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On Asymptotic Effects of Boundary Perturbations in Exponentially Shaped Josephson Junctions

Monica De Angelis · Pasquale Renno

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Abstract A parabolic integro differential operator $\mathcal{L}$, suitable to describe many phenomena in various physical fields, is considered. By means of equivalence between $\mathcal{L}$ and the third order equation describing the evolution inside an exponentially shaped Josephson junction (ESJJ), an asymptotic analysis for (ESJJ) is achieved, explicitly evaluating, boundary contributions related to the Dirichlet problem.

Keywords Superconductivity · Junctions · Laplace Transform · Initial-boundary problems for higher order parabolic equations

Mathematics Subject Classification (2010) 44A10 · 35A08 · 35K35 · 35E05

1 Introduction

Many equivalences among nonlinear operators and p.d.e. systems exist and an extensive bibliography is given. (see, f.i. [1–4]). Here, the semilinear equation, which characterizes exponentially shaped Josephson junctions in superconductivity (ESJJ) ([5–9] and references therein), is considered and equivalence with the following parabolic integro differential equation:

$$\mathcal{L}u \equiv u_t - \varepsilon u_{xx} + au + b\int_0^t e^{-\beta(t-\tau)}u(x,\tau)\,d\tau = f(x,t,u) \tag{1}$$

is proved. In this way, a priori estimates for (ESJJ) are obtained and, by means of a well known theorem on convolutions behavior, asymptotic effects of boundary perturbations are achieved for the solution of initial boundary value problem with Dirichlet conditions.

M. De Angelis (✉) · P. Renno
Dip. Mat. Appl. "R. Caccioppoli", Univ. of Naples "Federico II", Via Claudio n. 21, 80125, Naples, Italy
e-mail: modeange@unina.it

P. Renno
e-mail: renno@unina.it

Operator $\mathcal{L}$ defined in (1) can describe many linear and non linear physical phenomena and there is plenty of bibliography [10–17]. In particular, when $F = F(x, t, u)$, some non linear phenomena involve equation (1) both in superconductivity and biology and Dirichlet conditions in superconductivity refer to the phase boundary specifications [7–9, 18], while in excitable systems occur when the pulse propagation in heart cells is studied [19]. Besides, Dirichlet problem is also considered for stability analysis and asymptotic behavior of reaction-diffusion systems solutions, [20, 21], or in hyperbolic diffusion [22].

Previous analyses related to Eq. (1) have been developed in [23–26] assuming ε, a, b, β as positive constants. The fundamental solution K has been determined and many of its properties have been proved. Moreover, various boundary value problems have been considered as well as non linear integral equations have been determined, whose Green functions have numerous typical properties of the diffusion equation. Besides, the kernel $e^{-\beta(t-\tau)}u(x,\tau)$ in (1) can be modified as physical situations demand, and this particular choice has been made taking into consideration superconductive and biological models. As a matter of fact, it is possible to prove that (1) can characterize reaction diffusion models, like the FitzHugh-Nagumo system, suitable to model the propagation of nerve impulses. Furthermore, the perturbed sine Gordon equation (PSGE) can be deduced and, as proved further, the evolution in a Josephson junction with nonuniform width can be characterized, too.

In superconductivity it is well known that the Josephson effect is modeled by the (PSGE) given by (see, f.i. [27]):

$$\varepsilon u_{xxt} - u_{tt} + u_{xx} - \alpha u_t = \sin u - \gamma \tag{2}$$

where u represents the difference between the phase of the wave functions related to the two superconductors of the junction, $\gamma > 0$ is a forcing term that is proportional to a bias current, the α-term accounts for dissipation due to normal electrons crossing the junction, and the ε-term accounts for dissipation caused by normal electrons flowing parallel to the junction.

Besides, when the case of an exponentially shaped Josephson junction (ESJJ) is considered, denoting by λ a positive constant, the evolution of the phase inside the junction is described by the third order equation:

$$\varepsilon u_{xxt} + u_{xx} - u_{tt} - \varepsilon\lambda u_{xt} - \lambda u_x - \alpha u_t = \sin u - \gamma \tag{3}$$

where terms λu_x and $\lambda\varepsilon u_{xt}$ stand for the current as a consequence of the tapering and, more specifically, λu_x represents a geometrical force leading the fluxons from the wide edge to the narrow edge [28–30]. Exponentially shaped Josephson junctions provide several advantages compared to rectangular ones. Among the others, we can mention the possibility of getting a voltage which is not chaotic anymore, but rather periodic. This allows to exclude some among the possible causes of large spectral width and to avoid the problem of trapped flux [31–34].

2 Statement of the Problem

If T is an arbitrary positive constant and

$$\Omega_T \equiv \left\{(x,t) : 0 \le x \le L;\, 0 < t \le T\right\},$$

let us consider the following initial boundary value problem with Dirichlet boundary conditions:

$$\begin{cases} u_t - \varepsilon u_{xx} + au + b\int_0^t e^{-\beta(t-\tau)} u(x,\tau)\,d\tau = F(x,t,u), & (x,t) \in \Omega_T, \\ u(x,0) = u_0(x), & x \in [0,L], \\ u(0,t) = g_1(t), \quad u(L,t) = g_2(t), & 0 < t \le T. \end{cases} \tag{4}$$

Denoting by $K(x,t)$ the fundamental solution of the linear operator defined by (1), and considering ε, a, b, β as positive constants, it results [26]:

$$K(r,t) = \frac{1}{2\sqrt{\pi\varepsilon}}\left[\frac{e^{-\frac{r^2}{4t}-at}}{\sqrt{t}} - b\int_0^t \frac{e^{-\frac{r^2}{4y}-ay}}{\sqrt{t-y}} e^{-\beta(t-y)} J_1\big(2\sqrt{by(t-y)}\big)\,dy\right] \tag{5}$$

where $r = |x|/\sqrt{\varepsilon}$ and $J_n(z)$ denotes the Bessel function of first kind and order n. Moreover, the following theorem holds:

Theorem 1 *For all* $t > 0$, *the Laplace transform of* $K(r,t)$ *with respect to* t *converges absolutely in the half-plane* $\Re e\, s > \max(-a,-\beta)$ *and it results*:

$$\hat{K}(r,s) = \int_0^\infty e^{-st} K(r,t)\,dt = \frac{e^{-r\sigma}}{2\sqrt{\varepsilon}\sigma} \tag{6}$$

with $\sigma^2 = s + a + \frac{b}{s+\beta}$.

Now, let us consider the following Laplace transforms with respect to t:

$$\hat{u}(x,s) = \int_0^\infty e^{-st} u(x,t)\,dt, \qquad \hat{F}(x,s) = \int_0^\infty e^{-st} F\big[x,t,u(x,t)\big]\,dt,$$

and let $\hat{g}_1(s)$, $\hat{g}_2(s)$ be the L transforms of the data $g_i(t)$ $(i = 1,2)$.

Then the Laplace transform of the problem (4) is formally given by:

$$\begin{cases} \hat{u}_{xx} - \dfrac{\sigma^2}{\varepsilon}\hat{u} = -\dfrac{1}{\varepsilon}\big[\hat{F}(x,s) + u_0(x)\big], \\ \hat{u}(0,s) = \hat{g}_1(s), \quad \hat{u}(L,s) = \hat{g}_2(s). \end{cases} \tag{7}$$

If one introduces the following *theta function*

$$\begin{aligned} \hat{\theta}(y,\sigma) &= \frac{1}{2\sqrt{\varepsilon}\sigma}\left\{e^{-\frac{y}{\sqrt{\varepsilon}}\sigma} + \sum_{n=1}^\infty \big[e^{-\frac{2nL+y}{\sqrt{\varepsilon}}\sigma} + e^{-\frac{2nL-y}{\sqrt{\varepsilon}}\sigma}\big]\right\} \\ &= \frac{\cosh[\sigma/\sqrt{\varepsilon}(L-y)]}{2\sqrt{\varepsilon}\sigma\sinh(\sigma/\sqrt{\varepsilon}L)} \end{aligned} \tag{8}$$

then, by (7) and (8) one deduces:

$$\begin{aligned} \hat{u}(x,s) = &\int_0^L \big[\hat{\theta}(x+\xi,s) - \hat{\theta}(|x-\xi|,s)\big]\big[u_0(\xi) + \hat{F}(\xi,s)\big]\,d\xi \\ &- 2\varepsilon\hat{g}_1(s)\hat{\theta}_x(x,s) + 2\varepsilon\hat{g}_2(s)\hat{\theta}_x(L-x,s), \end{aligned} \tag{9}$$

where

$$\hat{\theta}_x(x,\sigma) = \frac{\sinh[\sigma/\sqrt{\varepsilon}(x-L)]}{2\varepsilon\sinh(\sigma/\sqrt{\varepsilon}L)}. \tag{10}$$

3 Explicit Solution

In order to obtain the inverse formula for (9), let us apply (6) to (8). Then, one deduces the following function which is similar to *theta functions*:

$$\theta(x,t) = K(x,t) + \sum_{n=1}^{\infty}\big[K(x+2nL,t) + K(x-2nL,t)\big]$$
$$= \sum_{n=-\infty}^{\infty} K(x+2nL,t). \tag{11}$$

So that, denoting by

$$G(x,\xi,t) = \theta\big(|x-\xi|,t\big) - \theta(x+\xi,t),$$

when $F = f(x,t)$, by (9) the explicit solution of the *linear* problem (4) is given by:

$$u(x,t) = \int_0^L G(x,\xi,t)u_0(\xi)\,d\xi + \int_0^t d\tau \int_0^L G(x,\xi,t)f(\xi,\tau)\,d\xi$$
$$- 2\varepsilon \int_0^t \theta_x(x,t-\tau)g_1(\tau)\,d\tau + 2\varepsilon \int_0^t \theta_x(x-L,t-\tau)g_2(\tau)\,d\tau. \tag{12}$$

So, owing to the basic properties of $K(x,t)$, it is easy to deduce the following theorem:

Theorem 2 *When the linear source $f(x,t)$ and the initial boundary data $u_0(x)$, g_i ($i = 1,2$) are continuous in Ω_T, then problem* (4) *admits a unique regular solution $u(x,t)$ given by* (12).

Furthermore, when the source term $F = F(x,t,u)$ depends on the unknown function $u(x,t)$, too, then problem (4) admits the following integral equation:

$$u(x,t) = \int_0^L G(x,\xi,t)u_0(\xi)d\xi + \int_0^t d\tau \int_0^L G(x,\xi,t)F\big(\xi,\tau,u(x,\tau)\big)\,d\xi$$
$$- 2\varepsilon \int_0^t \theta_x(x,t-\tau)g_1(\tau)\,d\tau + 2\varepsilon \int_0^t \theta_x(x-L,t-\tau)g_2(\tau)\,d\tau. \tag{13}$$

Indeed, if

$$F = F\big(x,t,u(x,t),u_x(x,t)\big),$$

let

$$D = \big\{(x,t,u,p) : (x,t) \in \Omega_T, -\infty < u < \infty - \infty < p < \infty\big\},$$

and let us assume the following Hypotheses A:

- The function $F(x,t,u,p)$ is defined and continuous on D and it is bounded for all u and p.
- For each $k > 0$ and for $|u|, |p| < k$, the function F is Lipschitz continuous in x and t for each compact subset of D.
- There exists a constant β_F such that:

$$\big|F(x,t,u_1,p_1) - F(x,t,u_2,p_2)\big| \le \beta_F\big\{|u_1-u_2| + |p_1-p_2|\big\}$$

  holds for all (u_i,p_i) $i = 1,2$.

Moreover, let $\|z\| = \sup_{\Omega_T} |z(x,t)|$, and let $\mathcal{B}_T$ denote the Banach space

$$\mathcal{B}_T \equiv \left\{ z(x,t) : z \in C(\Omega_T),\ \|z\| < \infty \right\}. \tag{14}$$

By means of standard methods related to integral equations and owing to properties of K and F, it is easy to prove that the mapping defined by (13) is a contraction of $\mathcal{B}_T$ in $\mathcal{B}_T$ and so it admits an unique fixed point $u(x,t) \in \mathcal{B}_T$ [35, 36]. Hence

Theorem 3 *When the data* (u_0, g_1, g_2) *are continuous functions, then the Dirichlet problem related to the non linear system* (4), *has a unique solution in the space of solutions which are regular in* Ω_T.

4 Equivalence with (ESJJ)

Among many others equivalences, a significant one occurs between operator $\mathcal{L}$ and Eq. (3) which describes the evolution inside an exponentially shaped Josephson junction.

Indeed, let us assume

$$\begin{aligned} &\beta = \frac{1}{\varepsilon}, \qquad b = \beta^2(1-\alpha\varepsilon), \qquad a\beta = \frac{\lambda^2}{4} - b, \\ &f = -\int_0^t e^{-\frac{1}{\varepsilon}(t-\tau)} f_1(x,\tau,u)\, d\tau, \end{aligned} \tag{15}$$

with

$$f_1 = e^{-\frac{\lambda}{2}x}\left[\sin\left(e^{x\lambda/2}u\right) - \gamma\right]. \tag{16}$$

From the integro differential equation (1) it follows that:

$$\varepsilon u_{xxt} - u_{tt} + u_{xx} - \left(\alpha + \varepsilon\frac{\lambda^2}{4}\right)u_t - \frac{\lambda^2}{4}u = f_1. \tag{17}$$

Therefore, assuming $e^{\frac{\lambda}{2}x}u = \bar{u}$, (17) turns into Eq. (3).

The diffusion effects due to the dissipative terms $\varepsilon(u_{xxt} - \lambda u_{xt})$ of (3) have already been investigated in [9]. Here, in Sect. 7, by means of this equivalence, the influences of data on the solution which is related to the Dirichlet problem, will be explicitly evaluated.

5 Some Properties on θ and θ_x Functions

In order to obtain a priori estimates and asymptotic effects related to boundary perturbations, some properties of the fundamental solution K and the theta function defined in (11) ought to be evaluated. In [7, 24–26] some of these have already been proved and it results:

$$\int_0^L \left|\theta\left(|x-\xi|,t\right)\right| d\xi \le (1+\sqrt{b}\pi t)e^{-\omega t}, \quad \omega = \min\{a,\beta\}, \tag{18}$$

$$\begin{aligned} \left|\int_0^L \theta\left(|x-\xi|,t\right) d\xi\right| &\le \sum_{n=-\infty}^{\infty} \int_0^L \left|K\left(|x-\xi+2nL|,t\right)\right| d\xi \\ &= \sum_{n=-\infty}^{\infty} \int_{x+(2n-1)L}^{x+2nL} \left|K(y,t)\right| dy \le \int_{\Re} \left|K(y,t)\right| dy. \end{aligned} \tag{19}$$

Moreover, letting:

$$K_1 \equiv \int_0^t e^{-\beta(t-\tau)} K(x,\tau)\, d\tau, \tag{20}$$

one has:

$$\int_{\Re} |K_1|\, d\xi \le E(t); \qquad \int_0^t d\tau \int_{\Re} |K_1|\, d\xi \le \beta_1 \tag{21}$$

being

$$E(t) = \frac{e^{-\beta t} - e^{-at}}{a-\beta} > 0 \quad \text{and} \quad \beta_1 = (a\beta)^{-1}.$$

Furthermore, as for θ_x, from (5), it is well-rendered that the x derivative of the integral term vanishes for $x \to 0$, while the first term represents the derivative with respect to x of the fundamental solution related to the heat equation. Indeed, since (11), it results:

$$\begin{aligned}\theta_r &= K_r + \sum_{n=1}^{\infty}\left[-\frac{x+2nl}{2t} K(x+2nL,t) - \frac{x-2nl}{2t} K(x-2nL,t)\right] \\ &= K_r + J(r,t)\end{aligned} \tag{22}$$

and, by means of classic calculations (see, f.i. [35]), one has:

$$\lim_{x\to 0} J(r,t) = 0. \tag{23}$$

So, classic theorems assure that conditions $(4)_3$ are surely satisfied.

Moreover, being:

$$\lim_{t\to\infty} \int_0^t \theta_x(x,\tau)\, d\tau = \lim_{s\to 0} \hat{\theta}_x(x,s), \qquad \mathrm{Re}\, s > \max\{-a,-\beta\} \tag{24}$$

from (10), it results:

$$\lim_{t\to\infty} \int_0^t \theta_x(x,\tau)\, d\tau = \frac{1}{2\varepsilon} \frac{\sinh \sigma_0 (x-L)}{\sinh(\sigma_0 L)} \tag{25}$$

where $\sigma_0 = \sqrt{(a+\frac{b}{\beta})\frac{1}{\varepsilon}}$.

Besides, for all $x > \delta > 0$ it results:

$$\sum_{n=-\infty}^{\infty} e^{-\frac{(x+2nL)^2}{4\varepsilon t}} \le t \frac{4\varepsilon\pi^2}{4l^2 e} \csc^2(\pi\delta/2L), \tag{26}$$

and one has

$$|K_r(r,t)| \le r \frac{e^{-\frac{r^2}{4t}}}{4\sqrt{\pi\varepsilon t^3}} \left[1 + 4bt^2\right] e^{-\omega t}. \tag{27}$$

So, the following theorem can be proved:

Theorem 4 *For all* $0 \le x \le L$, *there exists a positive constant* C *depending only on constants* a, ε, b, β *such that*:

$$\int_0^\infty |\theta_x(x,\tau)|\, d\tau \le C. \tag{28}$$

Moreover, it results:

$$\lim_{t\to\infty} \theta_x(x,t) = 0. \tag{29}$$

6 Analysis of Boundary Contributions

By means of the following well known theorem (see f.i. [25] and references therein), boundary contributions related to the Dirichlet problem (4) can be explicitly evaluated.

Theorem 5 *Let* $h(t)$ *and* $\chi(t)$ *be two continuous functions on* $[0, \infty[$ *satisfying the following hypotheses*

$$\exists \lim_{t\to\infty} h(t) = h(\infty), \qquad \exists \lim_{t\to\infty} \chi(t) = \chi(\infty), \tag{30}$$

$$\dot{h}(t) \in L_1[0, \infty). \tag{31}$$

Then, it results:

$$\lim_{t\to\infty} \int_o^t \chi(t-\tau)\dot{h}(\tau)\,d\tau = \chi(\infty)\big[h(\infty) - h(0)\big]. \tag{32}$$

According to this, the following theorem holds:

Theorem 6 *If data* g_i $(i = 1, 2)$ *are two continuous functions satisfying condition* (30) *then, one has*:

$$\lim_{t\to\infty} \int_0^t \theta_x(x,\tau) g_i(t-\tau)\,d\tau = g_{i,\infty} \frac{1}{2\varepsilon} \frac{\sinh \sigma_0(x-L)}{\sinh \sigma_0 L} \tag{33}$$

being $g_{i,\infty} = \lim_{t\to\infty} g_i$ $(i = 1, 2)$ *and* $\sigma_0 = \sqrt{(a + \frac{b}{\beta})\frac{1}{\varepsilon}}$.

Moreover, when data g_i $(i = 1, 2)$ *verify both condition* (30) *and condition* (31), *then it results*:

$$\lim_{t\to\infty} \int_0^t \theta_x(x,\tau) g_i(t-\tau)\,d\tau = 0 \quad (i = 1, 2). \tag{34}$$

Proof Let us assume, in (32), $h = \int_0^t \theta_x(x,\tau)\,d\tau$ and $\chi = g_i$ $(i = 1, 2)$. Then, (33) follows by (25) and (28).

Besides, when $\chi = \theta_x(x,t)$ and $\dot{h} = g_i$, since (29), (34) is deduced. □

7 Asymptotic Behaviours Related to (ESJJ)

According to the equivalence between (3) and the integro differential equation (1), previous results can be applied to (ESJJ).

Let us consider the following initial boundary value problem with Dirichlet conditions:

$$\begin{cases} \varepsilon u_{xxt} + u_{xx} - u_{tt} - \varepsilon\lambda u_{xt} - \lambda u_x - \alpha u_t = \sin u - \gamma, & (x,t) \in \Omega_T, \\ u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), & x \in [0, L], \\ u(0,t) = g_1(t), \quad u(L,t) = g_2(t), & 0 < t \le T. \end{cases} \tag{35}$$

When in (4) one assumes

$$F(x,t,u) = e^{-\frac{\lambda}{2}x}\left[\int_0^t e^{-\frac{1}{\varepsilon}(t-\tau)}\big[\sin\big(e^{x\lambda/2}u\big) - \gamma\big]\,d\tau - v_0(x)e^{-\frac{t}{\varepsilon}}\right], \tag{36}$$

and a, b, β, λ are given by (15)$_1$, system (35) can be given the form (4) and hypotheses A assure that the following integro equation states:

$$u(x,t) = \int_0^L G(x,\xi,t) e^{-\frac{\lambda}{2}x} u_0(\xi)\, d\xi + \int_0^t d\tau \int_0^L G(x,\xi,t) F\big(\xi,\tau,u(x,\tau)\big)\, d\xi - 2\varepsilon \int_0^t \theta_x(x,t-\tau) g_1(\tau)\, d\tau + 2\varepsilon \int_0^t \theta_x(x-L,t-\tau) e^{-\frac{\lambda L}{2}} g_2(\tau)\, d\tau. \quad (37)$$

So, a priori estimates of the solution and asymptotic effects of boundary perturbations can be obtained. Indeed, letting

$$\|u_0\| = \sup_{0\le x\le L} \big|u_0(x)\big|, \qquad \|v_0\| = \sup_{0\le x\le L} \big|v_0(x)\big|,$$

and

$$\|f\| = \sup_{\Omega_T} \big|f(x,t,u)\big|,$$

with f defined by (15)$_2$–(16), the following theorems hold:

Theorem 7 *When $g_1 = g_2 = 0$, the solution $u(x,t)$ related to (ESJJ) model, and determined in (37), satisfies the following estimate*:

$$|u| \le 2\big[\|u_0\|(1+\pi\sqrt{b}t)e^{-\omega t} + \|v_0\|E(t) + \beta_1\|f\|\big]. \quad (38)$$

Proof The initial disturbances can be evaluated by means of (18), (19) and (21)$_1$. Besides, since (21)$_2$, the behaviour of the source term is determined. □

Further, according to Theorem 6, if $u_0 = v_0 = 0$ and $f = 0$, one has:

Theorem 8 *When t tends to infinity and data g_i $(i = 1, 2)$ are two continuous functions satisfying condition (30), it results*:

$$u = -g_{1,\infty} \frac{\sinh\frac{\lambda}{2}(x-L)}{\sinh\frac{\lambda L}{2}} + g_{2,\infty} \frac{2\sinh[\frac{\lambda}{2}(x-2L)]}{e^{\lambda L}-1} \quad (39)$$

Otherwise, in the hypotheses (30), (31), the effects determined by boundary disturbance vanish.

Remarks When t tends to infinity, the effect due to the initial disturbances (u_0, v_0) is vanishing, while the effect of the non linear source is bounded for all t. Furthermore, for large t, the effects due to boundary disturbances g_1, g_2 are null or at least everywhere bounded.

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References

1. Rionero, S.: Asymptotic behaviour of solutions to a nonlinear third order P.D.E. modeling physical phenomena. Boll. Unione Matematica Italiana (2012)
2. Carillo, S., Valente, V., Caffarelli, G.V.: A Linear viscoelasticity problem with a singular memory kernel: An existence and uniqueness result. Differ. Integral Equ. **26**(9–10), 1115–1125 (2013)

3. Scott, A.C.: The Nonlinear Universe: Chaos, Emergence, Life. Springer, Berlin (2007)
4. Scott, A.C.: Neuroscience: A Mathematical Primer. Springer, Berlin (2002)
5. D'Anna, A., De Angelis, M., Fiore, G.: Existence and uniqueness for some 3rd order dissipative problems with various boundary conditions. Acta Appl. Math. **122**, 255–267 (2012)
6. De Angelis, M.: On exponentially shaped Josephson junctions. Acta Appl. Math. **122**, 179–189 (2012)
7. De Angelis, M.: On a model of superconductivity and biology. Adv. Appl. Math. Sci. **7**, 41–50 (2010)
8. Angelis, M.D., Fiore, G.: Existence and uniqueness of solutions of a class of third order dissipative problems with various boundary conditions describing the Josephson effect. J. Math. Anal. Appl. **404**(2), 477–490 (2013)
9. De Angelis, M., Fiore, G.: Diffusion effects in a superconductive model. Commun. Pure Appl. Anal. **13**(1), 217–223 (2014)
10. Bini, D., Cherubini, C., Filippi, S.: Viscoelastic Fizhugh-Nagumo models. Phys. Rev. E 041929 (2005)
11. Renardy, M.: On localized Kelvin-Voigt damping. ZAMM Z. Angew. Math. Mech. **84** (2004)
12. De Angelis, M., Renno, P.: Diffusion and wave behavior in linear Voigt model. C. R. Méc. **330**, 21–26 (2002)
13. Morro, A., Payne, L.E., Straughan, B.: Decay, growth,continuous dependence and uniqueness results of generalized heat theories. Appl. Anal. **38** (1990)
14. Flavin, J.N., Rionero, S.: Qualitative Estimates for Partial Differential Equations: An Introduction. CRC Press, Boca Raton (1996)
15. Lamb, H.: Hydrodynamics. Cambridge University Press, Cambridge (1971)
16. De Angelis, M., Monte, A.M., Renno, P.: On fast and slow times in models with diffusion. Math. Models Methods Appl. Sci. **12**(12), 1741–1749 (2012)
17. Straughan, B.: Heat Waves. Springer Series in Applied Mathematical Sciences, vol. 177 (2011)
18. De Angelis, M.: Asymptotic analysis for the strip problem related to a parabolic third order operator. Appl. Math. Lett. **14**, 425–430 (2001)
19. Keener, J.P., Sneyd, J.: Mathematical Physiology. Springer, New York (1998)
20. Torcicollo, I.: On the dynamics of the nonlinear duopoly game. Int. J. Non-Linear Mech. **57**, 31–38 (2013)
21. Capone, F., De Cataldis, V., De Luca, R.: On the nonlinear stability of an epidemic SEIR reaction-diffusion model. Ric. Mat. **62**, 161–181 (2013)
22. Gentile, M., Straughan, B.: Hyperbolic Diffusion with Christov-Morro Theory. Mathematics and Computers in Simulation (2012). doi:10.1016/j.matcom.2012.07.010
23. De Angelis, M.: A priori estimates for excitable models. Meccanica **48**(10), 2491–2496 (2013)
24. De Angelis, M.: Asymptotic estimates related to an integro differential equation. Nonlinear Dyn. Syst. Theory **13**(3), 217–228 (2013)
25. De Angelis, M., Renno, P.: Asymptotic effects of boundary perturbations in excitable systems. Accepted by Discrete Contin. Dyn. Syst.—Ser. B, http://arxiv.org/pdf/1304.3891v1.pdf
26. De Angelis, M., Renno, P.: Existence, uniqueness and a priori estimates for a non linear integro-differential equation. Ric. Mat. **57**, 95–109 (2008)
27. Barone Paterno': Physical and applications of the Josephson effect (1982)
28. Benabdallah, A., Caputo, J.G., Scott, A.C.: Exponentially tapered Josephson flux-flow oscillator. Phys. Rev. B **54**(22), 16139 (1996)
29. Benabdallah, A., Caputo, J.G., Scott, A.C.: Laminar phase flow for an exponentially tapered Josephson oscillator. J. Appl. Phys. **588**(6), 3527 (2000)
30. Carapella, G., Martucciello, N., Costabile, G.: Experimental investigation of flux motion in exponentially shaped Josephson junctions. Phys. Rev. B **66**, 134531 (2002)
31. Boyadjiev, T.L., Semerdjieva, E.G., Shukrinov, Yu.M.: Common features of vortex structure in long exponentially shaped Josephson junctions and Josephson junctions with inhomogeneities. Physica C 460–462 (2007)
32. Jaworski, M.: Exponentially tapered Josephson junction: some analytic results. Theor. Math. Phys. **144**, 1176–1180 (2005)
33. Shukrinov, Yu.M., Semerdjieva, E.G., Boyadjiev, T.L.: Vortex structure in exponentially shaped Josephson junctions. J. Low Temp Phys. **299** (2005)
34. Jaworski, M.: Fluxon dynamics in exponentially shaped Josephson junction. Phys. Rev. B **71**, 22 (2005)
35. Cannon, J.R.: The One-Dimensional Heat Equation. Addison-Wesley, Reading (1984)
36. De Angelis, M., Maio, A., Mazziotti, E.: Existence and uniqueness results for a class of non linear models. In: Mathematical Physics Models and Engineering Sciences, pp. 191–202 (2008) (eds. Liguori, Italy)