1) \tag{27}$$

where:

$$F(t, w_1) = E_2 L(w_1) + E_3 N(w_1) + E_3 G, \tag{28}$$

$$L(w_1) = -2\big[w_1 \partial_x \big(u^S + u^D\big)\big], \tag{29}$$

$$N(w_1) = -\left[w_1 \int_0^Y \partial_x w_1 dY' + \big(u^S + u^D\big) \int_0^Y \partial_x w_1 dY'\right] + w_1 \int_0^Y \partial_x \big(u^S + u^D\big) dY', \tag{30}$$

$$G = -\big(u^D v^S + u^S v^D + u^S v^S\big), \tag{31}$$

and E_2 and E_3 are inverse heat operators. E_2 satisfies the following heat equation with zero initial and boundary data:

$$\begin{aligned} (\partial_t - \partial_{YY}) E_2 f &= f \\ E_2 f|_{t=0} &= 0 \end{aligned} \tag{32}$$

$$\gamma E_2 f = 0,$$

and $E_3 f \equiv E_2 \partial_y f$, for those functions $f(x, Y, t)$ such that $f(x, Y = 0, t) = 0$. The explicit expressions of E_2 and E_3 are given in [8].

On w_1 one imposes the following boundary and initial data:

$$w_1(x, Y = 0, t) = 0 \tag{33}$$

$$w_1(x, Y, t = 0) = 0 \tag{34}$$

$$w_1(x, Y \to \infty, t) = 0. \tag{35}$$

Moreover, using the incompressibility condition, one has:

$$w_2(x, Y, t) = -\int_0^Y dY' \, \partial_x w_1\big(x, Y', t\big). \tag{36}$$

We shall prove the well-posedness of the above Initial Boundary Value Problem for w_1, namely we shall prove the following:

Theorem 3 *Suppose* $U \in \mathcal{H}^{\sigma_0}_{\beta_0,T}$ *and* $\bar{u}_0 \equiv u_0 - U_{t=0} \in \mathcal{H}^{\sigma_0,\alpha}$. *Then there exist* $0 < \rho_1 < \rho_0$, $\beta_1 > \beta_0 > 0$ *and* $0 < T_1 < T$ *such that Eqs.* (27)–(35) *admit a unique solution* $w_1 \in \mathcal{H}^{\sigma_1,\alpha}_{\beta_1,T_1}$. *The following estimate holds*:

$$|w_1|_{\sigma_1,\alpha,\beta_1,T_1} \le c \left(|u_0|_{\sigma_0,\alpha} + |U|_{\sigma_0,\beta_0,T}\right). \tag{37}$$

The remaining part of this section will therefore be devoted to prove that the operator $F(t, w_1)$, as defined by (28), satisfies the hypothesis of the ACK Theorem.

The first step consists in showing that $F(t, 0)$ can be bounded by the initial data of the Prandtl equation and the boundary data of the outer Euler flow. This involves the estimate of the following three terms: $|E_3(u^D v^S)|_{\sigma_1,\alpha,\beta_1,T_1}$, $|E_3(u^S v^D)|_{\sigma_1,\alpha,\beta_1,T_1}$, $|E_3(u^S v^S)|_{\sigma_1,\alpha,\beta_1,T_1}$.

The ideas underlying the three estimates are quite similar, we shall therefore give here only the estimate of $E_3(u^S v^S)$. In what follows we shall denote by $\chi(Y)$ a monotone bounded function going to zero linearly fast as $Y \to 0$ and we shall moreover set:

$$\mathcal{K}\big(Y, Y', t, s\big) = \frac{Y - Y'}{t - s} \frac{e^{-(Y-Y')^2/4(t-s)}}{[4\pi(t-s)]^{1/2}} + \frac{Y + Y'}{t - s} \frac{e^{-(Y+Y')^2/4(t-s)}}{[4\pi(t-s)]^{1/2}}. \tag{38}$$

One therefore has:

$$\begin{aligned}
&\big|E_3\big(u^S v^S\big)\big|_{\sigma',\alpha} \\
&\quad = 4 \sup_{Y \ge 0} \langle Y\rangle^\alpha \sum_{k\in\mathbb{Z}} e^{|k|\sigma'} \bigg| \int_0^t ds \int_0^\infty dY \, \mathcal{K}\big(Y, Y', t, s\big) \hat{u_0}(k) k \hat{u_0}(k) \\
&\qquad \times \int_{Y'}^\infty dY'' \frac{e^{-\frac{Y''^2}{4s}}}{\sqrt{\pi s}} \int_0^{Y'} dY''' \int_{Y'''}^\infty \frac{e^{-\frac{\tau^2}{4s}}}{\sqrt{\pi s}} \, d\tau \bigg| \\
&\quad \le c \frac{|u_0|_{\sigma'',\beta_1,T_1}}{\sigma'' - \sigma'} \sup_{Y \ge 0} \langle Y\rangle^\alpha \bigg| \int_0^t ds \int_0^\infty dY' \mathcal{K}\big(Y, Y', t, s\big) \frac{e^{-\frac{Y'^2}{4s}}}{\sqrt{\pi s}} \chi\big(Y'\big) \bigg|
\end{aligned}$$

$$
\begin{aligned}
&\le c\frac{|u_0|_{\sigma'',\beta_1,T_1}}{\sigma''-\sigma'} \sup_{Y\ge 0}\sup_{0\le t\le T_1}\int_0^t \frac{ds}{\sqrt{t-s}} \times \left(\int_{-\frac{Y}{\sqrt{4(t-s)}}}^{\infty} d\eta\eta\, e^{-\eta^2} + \int_{\frac{Y}{\sqrt{4(t-s)}}}^{\infty} d\eta\,\eta\, e^{-\eta^2}\right)\\
&\le c\frac{|u_0|_{\sigma'',\beta_1,T_1}}{\sigma''-\sigma'} \le c|u_0|_{\sigma'',\beta_0,T}
\end{aligned}
$$

where, in passing from the second to the third row we have used Proposition 1 to estimate the term which involve the x-derivative and the fact that

$$
\int_0^Y dY' \int_{Y'}^{\infty} d\sigma\, \frac{e^{-\frac{\sigma^2}{4t}}}{\sqrt{4\pi t}} \le \psi(Y).
$$

The proof of the estimates of the terms

$$
\left|\partial_Y E_3\left(u^S v^S\right)\right|_{\sigma',\alpha}, \quad \left|\partial_{YY} E_3\left(u^S v^S\right)\right|_{\sigma',\alpha} \quad \text{and} \quad \left|\partial_t E_3\left(u^S v^S\right)\right|_{\sigma',\alpha}
$$

proceeds along the same lines of the analogous estimates in [8], namely letting the derivatives act on the kernel of the operator E_3 and using the regularizing properties of the integration in time to control the term going as $1/\sqrt{t-s}$. This proves that $F(t,0)$ is bounded in $\mathcal{H}^{\sigma_1,\alpha}_{\beta_1,T_1}$.

To prove the well-posedness of Eq. (27) for w_1, we are left to prove the almost contractiveness property of the operator F, in the form given by (13).

We have to estimate the operators $E_2L(w_1)$ and $E_3N(w_1)$.

Of the two terms appearing in $E_2L(w_1)$, the most difficult is $E_2(w_1\partial_x u^S)$, owing to the presence of the singular term u^S. We shall give its estimate in the space $\mathcal{H}^{\sigma',\alpha}$. In particular we shall just show the estimate of the term $|\partial_{YY} E_2(w_1\partial_x u^S)|_{\rho',\alpha}$. Let us use the following notation for the heat kernel:

$$
\mathcal{E}\left(Y,Y',t-s\right) = \frac{e^{-(Y-Y')^2/4(t-s)}}{[4\pi(t-s)]^{1/2}} - \frac{e^{-(Y+Y')^2/4(t-s)}}{[4\pi(t-s)]^{1/2}}.
$$

One then has:

$$
\begin{aligned}
&\left|\partial_{YY} E_2\left(w_1\partial_x u^S\right)\right|_{\sigma',\alpha}\\
&\quad = 2\sup_{Y\ge 0}\langle Y\rangle^{\alpha}\sum_{k\in\mathbb{Z}} e^{|k|\sigma}\left|\int_0^t ds\int_0^{\infty} dY'\partial_{Y'Y'}\mathcal{E}\left(Y,Y',t-s\right)\right.\\
&\qquad \left.\times\, \hat{w}_1\left(k,Y',s\right)k\hat{u}_0(k)\int_{Y'}^{\infty} d\rho\, \frac{e^{-\frac{\rho^2}{4s}}}{\sqrt{\pi s}}\right|\\
&\quad \le c\sup_{Y\ge 0}\langle Y\rangle^{\alpha}\sum_{k\in\mathbb{Z}} e^{|k|\sigma}\left|\int_0^t ds\int_0^{\infty} dY'\partial_{Y'}\mathcal{E}\left(Y,Y',t-s\right)\right.\\
&\qquad \left.\times\, \partial_{Y'}\hat{w}_1\left(k,Y',s\right)k\hat{u}_0(k)\int_{Y'}^{\infty} d\rho\, \frac{e^{-\frac{\rho^2}{4s}}}{\sqrt{\pi s}}\right|\\
&\qquad + c\sup_{Y\ge 0}\langle Y\rangle^{\alpha}\sum_{k\in\mathbb{Z}} e^{|k|\sigma}\left|\int_0^t ds\int_0^{\infty} dY'\partial_{Y'}\mathcal{E}\left(Y,Y',t-s\right)\right.
\end{aligned}
$$

$$\times\, \hat{w}_1\big(k, Y', s\big) k\hat{u}_0(k) \frac{e^{-\frac{Y'^2}{4s}}}{\sqrt{\pi s}} \Bigg|$$

$$\leq c \sup_{Y\geq 0} \langle Y\rangle^{\alpha} \int_0^t ds \int_0^{\infty} dY' \big|\partial_{Y'}\, \mathcal{E}\big(Y, Y', t-s\big)\big|$$

$$\times\, \big|\partial_{Y'} \hat{w}_1\big(\cdot, Y', s\big)\ \partial_x \hat{u}_0(\cdot)\big|_{\sigma'} \int_{Y'}^{\infty} d\rho\, \frac{e^{-\frac{\rho^2}{4s}}}{\sqrt{\pi s}}$$

$$+ \sup_{Y\geq 0} \langle Y\rangle^{\alpha} \sum_{k\in\mathbb{Z}} e^{|k|\sigma} \Bigg| \int_0^t ds \int_0^{\infty} dY' \partial_{Y'}\, \mathcal{E}\big(Y, Y', t-s\big)$$

$$\times\, Y'^{-1} \hat{w}_1\big(k, Y', s\big)\ \partial_x \hat{u}_0(k) Y' \frac{e^{-\frac{Y'^2}{4s}}}{\sqrt{\pi s}} \Bigg|$$

$$\leq c \int_0^t ds \frac{|\partial_Y w_1|_{\sigma',\alpha}}{\sqrt{t-s}} + c \int_0^t ds \frac{|w_1|_{\rho'',\alpha}}{\sqrt{t-s}} \leq c \int_0^t ds \frac{|w_1|_{\rho'',\alpha}}{\sqrt{t-s}}.$$

In the above estimate we have used the fact that $w_1(x, Y=0, t) = 0$ so that $Y^{-1} w_1(x, Y, t)$ can be bounded in terms of the norm of $\partial_Y w_1$. This has allowed to get an extra Y that we used to compensate the singularity of $\partial_{YY} u^S$.

The estimate of the term $|\partial_t E_2(w_1 \partial_x u^S)|_{\sigma',\alpha}$ is obtained analogously to what has been done in [8].

This concludes the proof of the almost contractiveness of the operator $F(t, w_1)$ in Eq. (27) and the proof of Theorem 3 is achieved.

References

1. Asano, K.: A note on the abstract Cauchy-Kowalewski theorem. Proc. Jpn. Acad., Ser. A, Math. Sci. **64**(4), 102–105 (1988)
2. Bardos, C., Titi, E.S.: Euler equations for an ideal incompressible fluid. Usp. Mat. Nauk **62**(3(375)), 5–46 (2007)
3. Boyd, J.P., Flyer, N.: Compatibility conditions for time-dependent partial differential equations and the rate of convergence of Chebyshev and Fourier spectral methods. Comput. Methods Appl. Mech. Eng. **175**(3–4), 281–309 (1999)
4. Caflisch, R.E.: A simplified version of the abstract Cauchy-Kowalewski theorem with weak singularities: construction of the solution and the zero viscosity limit. Bull. Am. Math. Soc. **23**(2), 495–500 (1980)
5. Caflisch, R.E., Sammartino, M.: Navier-Stokes equations on an exterior circular domain: construction of the solution and the zero viscosity limit. C. R. Acad. Sci. Paris Sér. I Math. **324**(8), 861–866 (1997)
6. Caflisch, R.E., Sammartino, M.: Existence and singularities for Prandtl and boundary layer equations. Z. Angew. Math. Mech. **80**, 733–744 (2000). [Special issue on the occasion of the 125th anniversary of the birth of Ludwig Prandtl]
7. Cannone, M., Lombardo, M.C., Sammartino, M.: Existence and uniqueness for the Prandtl equations. C. R. Acad. Sci. Paris Sér. I Math. **332**(3), 277–282 (2001)
8. Cannone, M., Lombardo, M.C., Sammartino, M.: Well-posedness of Prandtl equations with non-compatible data. Nonlinearity **26**(12), 3077–3100 (2013)
9. Chemetov, N.V., Cipriano, F.: The inviscid limit for the Navier–Stokes equations with slip condition on permeable walls. J. Nonlinear Sci. **23**(5), 731–750 (2013)
10. Chen, Q., Qin, Z., Temam, R.: Numerical resolution near $t = 0$ of nonlinear evolution equations in the presence of corner singularities in space dimension 1. Commun. Comput. Phys. **9**(3), 568–586 (2011)
11. Chen, Q., Qin, Z., Temam, R.: Treatment of incompatible initial and boundary data for parabolic equations in higher dimension. Math. Comput. **80**(276), 2071–2096 (2011)

12. Cheng, W., Wang, X.: Discrete Kato-type theorem on inviscid limit of Navier-Stokes flows. J. Math. Phys. **48**(6), 065303 (2007)
13. Della Rocca, G., Lombardo, M.C., Sammartino, M., Sciacca, V.: Singularity tracking for Camassa-Holm and Prandtl's equations. Appl. Numer. Math. **56**(8), 1108–1122 (2006)
14. E, W., Engquist, B.: Blowup of the solutions to the unsteady Prandtl's equations. Commun. Pure Appl. Math. **50**, 1287–1293 (1997)
15. Flyer, N., Fornberg, B.: Accurate numerical resolution of transients in initial-boundary value problems for the heat equation. J. Comput. Phys. **184**(2), 526–539 (2003)
16. Gargano, F., Sammartino, M., Sciacca, V.: Singularity formation for Prandtl's equations. Physica D **238**(19), 1975–1991 (2009)
17. Gargano, F., Sammartino, M., Sciacca, V.: High Reynolds number Navier–Stokes solutions and boundary layer separation induced by a rectilinear vortex. Comput. Fluids **52**(0), 73–91 (2011)
18. Gargano, F., Sammartino, M., Sciacca, V., Cassel, K.: Analysis of complex singularities in high-Reynolds-number Navier-Stokes solutions. J. Fluid Mech. (2014, to appear). arXiv:1310.3943
19. Gérard-Varet, D., Dormy, E.: On the ill-posedness of the Prandtl equation. J. Am. Math. Soc. **23**(2), 591–609 (2010)
20. Gérard-Varet, D., Masmoudi, N.: Well-posedness for the Prandtl system without analyticity or monotonicity. Preprint (2013)
21. Gérard-Varet, D., Nguyen, T.: Remarks on the ill-posedness of the Prandtl equation. Asymptot. Anal. **77**(1–2), 71–88 (2012)
22. Gie, G.-M.: Asymptotic expansion of the Stokes solutions at small viscosity: the case of non-compatible initial data. Preprint (2013)
23. Guo, Y., Nguyen, T.: A note on Prandtl boundary layers. Commun. Pure Appl. Math. **64**(10), 1416–1438 (2011)
24. Jung, C., Temam, R.: Singular perturbations and boundary layer theory for convection-diffusion equations in a circle: The generic noncompatible case. SIAM J. Math. Anal. **44**(6), 4274–4296 (2012)
25. Kato, T.: Remarks on the zero viscosity limit for nonstationary Navier-Stokes flows with boundary. In: Seminar on Partial Differential Equations. Math. Sci. Res. Inst. Publ., pp. 85–98 (1984)
26. Kelliher, J.P.: On Kato's conditions for vanishing viscosity. Indiana Univ. Math. J. **56**(4), 1711–1721 (2007)
27. Kukavica, I., Vicol, V.: On the local existence of analytic solutions to the Prandtl boundary layer equations. Commun. Math. Sci. **11**(1), 269–292 (2013)
28. Lombardo, M.C., Caflisch, R.E., Sammartino, M.: Asymptotic analysis of the linearized Navier-Stokes equation on an exterior circular domain: explicit solution and the zero viscosity limit. Commun. Partial Differ. Equ. **26**(1–2), 335–354 (2001)
29. Lombardo, M.C., Cannone, M., Sammartino, M.: Well-posedness of the boundary layer equations. SIAM J. Math. Anal. **35**(4), 987–1004 (2003) (electronic)
30. Lombardo, M.C., Sammartino, M.: Zero viscosity limit of the Oseen equations in a channel. SIAM J. Math. Anal. **33**(2), 390–410 (2001)
31. Lopes Filho, M., Mazzucato, A., Lopes, H.N.: Vanishing viscosity limit for incompressible flow inside a rotating circle. Physica D: Nonlinear Phenom. **237**(10–12), 1324–1333 (2008)
32. Maekawa, Y.: Solution formula for the vorticity equations in the half plane with application to high vorticity creation at zero viscosity limit. Adv. Differ. Equ. **18**(1–2), 101–146 (2013)
33. Maekawa, Y.: On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half plane. Preprint (2013)
34. Nguyen, T., Sueur, F.: Boundary-layer interactions in the plane-parallel incompressible flows. Nonlinearity **25**(12), 3327 (2012)
35. Nirenberg, L.: An abstract form of the nonlinear Cauchy-Kowalewski theorem. J. Differ. Geom. **6**, 561–576 (1972)
36. Nishida, T.: A note on a theorem of Nirenberg. J. Differ. Geom. **12**, 629–633 (1977)
37. Oleinik, O.A., Samokhin, V.N.: Mathematical Models in Boundary Layer Theory. Applied Mathematics and Mathematical Computation., vol. 15. Chapman & Hall/CRC, Boca Raton (1999)
38. Sammartino, M., Caflisch, R.E.: Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations. Commun. Math. Phys. **192**(2), 433–461 (1998)
39. Sammartino, M., Caflisch, R.E.: Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. II. Construction of the Navier-Stokes solution. Commun. Math. Phys. **192**(2), 463–491 (1998)
40. Temam, R.: Behaviour at time $t = 0$ of the solutions of semilinear evolution equations. J. Differ. Equ. **43**(1), 73–92 (1982)
41. Temam, R.: Suitable initial conditions. J. Comput. Phys. **218**(2), 443–450 (2006)

42. Temam, R., Wang, X.: The convergence of the solutions of the Navier-Stokes equations to that of the Euler equations. Appl. Math. Lett. **10**(5), 29–33 (1997)
43. Treves, J.: Ovsjannikov theorem and hyperdifferential operators. *I.M.P.A.*, Rio-de-Janeiro (Brasil) (1969)
44. van Dommelen, L.L., Shen, S.F.: The spontaneous generation of the singularity in a separating laminar boundary layer. J. Comput. Phys. **38**, 125–140 (1980)
45. Xin, Z., Zhang, L.: On the global existence of solutions to the Prandtl's system. Adv. Math. **181**(1), 88–133 (2004)