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On the Dalgaard-Strulik Model with Logistic Population Growth Rate and Delayed-Carrying Capacity

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Abstract Recently Dalgaard and Strulik have proposed (in Resour. Energy Econ. 33:782–797, 2011) an energy model of capital accumulation based on the mathematical framework developed by Solow-Swan and coupled with Cobb-Douglas production function (Solow in Q. J. Economics 70:65–94, 1956; Swan in Econ. Rec. 32(63):334–361, 1956). The model is based on a constant rate of population growth assumption. The present paper, according to the analysis performed by Yukalov et al. (Physica D 238:1752–1767, 2009), improves the Dalgaard-Strulik model by introducing a logistic-type equation with delayed carrying capacity which alters the asymptotic stability of the relative steady state. Specifically, by choosing the time delay as a bifurcation parameter, it turns out that the steady state loses stability and a Hopf bifurcation occurs when time delay passes through critical values. The results are of great interest in the applied and theoretical economics.

Keywords Dalgaard-Strulik model · Energy · Time delay · Hopf bifurcation · Logistic model · Nonconstant carrying capacity

1 Introduction

The modern economic growth theory has its origin in the seminal papers by Solow [31] and Swan [32], who contemporaneously and independently have proposed a new theoretical framework for understanding world-wide growth of output and the persistence of geographical differences in per capita output. The model they provided is known as the Solow-Swan model, or in brief the Solow model after that the most famous of the two economists was awarded by the Nobel prize in economics for his contributions.

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A standard assumption in the Solow-Swan model is that population grows at a positive constant rate (Malthusian model [29]), giving rise to an exponential population growth curve. However, nowhere in nature it is possible to have unlimited exponential growth of any population over the long run. Claiming that the model proposed by Malthus was too simplistic, as it only included linear terms, led to replacing the linear population model of Malthus by a non-linear one. The first model of this type (nowadays called the logistic model) was proposed by Verhulst [33], who introduced an additional quadratic term with a negative coefficient in the Malthusian model, so that any population growth rate followed an elongated S-curve. The introduction of logistic-type population growth law, within the Solow-Swan model and its extension the Ramsey model [30], has recently attracted a great attention in the literature (see, e.g. Bucci and Guerrini [14], Ferrara [16–18], Ferrara and Guerrini [19], Guerrini [21–25]; applications also refer to mammary carcinoma-immune system competition, see, among others, Bianca and Pennisi [11] and the references therein).

In the logistic model it is assumed that the growth rate of a population at any time depends on the relative number of individuals at that time. But, in practice, the complex process of reproduction is not instantaneous in time. In particular in paper [27] Hutchinson pointed out that the logistic equation would be inappropriate for the description of population growth in the case where there is a lag in some of the processes involved. Hence, the logistic equation has been modified by introducing a time delay. The resulting model is known as the delayed logistic equation or Hutchinson's equation or, under a suitable change of variables, Wright's equation [34]. It is worth remarking that this model is one of the first examples of a delay differential equation that has been thoroughly examined.

Recently, following Banavar et al. [2], Dalgaard and Strulik [15] have developed a mathematical model of an economy viewed as a transport network for energy. Specifically energy originates from a power plant and is diffused across the economy to the sites at which it is used via a power grid. The network is space-filling and the number of transfer sites rises with the volume of the network. Since each transfer site uses energy, there exists a relation between the size of the network, and energy consumed at the transfer sites. As a result, changes in per capita energy consumption requires changes in the size of the network, i.e. every time a new piece of equipment is connected to an electricity outlet, a new transfer site emerges, and the network expands allowing for more energy consumption per capita. Hence capital is needed to transfer energy to the sites where capital uses energy. Dalgaard and Strulik assume that for a given size of the network, the total energy consumption is log-linearly related to population size. This choice is supported by the empirical studies [28]. In particular it is assumed that the population expands at a constant rate.

It is worth stressing that the law of motion for capital of Dalgaard and Strulik is mathematically isomorphic to the one emanating from a Solow-Swan model, where the aggregate production function is assumed to be Cobb–Douglas. Accordingly, its dynamics are formally the same as in the Solow-Swan model. In particular, there exists a unique stable equilibrium (or steady state in the language of the economical sciences) to which the economy adjusts.

Yukalov et al. [35] pointed out that many complex systems evolve according to multistep processes, where a period of fast growth is followed by a lasting period of stagnation or saturation, which is itself followed by another fast growth regime, and so on. To capture the previously described phenomenology, a further generalization of the logistic model has been proposed, in which the carrying capacity is nonconstant but it is a function of time (see, e.g., Banks [3] for time dependent carrying capacities). Specifically, the carrying capacity consists of two terms: the first term corresponds to a fixed carrying capacity, provided by nature; the second term is the carrying capacity created (or destroyed) by the system, which is naturally delayed, as far as any creation/destruction requires some time.

The main objective of this paper is to improve the Dalgaard and Strulik model by means of the coupling, within the same framework, of the two different research lines that have been proposed and analyzed separately in the recent past: the rediscovering of the Solow-Swan model via an energy network approach due to Dalgaard and Strulik [15], and the new variant of the logistic-type equation with nonconstant carrying capacity proposed by Yukalov et al. [35]. Accordingly the paper considers a logistic-type equation with delayed carrying capacity which alters the asymptotic stability of the relative steady state. Specifically, by choosing the time delay as a bifurcation parameter, it turns out that the steady state loses stability and a Hopf bifurcation occurs when time delay passes through critical values. Generally, time delay is introduced whenever the system's behavior is dependent at least in part on its history. The introduction of time delay is a common approach used in most complex systems, especially in biological systems, for instance in the modelling of gene expression, cell division, as well as cell differentiation and cell maturation, with the aim to be more consistent with the cell growth kinetics, see the review paper [1]. Recently, time delay has been introduced in energy-based models of capital accumulation, see [12, 13] and the reference therein. It is worth stressing that this work is motivated by economical applications, especially for the asymptotic economic growth [20].

The contents of this paper are organized into four more sections. In detail, after this introduction, Sect. 2 reviews the original model by Dalgaard and Strulik and deals with the generalization obtained by introducing the logistic growth and the time delay in the carrying capacity. Section 3 is concerned with the existence and the asymptotic analysis of steady states. Analytical investigations on the existence of Hopf bifurcations are performed in Sect. 4. Finally Sect. 5 concludes the paper with a critical analysis of the results and is concerned with discussions of future research perspective.

2 The Generalized Dalgaard-Strulik Model

The original model developed by Dalgaard and Strulik in [15] deals with the modelling of an economy viewed as a transportation network for electricity, where electricity is used to run, maintain, and create capital. As already mentioned in the introduction, the Dalgaard-Strulik model is derived under the following assumptions:

- the per capita electricity consumption is proportional to the size of the network;
- the total capital stock K is proportional to the total energy flow in the system.

Let $E = E(t)$ be the energy consumed at all the transfer sites, $K = K(t)$ the total capital stock and $L = L(t)$ the population density at time t. Under the above assumptions and applying the Banavar theorem [2] the following model for the electricity consumption per capita has been proposed:

$$e(t) = \epsilon[k(t)]^a, \tag{1}$$

where $e = E/L$ denotes electricity consumption per capita, $k = K/L$ is the per capita capital stock, $0 < a < 1$ is a real constant proportional to the dimension and efficiency of the network, and $\varepsilon > 0$ is a real constant that is independent of capital per worker.

Assuming that the energy costs of maintaining and running the generic capital good is μ, whereas the energy requirements to create a new capital good is ν, one may express the energy balance as follow:

$$e(t) = \mu k(t) + \nu \frac{\dot{K}(t)}{L(t)}, \tag{2}$$

where the dot denotes the first derivative of k with respect to time. According to Eq. (2) when the energy use shutted off, i.e. $e(t) = 0$, then the capital stock declines over time at the rate μ/ν, due to lack of maintenance and replacement. Hence, the ratio μ/ν captures the physical phenomenon of capital depreciation, commonly introduced in models of growth and capital accumulation. The first time-derivative of k reads

$$\frac{\dot{K}(t)}{L(t)} = \dot{k}(t) + n\,k(t), \tag{3}$$

where $n = \dot{L}(t)/L(t)$ denotes the constant population growth rate. Inserting Eq. (3) into Eq. (2) and by using the network equation (1), the following Dalgaard-Strulik law of motion for capital yields [15]:

$$\dot{k}(t) = \frac{\varepsilon}{v}\,[k(t)]^a - \left(\frac{\mu}{v} + n\right) k(t). \tag{4}$$

In what follows we consider a generalization of the mathematical model (4). Specifically, following Yukalov et al. [35], it is assumed that the population growth rate n in Eq. (3) is nonconstant, but it evolves according to a delayed logistic-type equation with a time dependent carrying capacity $\kappa(t)$, namely we have

$$\dot{L}(t) = \gamma L(t) - \frac{C[L(t)]^2}{\kappa(t)},$$

where

- $\gamma L(t)$ models the individual balance between birth and death;
- $C[L(t)]^2/\kappa(t)$ describes collective effects, with the coefficient C defining the balance between competition and cooperation;
- $\kappa(t)$ is the nonconstant carrying capacity, which reads:

$$\kappa(t) = A + BL(t - \tau), \tag{5}$$

  where $A > 0$ is the pre-existing carrying capacity, e.g., provided by nature, and B is the carrying capacity created if $B > 0$, or destroyed if $B < 0$, by the activity of the agents composing the considered society. Finally, τ represents the time delay.

It is clear that as far as any creation/destruction requires some time, the created (or destroyed) capacity is naturally delayed. Within this framework, the law of motion for capital is described by the following system of non-linear delay differential equations:

$$\begin{cases} \dot{k}(t) &= \dfrac{\varepsilon}{v}\,[k(t)]^a - \left[\dfrac{\mu}{v} + \gamma - \dfrac{CL(t)}{A + BL(t-\tau)}\right] k(t), \\ \dot{L}(t) &= \gamma L(t) - \dfrac{C[L(t)]^2}{A + BL(t-\tau)}, \end{cases} \tag{6}$$

for some initial function $L(t) = \varphi(t)$, $t \in [-\tau, 0]$.

It is worth precising that in contrast to classical dynamical systems with zero delay, the initial function $\varphi(t)$ is required, which is defined over the range of time delimited by the delay.

3 Existence and Stability Analysis of Steady States

This section deals with the asymptotic stability analysis of the equilibria of the system (6). It is well-known that equilibria of system (6), denoted by (k_*, L_*), coincide with the corresponding steady states for zero delay. In our analysis, we exclude the economically meaningless solution $k_* = 0$, $L_* = 0$.

Lemma 1 *If*

$$C - \gamma B \neq 0 \quad \textit{and} \quad \operatorname{sign}(\gamma) = \operatorname{sign}(C - \gamma B), \tag{7}$$

then there exists a unique non-trivial equilibrium (k_*, L_*) *of the system* (6) *such that*:

$$\varepsilon k_*^{a-1} = \mu, \qquad L_* = \frac{\gamma A}{C - \gamma B}. \tag{8}$$

Proof Equilibria of (6) are obtained by setting $\dot{k}(t) = \dot{L}(t) = 0$ and $\tau = 0$ in (2). These conditions yield $\varepsilon k_*^a = \mu k_*$ and $\gamma L_* = C L_*^2/[A + B L_*]$. Then we have the proof. □

Relation (7) is assumed throughout the whole paper.

Remark 1 The existence condition of steady states (7) is further specialized as follows:

1. Let $B > 0$. If $C > 0$ (resp. $C < 0$), then condition (7) reads $0 < \gamma < C/B$ (resp. $C/B < \gamma < 0$).
2. Let $B < 0$. If $C > 0$ (resp. $C < 0$), then condition (7) reads $\gamma < C/B$ and $\gamma > 0$ (resp. $\gamma < 0$ and $\gamma > C/B$).

Shifting the origin to the equilibrium point by introducing the new variable $x(t) = k(t) - k_*$, $y(t) = L(t) - L_*$ and linearizing the resulting system at the equilibrium (0, 0), we obtain

$$\begin{cases} \dot{x}(t) = \dfrac{(a-1)\mu}{v} x(t) + \gamma k_* L_*^{-1} y(t) - \gamma^2 B C^{-1} k_* L_*^{-1} y(t-\tau), \\ \dot{y}(t) = -\gamma y(t) + \gamma^2 B C y(t-\tau). \end{cases} \tag{9}$$

The characteristic equation related to (9) reads:

$$\left[\lambda - \frac{(a-1)\mu}{v}\right]\left(\lambda + \gamma - \gamma^2 B C e^{-\lambda\tau}\right) = 0. \tag{10}$$

It is well known that the trivial equilibrium of system (9) is locally asymptotically stable if each of the characteristic roots has negative real parts. Therefore, the marginal stability is determined by $\lambda = 0$ and $\lambda = i\omega$ $(\omega > 0)$. Substituting $\lambda = 0$ into Eq. (10), one obtains that the characteristic equation (10) may have a zero root. However, we will see later that $\lambda = 0$ is not a simple root for this equation. Hence, the system is a degenerated case and it is very difficult to determine the crossing direction of the characteristic roots through the imaginary axis. For simplicity, we will exclude this case in our analysis.

Before proceeding, we need to investigate the case $\tau = 0$.

Lemma 2 *Let* $\tau = 0$. *The trivial equilibrium of system* (9) *is locally asymptotically stable if* $\gamma(\gamma BC - 1) < 0$, *and unstable if* $\gamma(\gamma BC - 1) > 0$.

Proof When $\tau = 0$ Eq. (10) becomes $[\lambda - (a-1)\mu/v](\lambda + \gamma - \gamma^2 BC) = 0$, which has two real solutions, one negative, $\lambda = (a-1)\mu/v < 0$, and another, $\lambda = \gamma(\gamma BC - 1)$, whose sign is undetermined. □

Corollary 1

1. *Let* $B > 0$, $C > 0$:
   - *If* $C < 1$, *then trivial equilibrium of* (9) *is stable for* $0 < \gamma < 1/(BC)$, *and unstable for* $1/(BC) < \gamma < C/B$.
   - *If* $C > 1$, *then trivial equilibrium is always stable.*
2. *Let* $B < 0$, $C < 0$:
   - *If* $-1 < C < 0$, *then the trivial equilibrium of* (9) *is always unstable.*
   - *If* $C < -1$, *then the trivial equilibrium is stable for* $C/B < \gamma < 1/(BC)$, *and unstable for* $\gamma < 0$ *and* $\gamma > 1/BC$.
3. *Let* $B > 0$, $C < 0$:
   - *If* $C < -1$, *then the trivial equilibrium of* (9) *is always unstable.*
   - *If* $-1 < C < 0$, *then trivial equilibrium of* (9) *is stable if* $C/B < \gamma < 1/(BC)$, *unstable if* $1/(BC) < \gamma < 0$.
4. *Let* $B < 0$, $C > 0$:
   - *If* $C < 1$, *then the trivial equilibrium of* (9) *is stable for* $\gamma < 1/(BC)$ *and* $\gamma > 0$, *unstable if* $1/(BC) < \gamma < C/B$.
   - *If* $C > 1$, *then the trivial equilibrium is always stable.*

Proof It follows from the Remark 1 and Lemma 2. □

4 Hopf Bifurcations

In this section, choosing the delay τ as the bifurcation parameter, we discuss the local stability of equilibrium and provide the conditions under which Hopf bifurcations occur.

Let $\lambda = \pm i\omega$ ($\omega > 0$) denote the two purely imaginary roots of Eq. (10). Substituting $i\omega$ into Eq. (10), and separating the real and imaginary parts, we get the following equations:

$$\omega = -\gamma^2 BC \sin \omega\tau, \qquad \gamma = \gamma^2 BC \cos \omega\tau. \tag{11}$$

By squaring and adding them, we find that ω is solution of the following equation

$$\omega^2 = \gamma^2 \left(\gamma^2 B^2 C^2 - 1\right). \tag{12}$$

It is easy to show that Eq. (12) has exactly one positive root that is given by

$$\omega_0 = |\gamma| \sqrt{\gamma^2 B^2 C^2 - 1} \tag{13}$$

if

$$|\gamma BC| > 1. \tag{14}$$

Remark 2

1. Let $B > 0$, $C > 0$. Then Condition (14) is equivalent to $1/(BC) < \gamma < C/B$ and $C < 1$.
2. Let $B < 0$, $C < 0$. Then Condition (14) is equivalent to $\gamma < -1/(BC)$ or $\gamma > C/B$ and $-1 < C < 0$ or $\gamma > 1/(BC)$ and $C < -1$.
3. Let $B > 0$, $C < 0$. Then Condition (14) is equivalent to $C/B < \gamma < 1/(BC)$ and $C < -1$.
4. Let $B < 0$, $C > 0$. Then Condition (14) is equivalent to $\gamma > -1/(BC)$ or $\gamma < 1/(BC)$ and $C > 1$ or $\gamma < C/B$ and $C < 1$.

Using Eqs. (11), we can determine

$$\tau_j = \begin{cases} \dfrac{1}{\omega_0}\left\{\tan^{-1}\left(-\dfrac{\omega_0}{\gamma}\right) + 2j\pi\right\}, & \text{if } \gamma < 0, \\ \dfrac{1}{\omega_0}\left\{\tan^{-1}\left(-\dfrac{\omega_0}{\gamma}\right) + (2j+1)\pi\right\}, & \text{if } \gamma > 0, \end{cases} \quad j = 0, 1, 2, \ldots. \tag{15}$$

Lemma 3 *Let ω_0 be the unique positive root of Eq.* (12) *defined by* (13). *For $\tau = \tau_j$, Eq.* (10) *admits a pair of simple conjugate pure imaginary roots $\pm i\omega_0$. The crossing direction of the pair of simple conjugate pure imaginary roots through the imaginary axis is determined by*

$$\left.\frac{d\,(\mathrm{Re}\,\lambda)}{d\tau}\right|_{\tau=\tau_j} > 0. \tag{16}$$

Proof Let $\lambda(\tau) = \mu(\tau) + i\omega(\tau)$ be the root of (10) such that $\mu(\tau_j) = 0$ and $\omega(\tau_j) = \omega_0$. Differentiation of both sides of Eq. (10) with respect to τ gives

$$\left(1 + \gamma^2 BC\tau e^{-\lambda\tau}\right)\frac{d\lambda}{d\tau} = -\gamma^2 BC\lambda e^{-\lambda\tau}. \tag{17}$$

Note that $\lambda = \pm i\omega_0$ is a simple root, otherwise we would have $1 + \gamma^2 BC\tau e^{-i\omega_0\tau} = 0$, leading to the contradiction $\omega_0 = 0$. From Eq. (17), we get

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = -\frac{1}{\lambda\,(\lambda + \gamma)} - \frac{\tau}{\lambda},$$

so that

$$\mathrm{sign}\left\{\left.\frac{d\,(\mathrm{Re}\lambda)}{d\tau}\right|_{\tau=\tau_j}\right\} = \mathrm{sign}\left\{\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_j}\right\} = \mathrm{sign}\left\{\frac{1}{\gamma^2 + \omega_0^2}\right\} = 1.$$

□

Remark 3 $\lambda = 0$ is not a simple root for Eq. (10) since it does not satisfy Eq. (17).

Since the sign of $[d(\mathrm{Re}\lambda)/d\tau]_{\tau=\tau_j}$ is positive, each crossing of the real part of characteristic roots at τ_j must be from left to right. Hence, the crossing direction is always toward instability. We will need to distinguish the case where, without time delay, the equilibrium point would be stable, from the one where, without time delay, the equilibrium point would be unstable. In this latter case, there will be no stability switches.

According to the previous analysis, one has the following result on the stability and Hopf bifurcation of system (6).

Theorem 1

1. *Let* $B > 0$, $C > 0$:
   - *If* $C > 1$ *or* $C < 1$ *and* $0 < \gamma < 1/(BC)$, *then the positive equilibrium* (k_*, L_*) *of system* (6) *is locally asymptotically stable for all* $\tau \geq 0$.
   - *If* $C < 1$, *and* $1/(BC) < \gamma < C/B$, *then the positive equilibrium* (k_*, L_*) *is unstable for all* $\tau \geq 0$. *In this latter case, a sequence of Hopf bifurcations occur at* $\tau = \tau_j$ $(j = 0, 1, 2, \ldots)$.
2. *Let* $B < 0$, $C < 0$:
   - *If* $C > -1$ *and* $\gamma < C/B$, *or* $C < -1$ *and* $\gamma < 0$, *or* $C < -1$ *and* $\gamma > 1/(BC)$, *then the positive equilibrium* (k_*, L_*) *of system* (6) *is unstable for all* $\tau \geq 0$. *If* $\gamma < -1/(BC)$ *or* $C < -1$ *and* $\gamma > 1/(BC)$ *or* $C > -1$ *and* $\gamma > C/B$, *then a sequence of Hopf bifurcations occur at* $\tau = \tau_j$ $(j = 0, 1, 2, \ldots)$.
   - *If* $C < -1$ *and* $C/B < \gamma < 1/(BC)$, *then the positive equilibrium* (k_*, L_*) *of system* (6) *is locally asymptotically stable for all* $\tau \geq 0$.
3. *Let* $B > 0$, $C < 0$:
   - *If* $-1 < C < 0$ *and* $C/B < \gamma < 1/(BC)$, *then the positive equilibrium* (k_*, L_*) *of system* (6) *is locally asymptotically stable for all* $\tau \geq 0$.
   - *If* $C < -1$ *or* $-1 < C < 0$ *and* $1/(BC) < \gamma < 0$, *then the positive equilibrium is unstable for all* $\tau \geq 0$.
   - *If* $C < -1$ *and* $C/B < \gamma < 1/(BC)$, *a sequence of Hopf bifurcations occur at* $\tau = \tau_j$ $(j = 0, 1, 2, \ldots)$.
4. *Let* $B < 0$, $C > 0$. *Let* $\tau_0 = \min\{\tau_j\}$, *with* τ_j *given in* (15):
   - *If* $\gamma > -1/(BC)$ *or* $C > 1$ *and* $\gamma < 1/(BC)$, *then the positive equilibrium* (k_*, L_*) *of system* (6) *is locally asymptotically stable for* $\tau \in [0, \tau_0)$, *and unstable for* $\tau > \tau_0$. *Hopf bifurcations occur for* $\tau = \tau_j$ $(j = 0, 1, 2, \ldots)$.
   - *If* $C < 1$ *and* $\gamma < 1/(BC)$ *or* $C < 1$ *and* $0 < \gamma < -1/(BC)$ *or* $C > 1$ *and* $1/(BC) < \gamma < -1/(BC)$, *then the positive equilibrium* (k_*, L_*) *of system* (6) *is locally asymptotically stable for all* $\tau \geq 0$.
   - *If* $C < 1$ *and* $1/(BC) < \gamma < C/B$, *then the system is unstable for all* $\tau \geq 0$. *Hopf bifurcations occur at* $\tau = \tau_j$ $(j = 0, 1, 2, \ldots)$.

5 Research Perspective

This section is devoted to a summary of research perspective for the generalized Dalgaard and Strulik model proposed in the present paper. The model is based on the thermodynamic assumption according which the capital is generated and maintained by human and non-human energy. The main generalization refers to the introduction of the time-dependent population growth rate which allows to take into account birth and death phenomena in the populations that constitute the energy transfer sites. This is an important issue considering that the increasing in the number of individuals should encourage the development of the network and subsequently the economic growth. The introduction of time delay should improve the asymptotic analysis since allows to consider the events occurred previously.

A research perspective includes the problem of determining the bifurcating periodic solutions and the stability and directions of the Hopf bifurcation using the normal form theory and the center manifold reduction (see, e.g, Hassard et al. [26]).

As already mentioned in the introduction, some basic principles in biology can be used for the modelling of most complex economic phenomena. In particular the model presented in this paper can be further generalized by modelling the time evolution of populations by means of the methods of the mathematical kinetic theory for active particles, see [4] and the reference therein. According to this theory the overall system is decomposed into different functional subsystems constituted by particles which have the ability to perform autonomous strategies. The microscopic state of each particle includes, in addition to classical space and velocity variable, a scalar variable, called activity, which models the strategy expressed by each functional subsystems. The particles are able to interact each other according to two types of interactions: conservative interactions which modify only the strategy of the particles and non-conservative interactions which are able to model birth, death and competition processes (therefore they change the number of particles).

The energy-based method proposed by Dalgaard and Strulik in [15] can be further improved by taking into account the possibility to include the conservation of global resources by using the framework of the thermostatted kinetic theory for active particles recently proposed in [5, 6] and generalized in [7, 8]. This new framework allows to model complex systems where the global energy must be preserved. The framework has been proposed for the modelling of large systems in physics and life sciences, e.g. semiconductor devices, nanosciences, biological phenomena, vehicular traffic, social and economics systems, crowds and swarms dynamics, see the review paper [9] and allows the modelling of nonequilibrium stationary states [10]. Therefore perspective includes also the possibility of generalizing the Dalgaard and Strulik model within this new framework.

A challenging perspective is the comparison of the generalized model introduced in the present paper with the experimentally measurable quantities. Indeed the mathematical models should reproduce both qualitatively and quantitatively empirical data. The economic growth is a complex phenomenon from which emerges a collective behavior that cannot be explained by the analysis of the single elements. Therefore the model should reproduce, at least at a qualitative level, the relative emerging collective behaviors. Accordingly our model should be able to match the data on electricity consumption per capita, which is an observable variable.

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