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Grade Filtration of Linear Functional Systems

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Abstract The grade (purity) filtration of a finitely generated left module M over an Auslander regular ring D is a built-in classification of the elements of M in terms of their grades (or their (co)dimensions if D is also a Cohen-Macaulay ring). In this paper, we show how grade filtration can be explicitly characterized by means of elementary methods of homological algebra. Our approach avoids using sophisticated methods such as bidualizing complexes, spectral sequences, associated cohomology, or Spencer cohomology used in the literature of algebraic analysis. Efficient implementations dedicated to the computation of grade filtration can then be easily developed in the standard computer algebra systems. Moreover, this characterization of grade filtration is shown to induce a new presentation of the left D-module M which is defined by a block-triangular matrix formed by equidimensional diagonal blocks. The linear functional system associated with the left D-module M can then be integrated in cascade by successively solving inhomogeneous linear functional systems defined by equidimensional homogeneous linear systems of increasing dimension. This equivalent linear system generally simplifies the computation of closed-form solutions of the original linear system. In particular, many classes of underdetermined/overdetermined linear systems of partial differential equations can be explicitly integrated by the Maple package PURITYFILTRATION and the GAP package `homalg`, but not by the standard PDE solvers of computer algebra systems such as Maple.

Keywords Algebraic analysis · Grade (purity) filtration · Module theory · Homological algebra · Symbolic computation · Mathematical systems theory · Underdetermined or overdetermined linear functional systems · Linear systems of partial differential equations

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1 Introduction

The theory of *linear functional systems* such as linear systems of partial differential (PD)/time-delay/difference equations is a rich branch of mathematics which finds its foundation in mathematical physics. Different analytic methods can be used to study *determined* linear functional systems (see, e.g., [18]), namely linear functional systems containing as many unknown functions as functionally independent linear equations. *Overdetermined* (resp., *underdetermined*) linear functional systems, namely linear functional systems containing fewer (resp., more) unknown functions than functionally independent linear equations, also find important applications in mathematical physics (see, e.g., [13, 37]), in differential geometry (see, e.g., [23, 37]), or in mathematical systems theory (see, e.g., [14, 35, 37, 39]). Formal methods for the study of overdetermined linear systems of PD equations can be traced back to the works of Cartan, Riquier, and Janet [26]. A modern approach was developed in the sixties by Spencer and his collaborators (see, e.g., [37, 52]). *Gröbner bases* and *Janet bases* [12, 26] over a noncommutative polynomial ring of functional operators are nowadays two fundamental computational tools used for the formal study of overdetermined linear functional systems (see, e.g., [14, 30, 48]).

Despite these important computational methods, computer algebra systems still have many difficulties to find closed-form solutions of overdetermined or undetermined linear functional systems (when they exist), for instance of linear systems of PD equations. One of the main reasons is that linear functional systems generally mix together unknown functions which satisfy linear functional systems of different dimension. For instance, the integration of the unknown functions of an overdetermined linear systems of PD equations depends on arbitrary functions of a certain number of the independent variables (due to the *Cartan-Kähler-Janet theorem* which generalizes the well-known *Cauchy-Kowalevski theorem*) (see, e.g., [26, 37, 52]). The maximal number of independent variables which appear in these arbitrary functions (sometimes plus the number of independent variables) is called the *dimension* of the system. Hence, an important issue for the study of overdetermined linear functional systems is to determine the unknown functions or their linear functional combinations which satisfy a linear functional system of a given dimension. This problem, related to the *equidimensional decomposition* of algebraic varieties (see, e.g., [19, 24, 49]), has lengthly been studied within *algebraic analysis* and *algebraic/analytic D-module theory* [9–11, 32] by Roos [49], Sato and Kashiwara [28, 29], Björk [9, 10], Ginsburg [22], and others. This problem corresponds to the so-called *grade filtration* $\{M_i\}_{i\geq 0}$ (also called *bidualizing* or *purity filtration*) of the finitely generated left D-module M which defines the linear system of PD equations, where D is a noncommutative polynomial ring of PD operators satisfying certain regularity conditions (e.g., D is an *Auslander regular ring*). This descending filtration of M is defined by the left D-submodules M_i's of M formed by the elements of M having a *codimension* (or a *grade*) greater or equal to i. The existence of the grade filtration of a finitely generated left/right module M over an Auslander regular ring D is proved in [9, 10, 22, 31, 49] (resp., in [28, 29]) using *bidualizing complexes* and *spectral sequence* arguments (resp., *derived categories*, *derived functors*, and *associated cohomology* [24]), i.e., by means of sophisticated homological algebra techniques (resp., modern developments of category theory). See also [37, 38] (resp., [36]) for a recent study of grade filtration based on *Spencer cohomology* and *Spencer sequences* (resp., *Gabriel localization* for commutative polynomial rings). Despite the difficulties to compute the spectral sequences defining the grade filtration, these were recently made constructive in [2, 3] thanks to the new concept of *generalized morphisms*, and they were implemented in the `homalg` package [8] of the system GAP [21] (`homalg` is a package dedicated to homological algebra oriented computations). To our knowledge, it is the first implementation of the computation of the grade filtration in a computer algebra system. We refer the reader to [19, 24, 49] (resp., [9, 10, 22, 28]) for applications of grade filtration to algebraic geometry (resp., algebraic analysis). Finally, techniques based on grade filtration have recently been introduced in mathematical systems theory (see [4, 36–42, 44]).

The purpose of this paper is to develop a new algorithm which computes the grade filtration of a finitely generated left module M over a *regular domain* D satisfying a slightly weaker condition (see (38)) than the standard *Auslander condition* (see, e.g., [9, 10]). In particular, many important classes of noncommutative polynomial rings of functional systems satisfy these conditions. The first benefit of this new algorithm is that it is an extension of the methods developed in [1, 14, 29, 37, 39] for the classification of modules (torsion modules, modules with torsion submodules, torsion-free/reflexive/projective modules). These methods have recently been applied to solve the problem of parametrizing underdetermined linear functional systems by means of arbitrary functions (*potentials*) studied in mathematical physics and in control theory (see [14, 15, 20, 37, 39, 54]). The second benefit of this algorithm is that it is conceptually much simpler than the algorithms based on bidualizing complexes, spectral sequences, and associated cohomology. In particular, it can be easily implemented in any computer algebra system in which Gröbner basis techniques are available (e.g., Maple, Mathematica, Singular, Macaulay2, Magma). The corresponding algorithm was implemented by the author in the Maple package PURITYFILTRATION [45] built upon OREMODULES [15]. Using the PURITYFILTRATION package, classes of overdetermined/underdetermined linear systems of PD equations which cannot be directly integrated by Maple can be explicitly solved [45] (see also the forthcoming `homalg` based package `D-modules`). Moreover, the algorithm has also been implemented recently in the `homalg` project package `AbelianSystems` [7] developed in collaboration with M. Barakat (University of Kaiserslautern). This implementation is much faster than the original `homalg` command based on spectral sequence computation, and thus it can be used to study larger examples. More recently, the algorithms developed in this paper were implemented in the Singular package `purityfiltration.lib` [51]. We hope that the results developed in this paper and demonstrated by the PURITYFILTRATION, `AbelianSystems`, and `purityfiltration.lib` packages will be used in the future to improve standard computer algebra systems such as Maple or Mathematica for the symbolic integration of overdetermined/underdetermined linear functional systems. The third benefit of this new approach is that it gives a filtration-adapted presentation matrix which has a remarkably simple form (block-diagonal and single off-diagonal). It does not seem that it can easily be obtained from the classical black-box spectral sequence approach [2, 9, 10, 22, 49]. The last benefit is that this algorithm holds for *computable abelian categories* [6], and thus it can be used in different contexts such as the computation of the grade filtration of coherent sheaves over projective schemes as shown in the `homalg` project package `Sheaves` [5].

Since techniques of module theory, homological algebra, and algebraic analysis are not largely well-known, they are summarized in Sect. 2. The main results about grade filtration are developed in Sect. 3. In Sect. 4, we show how the concept of grade filtration can be used to compute an equivalent block-triangular form of a linear functional system whose diagonal blocks define equidimensional linear functional systems. The integration of the original system is then equivalent to a cascade integration of inhomogeneous linear functional systems, the corresponding homogeneous linear systems being equidimensional and of increasing dimension (e.g., we first integrate a 0-dimensional/*holonomic* homogeneous linear system, then an inhomogeneous linear systems defined by a 1-dimensional/*subholonomic* homogeneous linear system, ...). Finally, in Sect. 5, we briefly give a few extensions of the results obtained in Sect. 3.

The paper was written in a self-contained way so that everyone willing to implement the computation of the grade filtration in a computer algebra system will find there all the necessary materials. To emphasize the main results, we shortly summarize the main ideas and results. Within the algebraic analysis approach (see Sect. 2), a linear system defines a finitely presented left module M over a ring D that we shall suppose to be an Auslander regular of global dimension n (see Definition 8). To compute the grade filtration $\{t_i(M)\}_{i=0,\ldots,n+1}$ of M defined by the left D-submodule $t_i(M)$ of M formed by the elements of M of grade greater than or equal to i (see (21)), we first need to consider a free resolution (see 6 of Definition 2) of M of the form (24), then dualize it to get the complex (25), and finally consider the beginning of a free resolution of the cokernel of each homomorphism $R_{ii}.$ defining (25) (see (27)). The complex (25) then induces a chain complex (32) between the free resolutions of two consecutive right D-modules $N_{ii} = \operatorname{coker}_D(R_{ii}.)$ (the so-called *Auslander transpose D-modules*). By truncating (32) to get the commutative diagram (33) and by dualizing it, we obtain the commutative diagram (47) formed by horizontal complexes. The defect of exactness of the horizontal complex at $D^{1\times p_{0i}}$ is the left D-module T_i defined by (48), and (47) induces a left D-homomorphism $\gamma_{(i+1)i} : T_{i+1} \longrightarrow T_i$ defined by (49) for $i = 1, \ldots, n$, and $\gamma_{10} : T_1 \longrightarrow M$ defined by (50). The constructions of the left D-modules T_i's and of the left D-homomorphisms $\gamma_{(i+1)i}$'s are summarized in Algorithm 1. The condition (38) of the Auslander regular ring D implies that the $\gamma_{(i+1)i}$'s are injective for $i = 0, \ldots, n$. We can then consider the left D-submodule $M_i = (\gamma_{10} \circ \gamma_{21} \circ \gamma_{32} \circ \cdots \circ \gamma_{i(i-1)})(T_i)$ of $M = M_0$ for $i = 1, \ldots, n$ (see (56)). Theorem 11 proves that $t_i(M) = M_i$ for $i = 0, \ldots, n$, which gives a complete characterization of the elements $t_i(M)$'s of the grade filtration (58) of M as images of injective homomorphisms. The construction of the grade filtration of M is summarized in Algorithm 2. Now, if we want to characterize the M_i's by means of finite presentations, we first need to consider the first two syzygy modules of $\operatorname{coker}_D(.R_{0i})$ of the left D-homomorphisms $.R_{0i}$ defined in (47) to get the commutative diagram (70) formed by exact horizontal sequences. Combining (47) and (70), we obtain Fig. 1, which induces a sequence of left D-modules L_i's isomorphic to T_i defined by (66), and the left D-homomorphisms $\gamma_{(i+1)i} : T_{i+1} \longrightarrow T_i$ induce the injective left D-homomorphisms $\overline{\gamma}_{(i+1)i} : L_{i+1} \longrightarrow L_i$ and $\overline{\gamma}_{10} : L_1 \longrightarrow M$ respectively defined by (73) and (77). Finally, using Baer's extension techniques (see Sect. 2.2) and the presentation of the i-pure left D-module $\operatorname{coker} \overline{\gamma}_{(i+1)i}$ (see Definition 7), which is isomorphic to $t_i(M)/t_{i+1}(M)$, we obtain a new presentation matrix $\overline{R}$ of M defined by (91). The equidimensional decomposition of M clearly appears on this filtration-adapted presentation (block-diagonal and single off-diagonal) and it is particularly interesting for the symbolic integration of the linear PD system defined by M when D is a ring of PD operators.

2 Algebraic Analysis Approach to Linear Functional Systems

In what follows, D will always be a *noetherian ring*, i.e., a ring D that is both a left and a right noetherian ring (see, e.g., [50]). Moreover, the set of $q \times p$ matrices with entries in D is denoted by $D^{q\times p}$, and the unit of the ring $D^{p\times p}$ by I_p. If $\mathcal{F}$ is a left D-module (e.g., $\mathcal{F} = D$) and $R \in D^{q\times p}$, then $.R$ and $R.$ are respectively the left D-homomorphism (i.e., the left D-linear map) and the abelian group homomorphism (i.e., $\mathbb{Z}$-homomorphism) defined by:

$$\begin{array}{llll} .R : D^{1\times q} \longrightarrow D^{1\times p} & & R. : \mathcal{F}^p \longrightarrow \mathcal{F}^q \\ \lambda = (\lambda_1 \ldots \lambda_q) \longmapsto \lambda R, & & \eta = (\eta_1 \ldots \eta_p)^T \longmapsto R\eta. \end{array}$$

With the above notations, we call *linear system* an abelian group of the form:

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\eta = 0\}.$$

The study of $\ker_{\mathcal{F}}(R.)$ in terms of the *finitely presented left D-module*

$$M = D^{1\times p}/(D^{1\times q}\, R)$$

and of the left D-module $\mathcal{F}$ was first developed in [33]. This approach is nowadays the cornerstone of the *algebraic D-module theory* (or *algebraic analysis*), developed by Bernstein and Sato's school (particularly by Kashiwara), in which D stands for a noncommutative ring of partial differential (PD) operators with coefficients in a *differential ring* (see, e.g., [9–11, 29, 32]). More precisely, if A is a ring and $\{\delta_i\}_{i=1,\dots,n}$ are n *commuting derivations* of A, namely, $\delta_i : A \longrightarrow A$ satisfies

$$\forall a_1, a_2 \in A, \quad \begin{cases} \delta_i(a_1 + a_2) = \delta_i(a_1) + \delta_i(a_2), \\ \delta_i(a_1 a_2) = \delta_i(a_1) a_2 + a_1 \delta_i(a_2), \end{cases}$$

for all $i = 1, \dots, n$, and $\delta_i \circ \delta_j = \delta_j \circ \delta_i$ for all $i, j = 1, \dots, n$, then the ring $D = A\langle \partial_1, \dots, \partial_n \rangle$ of PD operators with coefficients in A is the noncommutative polynomial ring in $\partial_1, \dots, \partial_n$ with coefficients in A which satisfies the relations:

$$\forall i = 1, \dots, n,\ \forall a \in A, \quad \partial_i a = a \partial_i + \delta_i(a), \qquad \forall i, j = 1, \dots, n, \quad \partial_i \partial_j = \partial_j \partial_i.$$

Prototypical examples of a ring D of PD operators are the so-called *Weyl algebras* $A_n(k)$ and $B_n(k)$ of PD operators with respectively coefficients in $A = k[x_1, \dots, x_n]$ and in $A = k(x_1, \dots, x_n)$, where k is a field (that we shall suppose to be of characteristic 0), $\hat{\mathcal{D}}_n(k)$, or $\mathcal{D}_n(k')$ the rings of PD operators with coefficients in the ring of formal power series $A = k[\![x_1, \dots, x_n]\!]$ or in the ring of locally convergent power series $A = k'\{x_1, \dots, x_n\}$, where $k' = \mathbb{R}$ or $\mathbb{C}$. These rings are noetherian domains (see, e.g., [9, 11, 32]). If D is a ring of PD operators and $\mathcal{F}$ a left D-module (e.g., $\mathcal{F} = A$), then $R \in D^{q\times p}$ is a matrix of PD operators and the linear system $\ker_{\mathcal{F}}(R.)$ is the k-vector space formed by the $\mathcal{F}$-solutions of the linear system of PD equations $R\eta = 0$. Within algebraic analysis, more general classes of noncommutative polynomial rings of functional operators can be considered such as *Ore algebras* as explained in [14], which allows one to consider a more general class of linear functional systems.

Let us now explain basic ideas of algebraic analysis. Let $\pi : D^{1\times p} \longrightarrow M$ be the left D-homomorphism which maps $\lambda \in D^{1\times p}$ to its residue class $\pi(\lambda) \in M$, and $\{f_j\}_{j=1,\dots,p}$ the *standard basis* of $D^{1\times p}$, namely, f_j is the row vector of length p with 1 at the jth position and 0 elsewhere. Then, $\{y_j = \pi(f_j)\}_{j=1,\dots,p}$ is a family of generators of M since for every $m \in M$, there exists $\lambda = (\lambda_1 \dots \lambda_p) \in D^{1\times p}$ such that $m = \pi(\lambda)$, which yields:

$$m = \pi(\lambda) = \pi\left(\sum_{j=1}^{p} \lambda_j f_j\right) = \sum_{j=1}^{p} \lambda_j \pi(f_j) = \sum_{j=1}^{p} \lambda_j y_j.$$

The family of generators $\{y_j\}_{j=1,\dots,p}$ of M satisfies D-linear relations: if $R_{i\bullet}$ denotes the ith row of R, then $R_{i\bullet} \in D^{1\times q} R$, which yields $\pi(R_{i\bullet}) = 0$, and thus:

$$\forall i = 1, \dots, q, \quad \pi(R_{i\bullet}) = \pi\left(\sum_{j=1}^{p} R_{ij} f_j\right) = \sum_{j=1}^{p} R_{ij} \pi(f_j) = \sum_{j=1}^{p} R_{ij} y_j = 0.$$

If $y = (y_1 \dots y_p)^T \in M^p$, then the above relations can be rewritten as $Ry = 0$.

If $\mathcal{F}$ is a left D-module, $\hom_D(M, \mathcal{F})$ the abelian group of left D-homomorphisms from M to $\mathcal{F}$, and $\phi \in \hom_D(M, \mathcal{F})$, then $\eta = (\phi(y_1) \dots \phi(y_p))^T \in \mathcal{F}^p$ and

$$\forall i = 1, \dots, q, \quad \sum_{j=1}^{p} R_{ij} \eta_j = \sum_{j=1}^{p} R_{ij} \phi(y_j) = \phi\left(\sum_{j=1}^{p} R_{ij} y_j\right) = \phi(0) = 0,$$

i.e., $\eta \in \ker_{\mathcal{F}}(R.)$. Conversely, if $\eta \in \ker_{\mathcal{F}}(R.)$, then we can define the map $\phi_\eta : M \longrightarrow \mathcal{F}$ by $\phi_\eta(\pi(\lambda)) = \lambda\eta$ for all $\lambda \in D^{1\times p}$. Indeed, ϕ_η is well-defined: if $\pi(\lambda) = \pi(\lambda')$, then $\lambda = \lambda' + \mu R$ for a certain $\mu \in D^{1\times q}$, which yields:

$$\phi_\eta\big(\pi(\lambda)\big) = \lambda\eta = \lambda'\eta + \mu R\eta = \lambda'\eta.$$

The map ϕ_η is clearly left D-linear and $\phi_\eta(0) = 0$ since $\phi_\eta(\pi(\mu R)) = \mu(R\eta) = 0$ for all $\mu \in D^{1\times q}$, and thus $\phi_\eta \in \hom_D(M, \mathcal{F})$. If we introduce the following abelian group homomorphisms

$$\begin{aligned} \sigma : \ker_{\mathcal{F}}(R.) &\longrightarrow \hom_D(M, \mathcal{F}) \\ \eta &\longmapsto \phi_\eta, \end{aligned} \qquad \begin{aligned} \chi : \hom_D(M, \mathcal{F}) &\longrightarrow \ker_{\mathcal{F}}(R.) \\ \phi &\longmapsto \big(\phi(y_1) \dots \phi(y_p)\big)^T, \end{aligned}$$

then $\chi \circ \sigma = \mathrm{id}_{\ker_{\mathcal{F}}(R.)}$ since $\phi_\eta(y_j) = \eta_j$ for all $j = 1, \dots, p$, and $\sigma \circ \chi = \mathrm{id}_{\hom_D(M,\mathcal{F})}$ since $(\sigma \circ \chi)(\phi) = \phi_{(\phi(y_1)\dots\phi(y_p))^T} = \phi$, which shows that $\chi^{-1} = \sigma$, and proves that $\ker_{\mathcal{F}}(R.)$ and $\hom_D(M, \mathcal{F})$ are isomorphic as abelian groups, which is denoted by $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$.

Theorem 1 ([33]) *With the previous notations, we have*:

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}).$$

Theorem 1 shows that the linear system $\ker_{\mathcal{F}}(R.)$ can be intrinsically studied by means of the two left D-modules $M = D^{1\times p}/(D^{1\times q}\, R)$ and $\mathcal{F}$. The matrix R is a particular *finite presentation* of the left D-module M defined up to isomorphism (see, e.g., [50]). Hence, we can study the solution space $\hom_D(M, \mathcal{F})$ independently of the particular embedding of $\ker_{\mathcal{F}}(R.)$ into $\mathcal{F}^p$. A second benefit of Theorem 1 is that the linear system $\ker_{\mathcal{F}}(R.)$ can be studied by means of the properties of the left D-modules M and $\mathcal{F}$.

Definition 1 ([50]) Let D be a noetherian ring and M a finitely generated left D-module.

1. M is *free* if there exists $r \in \mathbb{N} = \{0, 1, 2, \dots\}$ such that $M \cong D^{1\times r}$. Then, r is then called the *rank* of M.
2. M is *projective* if there exist $r \in \mathbb{N}$ and a left D-module N such that

$$M \oplus N \cong D^{1\times r},$$

   where $\oplus$ denotes the direct sum of left D-modules.
3. M is *reflexive* if the left D-homomorphism $\varepsilon : M \longrightarrow \hom_D(\hom_D(M, D), D)$, defined by $\varepsilon(m)(f) = f(m)$ for all $m \in M$ and for all $f \in \hom_D(M, D)$, is an isomorphism.
4. If D is a domain, then M is *torsion-free* if the *torsion left D-submodule* of M defined by

$$t(M) = \big\{m \in M \mid \exists d \in D \setminus \{0\} : dm = 0\big\}$$

   is reduced to 0, i.e., if $t(M) = 0$.

5. If D is a domain, then M is *torsion* if $t(M) = M$, i.e., if every element of M is a torsion element.

Theorem 2 ([50]) *A free module is projective, a projective module is reflexive, and a reflexive module is torsion-free.*

In the next sections, we summarize basic homological techniques which will be used to algorithmically test whether or not M admits torsion elements or is torsion-free, reflexive, or projective (see Theorem 5 thereafter). These techniques will then be generalized in Sect. 3 to obtain an explicit characterization of the so-called *grade filtration* of M.

2.1 Basic Homological Algebra

Let us shortly recall a few definitions of homological algebra (see, e.g., [50]).

Definition 2

1. A *complex*, denoted by
$$M_\bullet \cdots \xrightarrow{d_{i+2}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \cdots, \tag{1}$$
is a sequence of left (resp., right) D-modules M_i and of left (resp., right) D-homomorphisms $d_i : M_i \longrightarrow M_{i-1}$ that satisfy $\operatorname{im} d_{i+1} \subseteq \ker d_i$, i.e.:
$$\forall i \in \mathbb{Z}, \quad d_i \circ d_{i+1} = 0.$$
2. The *defect of exactness* of (1) at M_i is the left/right D-module defined by:
$$H_i(M_\bullet) \triangleq \ker d_i / \operatorname{im} d_{i+1}.$$
3. The complex (1) is *exact at* M_i if $H_i(M_\bullet) = 0$, i.e., if $\ker d_i = \operatorname{im} d_{i+1}$, and *exact* if $\ker d_i = \operatorname{im} d_{i+1}$ for all $i \in \mathbb{Z}$. An exact complex is called an *exact sequence*.
4. An exact sequence of the form
$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0, \tag{2}$$
i.e., f is injective, $\ker g = \operatorname{im} f$ and g is surjective, is called a *short exact sequence*.
5. A *projective resolution* of a left D-module M is an exact sequence of the form
$$\cdots \xrightarrow{d_4} P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0,$$
where the P_i's are projective left D-modules, and $d_i \in \hom_D(P_i, P_{i-1})$ for $i \geq 1$, and $\hom_D(P_0, M)$. The smallest $n \in \mathbb{N}$ such that $P_m = 0$ for all $m > n$ is called the *length* of the projective resolution of M. Similarly for right D-modules.
6. A *free resolution* of a finitely generated left D-module M is an exact sequence of the form
$$\cdots \xrightarrow{.R_3} D^{1\times p_2} \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\pi} M \longrightarrow 0, \tag{3}$$
where $R_i \in D^{p_i \times p_{i-1}}$ and $.R_i : D^{1\times p_i} \longrightarrow D^{1\times p_{i-1}}$ is defined by $(.R_i)(\lambda) = \lambda R_i$.

7. A *free resolution* of a finitely generated right D-module N is an exact sequence of the form

$$0 \longleftarrow N \overset{\kappa}{\longleftarrow} D^{q_0} \overset{S_1 .}{\longleftarrow} D^{q_1} \overset{S_2 .}{\longleftarrow} D^{q_2} \overset{S_3 .}{\longleftarrow} \cdots, \tag{4}$$

where $S_i \in D^{q_{i-1} \times q_i}$ and $S_i .: D^{q_i} \longrightarrow D^{q_{i-1}}$ is defined by $(S_i .)(\eta) = S_i \eta$.

Example 1 If D is a noetherian domain and M a finitely generated left D-module, then we have the following short exact sequence of left D-modules:

$$0 \longrightarrow t(M) \overset{j}{\longrightarrow} M \overset{\rho}{\longrightarrow} M/t(M) \longrightarrow 0. \tag{5}$$

Remark 1 A module M is not defined by a unique projective/free resolution: if

$$0 \longrightarrow \ker \pi \longrightarrow P \overset{\pi}{\longrightarrow} M \longrightarrow 0, \qquad 0 \longrightarrow \ker \pi' \longrightarrow P' \overset{\pi'}{\longrightarrow} M \longrightarrow 0$$

are two exact sequences, where P and P' are projective modules, then *Fitting's lemma* asserts that $\ker \pi \oplus P' \cong \ker \pi' \oplus P$ (see, e.g., [50]). This isomorphism does not generally imply that $\ker \pi \cong \ker \pi'$. We say that $\ker \pi$ depends on M up to a *projective equivalence* (see, e.g., [50]). Similarly, if we consider two finite presentations of M,

$$D^{1 \times p_1} \overset{.R_1}{\longrightarrow} D^{1 \times p_0} \overset{\pi}{\longrightarrow} M \longrightarrow 0, \qquad D^{1 \times p_1'} \overset{.R_1'}{\longrightarrow} D^{1 \times p_0'} \overset{\pi'}{\longrightarrow} M \longrightarrow 0,$$

then $\ker_D(.R_1) \oplus D^{1 \times (p_1' + p_0)} \cong \ker_D(.R_1') \oplus D^{1 \times (p_1 + p_0')}$. See, e.g., [50], and [17] for a constructive proof. Similar results hold for the *syzygy modules* $\ker_D(.R_i)$'s of M.

Since D is a noetherian ring, one can easily prove that every finitely generated left (resp. right) D-module M admits a free resolution (see, e.g., [50]). Now, if $\mathcal{F}$ is a left D-module, then using a free resolution (3) of a finitely generated left D-module M, we can define the *extension abelian groups* $\mathrm{ext}_D^i(M, \mathcal{F})$'s for $i \geq 0$ as follows. Up to abelian group isomorphism, they are defined by the defects of exactness of the following complex of abelian groups

$$\cdots \overset{R_{i+1} .}{\longleftarrow} \mathcal{F}^{p_i} \overset{R_i .}{\longleftarrow} \mathcal{F}^{p_{i-1}} \overset{R_{i-1} .}{\longleftarrow} \cdots \overset{R_3 .}{\longleftarrow} \mathcal{F}^{p_2} \overset{R_2 .}{\longleftarrow} \mathcal{F}^{p_1} \overset{R_1 .}{\longleftarrow} \mathcal{F}^{p_0} \longleftarrow 0, \tag{6}$$

where $R_i .: \mathcal{F}^{p_{i-1}} \longrightarrow \mathcal{F}^{p_i}$ is defined by $(R_i .)(\eta) = R_i \eta$ for all $\eta \in \mathcal{F}^{p_{i-1}}$, namely:

$$\begin{cases} \mathrm{ext}_D^0(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_1 .), \\ \mathrm{ext}_D^i(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_{i+1} .) / \mathrm{im}_{\mathcal{F}}(R_i .), \quad i \geq 1. \end{cases} \tag{7}$$

Theorem 1 shows that $\mathrm{ext}_D^0(M, \mathcal{F}) = \hom_D(M, \mathcal{F})$. See also, e.g., [50].

We say that the complex (6) is obtained by application of the *contravariant left exact functor* $\hom_D(\cdot, \mathcal{F})$ to the *reduced (truncated) free resolution* of M, namely, to the complex obtained by removing M from the finite free resolution (3) as follows:

$$\cdots \overset{.R_4}{\longrightarrow} D^{1 \times p_3} \overset{.R_3}{\longrightarrow} D^{1 \times p_2} \overset{.R_2}{\longrightarrow} D^{1 \times p_1} \overset{.R_1}{\longrightarrow} D^{1 \times p_0} \longrightarrow 0. \tag{8}$$

A fundamental theorem of homological algebra asserts that the abelian groups $\mathrm{ext}_D^i(M, \mathcal{F})$'s depend only on the left D-modules M and $\mathcal{F}$ (up to abelian group isomorphism), i.e., they do not depend on the choice of the free resolution (3) of M (see, e.g., [50]). The $\mathrm{ext}_D^i(M, \mathcal{F})$'s can also be defined using projective resolutions of M (see, e.g., [50]). But this approach is less constructive than the one based on free resolutions. In what follows, we shall only consider free resolutions and we let the reader reformulate the different results based on projective resolutions.

The idea of replacing a rather complicated left D-module M by the complex (8) formed by the left D-modules $D^{1\times p_i}$'s (free modules) and trivial left D-homomorphisms $.R_i$'s (defined by matrices) is of paramount importance in the theory of *derived category* developed by Grothendieck and Verdier (see, e.g., [24]). In this paper, we shall show how the grade filtration of M, which is difficult to compute directly on M, can be explicitly characterized by many but simple (matrix) computations related to the computation of $\mathrm{ext}^i_D(M,D)$ and $\mathrm{ext}^j_D(\mathrm{ext}^i_D(M,D),D)$.

Similarly, if N a finitely generated right D-module and $\mathcal{G}$ a right D-module, then using a free resolution (4) of N, we can define the following abelian groups:

$$\begin{cases} \mathrm{ext}^0_D(N,\mathcal{G}) = \hom_D(N,\mathcal{G}) \cong \ker_{\mathcal{G}}(.S_1), \\ \mathrm{ext}^i_D(N,\mathcal{G}) \cong \ker_{\mathcal{G}}(.S_{i+1})/\operatorname{im}_{\mathcal{G}}(.S_i), \quad i \geq 1. \end{cases}$$

We note that if M is a left (resp., right) D-module, then $\mathrm{ext}^i_D(M,D)$ is a right (resp., left) D-module due to the $D-D$-bimodule structure of D (see, e.g., [50]).

Definition 3 ([50]) A left D-module $\mathcal{F}$ is called *injective* if $\mathrm{ext}^i_D(M,\mathcal{F}) = 0$ for all left D-modules M and for all $i \geq 1$.

Example 2 If Ω is an open convex subset of $\mathbb{R}^n$ and $k = \mathbb{R}$ or $\mathbb{C}$, then the space $C^\infty(\Omega)$ (resp., $\mathcal{D}'(\Omega)$, $\mathcal{S}'(\Omega)$, $\mathcal{A}(\Omega)$, $\mathcal{O}(\Omega)$) of smooth functions (resp., distributions/temperate distributions, real analytic/holomorphic functions) on Ω is an injective $D = k[\partial_1,\ldots,\partial_n]$-module [33, 35, 54].

If M is a finitely generated left D-module and $\mathcal{F}$ an injective left D-module, then applying the contravariant left exact functor $\hom_D(\cdot,\mathcal{F})$ to (3), using Theorem 1, and the fact that $\mathrm{ext}^i_D(\cdot,\mathcal{F}) = 0$ for all $i \geq 1$, we obtain the following exact sequence of abelian groups:

$$\cdots \xleftarrow{R_3.} \mathcal{F}^{p_2} \xleftarrow{R_2.} \mathcal{F}^{p_1} \xleftarrow{R_1.} \mathcal{F}^{p_0} \longleftarrow \hom_D(M,\mathcal{F}) \longleftarrow 0.$$

The contravariant functor $\hom_D(\cdot,\mathcal{F})$ is then said to be *exact*. Within mathematical systems theory, the linear system $\ker_{\mathcal{F}}(R_{i+1}.)$ is *parametrized* by R_i (called a *parametrization*) since $\ker_{\mathcal{F}}(R_{i+1}.) = R_i\,\mathcal{F}^{p_{i-1}}$ for all $i \geq 1$.

Let us now state two results which will be used in Sect. 3.

Theorem 3 ([50]) *Let* (2) *be a short exact sequence of left* (*resp., right*) *D-modules and N a left* (*resp., right*) *D-module. Then, the following long exact sequence holds*

$$\begin{aligned} 0 \longrightarrow \mathrm{ext}^0_D\big(M'',N\big) &\xrightarrow{g^\star} \mathrm{ext}^0_D(M,N) \xrightarrow{f^\star} \mathrm{ext}^0_D\big(M',N\big) \\ \longrightarrow \mathrm{ext}^1_D\big(M'',N\big) &\longrightarrow \mathrm{ext}^1_D(M,N) \longrightarrow \mathrm{ext}^1_D\big(M',N\big) \\ \longrightarrow \mathrm{ext}^2_D\big(M'',N\big) &\longrightarrow \mathrm{ext}^2_D(M,N) \longrightarrow \cdots, \end{aligned}$$

where $f^\star$ and $g^\star$ are respectively defined by:

$$\forall \phi \in \hom_D(M,N), \quad f^\star(\phi) = \phi \circ f, \quad \forall \psi \in \hom_D\big(M'',N\big), \quad g^\star(\psi) = \psi \circ g.$$

Remark 2 One can prove that a left D-module M is projective iff $\text{ext}_D^i(M, N) = 0$ for all left D-module N and for all $i \geq 1$ (see, e.g., [50]). If P and P' are the two projective left D-modules considered in Remark 1, the *additivity* of the functor $\text{ext}_D^i(\cdot, N)$ (see, e.g., [50]) then yields

$$\forall i \geq 1, \quad \begin{cases} \text{ext}_D^i\big(\ker\pi \oplus P', N\big) \cong \text{ext}_D^i(\ker\pi, N) \oplus \text{ext}_D^i\big(P', N\big) = \text{ext}_D^i(\ker\pi, N), \\ \text{ext}_D^i\big(\ker\pi' \oplus P, N\big) \cong \text{ext}_D^i\big(\ker\pi', N\big) \oplus \text{ext}_D^i(P, N) = \text{ext}_D^i\big(\ker\pi', N\big), \end{cases}$$

and thus $\text{ext}_D^i(\ker\pi, N) \cong \text{ext}_D^i(\ker\pi', N)$ for $i \geq 1$, which shows that $\text{ext}_D^i(\ker\pi, N)$ depends only on M and N (up to isomorphism) for $i \geq 1$.

Combining Remark 2 with Theorem 3, we obtain the following result.

Proposition 1 ([50]) *Let* (2) *be a short exact sequence of left* (*resp*., *right*) *D-modules and M a projective left* (*resp*., *right*) *D-module. Then, for every left* (*resp*., *right*) *D-module N, we have* $\text{ext}_D^{i+1}(M'', N) \cong \text{ext}_D^i(M', N)$ *for* $i \geq 1$.

Let us introduce important invariants of modules and rings.

Definition 4 ([50])

1. The *left projective dimension* of a left D-module M, denoted by $\text{lpd}_D(M)$, is the minimum of the lengths of projective resolutions of M. If no such integer exists, then we set $\text{lpd}_D(M) = \infty$. Similarly for the *right projective dimension* $\text{rpd}_D(N)$ of a right D-module N.
2. The *left global dimension* (resp., *right global dimension*) of a ring D, denoted by $\text{lgd}(D)$ (resp., $\text{rgd}(D)$), is the supremum of $\text{lpd}_D(M)$ (resp., $\text{rpd}_D(N)$) for all left D-modules M (resp., all right D-modules N).
3. If the left and the right global dimension of D coincide, then the common value is called the *global dimension* of D and is denoted by $\text{gld}(D)$.

Proposition 2 ([10]) *Let D be a noetherian ring and M a finitely generated left D-module. Then, we have*:

$$\text{lpd}_D(M) = \sup\big\{i \in \mathbb{N} \mid \text{ext}_D^i(M, D) \neq 0\big\}.$$

Similarly for the right projective dimension $\text{rpd}_D(N)$ *of a right D-module N.*

Proposition 3 ([50]) $\text{lgd}(D) \leq n$ *iff* $\text{ext}_D^i(M, N) = 0$ *for all left D-modules M and N, and for all $i > n$.*

Theorem 4 ([50]) *If D is a noetherian ring, then* $\text{lgld}(D) = \text{rgld}(D)$.

Example 3 If k is a field, then $\text{gld}(k[x_1, \ldots, x_n]) = n$ (see, e.g., [50]). If k is a field of characteristic 0, $k' = \mathbb{R}$ or $\mathbb{C}$, and $D = A_n(k)$, $B_n(k)$, $\hat{\mathcal{D}}_n(k)$, or $\mathcal{D}_n(k')$, then $\text{gld}(D) = n$ (see, e.g., [9, 10, 29]).

We are now in a position to recall how the properties stated in Definition 1 can be checked by means of homological techniques for a *regular domain* D, namely a noetherian domain D of finite global dimension $\text{gld}(D)$.

Theorem 5 ([1, 14, 29, 37, 39]) *Let D be a noetherian domain with a finite global dimension* $\mathrm{gld}(D) = n$, $M = D^{1\times p}/(D^{1\times q} R)$ *a finitely presented left D-module, and* $N = D^q/(R D^p)$ *the so-called* Auslander transpose *right D-module of M.*

1. *The following left D-isomorphism holds*:

$$t(M) \cong \mathrm{ext}_D^1(N, D). \tag{9}$$

2. *M is torsion-free iff* $\mathrm{ext}_D^1(N, D) = 0$.
3. *The following long exact sequence holds*

$$0 \longrightarrow \mathrm{ext}_D^1(N, D) \longrightarrow M \xrightarrow{\varepsilon} \hom_D\big(\hom_D(M, D), D\big) \longrightarrow \mathrm{ext}_D^2(N, D) \longrightarrow 0, \tag{10}$$

where ε is defined in 3 *of Definition* 1.

4. *M is reflexive iff* $\mathrm{ext}_D^i(N, D) = 0$ *for* $i = 1, 2$.
5. *M is projective iff* $\mathrm{ext}_D^i(N, D) = 0$ *for* $i = 1, \ldots, n$.

Remark 3 The Auslander transpose right D-module $N = D^q/(R D^p)$ depends on the left D-module $M = D^{1\times p}/(D^{1\times q} R)$ up to a projective equivalence. Indeed, if $M \cong M' = D^{1\times p'}/(D^{1\times q'} R')$, then we get $N \oplus D^{(p+q')} \cong N' \oplus D^{(p'+q)}$, where $N' = D^{q'}/(R' D^{p'})$ is the Auslander transpose of M' [1]. See [17] for a constructive proof. Using Remark 2, the additivity of the functor $\mathrm{ext}_D^i(\cdot, \mathcal{F})$ (see, e.g., [50]) then yields $\mathrm{ext}_D^i(N, \mathcal{F}) \cong \mathrm{ext}_D^i(N', \mathcal{F})$ for all left D-modules $\mathcal{F}$ and for $i \geq 1$. Therefore, the results stated in Theorem 5 do not depend on the chosen presentation of M.

The results of Theorem 5 were implemented in the OREMODULES package [15] for the class of Ore algebras of functional operators implemented in the Maple package `Ore_algebra` (e.g., PD, shift, difference, time-delay operators) for which Buchberger's algorithm terminates for any admissible term order and which computes a Gröbner basis [14]. Using the OREMODULES package, we can effectively check whether or not the left D-module $M = D^{1\times p}/(D^{1\times q} R)$ admits torsion elements, or is torsion-free, reflexive or projective. For applications of Theorem 5 to mathematical systems theory and mathematical physics, see [15].

Let us now explain how to compute the torsion left D-submodule $t(M)$ of the $M = D^{1\times p}/(D^{1\times q} R)$. We first consider $Q \in D^{p\times m}$ such that $\ker_D(R.) = Q\, D^m$. We get the exact sequence $0 \longleftarrow N \longleftarrow D^q \xleftarrow{R.} D^p \xleftarrow{Q.} D^m$. Then, 1 of Theorem 5 shows that the defect of exactness of the complex $D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{.Q} D^{1\times m}$ at $D^{1\times p}$ is defined by

$$t(M) \cong \mathrm{ext}_D^1(N, D) \cong \ker_D(.Q)/\mathrm{im}_D(.R) = \big(D^{1\times q'} R'\big)/\big(D^{1\times q} R\big), \tag{11}$$

where $R' \in D^{q'\times p}$ is any matrix such that $\ker_D(.Q) = D^{1\times q'} R'$. Moreover, the standard *third isomorphism theorem* (see, e.g., [50]) then yields:

$$M/t(M) \cong \big[D^{1\times p}/\big(D^{1\times q} R\big)\big]/\big[\big(D^{1\times q'} R'\big)/\big(D^{1\times q} R\big)\big] \cong D^{1\times p}/\big(D^{1\times q'} R'\big). \tag{12}$$

We note that an analogous to Theorem 1 for right D-modules asserts that $\hom_D(M, D) \cong \ker_D(R.)$. Hence, if $\hom_D(M, D) = 0$, then we get the following exact sequence

$$0 \longleftarrow N \longleftarrow D^q \xleftarrow{R.} D^p \longleftarrow 0,$$

and thus the defect of exactness of the complex $D^{1\times q} \xrightarrow{.R} D^{1\times p} \longrightarrow 0$ at $D^{1\times p}$ is $t(M) \cong \mathrm{ext}^1_D(N, D) \cong D^{1\times p}/(D^{1\times q} R) = M$ by (9), i.e., M is a torsion left D-module. Conversely, if M is a torsion left D-module and $f \in \hom_D(M, D)$, then for every $m \in M$, there exists $d \in D \setminus \{0\}$ such that $dm = 0$, which yields $d f(m) = f(dm) = 0$, and thus $f(m) = 0$ since D is a domain and $f(m) \in D$. Thus, $f = 0$, i.e., $\hom_D(M, D) = 0$.

Corollary 1 ([14]) *Let M be a finitely generated left module over a noetherian domain D. Then, M is a torsion left D-module iff* $\hom_D(M, D) = 0$.

The next proposition gives a finite presentation of a factor module.

Proposition 4 ([16]) *Let $R \in D^{q\times p}$ and $R' \in D^{q'\times p}$ satisfy $D^{1\times q} R \subseteq D^{1\times q'} R'$, i.e., are such that $R = R'' R'$ for a certain $R'' \in D^{q\times q'}$. Moreover, let $R'_2 \in D^{r'\times q'}$ be a matrix such that* $\ker_D(.R') = D^{1\times r'} R'_2$, *and let π and π' be respectively the following canonical projections*:

$$\pi : D^{1\times q'} R' \longrightarrow \big(D^{1\times q'} R'\big)/\big(D^{1\times q} R\big),$$

$$\pi' : D^{1\times q'} \longrightarrow D^{1\times q'}/\big(D^{1\times q} R'' + D^{1\times r'} R'_2\big).$$

Then, the left D-homomorphism ι defined by

$$\begin{aligned} D^{1\times q'}/\big(D^{1\times q} R'' + D^{1\times r'} R'_2\big) &\xrightarrow{\iota} \big(D^{1\times q'} R'\big)/\big(D^{1\times q} R\big) \\ \pi'(\lambda) &\longmapsto \pi\big(\lambda R'\big), \end{aligned} \tag{13}$$

is an isomorphism and its inverse ι^{-1} is defined by:

$$\begin{aligned} \big(D^{1\times q'} R'\big)/\big(D^{1\times q} R\big) &\xrightarrow{\iota^{-1}} D^{1\times q'}/\big(D^{1\times q} R'' + D^{1\times r'} R'_2\big) \\ \pi\big(\lambda R'\big) &\longmapsto \pi'(\lambda). \end{aligned}$$

Applying Proposition 4 to $t(M) \cong (D^{1\times q'} R')/(D^{1\times q} R)$, we obtain

$$t(M) \cong D^{1\times q'}/\big(D^{1\times q} R'' + D^{1\times r'} R'_2\big) = D^{1\times q'}/\big(D^{1\times (q+r')} \big(R''^T \quad R'^T_2\big)^T\big), \tag{14}$$

where $R'' \in D^{q\times q'}$ and $R'_2 \in D^{r'\times q'}$ are respectively defined by $R = R'' R'$ and $\ker_D(.R') = D^{1\times r'} R'_2$.

If $t(M) = 0$, then using (11), the complex $D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{.Q} D^{1\times m}$ is exact at $D^{1\times p}$, and thus it defines the beginning of a free resolution of the left D-module $L = D^{1\times m}/(D^{1\times q} Q)$. Up to isomorphism, a finitely generated torsion-free left D-module M can then be embedded into a finitely generated free left D-module since $M = D^{1\times p}/(D^{1\times q} R) \cong \mathrm{im}_D(.Q) \subseteq D^{1\times m}$. If $\mathcal{F}$ is an injective left D-module, then applying the exact functor $\hom_D(\cdot, \mathcal{F})$ to the above beginning of a free resolution of L, we obtain the exact sequence $\mathcal{F}^q \xleftarrow{R.} \mathcal{F}^p \xleftarrow{Q.} \mathcal{F}^m$, i.e., $\ker_{\mathcal{F}}(R.) = Q\mathcal{F}^m$, and thus Q is a parametrization of $\ker_{\mathcal{F}}(R.)$. The computation of parametrizations is implemented in the OREMODULES package [15]. This package allows one to explicitly parametrize underdetermined linear functional systems appearing in mathematical physics and in control theory (see [15]).

The above techniques will be generalized in Sect. 3 to determine the so-called *grade filtration* of a finitely generated left D-module M.

To finish with this section, we shortly recall a few classical results on homomorphisms of finitely presented modules that will be used in the next sections.

Proposition 5 ([16, 17]) *Let $M = D^{1\times p}/(D^{1\times q} R)$ (resp., $M' = D^{1\times p'}/(D^{1\times q'} R')$) be a left D-module finitely presented by $R \in D^{q\times p}$ (resp., by $R' \in D^{q'\times p'}$) and $\pi : D^{1\times p} \longrightarrow M$ (resp., $\pi' \colon D^{1\times p'} \longrightarrow M'$) the canonical projection onto M (resp., M'). Then, $f \in \hom_D(M, M')$ is defined by $f(\pi(\lambda)) = \pi'(\lambda P)$ for all $\lambda \in D^{1\times p}$, where $P \in D^{p\times p'}$ satisfies $RP = QR'$ for a certain $Q \in D^{q\times q'}$. Moreover, we have:*

1. $\ker f = (D^{1\times r} S)/(D^{1\times q} R)$, *where the matrix $S \in D^{r\times p}$ is defined by:*
$$\ker_D\big(.\big(P^T \quad R'^T\big)^T\big) = D^{1\times r}(S \quad -T), \quad T \in D^{r\times q'}.$$
In particular, f is injective iff there exists a matrix $F \in D^{r\times q}$ such that $S = FR$.
2. $\operatorname{im} f = (D^{1\times p} P + D^{1\times q'} R')/(D^{1\times q'} R') \cong \operatorname{coim} f = D^{1\times p}/(D^{1\times r} S)$.
3. $\operatorname{coker} f = D^{1\times p'}/(D^{1\times p} P + D^{1\times q'} R')$. *Thus, f is surjective iff $(P^T \quad R'^T)^T$ admits a left inverse, i.e., $X \in D^{p'\times p}$ and $Y \in D^{p'\times q'}$ exist such that $XP + YR' = I_{p'}$.*
4. *f is an isomorphism, i.e., $M \cong M'$, iff there exists $F \in D^{r\times q}$ such that $S = FR$ and the matrix $(P^T \; R'^T)^T$ admits a left inverse. If $X \in D^{p'\times p}$ is defined as in 3, then $f^{-1} \in \hom_D(M', M)$ is defined by $f^{-1}(\pi'(\lambda')) = \pi(\lambda' X)$ for all $\lambda' \in D^{1\times p'}$.*

2.2 Baer's Extensions

In this section, we give another interpretation of the abelian group $\operatorname{ext}^1_D(M, N)$ which will be used in Sect. 4. To do that, let us introduce a few more definitions.

Definition 5 ([50])

1. Let M and N be two left D-modules. An *extension of M by N* is a short exact sequence of left D-modules of the form:
$$e : 0 \longrightarrow N \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0. \tag{15}$$
2. Two extensions $e_i : 0 \longrightarrow N \xrightarrow{\alpha_i} E_i \xrightarrow{\beta_i} M \longrightarrow 0$ of M by N for $i = 1, 2$ are said to be *equivalent*, which is denoted by $e_1 \sim e_2$, if there exists a left D-isomorphism $\phi \colon E_1 \longrightarrow E_2$ such that $\alpha_2 = \phi \circ \alpha_1$ and $\beta_1 = \beta_2 \circ \phi$, or equivalently, such that the following commutative exact diagram holds:
$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\alpha_1"] \arrow[d, equal] & E_1 \arrow[r, "\beta_1"] \arrow[d, "\phi"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r, "\alpha_2"] & E_2 \arrow[r, "\beta_2"] & M \arrow[r] & 0. \end{tikzcd}$$
3. Let $[e]$ be the equivalence class of the extension e for the above equivalence relation $\sim$. The set of all equivalence classes of extensions of M by N is denoted by $\mathrm{e}_D(M, N)$.

The next theorem, which can be traced back to Baer's work, plays an important role in homological algebra. In particular, it explains the terminology *extension* used for $\operatorname{ext}^1_D(M, N)$.

Theorem 6 ([50]) *Let M and N be two left D-modules. Then, we have*:

$$\mathrm{ext}_D^1(M,N) \cong \mathrm{e}_D(M,N).$$

The next theorem gives an explicit description of the isomorphism stated in Theorem 6 in the case where M and N are two finitely presented left D-modules.

Theorem 7 ([46, 47]) *Let $M = D^{1\times p}/(D^{1\times q}R)$ and $N = D^{1\times s}/(D^{1\times t}S)$ be two finitely presented left D-modules, $\pi: D^{1\times p} \longrightarrow M$ (resp., $\delta: D^{1\times s} \longrightarrow N$) the canonical projection onto M (resp., N), $R_2 \in D^{r\times q}$ such that $\ker_D(.R) = D^{1\times r}R_2$, and*:

$$\Omega = \big\{X \in D^{q\times s} \mid \exists Y \in D^{r\times t}:\ R_2 X = Y S\big\}.$$

Then, every equivalence class of extensions of M by N is defined by the following short exact sequence

$$e: 0 \longrightarrow N \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0, \tag{16}$$

where the left D-module $E = D^{1\times(p+s)}/(D^{1\times(q+t)}L)$ is finitely presented by

$$L = \begin{pmatrix} R & -A \\ 0 & S \end{pmatrix} \in D^{(q+t)\times(p+s)},$$

for a certain $A \in \Omega$, $\alpha \in \hom_D(N,E)$ and $\beta \in \hom_D(E,M)$ are defined by

$$\begin{array}{rcl} N & \xrightarrow{\alpha} & E \\ \delta(\mu) & \longmapsto & \varrho\big(\mu(0 \quad I_s)\big), \end{array} \qquad \begin{array}{rcl} E & \xrightarrow{\beta} & M \\ \varrho(\nu) & \longmapsto & \pi\big(\nu(I_p \quad 0)^T\big), \end{array}$$

and $\varrho: D^{1\times(p+s)} \longrightarrow E$ is the canonical projection onto E. Finally, the equivalence class $[e]$ depends only on the residue class $\epsilon(A)$ of A in the following abelian group:

$$\Omega/\big(R D^{p\times s} + D^{q\times t} S\big) \cong \mathrm{ext}_D^1(M,N). \tag{17}$$

Remark 4 The extension e of Theorem 7 is *trivial*, i.e., $E \cong N \oplus M$, iff there exist $U \in D^{p\times s}$ and $V \in D^{q\times t}$ such that $A = RU + VS$, i.e., iff $\epsilon(A) = 0$. If D is a commutative polynomial ring over a *computable field* k, then using *Kronecker product* and Gröbner/Janet bases, we can check whether or not this identity holds and if so, compute solutions U and V. See, e.g., [47, 55].

The next corollary shows how to determine $\epsilon(A)$ for a given extension e.

Corollary 2 ([47]) *With the notations of Theorem 7, let*

$$e': 0 \longrightarrow N \xrightarrow{u} F \xrightarrow{v} M \longrightarrow 0$$

be an extension of the finitely presented left D-module $M = D^{1\times p}/(D^{1\times q}R)$ by the finitely presented left D-module $N = D^{1\times s}/(D^{1\times t}S)$, $\{f_j\}_{j=1,\ldots,p}$ (resp., $\{e_i\}_{i=1,\ldots,q}$) the standard basis of $D^{1\times p}$ (resp., $D^{1\times q}$), $y_j = \pi(f_j)$, and $z_j \in F$ a pre-image of y_j under v for all $j = 1,\ldots,p$. Then, we have $\sum_{j=1}^{p} R_{ij} z_j \in \operatorname{im} u$ for all $i = 1,\ldots,q$, and, since u is injective, there exists a unique $n_i \in N$ satisfying $u(n_i) = \sum_{j=1}^{p} R_{ij} z_j$. If we consider a pre-image

$a_i \in D^{1\times s}$ of n_i under δ, i.e., $n_i = \delta(a_i)$ for all $i = 1, \dots, q$, then the extensions e' and (16) are equivalent, where $E = D^{1\times(p+s)}/(D^{1\times(q+t)} L)$ and:

$$L = \begin{pmatrix} R & -A \\ 0 & S \end{pmatrix} \in D^{(q+t)\times(p+s)}, \quad A = \begin{pmatrix} a_1 \\ \vdots \\ a_q \end{pmatrix} \in D^{q\times s}.$$

Equivalently, the following commutative exact diagram holds

$$\begin{tikzcd} & D^{1\times q} \arrow[r, ".R"] \arrow[d, "\phi"] & D^{1\times p} \arrow[r, "\pi"] \arrow[d, "\psi"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r, "u"] & F \arrow[r, "v"] & M \arrow[r] & 0, \end{tikzcd}$$

where ψ and ϕ are respectively defined by:

$$\begin{aligned} \psi \colon D^{1\times p} &\longrightarrow F & \qquad \phi \colon D^{1\times q} &\longrightarrow N \\ f_j &\longmapsto z_j, \ j = 1, \dots, p, & e_i &\longmapsto n_i = \delta(a_i), \ i = 1, \dots, q. \end{aligned}$$

Theorem 7 and Corollary 2 will be abundantly used in Sect. 4. For more results on Baer's extensions, examples, and applications to mathematical systems theory, see [4, 46, 47, 50, 55].

The next proposition shows how the presentation of the left D-module E defining the extension of M by N (see Theorem 7) changes with the presentations of M and N.

Proposition 6 *With the notations of Theorem 7, let*

$$M = D^{1\times p}/\big(D^{1\times q} R\big), \qquad N = D^{1\times s}/\big(D^{1\times t} S\big), \qquad E = D^{1\times(p+s)}/\big(D^{1\times(q+t)} L\big)$$

be three left D-modules defining the extension e of M by N (16). Moreover, let f and g be two left D-isomorphisms defined by

$$\begin{aligned} f \colon M = D^{1\times p}/\big(D^{1\times q} R\big) &\longrightarrow M' = D^{1\times p'}/\big(D^{1\times q'} R'\big) \\ \pi(\lambda) &\longmapsto \pi'(\lambda P), \\ g \colon N = D^{1\times s}/\big(D^{1\times t} S\big) &\longrightarrow N' = D^{1\times s'}/\big(D^{1\times t'} S'\big) \\ \delta(\mu) &\longmapsto \delta'(\mu X), \end{aligned}$$

where π' (resp., δ') is the canonical projection onto M' (resp., N'), i.e., $P \in D^{p\times p'}$, $X \in D^{s\times s'}$ are such that there exist $Q \in D^{q\times q'}$, $P' \in D^{p'\times p}$, $Q' \in D^{q'\times q}$, $Y \in D^{t\times t'}$, $X' \in D^{s'\times s}$, $Y' \in D^{t'\times t}$, $T \in D^{p\times q}$, $T' \in D^{p'\times q'}$, $Z \in D^{s\times t}$, and $Z' \in D^{s'\times t'}$ satisfying the following identities:

$$\begin{cases} RP = QR', \\ R'P' = Q'R, \\ I_p = PP' + TR, \\ I_{p'} = P'P + T'R', \end{cases} \qquad \begin{cases} SX = YS', \\ S'X' = Y'S, \\ I_s = XX' + ZS, \\ I_{s'} = X'X + Z'S'. \end{cases} \tag{18}$$

Then, the extension e yields the following extension of M' by N'

$$e' \colon 0 \longrightarrow N' \xrightarrow{\alpha \circ g^{-1}} E \xrightarrow{f \circ \beta} M' \longrightarrow 0, \tag{19}$$

which implies that the left D-module E admits the following presentation

$$L' = \begin{pmatrix} R' & -Q' A X \\ 0 & S' \end{pmatrix} \in D^{(q'+t') \times (p'+s')},$$

i.e., $E \cong E' = D^{1 \times (p'+s')}/(D^{1 \times (q'+t')} L')$, where this left D-isomorphism is defined by

$$\begin{aligned} \varphi \colon E &\longrightarrow E' & \qquad \varphi^{-1} \colon E' &\longrightarrow E \\ \varrho(\nu) &\longmapsto \varrho'(\nu U), & \varrho'\big(\nu'\big) &\longmapsto \varrho\big(\nu' U'\big), \end{aligned}$$

$$U = \begin{pmatrix} P & T A X \\ 0 & X \end{pmatrix} \in D^{(p+s) \times (p'+s')}, \qquad U' = \begin{pmatrix} P' & 0 \\ 0 & X' \end{pmatrix} \in D^{(p'+s') \times (p+s)},$$

and $\varrho' \colon D^{1 \times (p'+s')} \longrightarrow E'$ is the canonical projection onto E'.

Proof With the notations (18), 4 of Proposition 5 yields:

$$\begin{aligned} f^{-1} \colon M' = D^{1 \times p'}/\big(D^{1 \times q'} R'\big) &\longrightarrow M = D^{1 \times p}/\big(D^{1 \times q} R\big) \\ \pi'\big(\lambda'\big) &\longmapsto \pi\big(\lambda' P'\big), \\ g^{-1} \colon N' = D^{1 \times s'}/\big(D^{1 \times t'} S'\big) &\longrightarrow N = D^{1 \times s}/\big(D^{1 \times t} S\big) \\ \delta'\big(\mu'\big) &\longmapsto \delta\big(\mu' X'\big). \end{aligned}$$

Using (18), we get $(I_q - Q Q' - R T) R = R - Q Q' R - R T R = R - R P P' - R T R = 0$. Thus, if $\ker_D(.R) = D^{1 \times r} R_2$, then there exists $T_2 \in D^{q \times r}$ such that:

$$I_q = Q Q' + R T + T_2 R_2. \tag{20}$$

Now, (16) yields (19). Moreover, since $A \in \Omega$ (see Theorem 7), there exists $B \in D^{r \times t}$ such that $R_2 A = B S$. Hence, using this identity, (18), and (20), we get

$$\begin{aligned} L U &= \begin{pmatrix} R & -A \\ 0 & S \end{pmatrix} \begin{pmatrix} P & T A X \\ 0 & X \end{pmatrix} = \begin{pmatrix} R P & (R T - I_q) A X \\ 0 & S X \end{pmatrix} \\ &= \begin{pmatrix} Q R' & -(Q Q' A + T_2 (R_2 A)) X \\ 0 & Y S' \end{pmatrix} = \begin{pmatrix} Q R' & -(Q Q' A + T_2 B S) X \\ 0 & Y S' \end{pmatrix} \\ &= \begin{pmatrix} Q R' & -Q Q' A X - T_2 B Y S' \\ 0 & Y S' \end{pmatrix} = \begin{pmatrix} Q & -T_2 B Y \\ 0 & Y \end{pmatrix} \begin{pmatrix} R' & -Q' A X \\ 0 & S' \end{pmatrix} = V L', \end{aligned}$$

where V is the first matrix appearing in the last but one equality, which shows that φ is well-defined by Proposition 5. Similarly, using (18), we get

$$L'U' = \begin{pmatrix} R' & -Q'AX \\ 0 & S' \end{pmatrix} \begin{pmatrix} P' & 0 \\ 0 & X' \end{pmatrix} = \begin{pmatrix} R'P' & -Q'AXX' \\ 0 & S'X' \end{pmatrix}$$
$$= \begin{pmatrix} Q'R & -Q'A(I_s - ZS) \\ 0 & Y'S \end{pmatrix} = \begin{pmatrix} Q' & Q'AZ \\ 0 & Y' \end{pmatrix} \begin{pmatrix} R & -A \\ 0 & S \end{pmatrix} = V'L,$$

where V' is the first matrix appearing in the last but one equality, which yields $\phi \in \hom_D(E', E)$ defined by $\phi(\varrho'(\nu')) = \varrho(\nu'U')$ for all $\nu' \in D^{1\times(p'+s')}$ by Proposition 5. Using (18), we also have

$$UU' = \begin{pmatrix} P & TAX \\ 0 & X \end{pmatrix} \begin{pmatrix} P' & 0 \\ 0 & X' \end{pmatrix} = \begin{pmatrix} PP' & TAXX' \\ 0 & XX' \end{pmatrix}$$
$$= \begin{pmatrix} I_p - TR & TA(I_s - ZS) \\ 0 & I_s - ZS \end{pmatrix} = I_{p+s} - \begin{pmatrix} T & TAZ \\ 0 & Z \end{pmatrix} \begin{pmatrix} R & -A \\ 0 & S \end{pmatrix},$$

which shows that $\phi \circ \varphi = \mathrm{id}_E$. Moreover, using (18), we obtain

$$\big(P'T - T'Q'\big)R = P'TR - T'Q'R = P'TR - T'R'P'$$
$$= P'\big(I_p - PP'\big) - \big(I_{p'} - P'P\big)P' = 0,$$

which shows that there exists $W \in D^{p'\times r}$ such that $P'T - T'Q' = WR_2$. Using $R_2A = BS$ and $SX = YS'$ (see (18)), $(P'T - T'Q')AX = W(R_2A)X = WBSX = WBYS'$, and thus $P'TAX = T'Q'AX + WBYS'$, and then

$$UU' = \begin{pmatrix} P' & 0 \\ 0 & X' \end{pmatrix} \begin{pmatrix} P & TAX \\ 0 & X \end{pmatrix} = \begin{pmatrix} P'P & P'TAX \\ 0 & X'X \end{pmatrix}$$
$$= \begin{pmatrix} I_{p'} - T'R' & P'TAX \\ 0 & I_{s'} - Z'S' \end{pmatrix} = I_{p'+s'} - \begin{pmatrix} T' & -WBY \\ 0 & Z' \end{pmatrix} \begin{pmatrix} R' & -Q'AX \\ 0 & S' \end{pmatrix},$$

which shows that $\varphi \circ \phi = \mathrm{id}_{E'}$, and proves that φ is a left D-isomorphism, $\phi = \varphi^{-1}$. □

2.3 Pure Modules and Grade Filtration

Let us introduce the concept of *grade number*.

Definition 6 ([9, 10]) The *grade number* of a nonzero finitely generated left D-module M is defined by $j_D(M) = \inf\{i \in \mathbb{N} \mid \mathrm{ext}_D^i(M, D) \neq 0\}$. If $M = 0$, then we set $j_D(M) = \infty$. A similar definition holds for right D-modules.

If $M \neq 0$, then $j_D(M)$ is then the smallest nonnegative integer such that:

$$\mathrm{ext}_D^{j_D(M)}(M, D) \neq 0.$$

Remark 5 If $\mathrm{gld}(D)$ is finite and M is a nonzero left D-module, then using Proposition 3, $\mathrm{ext}^i_D(M, D) = 0$ for all $i > \mathrm{gld}(D)$, which yield $0 \leq j_D(M) \leq \mathrm{gld}(D)$.

Let us introduce the concept of *pure module* that will play an important role.

Definition 7 ([10]) A finitely generated left D-module M is said to be *pure* or $j_D(M)$*-pure* if $j_D(N) = j_D(M)$ for all nonzero left D-submodules N of M.

Remark 6 If M is a pure left D-module, then the cyclic left D-module Dm generated by $m \in M \setminus \{0\}$ satisfies $j_D(Dm) = j_D(M)$. More generally, if N is a left D-submodule of a $j_D(M)$-pure left D-module M, then N is also $j_D(M)$-pure since every left D-submodule of N is a left D-submodule of M and $j_D(N) = j_D(M)$.

In what follows, we shall mainly focus on the class of *Auslander regular rings*.

Definition 8 ([10]) We have:

1. A ring D is called an *regular ring* if D is a noetherian ring of finite global dimension $\mathrm{gld}(D)$.
2. A ring D is called an *Auslander regular ring* if D is a regular ring which satisfies the *Auslander condition*, namely, for every $i \in \mathbb{N}$, for every finitely generated left (resp., right) D-module M, and for every right (resp., left) D-submodule N of $\mathrm{ext}^i_D(M, D)$, then $j_D(N) \geq i$.

Remark 7 If D is an Auslander regular ring, then for a nonzero finitely generated left D-module M, taking $N = \mathrm{ext}^i_D(M, D)$ in Definition 8, $j_D(\mathrm{ext}^i_D(M, D)) \geq i$, i.e., $\mathrm{ext}^j_D(\mathrm{ext}^i_D(M, D), D) = 0$ for $0 \leq j < i$. Considering $\mathrm{ext}^i_D(M, D)$ instead of M in Definition 8, then we get that $N \subseteq \mathrm{ext}^i_D(\mathrm{ext}^i_D(M, D), D)$ yields $j_D(N) \geq i$.

Theorem 8 ([10]) *Let D be an Auslander regular ring and M a nonzero finitely generated left D-module. Then, we have*:

1. *M is pure iff M is a left D-submodule of* $\mathrm{ext}^{j_D(M)}_D(\mathrm{ext}^{j_D(M)}_D(M, D), D)$.
2. *M is pure iff* $\mathrm{ext}^i_D(\mathrm{ext}^i_D(M, D), D) = 0$ *for* $i \neq j_D(M)$.
3. *If* $\mathrm{ext}^i_D(\mathrm{ext}^i_D(M, D), D) \neq 0$, *then* $\mathrm{ext}^i_D(\mathrm{ext}^i_D(M, D), D)$ *is a pure left D-module with grade number i, i.e.,* $j_D(\mathrm{ext}^i_D(\mathrm{ext}^i_D(M, D), D)) = i$.

Example 4 By 1 of Theorem 8, M is 0-pure iff M is a left D-submodule of $\hom_D(\hom_D(M, D), D)$. If D is a domain, then using 3 of Theorem 5, we deduce that M is 0-pure iff M is a torsion-free left D-module. In particular, the left D-module $M/t(M)$ is either zero or 0-pure.

Let us now show that pure modules naturally appear in the study of a finitely generated left module M over an Auslander regular ring D. Let us consider:

$$t_i(M) = \big\{m \in M \mid j_D(Dm) \geq i\big\}, \quad i = 0, \ldots, n = \mathrm{gld}(D), \qquad t_{n+1}(M) = 0. \tag{21}$$

To prove that the $t_i(M)$'s are left D-modules, we need the following result.

Proposition 7 ([10]) *If* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *is a short exact sequence of left modules over an Auslander regular ring* D, *then*:

$$j_D(M) = \inf\{j_D(M'), j_D(M'')\}.$$

Remark 8 If $\mathrm{ext}_D^i(M', D) = 0$ and $\mathrm{ext}_D^i(M'', D) = 0$ for $0 \leq i \leq j$, then Theorem 3 yields $\mathrm{ext}_D^i(M, D) = 0$ for $0 \leq i \leq j$, and thus $j_D(M) \geq \inf\{j_D(M'), j_D(M'')\}$. Thus, the Auslander regularity condition is only used to prove the other inequality.

Let us now explain why $t_i(M)$ is a left D-module. Firstly, if $m \in t_i(M)$ and $d \in D$, then $dm \in Dm$, i.e., $D(dm) \subseteq Dm$. Then, applying Proposition 7 to the short exact sequence $0 \longrightarrow D(dm) \longrightarrow Dm \longrightarrow Dm/D(dm) \longrightarrow 0$, we get $j_D(D(dm)) \geq j_D(Dm) \geq i$, i.e., $dm \in t_i(M)$. Secondly, let m_1 and $m_2 \in t_i(M)$. Then, we have $m_1 + m_2 \in Dm_1 + Dm_2$. Since $D(m_1 + m_2) \subseteq Dm_1 + Dm_2$, similarly as previously, Proposition 7 yields $j_D(D(m_1 + m_2)) \geq j_D(Dm_1 + Dm_2)$. Now, applying again Proposition 7 to the following two standard short exact sequences

$$0 \longrightarrow Dm_1 \cap Dm_2 \longrightarrow Dm_1 \oplus Dm_2 \longrightarrow Dm_1 + Dm_2 \longrightarrow 0,$$
$$0 \longrightarrow Dm_1 \longrightarrow Dm_1 \oplus Dm_2 \longrightarrow Dm_2 \longrightarrow 0,$$

(see, e.g., [50]), we then obtain the following inequality and equality

$$\begin{cases} j_D(Dm_1 + Dm_2) \geq j_D(Dm_1 \oplus Dm_2), \\ j_D(Dm_1 \oplus Dm_2) = \inf\{j_D(Dm_1), j_D(Dm_2)\} = i, \end{cases}$$

which yields $j_D(D(m_1 + m_2)) \geq i$, i.e., $m_1 + m_2 \in t_i(M)$.

If M' is a left D-submodule of M such that $j_D(M') \geq i$ and if $m' \in M' \setminus \{0\}$, then applying Proposition 7 to the short exact sequence

$$0 \longrightarrow Dm' \longrightarrow M' \longrightarrow M'/(Dm') \longrightarrow 0,$$

we get $j_D(Dm') \geq j_D(M') \geq i$, i.e., $m' \in t_i(M)$, and thus $M' \subseteq t_i(M)$, which proves that $t_i(M)$ is the largest left D-submodule of M (D is a noetherian ring) which satisfies $j_D(t_i(M)) \geq i$.

Note that $t_0(M) = \{m \in M \mid j_D(Dm) \geq 0\} = M$. Thus, the following descending filtration of M holds:

$$0 = t_{n+1}(M) \subseteq t_n(M) \subseteq t_{n-1}(M) \subseteq \cdots \subseteq t_1(M) \subseteq t_0(M) = M. \tag{22}$$

If D is a domain, then using Corollary 1, we get $t_1(M) = t(M)$ since:

$$m \in t(M) \quad \Leftrightarrow \quad \mathrm{ext}_D^0(Dm, D) = 0 \quad \Leftrightarrow \quad j_D(Dm) \geq 1 \quad \Leftrightarrow \quad m \in t_1(M).$$

It can been seen that a module M is i-pure iff $t_i(M) = M$ and $t_{i+1}(M) = 0$.

Lemma 1 *The left* D*-module* $t_i(M)/t_{i+1}(M)$ *is either zero or is* i*-pure.*

Proof Let us suppose that $P = t_i(M)/t_{i+1}(M)$ is nonzero. Applying Proposition 7 to the short exact sequence $0 \longrightarrow t_{i+1}(M) \longrightarrow t_i(M) \longrightarrow P \longrightarrow 0$, we get $j_D(P) \geq j_D(t_i(M)) \geq i$, and thus $P \subseteq t_i(P) \subseteq P$, i.e., $t_i(P) = P$. Let us now check that $t_{i+1}(P) = 0$, which will prove the result. Composing the two canonical projections $\alpha\colon t_i(M) \longrightarrow P = t_i(M)/t_{i+1}(M)$ and $\beta\colon P \longrightarrow P/t_{i+1}(P)$, we get the following commutative exact diagram:

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
& & t_{i+1}(M) \arrow[d] & \ker(\beta \circ \alpha) \arrow[d] & \\
& 0 \arrow[r] & t_i(M) \arrow[r, equal] \arrow[d, "\alpha"] & t_i(M) \arrow[r] \arrow[d, "\beta \circ \alpha"] & 0 \\
0 \arrow[r] & t_{i+1}(P) \arrow[r] & P \arrow[r, "\beta"] \arrow[d] & P/t_{i+1}(P) \arrow[r] \arrow[d] & 0. \\
& & 0 & 0 &
\end{tikzcd}$$

The *snake lemma* (see, e.g., [50]) then yields the following short exact sequence:

$$0 \longrightarrow t_{i+1}(M) \longrightarrow \ker(\beta \circ \alpha) \longrightarrow t_{i+1}(P) \longrightarrow 0.$$

Using Proposition 7, $j_D(\ker(\beta \circ \alpha)) = \inf\{j_D(t_{i+1}(M)), j_D(t_{i+1}(P))\} \geq i + 1$. Since $t_{i+1}(M) \subseteq \ker(\beta \circ \alpha) \subseteq t_i(M) \subseteq M$, we obtain $\ker(\beta \circ \alpha) = t_{i+1}(M)$, and thus $t_{i+1}(P) = 0$ by the above short exact sequence. □

According to Lemma 1, (22) is called the *grade (purity) filtration* of M (see [10]).

Theorem 9 ([9–11]) *Let D be a ring equipped with a filtration $\{D_r\}_{r \geq -1}$, where $D_{-1} = 0$, such that the associated graded ring $\operatorname{gr}(D) = \bigoplus_{r \in \mathbb{N}} D_r/D_{r-1}$ satisfies the following three properties*:

1. $\operatorname{gr}(D)$ *is a commutative ring.*
2. $\operatorname{gr}(D)$ *is a noetherian ring.*
3. $\operatorname{gr}(D)$ *is a* regular ring of pure dimension $d \in \mathbb{N}$, *namely,* $\operatorname{gld}(\operatorname{gr}(D)_{\mathfrak{m}})$ *is equal to d for all localizations* $\operatorname{gr}(D)_{\mathfrak{m}}$ *of* $\operatorname{gr}(D)$ *at maximal ideals* $\mathfrak{m}$ *of* $\operatorname{gr}(D)$.

Then, the following results hold:

1. $\operatorname{gld}(\operatorname{gr}(D)_{\mathfrak{m}})$ *is equal to the* Krull dimension $\operatorname{Kdim}(\operatorname{gr}(D)_{\mathfrak{m}})$ *of the noetherian local ring* $\operatorname{gr}(D)_{\mathfrak{m}}$, *which also equal to the dimension* $\dim_{\operatorname{gr}(D)_{\mathfrak{m}}/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$ *of* $\mathfrak{m}/\mathfrak{m}^2$ *as a* $\operatorname{gr}(D)_{\mathfrak{m}}/\mathfrak{m}$*-vector space. This common value d for all maximal ideals $\mathfrak{m}$ of* $\operatorname{gr}(D)$ *is denoted by* $\dim(D)$.
2. *If $M \neq 0$ is a left D-module M, then the* characteristic ideal $J(M)$ *of* $\operatorname{gr}(D)$, *defined by*

$$J(M) = \sqrt{\operatorname{ann}_{\operatorname{gr}(D)}\big(\operatorname{gr}(M)\big)} = \big\{a \in \operatorname{gr}(D) \mid \exists\, k \in \mathbb{N}\colon a^k \operatorname{gr}(M) = 0\big\},$$

does not depend on any good filtration of M (e.g., if $M = \sum_{i=j}^{p} D y_j$ then $\{M_r\}_{r \in \mathbb{N}}$ defined by $M_r = \sum_{j=1}^{p} D_r y_j$ for all $r \in \mathbb{N}$ is a good filtration of M and we have $\operatorname{gr}(M) = \sum_{j=1}^{p} \operatorname{gr}(D) y_j$).

3. *If the* dimension *of M is defined by* $\dim_D(M) = \mathrm{Kdim}(\mathrm{gr}(D)/J(M))$ *and the* codimension *of M by* $\mathrm{codim}_D(M) = \dim(D) - \dim_D(M)$, *then we have*:

$$j_D(M) = \mathrm{codim}_D(M). \tag{23}$$

A ring D satisfying (23) for all modules M is called a *Cohen-Macaulay ring*. A natural substitute for $\dim_D(\cdot)$ for more general k-algebras is the so-called *Gel'fand-Kirillov dimension* GKdim (see, e.g., [34]).

If D satisfies the hypotheses of Theorem 9, then $\dim(D) = \mathrm{gld}(\mathrm{gr}(D))$ since $\mathrm{gld}(\mathrm{gr}(D)) = \sup_{\mathfrak{m}\in \mathrm{Max}(\mathrm{gr}(D))} \mathrm{gld}(\mathrm{gr}(D)_{\mathfrak{m}})$, where $\mathrm{Max}(\mathrm{gr}(D))$ is the set of the maximal ideals of $\mathrm{gr}(D)$ (see, e.g., [50]).

Example 5 If k is a field of characteristic 0 and A a *differential field* (namely, a field with a differential ring structure) of characteristic 0 (e.g., k, $k(x_1, \dots, x_n)$), or $k[x_1, \dots, x_n]$, $k[\![x_1, \dots, x_n]\!]$, $k'\{x_1, \dots, x_n\}$ where $k' = \mathbb{R}$ or $\mathbb{C}$, then the ring $D = A\langle \partial_1, \dots, \partial_n\rangle$ of PD operators with coefficients in A is Auslander regular and Cohen-Macaulay (see [9–11]). In particular, if $\{D_i\}_{i\geq -1}$ is the *order filtration* of D, namely D_i is the A-submodule of D formed by the PD operators of order less than or equal to i, and χ_i is the class of ∂_i in D_1/D_0, then $\mathrm{gr}(D) = A[\chi_1, \dots, \chi_n]$. Thus, if A is a differential field of characteristic 0 (e.g., k, $k(x_1, \dots, x_n)$), then $\dim(D) = n$, and if $A = k[x_1, \dots, x_n]$, $k[\![x_1, \dots, x_n]\!]$, or $k'\{x_1, \dots, x_n\}$, then $\dim(A) = n$ and $\dim(D) = 2n$.

Corollary 3 ([9–11]) *Let D be an Auslander regular ring and a Cohen-Macaulay ring*, *and M a nonzero finitely generated left D-module. Then*, *we have*:

1. $\dim_D(\mathrm{ext}^i_D(M, D)) \leq \dim(D) - i$.
2. $\dim_D(\mathrm{ext}^{j_D(M)}_D(M, D)) = \dim(D) - j_D(M)$.
3. *If* $\mathrm{ext}^i_D(\mathrm{ext}^i_D(M, D), D) \neq 0$, *then* $\dim_D(\mathrm{ext}^i_D(\mathrm{ext}^i_D(M, D), D)) = \dim(D) - i$.
4. *If M is an i-pure left D-module*, *then* $\dim_D(M) = \dim(D) - i$.

If D is an Auslander regular ring with $\mathrm{gld}(D) = n$, then a nonzero finitely generated left D-module M is called *holonomic* (resp., *subholonomic*) if $j_D(M) = n$ (resp., $j_D(M) \geq n - 1$). It is convenient to assume that $M = 0$ is also holonomic so that M is holonomic if $j_D(M) \geq n$. If D is also a Cohen-Macaulay ring, then $M \neq 0$ is holonomic (resp., subholonomic) iff $\dim_D(M) = \dim(D) - n$ (resp., $\dim_D(M) \leq \dim(D) - n + 1$). In particular, if D is one of the rings of PD operators defined in Example 5, then we find again the classical definitions of holonomic and subholonomic modules over a ring of PD operators (see, e.g., [9–11, 32]).

Let us state a few remarks on holonomic modules. If

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence and $j_D(M') = j_D(M'') = i$, then $j_D(M) = i$ by Proposition 7. In particular, if M' and M'' are two holonomic left D-modules, so is M. The converse result also holds since Proposition 7 and $j_D(M) \geq n$ yield $j_D(M') \geq n$ and $j_D(M'') \geq n$. Thus, M is a holonomic left D-module iff M' and M'' are two holonomic left D-modules. Finally, a *simple* module (i.e., a module containing no nontrivial submodules) left $A_n(k)$-module is not necessarily holonomic as shown in [53]. But a simple module over an Auslander regular ring D is pure.

3 Grade Filtration

The goal of the section is to show how the grade filtration (22) of a finitely generated left module M over an Auslander regular ring D can be explicitly computed. Since we are motivated by developing an effective algorithm which can be implemented in computer algebra systems, in what follows, we shall only use free resolutions of modules and not the more general projective resolutions. This extension can easily be done and it is left to the interested reader.

Let D be a regular ring, i.e., a noetherian domain D with a finite global dimension $\mathrm{gld}(D) = n$, and M a finitely generated left D-module. Let us consider a free resolution of M:

$$0 \longleftarrow M \xleftarrow{\pi} D^{1\times p_0} \xleftarrow{.R_1} D^{1\times p_1} \xleftarrow{.R_2} \cdots \xleftarrow{.R_{i-1}} D^{1\times p_{i-1}} \xleftarrow{.R_i} D^{1\times p_i} \xleftarrow{.R_{i+1}} \cdots \tag{24}$$

Using (7) and Proposition 3, the defects of exactness of the following complex

$$0 \longrightarrow D^{p_0} \xrightarrow{R_1.} D^{p_1} \xrightarrow{R_2.} \cdots \xrightarrow{R_{i-1}.} D^{p_{i-1}} \xrightarrow{R_i.} D^{p_i} \xrightarrow{R_{i+1}.} D^{p_{i+1}} \xrightarrow{R_{i+2}.} \cdots \tag{25}$$

are the right D-modules defined by:

$$\begin{cases} \mathrm{ext}_D^0(M,D) \cong \ker_D(R_1.), \\ \mathrm{ext}_D^i(M,D) \cong \ker_D(R_{i+1}.)/(R_i D^{p_{i-1}}), \quad 1 \le i \le n, \\ \mathrm{ext}_D^i(M,D) = 0, \quad i > n. \end{cases} \tag{26}$$

To characterize the $\mathrm{ext}_D^i(M,D)$'s for all $0 \le i \le n$, we need to study $\ker_D(R_{i+1}.)$. For $1 \le k \le n+1$, considering the beginning of a free resolution of the finitely generated right D-module $\ker_D(R_k.)$, we obtain the following long exact sequence of right D-modules

$$D^{p_{(-1)k}} \xrightarrow{R_{0k}.} D^{p_{0k}} \xrightarrow{R_{1k}.} D^{p_{1k}} \xrightarrow{R_{2k}.} \cdots \xrightarrow{R_{(k-1)k}.} D^{p_{(k-1)k}} \xrightarrow{R_{kk}.} D^{p_{kk}} \xrightarrow{\kappa_{kk}} N_{kk} \longrightarrow 0, \tag{27}$$

where for k from 1 to $n+1$, we have set $R_{kk} = R_k$, $p_{kk} = p_k$, $p_{(k-1)k} = p_{k-1} = p_{(k-1)(k-1)}$ and:

$$N_{kk} = \mathrm{coker}_D(R_{kk}.) = D^{p_{kk}} / \big(R_{kk} D^{p_{(k-1)k}}\big).$$

Let us explain why this choice of the notations is natural. If we consider a squared-line paper sheet where each square has coordinates $(j,k) \in \mathbb{N}^2$, and if the long exact sequence (27) is placed at k^{th} level with $D^{p_{jk}}$ at position (j,k), then the horizontal arrow of the right D-homomorphism $R_{jk}.$ arrives at $D^{p_{jk}}$ with $j \le k$ (a good mnemonic device). For instance, the first three horizontal exact sequences can be arranged as follows:

$$\begin{array}{l} D^{p_{-13}} \xrightarrow{R_{03}.} D^{p_{03}} \xrightarrow{R_{13}.} D^{p_{13}} \xrightarrow{R_{23}.} D^{p_{23}} \xrightarrow{R_{33}.} D^{p_{33}} \xrightarrow{\kappa_{33}} N_{33} \longrightarrow 0, \\ D^{p_{-12}} \xrightarrow{R_{02}.} D^{p_{02}} \xrightarrow{R_{12}.} D^{p_{12}} \xrightarrow{R_{22}.} D^{p_{22}} \xrightarrow{\kappa_{22}} N_{22} \longrightarrow 0, \\ D^{p_{-11}} \xrightarrow{R_{01}.} D^{p_{01}} \xrightarrow{R_{11}.} D^{p_{11}} \xrightarrow{\kappa_{11}} N_{11} \longrightarrow 0. \end{array}$$

Since (25) is a complex, $R_{kk} R_{(k-1)(k-1)} = R_k R_{k-1} = 0$ for all $k = 2, \dots, n+1$, and thus $R_{(k-1)(k-1)} D^{p_{(k-2)(k-1)}} \subseteq \ker_D(R_{kk}.) = R_{(k-1)k} D^{p_{(k-2)k}}$, which shows the existence of a matrix $F_{(k-2)k} \in D^{p_{(k-2)k} \times p_{(k-2)(k-1)}}$ such that:

$$\forall k = 2, \dots, n+1, \quad R_{(k-1)(k-1)} = R_{(k-1)k} F_{(k-2)k}. \tag{28}$$

Using (28), $R_{(k-1)k}\,F_{(k-2)k}\,R_{(k-2)(k-1)} = R_{(k-1)(k-1)}\,R_{(k-2)(k-1)} = 0$, i.e.,

$$F_{(k-2)k}\,R_{(k-2)(k-1)}\,D^{p_{(k-3)(k-1)}} \subseteq \ker_D(R_{(k-1)k}\,.) = R_{(k-2)k}\,D^{p_{(k-3)k}},$$

and thus there exists a matrix $F_{(k-3)k} \in D^{p_{(k-3)k} \times p_{(k-3)(k-1)}}$ such that:

$$\forall k = 2, \ldots, n+1, \quad F_{(k-2)k}\,R_{(k-2)(k-1)} = R_{(k-2)k}\,F_{(k-3)k}. \tag{29}$$

For $k = 3, \ldots, n+1$, we can similarly show that matrices $F_{(k-j)k} \in D^{p_{(k-j)k} \times p_{(k-j)(k-1)}}$ exist with $j = 3, \ldots, k$ such that:

$$F_{(k-j)k}\,R_{(k-j)(k-1)} = R_{(k-j)k}\,F_{(k-j-1)k}. \tag{30}$$

Let us denote by:

$$R_{00} = 0, \qquad N_{00} = D^{p_{00}}/0 \cong D^{p_{00}}, \qquad p_{01} = p_{00}, \qquad F_{01} = I_{p_{01}}, \qquad p_{-10} = 0. \tag{31}$$

Using (27), (28), (29), (30), and (31), we get the following commutative diagram formed by $n+2$ horizontal exact sequences (where to reduce the size of the diagram, we set $m = n+1$):

$$\begin{tikzcd}
D^{p_{-1m}} \arrow[r, "R_{0m}\cdot"] & D^{p_{0m}} \arrow[r, "R_{1m}\cdot"] & D^{p_{1m}} \arrow[r, "R_{2m}\cdot"] & D^{p_{2m}} \arrow[r, "R_{3m}\cdot"] & D^{p_{3m}} \arrow[r, "R_{4m}\cdot"] & D^{p_{4m}} \arrow[r, "R_{5m}\cdot"] & \ldots \\
D^{p_{-1n}} \arrow[r, "R_{0n}\cdot"] \arrow[u, "F_{-1m}\cdot"'] & D^{p_{0n}} \arrow[r, "R_{1n}\cdot"] \arrow[u, "F_{0m}\cdot"'] & D^{p_{1n}} \arrow[r, "R_{2n}\cdot"] \arrow[u, "F_{1m}\cdot"'] & D^{p_{2n}} \arrow[r, "R_{3n}\cdot"] \arrow[u, "F_{2m}\cdot"'] & D^{p_{3n}} \arrow[r, "R_{4n}\cdot"] \arrow[u, "F_{3m}\cdot"'] & D^{p_{4n}} \arrow[r, "R_{5n}\cdot"] \arrow[u, "F_{4m}\cdot"'] & \ldots \\
\vdots \arrow[u, "F_{-1n}\cdot"'] & \vdots \arrow[u, "F_{0n}\cdot"'] & \vdots \arrow[u, "F_{1n}\cdot"'] & \vdots \arrow[u, "F_{2n}\cdot"'] & \vdots \arrow[u, "F_{3n}\cdot"'] & \vdots \arrow[u, "F_{4n}\cdot"'] & \\
D^{p_{-13}} \arrow[r, "R_{03}\cdot"] \arrow[u, "F_{-14}\cdot"'] & D^{p_{03}} \arrow[r, "R_{13}\cdot"] \arrow[u, "F_{04}\cdot"'] & D^{p_{13}} \arrow[r, "R_{23}\cdot"] \arrow[u, "F_{14}\cdot"'] & D^{p_{23}} \arrow[r, "R_{33}\cdot"] \arrow[u, "F_{24}\cdot"'] & D^{p_{33}} \arrow[r, "\kappa_{33}"] \arrow[u, equal] & N_{33} \arrow[r] & 0 \\
D^{p_{-12}} \arrow[r, "R_{02}\cdot"] \arrow[u, "F_{-13}\cdot"'] & D^{p_{02}} \arrow[r, "R_{12}\cdot"] \arrow[u, "F_{03}\cdot"'] & D^{p_{12}} \arrow[r, "R_{22}\cdot"] \arrow[u, "F_{13}\cdot"'] & D^{p_{22}} \arrow[r, "\kappa_{22}"] \arrow[u, equal] & N_{22} \arrow[r] & 0 & \\
D^{p_{-11}} \arrow[r, "R_{01}\cdot"] \arrow[u, "F_{-12}\cdot"'] & D^{p_{01}} \arrow[r, "R_{11}\cdot"] \arrow[u, "F_{02}\cdot"'] & D^{p_{11}} \arrow[r, "\kappa_{11}"] \arrow[u, equal] & N_{11} \arrow[r] & 0 & & \\
0 \arrow[r] & D^{p_{00}} \arrow[r, "\kappa_{00}"] \arrow[u, equal] & N_{00} \arrow[r] & 0. & & &
\end{tikzcd} \tag{32}$$

Now, if we denote by $N_{(k-j)k}$ the finitely presented right D-module defined by

$$N_{(k-j)k} = \operatorname{coker}_D(R_{(k-j)k}\,.) = D^{p_{(k-j)k}}/\big(R_{(k-j)k}\,D^{p_{(k-j-1)k}}\big),$$

then (32) can be truncated to get the following commutative diagram formed by horizontal exact sequences:

$$
\begin{tikzcd}
D^{p_{-1(n+1)}} \arrow[r, "R_{0(n+1)}."] & D^{p_{0(n+1)}} \arrow[r, "R_{1(n+1)}."] & D^{p_{1(n+1)}} \arrow[r, "\kappa_{1(n+1)}"] & N_{1(n+1)} \arrow[r] & 0 \\
D^{p_{-1n}} \arrow[r, "R_{0n}."] \arrow[u, "F_{-1(n+1)}."'] & D^{p_{0n}} \arrow[r, "R_{1n}."] \arrow[u, "F_{0(n+1)}."'] & D^{p_{1n}} \arrow[r, "\kappa_{1n}"] \arrow[u, "F_{1(n+1)}."'] & N_{1n} \arrow[r] & 0 \\
\vdots \arrow[u, "F_{-1n}."'] & \vdots \arrow[u, "F_{0n}."'] & \vdots \arrow[u, "F_{1n}."'] & \vdots & \\
D^{p_{-13}} \arrow[r, "R_{03}."] \arrow[u, "F_{-14}."'] & D^{p_{03}} \arrow[r, "R_{13}."] \arrow[u, "F_{04}."'] & D^{p_{13}} \arrow[r, "\kappa_{13}"] \arrow[u, "F_{14}."'] & N_{13} \arrow[r] & 0 \\
D^{p_{-12}} \arrow[r, "R_{02}."] \arrow[u, "F_{-13}."'] & D^{p_{02}} \arrow[r, "R_{12}."] \arrow[u, "F_{03}."'] & D^{p_{12}} \arrow[r, "\kappa_{12}"] \arrow[u, "F_{13}."'] & N_{12} \arrow[r] & 0 \\
D^{p_{-11}} \arrow[r, "R_{01}."] \arrow[u, "F_{-12}."'] & D^{p_{01}} \arrow[r, "R_{11}."] \arrow[u, "F_{02}."'] & D^{p_{11}} \arrow[r, "\kappa_{11}"] \arrow[u, equal] & N_{11} \arrow[r] & 0 \\
0 \arrow[r] & D^{p_{00}} \arrow[r, "\kappa_{00}"] \arrow[u, equal] & N_{00} \arrow[r] & 0. &
\end{tikzcd}
\tag{33}
$$

For $k = 1, \ldots, n+1$ and $j = 0, \ldots, k-1$, using the exactness of the complex

$$
D^{p_{(k-j-2)k}} \xrightarrow{R_{(k-j-1)k}.} D^{p_{(k-j-1)k}} \xrightarrow{R_{(k-j)k}.} D^{p_{(k-j)k}}
$$

at $D^{p_{(k-j-1)k}}$, we get $N_{(k-j-1)k} = \operatorname{coker}_D(R_{(k-j-1)k}.) \cong \operatorname{im}_D(R_{(k-j)k}.)$ which, when combined with the short exact sequence

$$
0 \longrightarrow \operatorname{im}_D(R_{(k-j)k}.) \longrightarrow D^{p_{(k-j)k}} \xrightarrow{\kappa_{(k-j)k}} N_{(k-j)k} \longrightarrow 0,
$$

yields the following short exact sequence of right D-modules:

$$
0 \longrightarrow N_{(k-j-1)k} \longrightarrow D^{p_{(k-j)k}} \longrightarrow N_{(k-j)k} \longrightarrow 0. \tag{34}
$$

Using (26), we obtain the following characterization of the right D-modules $\operatorname{ext}^i_D(M, D)$'s:

$$
\begin{cases}
\operatorname{ext}^i_D(M, D) \cong \ker_D(R_{(i+1)(i+1)}.)/\operatorname{im}_D(R_{ii}.) = \big(R_{i(i+1)}\, D^{p_{(i-1)(i+1)}}\big)/\big(R_{ii}\, D^{p_{(i-1)i}}\big), \\
\quad 0 \leq i \leq n, \\
\operatorname{ext}^i_D(M, D) = 0, \quad i > n.
\end{cases}
\tag{35}
$$

Since $N_{ii} = D^{p_{ii}}/(R_{ii}\,D^{p_{(i-1)i}})$, $N_{i(i+1)} = D^{p_{i(i+1)}}/(R_{i(i+1)}\,D^{p_{(i-1)(i+1)}})$, $p_{i(i+1)} = p_{ii}$, and $N_{00} = D^{p_{00}}$, (35) and the *third isomorphism theorem* of module theory (see, e.g., [50]) yield the following short exact sequence of right D-modules:

$$0 \longrightarrow \mathrm{ext}_D^i(M, D) \longrightarrow N_{ii} \longrightarrow N_{i(i+1)} \longrightarrow 0, \quad i = 0, \ldots, n. \tag{36}$$

Applying the contravariant left exact functor $\hom_D(\cdot\,, D)$ to the short exact sequence of (36) and using Theorem 3, we obtain the following long exact sequences:

$$\begin{array}{rlll} 0 \longrightarrow & \mathrm{ext}_D^0(N_{01}, D) & \longrightarrow \mathrm{ext}_D^0(N_{00}, D) \longrightarrow & \mathrm{ext}_D^0\big(\mathrm{ext}_D^0(M, D), D\big) \\ \longrightarrow & \mathrm{ext}_D^1(N_{01}, D) & \longrightarrow \mathrm{ext}_D^1(N_{00}, D), & \\ \cdots \longrightarrow & \mathrm{ext}_D^{i-1}(N_{i(i+1)}, D) & \longrightarrow \mathrm{ext}_D^{i-1}(N_{ii}, D) \longrightarrow & \mathrm{ext}_D^{i-1}\big(\mathrm{ext}_D^i(M, D), D\big) \\ \longrightarrow & \mathrm{ext}_D^i(N_{i(i+1)}, D) & \longrightarrow \mathrm{ext}_D^i(N_{ii}, D) \longrightarrow & \mathrm{ext}_D^i\big(\mathrm{ext}_D^i(M, D), D\big) \\ \xrightarrow{\tau_{i+1}} & \mathrm{ext}_D^{i+1}(N_{i(i+1)}, D) & \longrightarrow \mathrm{ext}_D^{i+1}(N_{ii}, D) \longrightarrow & \cdots, \quad i = 1, \ldots, n. \end{array} \tag{37}$$

In what follows, we shall assume that D satisfies the following property

$$\forall i \geq 1, \quad \mathrm{ext}_D^{i-1}\big(\mathrm{ext}_D^i(M, D), D\big) = 0, \tag{38}$$

for all finitely generated left D-modules M. In particular, by Remark 7, this condition holds if D is an Auslander regular ring (see Definition 8). The importance of (38) was already noticed in Sect. 9.1.4 of [2].

We note that $\mathrm{ext}_D^1(N_{00}, D)$ is reduced to 0 since $N_{00} = D^{p_{00}}$ is a free, and thus a projective right D-module (see Remark 2). Using (38), the above long exact sequences then yield the following long exact sequences of left D-modules:

$$\begin{array}{c} 0 \longrightarrow \mathrm{ext}_D^0(N_{01}, D) \longrightarrow \mathrm{ext}_D^0(N_{00}, D) \longrightarrow \mathrm{ext}_D^0\big(\mathrm{ext}_D^0(M, D), D\big) \longrightarrow \mathrm{ext}_D^1(N_{01}, D) \longrightarrow 0, \\ 0 \longrightarrow \mathrm{ext}_D^i(N_{i(i+1)}, D) \longrightarrow \mathrm{ext}_D^i(N_{ii}, D) \longrightarrow \mathrm{ext}_D^i\big(\mathrm{ext}_D^i(M, D), D\big), \quad i = 1, \ldots, n. \end{array} \tag{39}$$

Applying Proposition 1 to (34) for $k = i + 1$ and $j = 0, \ldots, i - 1$, i.e., to the short exact sequence $0 \longrightarrow N_{(i-j)(i+1)} \longrightarrow D^{p_{(i-j+1)(i+1)}} \longrightarrow N_{(i-j+1)(i+1)} \longrightarrow 0$, we obtain:

$$\forall i = 1, \ldots, n, \quad \mathrm{ext}_D^{i+1}(N_{(i+1)(i+1)}, D) \cong \mathrm{ext}_D^i(N_{i(i+1)}, D) \cong \cdots \cong \mathrm{ext}_D^1(N_{1(i+1)}, D). \tag{40}$$

Similarly, applying Proposition 1 to (34) for $k = i + 1$ and $j = 0$ gives:

$$\mathrm{ext}_D^{i+2}(N_{(i+1)(i+1)}, D) \cong \mathrm{ext}_D^{i+1}(N_{i(i+1)}, D). \tag{41}$$

Applying Proposition 1 to the above short exact sequence with $i = 0$ and $j = 0$, we get:

$$\mathrm{ext}_D^2(N_{11}, D) \cong \mathrm{ext}_D^1(N_{01}, D).$$

Thus, the first long exact sequence of (39) yields the following one

$$\begin{array}{l} 0 \longrightarrow \mathrm{ext}_D^0(N_{01}, D) \xrightarrow{\gamma_{10}} \mathrm{ext}_D^0(N_{00}, D) \xrightarrow{\gamma_{00}} \mathrm{ext}_D^0\big(\mathrm{ext}_D^0(M, D), D\big) \\ \longrightarrow \mathrm{ext}_D^2(N_{11}, D) \longrightarrow 0, \end{array} \tag{42}$$

and (39) and (40) yield the following exact sequence of left D-modules

$$\begin{array}{l} 0 \longrightarrow \mathrm{ext}_D^{i+1}(N_{(i+1)(i+1)}, D) \xrightarrow{\gamma_{(i+1)i}} \mathrm{ext}_D^i(N_{ii}, D) \xrightarrow{\gamma_{ii}} \mathrm{ext}_D^i\big(\mathrm{ext}_D^i(M, D), D\big) \\ \longrightarrow \operatorname{coker} \gamma_{ii} \longrightarrow 0, \end{array} \tag{43}$$

where, using (41), we have:

$$\forall i = 1, \ldots, n, \quad \operatorname{coker} \gamma_{ii} \cong \operatorname{im} \tau_{i+1} \subseteq \operatorname{ext}_D^{i+1}(N_{i(i+1)}, D) \cong \operatorname{ext}_D^{i+2}(N_{(i+1)(i+1)}, D). \tag{44}$$

Hence, if we introduce the following finitely generated left D-modules

$$\forall i = 0, \ldots, n+1, \quad T_i \triangleq \operatorname{ext}_D^i(N_{ii}, D), \tag{45}$$

then (43) can be rewritten as the following exact sequences:

$$0 \longrightarrow T_{i+1} \xrightarrow{\gamma_{(i+1)i}} T_i \xrightarrow{\gamma_{ii}} \operatorname{ext}_D^i\big(\operatorname{ext}_D^i(M, D), D\big) \longrightarrow \operatorname{coker} \gamma_{ii} \longrightarrow 0, \quad i = 1, \ldots, n. \tag{46}$$

Remark 9 If D is an Auslander regular ring, then using (45) and Remark 7, T_i is either zero or $j_D(T_i) \geq i$. Moreover, by 3 of Theorem 8, $\operatorname{ext}_D^i(\operatorname{ext}_D^i(M, D), D)$ is either zero or is i-pure. In particular, $\operatorname{coker} \gamma_{(i+1)i} = T_i / \gamma_{(i+1)i}(T_{i+1})$ is isomorphic to a left D-submodule $\operatorname{im} \gamma_{ii}$ of $\operatorname{ext}_D^i(\operatorname{ext}_D^i(M, D), D)$, and thus it is either zero or is i-pure by Remark 7. Finally, using Definition 8 and (44), we find that $\operatorname{coker} \gamma_{ii}$ is either zero or $j_D(\operatorname{coker} \gamma_{ii}) \geq i + 2$.

Using (40), up to isomorphism, the left D-modules T_i's are the defects of exactness at $D^{1 \times p_{0i}}$ (marked in red (color version online)) of the horizontal complexes of the following commutative diagram

$$\begin{tikzcd}
D^{1\times p_{-1(n+1)}} \arrow[d, ".F_{-1(n+1)}"] & D^{1\times p_{0(n+1)}} \arrow[l, ".R_{0(n+1)}"'] \arrow[d, ".F_{0(n+1)}"] & D^{1\times p_{1(n+1)}} \arrow[l, ".R_{1(n+1)}"'] \arrow[d, ".F_{1(n+1)}"] \\
D^{1\times p_{-1n}} \arrow[d, ".F_{-1n}"] & D^{1\times p_{0n}} \arrow[l, ".R_{0n}"'] \arrow[d, ".F_{0n}"] & D^{1\times p_{1n}} \arrow[l, ".R_{1n}"'] \arrow[d, ".F_{1n}"] \\
\vdots \arrow[d, ".F_{-14}"] & \vdots \arrow[d, ".F_{04}"] & \vdots \arrow[d, ".F_{14}"] \\
D^{1\times p_{-13}} \arrow[d, ".F_{-13}"] & D^{1\times p_{03}} \arrow[l, ".R_{03}"'] \arrow[d, ".F_{03}"] & D^{1\times p_{13}} \arrow[l, ".R_{13}"'] \arrow[d, ".F_{13}"] \\
D^{1\times p_{-12}} \arrow[d, ".F_{-12}"] & D^{1\times p_{02}} \arrow[l, ".R_{02}"'] \arrow[d, ".F_{02}"] & D^{1\times p_{12}} \arrow[l, ".R_{12}"'] \arrow[d, equal] \\
D^{1\times p_{-11}} \arrow[d] & D^{1\times p_{01}} \arrow[l, ".R_{01}"'] \arrow[d, equal] & D^{1\times p_{11}} \arrow[l, ".R_{11}"'] \\
0 & D^{1\times p_{00}} \arrow[l] & 0, \arrow[l]
\end{tikzcd} \tag{47}$$

i.e., we have:

$$T_0 = D^{1 \times p_{00}}, \qquad T_i = \ker_D(.R_{0i}) / \operatorname{im}_D(.R_{1i}), \quad i = 1, \ldots, n+1. \tag{48}$$

If $\rho_i : \ker_D(.R_{0i}) \longrightarrow T_i = \ker_D(.R_{0i})/(D^{1\times p_{1i}} R_{1i})$ is the canonical projection onto the D-module T_i for $i = 1, \ldots, n+1$, then $\gamma_{(i+1)i} \in \hom_D(T_{i+1}, T_i)$ (see (46)) is defined by:

$$\forall \lambda \in \ker_D(.R_{0(i+1)}), \quad \gamma_{(i+1)i}\big(\rho_{i+1}(\lambda)\big) = \rho_i(\lambda F_{0(i+1)}), \quad i = 1, \ldots, n. \tag{49}$$

The inclusion $\ker_D(.R_{01}) \subseteq D^{1\times p_{01}}$ yields the commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & D^{1\times p_{11}} R_{11} \arrow[r] \arrow[d, equal] & \ker_D(.R_{01}) \arrow[r, "\rho_1"] \arrow[d] & T_1 \arrow[r] \arrow[d, "\gamma_{10}"] & 0 \\
0 \arrow[r] & D^{1\times p_{11}} R_{11} \arrow[r] \arrow[d] & D^{1\times p_{01}} \arrow[r, "\pi"] & M \arrow[r] & 0, \\
 & 0 & & &
\end{tikzcd}$$

where $\gamma_{10} \in \hom_D(T_1, M)$ is defined by

$$\forall \lambda \in \ker_D(.R_{01}), \quad \gamma_{10}\big(\rho_1(\lambda)\big) = \pi(\lambda), \tag{50}$$

and π is the canonical projection onto $M = D^{1\times p_{01}}/(D^{1\times p_{11}} R_{11})$, i.e., $\gamma_{10} = \mathrm{id}_{T_1}$. In particular, γ_{10} is injective. Moreover, using the following inclusion

$$T_1 = \ker_D(.R_{01})/\big(D^{1\times p_{11}} R_{11}\big) \subseteq M = D^{1\times p_{01}}/\big(D^{1\times p_{11}} R_{11}\big),$$

the third isomorphism theorem of module theory (see, e.g., [50]) gives:

$$M/T_1 \cong D^{1\times p_{01}}/\ker_D(.R_{01}). \tag{51}$$

If D is a domain, then 1 of Theorem 5 shows that $T_1 = t(M)$ and $M/T_1 = M/t(M)$.

Let us now study the long exact sequences (42) and (46) for $i = n-1, n$.

A right D-module analogous of Theorem 1 shows that $\mathrm{ext}^0_D(N_{01}, D) \cong \ker_D(.R_{01})$. Using (31), $T_0 = \mathrm{ext}^0_D(N_{00}, D) = \hom_D(D^{p_{00}}, D) \cong D^{1\times p_{00}} = D^{1\times p_{01}}$ (see (48)). The long exact sequence (42) then becomes the following one:

$$0 \longrightarrow \ker_D(.R_{01}) \xrightarrow{\gamma_{10}} D^{1\times p_{01}} \xrightarrow{\gamma_{00}} \mathrm{ext}^0_D\big(\mathrm{ext}^0_D(M, D), D\big) \longrightarrow \mathrm{ext}^2_D(N_{11}, D) \longrightarrow 0.$$

Proposition 3, $\mathrm{gld}(D) = n$, and (44) yield

$$\operatorname{coker} \gamma_{(n-1)(n-1)} \subseteq \mathrm{ext}^{n+1}_D(N_{nn}, D) = 0,$$

i.e., $\operatorname{coker} \gamma_{(n-1)(n-1)} = 0$. Thus, setting $i = n-1$ in (46), we get the following short exact sequence

$$0 \longrightarrow T_n \xrightarrow{\gamma_{n(n-1)}} T_{n-1} \xrightarrow{\gamma_{(n-1)(n-1)}} \mathrm{ext}^{n-1}_D\big(\mathrm{ext}^{n-1}_D(M, D), D\big) \longrightarrow 0,$$

which shows that:

$$\operatorname{coker} \gamma_{n(n-1)} = T_{n-1}/\big(\gamma_{n(n-1)}(T_n)\big) \cong \mathrm{ext}^{n-1}_D\big(\mathrm{ext}^{n-1}_D(M, D), D\big). \tag{52}$$

Proposition 3, $\mathrm{gld}(D) = n$, and (44) imply that

$$\mathrm{coker}\,\gamma_{nn} \subseteq \mathrm{ext}_D^{n+2}(N_{(n+1)(n+1)}, D) = 0,$$

i.e., $\mathrm{coker}\,\gamma_{nn} = 0$. By Proposition 3, we also have:

$$T_{n+1} = \mathrm{ext}_D^{n+1}\big(N_{(n+1)(n+1)}, D\big) = 0.$$

Thus, setting $i = n$ in (46), we obtain the following short exact sequence

$$0 \longrightarrow T_n \xrightarrow{\gamma_{nn}} \mathrm{ext}_D^n\big(\mathrm{ext}_D^n(M, D), D\big) \longrightarrow 0,$$

which shows that:

$$T_n \cong \mathrm{ext}_D^n\big(\mathrm{ext}_D^n(M, D), D\big). \tag{53}$$

Therefore, the following exact sequences of left D-modules hold

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & T_n & \xrightarrow{\gamma_{nn}} & \mathrm{ext}_D^n\big(\mathrm{ext}_D^n(M, D), D\big) & \longrightarrow & 0, \\
0 & \longrightarrow & T_n & \xrightarrow{\gamma_{n(n-1)}} & T_{n-1} & \longrightarrow & \mathrm{coker}\,\gamma_{n(n-1)} & \longrightarrow & 0, \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & T_i & \xrightarrow{\gamma_{i(i-1)}} & T_{i-1} & \longrightarrow & \mathrm{coker}\,\gamma_{i(i-1)} & \longrightarrow & 0, \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & T_2 & \xrightarrow{\gamma_{21}} & T_1 & \longrightarrow & \mathrm{coker}\,\gamma_{21} & \longrightarrow & 0, \\
0 & \longrightarrow & T_1 & \xrightarrow{\gamma_{10}} & M & \xrightarrow{\rho} & M/T_1 & \longrightarrow & 0, \\
0 & \longrightarrow & \ker_D(.R_{01}) & \longrightarrow & D^{1\times p_{01}} & \xrightarrow{\pi'} & M/T_1 & \longrightarrow & 0, \\
0 & \longrightarrow & M/T_1 & \longrightarrow & \mathrm{ext}_D^0\big(\mathrm{ext}_D^0(M, D), D\big) & \longrightarrow & \mathrm{ext}_D^2(N_{11}, D) & \longrightarrow & 0,
\end{array} \tag{54}$$

where:

$$\forall i = 2, \dots, n, \quad \mathrm{coker}\,\gamma_{i(i-1)} \subseteq \mathrm{ext}_D^i\big(\mathrm{ext}_D^i(M, D), D\big). \tag{55}$$

Since the $\gamma_{i(i-1)}$'s are injective left D-homomorphisms and $\gamma_{10} = \mathrm{id}_{T_1}$, we can define the following sequence $\{M_i\}_{i=0,\dots,n}$ of left D-submodules of M as follows:

$$\begin{cases} M_0 = M, \\ M_1 = \gamma_{10}(T_1) = T_1, \\ M_i = (\gamma_{10} \circ \dots \circ \gamma_{i(i-1)})(T_i) \cong T_i, \quad i = 2, \dots, n. \end{cases} \tag{56}$$

Using (49) and (50), the left D-module M_i can be explicitly characterized by:

$$\begin{cases} M_1 = \pi\big(\ker_D(.R_{01})\big), \\ M_i = \pi\big(\ker_D(.R_{0i})(F_{0i} \dots F_{01})\big), \quad i = 2, \dots, n. \end{cases} \tag{57}$$

The inclusion $\gamma_{i(i-1)}(T_i) \subseteq T_{i-1}$ yields $M_i \subseteq M_{i-1}$, and we get the following descending filtration of M:

$$0 = M_{n+1} \subseteq M_n \subseteq M_{n-1} \subseteq \dots \subseteq M_2 \subseteq M_1 \subseteq M_0 = M. \tag{58}$$

Remark 10 Let us explain why the left D-modules M_i's depend only on M and not on the free resolution (24) of M. Using Remark 3, the Auslander transpose right D-module $N_{ii} = D^{p_{ii}}/(R_{ii}\,D^{p_{(i-1)i}})$ of the left D-module $\operatorname{coker}_D(.R_{ii}) = D^{1\times p_{ii}}/(D^{1\times p_{(i-1)i}}\,R_{ii})$ depends only on $\operatorname{coker}_D(.R_{ii})$ up to projective equivalence. Using Remark 1 and the exactness of the free resolution (24) of M, we find that the right D-modules

$$\begin{cases} \operatorname{coker}_D(.R_{ii}) = \operatorname{coker}_D(.R_i) \cong D^{1\times p_{i-1}}\,R_{i-1} = \ker_D(.R_{i-2}), & i \geq 3, \\ \operatorname{coker}_D(.R_{22}) = \operatorname{coker}_D(.R_2) = D^{1\times p_1}\,R_1 = \ker\pi, \\ \operatorname{coker}_D(.R_{11}) = \operatorname{coker}_D(.R_1) = M, \end{cases}$$

depend on M up to projective equivalence. Thus, the right D-module N_{ii} depends only on M up to a projective equivalence for $i \geq 1$. Finally, using Remark 2, $M_i \cong T_i = \operatorname{ext}^i_D(N_{ii}, D)$ depends only on M for $i \geq 1$.

We obtain the following results.

Theorem 10 *Let D be a regular ring of global dimension $\operatorname{gld}(D) = n$ which satisfies*

$$\forall i \geq 1, \quad \operatorname{ext}^{i-1}_D\big(\operatorname{ext}^i_D(M, D), D\big) = 0,$$

for all finitely generated left D-modules M. Then, with the above notations, the following results hold:

1. *The following long exact sequences of left D-modules hold*

$$0 \longrightarrow M_{i+1} \xrightarrow{\iota_{i+1}} M_i \xrightarrow{\varepsilon_i} \operatorname{ext}^i_D\big(\operatorname{ext}^i_D(M, D), D\big) \longrightarrow C_i \longrightarrow 0, \quad i = 0, \ldots, n, \tag{59}$$

where $C_i = \operatorname{coker}\varepsilon_i$ is isomorphic to a left D-submodule of $\operatorname{ext}^{i+2}_D(N_{(i+1)(i+1)}, D)$ for all $i = 0, \ldots, n-2$ (with equality for $i = 0$), $C_{n-1} = 0$, and $C_n = 0$. In particular:

$$M_n \cong \operatorname{ext}^n_D\big(\operatorname{ext}^n_D(M, D), D\big), \qquad M_{n-1}/M_n \cong \operatorname{ext}^{n-1}_D\big(\operatorname{ext}^{n-1}_D(M, D), D\big).$$

2. *The following descending filtration $\{M_i\}_{i=0,\ldots,n+1}$ of M holds*:

$$0 = M_{n+1} \subseteq M_n \subseteq M_{n-1} \subseteq \cdots \subseteq M_2 \subseteq M_1 \subseteq M_0 = M.$$

In particular, if $M_i = 0$, then $M_i = M_{i+1} = \cdots = M_n = 0$.

3. $M = M_{j_D(M)}$.

Proof 1. Using the last short exact sequence of (54), $M = M_0$, and $M_1 = T_1$, we obtain (59) for $i = 0$, where $C_0 = \operatorname{ext}^2_D(N_{11}, D)$. Let us now suppose that $i = 1, \ldots, n$ and let $\alpha_i = \gamma_{10} \circ \gamma_{21} \circ \gamma_{32} \circ \cdots \circ \gamma_{i(i-1)}$ be the left D-isomorphism from T_i to M_i (see (56)). Then, the long exact sequence (46) yields (59) where $\iota_{i+1} = \alpha_i \circ \gamma_{(i+1)i} \circ \alpha^{-1}_{i+1} = \operatorname{id}_{M_{i+1}}$, $\varepsilon_i = \gamma_{ii} \circ \alpha_i^{-1}$, and $C_i = \operatorname{coker}\varepsilon_i \cong \operatorname{coker}\gamma_{ii} \subseteq \operatorname{ext}^{i+2}_D(N_{(i+1)(i+1)}, D)$ by (44). Since $\operatorname{gld}(D) = n$, we get $C_{n-1} = C_n = 0$. Finally, (59) for $i = n$, $M_{n+1} = 0$ and C_n yield $M_n \cong \operatorname{ext}^n_D(\operatorname{ext}^n_D(M, D), D)$, and (59) for $i = n-1$ and $C_{n-1} = 0$ implies that $M_{n-1}/M_n \cong \operatorname{ext}^{n-1}_D(\operatorname{ext}^{n-1}_D(M, D), D)$.

2. The equality is a direct consequence of (58).

3. If $j_D(M) = 0$, then the result holds since $M = M_0$. Let us suppose that $j_D(M) \geq 1$. Then, we have $\operatorname{ext}^i_D(\operatorname{ext}^i_D(M, D), D) = 0$ for $i = 0, \ldots, j_D(M) - 1$ since $\operatorname{ext}^i_D(M, D) = 0$ for $i = 0, \ldots, j_D(M) - 1$. Using (59), we get $M_{i+1} = M_i$ for $i = 0, \ldots, j_D(M) - 1$. □

Let us give consequences of the above results for an Auslander regular ring D.

Proposition 8 *If D is an Auslander regular ring and* $\mathrm{gld}(D)=n$, *then we have*:

1. *If M_i is nonzero, then $j_D(M_i)\geq i$ for $i=0,\ldots,n$.*
2. *If M_i/M_{i+1} is nonzero, then M_i/M_{i+1} is an i-pure left D-module for $i=0,\ldots,n$. Moreover, if $M_{i+1}=0$, then M_i is either zero or an i-pure left D-submodule of M. In particular, M_n is either zero or a n-pure left D-module.*
3. *If C_i is nonzero, then $j_D(C_i)\geq i+2$ for $i=0,\ldots,n-2$.*
4. *$M_i=M_{i+1}$ iff* $\mathrm{ext}^i_D(\mathrm{ext}^i_D(M,D),D)=0$.

Proof 1. Since $M_i\cong T_i=\mathrm{ext}^i_D(N_{ii},D)$ for $i=1,\ldots,n$, Remark 7 then shows that $j_D(M_i)\geq i$. Moreover, $M_0=M$, and thus $j_D(M_0)\geq 0$.

2. By 3 of Theorem 8, $\mathrm{ext}^i_D(\mathrm{ext}^i_D(M,D),D)$ is either zero or i-pure, and so is the left D-module $M_i/M_{i+1}\cong \mathrm{im}\,\varepsilon_i\subseteq \mathrm{ext}^i_D(\mathrm{ext}^i_D(M,D),D)$ (see Remark 6). In particular, if $M_{i+1}=0$, then M_i is either zero or an i-pure left D-submodule of M. Finally, $M_n\cong \mathrm{ext}^n_D(\mathrm{ext}^n_D(M,D),D)$ (see 1 of Theorem 10) implies that M_n is either zero or n-pure.

3. $C_i=\mathrm{coker}\,\varepsilon_i$ is isomorphic to a left D-submodule of $\mathrm{ext}^{i+2}_D(N_{(i+1)(i+1)},D)$ for $i=0,\ldots,n-2$ (see 1 of Theorem 10). Then, using 2 of Definition 8, we get $j_D(C_i)\geq i+2$ for $i=0,\ldots,n-2$.

4. If $M_i=M_{i+1}$, then (59) gives $C_i\cong \mathrm{ext}^i_D(\mathrm{ext}^i_D(M,D),D)$. On the one hand, by 3 of Theorem 8, C_i is either zero or i-pure, and thus we either have $C_i=0$ or $j_D(C_i)=i$. On the other hand, using 3, if $C_i\neq 0$, then $j_D(C_i)\geq i+2$, which shows that $C_i=0$. Conversely, if $\mathrm{ext}^i_D(\mathrm{ext}^i_D(M,D),D)=0$, then (59) yields $M_i=M_{i+1}$. □

If D is also a Cohen-Macaulay ring, then Corollary 3 yields:

$$\begin{aligned}&\forall i=0,\ldots,n,\quad \dim_D(M_i)\leq \dim(D)-i,\\ &M_i/M_{i+1}\neq 0\quad\Rightarrow\quad \dim_D(M_i/M_{i+1})=\dim(D)-i.\end{aligned}\tag{60}$$

If D is an Auslander regular ring, then let us now show that the filtration $\{M_i\}_{i=0,\ldots,n}$ of M defined by (56) is exactly the grade filtration $\{t_i(M)\}_{i=0,\ldots,n}$ of M defined in (21).

Theorem 11 *Let D be an Auslander regular ring and M a finitely generated left D-module. Then, we have $t_i(M)=M_i$ for all $i=0,\ldots,n=$* $\mathrm{gld}(D)$, *i.e., the grade filtration* (22) *of M and the filtration* (58) *of M coincide.*

Proof 1. Let us first prove that $0\neq M_i\subseteq t_i(M)$. By 1 of Proposition 8, $j_D(M_i)\geq i$. If $m\in M_i$, then applying Proposition 7 to the following short exact sequence

$$0\longrightarrow Dm\longrightarrow M_i\longrightarrow M_i/(Dm)\longrightarrow 0,$$

we obtain $j_D(Dm)\geq j_D(M_i)=i$, and thus $m\in t_i(M)$, i.e., $M_i\subseteq t_i(M)$.

2. Following [9], let us prove $t_i(M)\subseteq M_i$ by induction on i, i.e., $t_i(M)=M_i$. We first note that $t_0(M)=M=M_0$, which proves the result for $i=0$. Let us now assume that $t_i(M)=M_i$ and let us show that it yields $t_{i+1}(M)=M_{i+1}$. Since $M_{i+1}\subseteq t_{i+1}(M)\subseteq t_i(M)$ by 1, we get $t_{i+1}(M)/M_{i+1}\subseteq t_i(M)/M_{i+1}=M_i/M_{i+1}$. Using 2 of Proposition 8, M_i/M_{i+1} is either zero or an i-pure left D-module. If $M_i/M_{i+1}=0$, then $t_{i+1}(M)/M_{i+1}=0$, i.e., $t_{i+1}(M)=M_{i+1}$, which proves the result. Hence, let us assume that M_i/M_{i+1} is an i-pure left D-module. Then, by definition of a pure module, its left D-submodule $t_{i+1}(M)/M_{i+1}$

is also either zero or i-pure. If $t_{i+1}(M)/M_{i+1}$ is i-pure, then $j_D(t_{i+1}(M)/M_{i+1}) = i$. But applying Proposition 7 to the following short exact sequence

$$0 \longrightarrow M_{i+1} \longrightarrow t_{i+1}(M) \longrightarrow t_{i+1}(M)/M_{i+1} \longrightarrow 0$$

gives $j_D(t_{i+1}(M)/M_{i+1}) \geq j_D(t_{i+1}(M)) \geq i+1$, which yields a contradiction. Thus, we obtain $t_{i+1}(M)/M_{i+1} = 0$, i.e., $t_{i+1}(M) = M_{i+1}$, which proves the result by induction. □

Remark 11 We can combine Theorem 11 and Proposition 8 to find again 2 of Theorem 8. Indeed, using Theorem 11 and the fact that M is i-pure iff $t_i(M) = M$ and $t_{i+1}(M) = 0$, we get that $M \neq 0$ is i-pure iff $M = M_1 = \cdots = M_i \neq 0$ and $M_{i+1} = M_{i+2} = \cdots = M_{n+1} = 0$. By 4 of Proposition 8, the equalities are equivalent to $\mathrm{ext}_D^k(\mathrm{ext}_D^k(M, D), D) = 0$ for $k = 0, \ldots, i-1$ and $k = i+1, \ldots, n$. Now, combining $M_i \neq 0$, $M_{i+1} = 0$, and (59), $\mathrm{ext}_D^i(\mathrm{ext}_D^i(M, D), D)$ contains the nonzero left D-submodule M_i, which shows that $\mathrm{ext}_D^i(\mathrm{ext}_D^i(M, D), D) \neq 0$. Conversely, if $\mathrm{ext}_D^k(\mathrm{ext}_D^k(M, D), D) = 0$ for $k \neq i$ and $\mathrm{ext}_D^i(\mathrm{ext}_D^i(M, D), D) \neq 0$, then $M = M_1 = \cdots = M_i$ and $M_{i+1} = M_{i+2} = \cdots = M_{n+1} = 0$ by 4 of Proposition 8. Finally, since $\mathrm{ext}_D^i(\mathrm{ext}_D^i(M, D), D) \neq 0$ yields $M \neq 0$, $M \neq 0$ is i-pure, which proves the result.

The existence of the filtration (58) only requires that D is a regular ring which satisfies (38). If D is an Auslander regular ring, then Theorem 11 proves that (58) is exactly the grade filtration (22) of M. If D is also a Cohen-Macaulay ring, then using (60), the descending filtration $\{M_i\}_{i=0,\ldots,n}$ of M gives a built-in classification of the elements of M by means of their (co)dimensions, i.e.:

$$\begin{aligned} i = 0, \ldots, n, \quad M_i = t_i(M) &= \big\{m \in M \mid \mathrm{codim}_D(Dm) \geq i\big\} \\ &= \big\{m \in M \mid \dim_D(Dm) \leq \dim(D) - i\big\}. \end{aligned}$$

This filtration is sometimes called the *codimension filtration* of M (or *equidimensional decomposition* in algebraic geometry).

Remark 12 If D satisfies the hypotheses of Theorem 9, then Theorem 9 shows that the *characteristic ideal* $J(M)$ of $\mathrm{gr}(D)$ does not depend on the choice of a good filtration of M. The *characteristic variety* of M is then defined by

$$\mathrm{char}(M) = \big\{\mathfrak{p} \in \mathrm{Spec}\big(\mathrm{gr}(D)\big) \mid J(M) \subseteq \mathfrak{p}\big\},$$

where $\mathrm{Spec}(\mathrm{gr}(D))$ is the set of prime ideals of $\mathrm{gr}(D)$ endowed with the *Zariski topology*. A well-known result in algebraic analysis states that a short exact sequence of left D-modules $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ yields the equality $\mathrm{char}(M) = \mathrm{char}(M') \cup \mathrm{char}(M'')$ (see [29, 32]). Applying this result to the short exact sequences

$$0 \longrightarrow M_{i+1} \longrightarrow M_i \longrightarrow M_i/M_{i+1} \longrightarrow 0, \quad i = 0, \ldots, n,$$

we get:

$$\mathrm{char}(M) = \bigcup_{i=0,\ldots,n} \mathrm{char}(M_i/M_{i+1}). \tag{61}$$

It can be proved that the characteristic variety $\mathrm{char}(P)$ of an i-pure module P is *equidimensional* in the sense that every irreducible component of $\mathrm{char}(P)$ has dimension $\dim(D) - i$ (see, e.g., [10]). Hence, (61) is an equidimensional decomposition of the affine algebraic variety $\mathrm{char}(M)$.

Theorem 11 shows that the grade filtration of M can be computed by means of elementary methods of module theory and homological algebra. In particular, we do not need to compute a *Cartan-Eilenberg resolution* $P^{\bullet\bullet}$ (see, e.g., [50]) of the complex (25) (called $\mathrm{Rhom}(M, D)$ in derived categories (see, e.g., [24])), the *total complex* $\mathrm{Tot}(\hom_D(P^{\bullet\bullet}, D))$ of the double complex $\hom_D(P^{\bullet\bullet}, D)$, and the spectral sequence associated with the first filtration of $\mathrm{Tot}(\hom_D(P^{\bullet\bullet}, D))$. For more details, see [2, 9–11, 22, 24, 31, 50]. Our approach has then the advantage to be easily implementable in any computer algebra system containing an implementation of Gröbner bases for (noncommutative) polynomial rings (e.g., Maple, Singular, Macaulay2, Magma, Mathematica).

The filtration (58) is a particular case of the more general *bidualizing filtration* $\{M_i\}_{i=0,\dots,n}$ of a finitely generated module M over a regular ring D [9, 10], of which the existence can be proved by means of a spectral sequence argument. In this case, M_i/M_{i+1} is then a left D-*subquotient* (i.e., a quotient of a left D-submodule) of $\mathrm{ext}^i_D(\mathrm{ext}^i_D(M, D), D)$, and not simply a left D-submodule as shown above for a regular ring D satisfying (38) (e.g., an Auslander regular ring). Finally, the results developed in [9, 49] were extended in [31] for *an Auslander-Gorenstein ring* D, namely a noetherian ring of finite *injective dimension* m as a left/right D-module (i.e., $\mathrm{ext}^i_D(M, D) = 0$ for $i > m$ and for all left/right D-modules M) [50] which satisfies the Auslander condition (see 2 of Definition 8).

Let us sum up the above results in the following algorithm.

Algorithm 1

Input: A regular ring D satisfying (38), $\mathrm{gld}(D) = n$, and a matrix $R \in D^{q\times p}$.
Output: A sequence $\{T_i\}_{i=1,\dots,n}$ of finitely generated left D-modules defined by (45) and a sequence $\{\gamma_{10} \in \hom_D(T_1, M)\} \cup \{\gamma_{(i+1)i} \in \hom_D(T_{i+1}, T_i)\}_{i=1,\dots,n}$ of left D-homomorphisms defined by (50) and (49) such that

$$\left\{M_1 = \gamma_{10}(T_1),\ M_i = (\gamma_{10} \circ \cdots \circ \gamma_{i(i-1)})(T_i),\ i = 2, \dots, n\right\}$$

is a descending filtration of M (which is the grade filtration when D is an Auslander regular ring).

1. Set $R_1 = R$, $p_1 = p$, $p_2 = q$, and $M = D^{1\times p_1}/(D^{1\times p_2} R_1)$.
2. Compute matrices $R_k \in D^{p_k \times p_{k-1}}$ for $k = 2, \dots, n$ such that (24) is an exact sequence.
3. Set $p_{kk} = p_k$, $p_{(k-1)k} = p_{k-1} = p_{(k-1)(k-1)}$, $R_{kk} = R_k$, and:
$$N_{kk} = D^{p_{kk}}/\left(R_{kk} D^{p_{(k-1)k}}\right).$$
4. For $k = 1, \dots, n$ and for $j = 1, \dots, k$, compute $R_{(k-j)k} \in D^{p_{(k-j)k}\times p_{(k-j-1)k}}$ such that (27) is an exact sequence.
5. For $k = 2, \dots, n$, compute matrices $F_{(k-2)k} \in D^{p_{(k-2)k}\times p_{(k-2)(k-1)}}$ such that:
$$R_{(k-1)(k-1)} = R_{(k-1)k} F_{(k-2)k}.$$
6. For $k = 2, \dots, n$ and for $j = 2, \dots, k$, compute $F_{(k-j)k} \in D^{p_{(k-j)k}\times p_{(k-j)(k-1)}}$ satisfying:
$$F_{(k-j)k} R_{(k-j)(k-1)} = R_{(k-j)k} F_{(k-j-1)k}.$$
7. Return the matrices R_{0i}, R_{1i}, and F_{0i} defining the left D-modules
$$\forall i = 1, \dots, n, \quad T_i = \ker_D(.R_{0i})/\operatorname{im}_D(.R_{1i}),$$

$\gamma_{10} = \mathrm{id}_{T_1}\colon T_1 = \ker_D(.R_{01})/\operatorname{im}_D(.R_{11}) \longrightarrow M = D^{1\times p_{01}}/\operatorname{im}_D(.R_{11})$, and $\gamma_{i(i-1)} \in \hom_D(T_i, T_{i-1})$ by (49) for $i = 2, \ldots, n$.

Remark 13 Using 3 of Theorem 10, i.e., $M = M_{j_D(M)}$, let us explain how Algorithm 1 can then be speeded up when $j_D(M) \geq 1$ by avoiding the computation of the left D-modules T_i's for $i = 1, \ldots, j_D(M)$. Since $\mathrm{ext}_D^i(M, D) = 0$ for $i = 0, \ldots, j_D(M) - 1$, then (25) yields the following free resolution of $N_{j_D(M) j_D(M)}$:

$$D^{p_0} \xrightarrow{R_1.} D^{p_1} \xrightarrow{R_2.} \cdots \xrightarrow{R_{j_D(M)}.} D^{p_{j_D(M)}} \xrightarrow{\kappa_{j_D(M)j_D(M)}} N_{j_D(M)j_D(M)} \longrightarrow 0. \tag{62}$$

Applying Proposition 1 to (62), $\mathrm{ext}_D^{j_D(M)}(N_{j_D(M)j_D(M)}, D) \cong \mathrm{ext}_D^1(N_{11}, D) = M_1$, where $N_{11} = D^{p_1}/(R_1 D^{p_0})$. Since $j_D(M) \geq 1$, using Theorem 1, $\ker_D(R_1.) \cong \mathrm{ext}_D^0(M, D) = 0$, and thus $M = M_1 \cong \mathrm{ext}_D^1(N_{11}, D)$. Hence, we do not need to compute the beginning of a free resolution of N_{kk} for $k = 1, \ldots, j_D(M)$, i.e., in 4 of Algorithm 1, we can only consider $k = j_D(M) + 1, \ldots, n$.

If D admits an *involution* θ, namely, $\theta\colon D \longrightarrow D$ satisfies $\theta^2 = \mathrm{id}_D$ and

$$\forall d_1, d_2 \in D, \quad \theta(d_1 + d_2) = \theta(d_1) + \theta(d_2), \qquad \theta(d_1 d_2) = \theta(d_2)\theta(d_1),$$

then we can compute the matrices $R_{(k-j)k}$ defined in 4 of Algorithm 1 by left Gröbner basis techniques. For more details, see [14].

Algorithm 1 and its improvement given in Remark 13 are implemented in the PURITYFILTRATION package [45] built upon the Maple package OREMODULES [15]. For more details, see [43]. The PURITYFILTRATION package can be used to compute the grade filtration of a finitely generated left D-module M, where D is one of the Ore algebras supported by the OREMODULES package. Algorithm 1 has also been implemented in the `homalg` based package `AbelianSystems` [7] by M. Barakat (University of Kaiserslautern) and the author, and more recently by C. Schilli (University of Aachen) in the Singular package `purityfiltration.lib` [51].

Let us now determine a finite presentation of the left D-modules T_i's defined by (45). To do that, we first consider the beginning of a finite free resolution of $P_i = D^{1\times p_{-1i}}/(D^{1\times p_{0i}} R_{0i})$, namely, matrices $R'_{1i} \in D^{p'_{1i}\times p_{0i}}$ and $R'_{2i} \in D^{p'_{2i}\times p'_{1i}}$ such that $\ker_D(.R_{0i}) = D^{1\times p'_{1i}} R'_{1i}$ and $\ker_D(.R'_{1i}) = D^{1\times p'_{2i}} R'_{2i}$ for $i = 1, \ldots, n$. We obtain the commutative diagram (69) formed by horizontal exact sequences.

Remark 14 If $R_{0k} = 0$, i.e., $\ker_D(R_{1k}.) = 0$, then applying the functor $\hom_D(\cdot, D)$ to the short exact sequence $0 \longrightarrow D^{p_{0k}} \xrightarrow{R_{1k}.} D^{p_{1k}} \xrightarrow{\kappa_{1k}} N_{1k} \longrightarrow 0$, we get the following complex

$$0 \longleftarrow D^{1\times p_{0k}} \xleftarrow{.R_{1k}} D^{1\times p_{1k}}.$$

Hence, we have $\ker_D(.R_{0k}) = D^{1\times p_{0k}}$, i.e., $R'_{1k} = I_{p_{0k}}$, $p'_{1k} = p_{0k}$, and $R'_{2k} = 0$.

Combining (57) with $\ker_D(.R_{0i}) = D^{1\times p'_{1i}} R'_{1i}$, we obtain the following characterization of the M_i's, i.e., of the $t_i(M)$'s when D is an Auslander regular ring:

$$\begin{cases} M_1 = \left(D^{1\times p'_{11}} R'_{11}\right)/\left(D^{1\times p_{11}} R_{11}\right), \\ M_i = \left(D^{1\times p'_{1i}} \left(R'_{1i} F_{0i} \ldots F_{01}\right)\right)/\left(D^{1\times p_{11}} R_{11}\right), \quad i = 2, \ldots, n. \end{cases} \tag{63}$$

Hence (63) shows that the residue classes of the rows of the matrix $R'_{1i} F_{0i} \dots F_{01}$ in the left D-module $M = D^{1\times p_{01}}/(D^{1\times p_{11}} R_{11})$ generate the left D-module M_i.

Algorithm 2

Input: A regular ring D satisfying (38), $\mathrm{gld}(D) = n$, and a matrix $R \in D^{q\times p}$.
Output: A sequence $\{M_i\}_{i=1,\dots,n}$ of left D-submodules of M defined by (63), i.e., the grade filtration (58) of M when D is an Auslander regular ring.

1. Apply Algorithm 1 to D and $R \in D^{q\times p}$ to get $R_{0i} \in D^{p_{0i}\times p_{-1i}}$ for $i = 1,\dots,n$, and $F_{0i} \in D^{p_{0i}\times p_{0(i-1)}}$ for $i = 2,\dots,n$.
2. Compute $R'_{1i} \in D^{p'_{1i}\times p_{0i}}$ such that $\ker_D(.R_{0i}) = D^{1\times p'_{1i}} R'_{1i}$ for $i = 1,\dots,n$.
3. Return the matrices $R'_{1i} F_{0i} \dots F_{01}$ (or their reductions with respect to the left D-module $D^{1\times p_{11}} R_{11}$) for $i = 1,\dots,n$, where $F_{01} = I_{p_{01}}$.

Algorithm 2 is implemented in the PURITYFILTRATION package [45].

Let us now compute a finite presentation of the left D-module M_i's. The identity $R_{1i} R_{0i} = 0$ yields $D^{1\times p_{1i}} R_{1i} \subseteq \ker_D(.R_{0i}) = D^{1\times p'_{1i}} R'_{1i}$, and thus there exists $R''_{1i} \in D^{p_{1i}\times p'_{1i}}$ such that:

$$\forall i = 1,\dots,n, \quad R_{1i} = R''_{1i} R'_{1i}. \tag{64}$$

Applying Proposition 4 to the left D-module T_i, we obtain

$$\begin{aligned}\forall i = 1,\dots,n, \quad T_i = \ker_D(.R_{0i})/\operatorname{im}_D(.R_{1i}) &= \big(D^{1\times p'_{1i}} R'_{1i}\big)/\big(D^{1\times p_{1i}} R_{1i}\big)\\ &\cong L_i \triangleq D^{1\times p'_{1i}}/\big(D^{1\times p_{1i}} R''_{1i} + D^{1\times p'_{2i}} R'_{2i}\big),\end{aligned} \tag{65}$$

where the above left D-isomorphism χ_i is defined by

$$\begin{aligned} L_i = D^{1\times p'_{1i}}/\big(D^{1\times p_{1i}} R''_{1i} + D^{1\times p'_{2i}} R'_{2i}\big) &\xrightarrow{\chi_i} T_i = \big(D^{1\times p'_{1i}} R'_{1i}\big)/\big(D^{1\times p_{1i}} R_{1i}\big)\\ \rho'_i(\lambda) &\longmapsto \rho_i\big(\lambda R'_{1i}\big),\end{aligned} \tag{66}$$

and $\rho'_i : D^{1\times p'_{1i}} \longrightarrow L_i$ is the canonical projection onto the left D-module L_i. The inverse $\chi_i^{-1} \in \hom_D(T_i, L_i)$ is defined by $\chi_i^{-1}(\rho_i(\lambda R'_{1i})) = \rho'_i(\lambda)$ for all $\lambda \in D^{1\times p'_{1i}}$.

Let us now complete the commutative diagram (69) to determine the left D-homomorphism $\overline{\gamma}_{(i+1)i}$ induced by the left D-homomorphism $\gamma_{(i+1)i}$ and the left D-isomorphisms χ_i and χ_{i+1}. Using (30) with $k = j = i$ and $i = 2,\dots,n$, we obtain $F_{0i} R_{0(i-1)} = R_{0i} F_{-1i}$. Pre-multiplying this identity by R'_{1i}, we get $R'_{1i} F_{0i} R_{0(i-1)} = R'_{1i} R_{0i} F_{-1i} = 0$, and thus $D^{1\times p'_{1i}}(R'_{1i} F_{0i}) \subseteq \ker_D(.R_{0(i-1)}) = D^{1\times p'_{1(i-1)}} R'_{1(i-1)}$, which proves the existence of a matrix $F'_{1i} \in D^{p'_{1i}\times p'_{1(i-1)}}$ such that:

$$\forall i = 2,\dots,n, \quad R'_{1i} F_{0i} = F'_{1i} R'_{1(i-1)}. \tag{67}$$

Similarly, we can prove the existence of a matrix $F'_{2i} \in D^{p'_{2i}\times p'_{2(i-1)}}$ such that:

$$\forall i = 2,\dots,n, \quad R'_{2i} F'_{1i} = F'_{2i} R'_{2(i-1)}. \tag{68}$$

Thus, the commutative diagram (70) formed by horizontal exact sequences holds.

$$
\begin{tikzcd}
D^{1\times p_{-1n}} \arrow[d, ".F_{-1n}"] & D^{1\times p_{0n}} \arrow[l, ".R_{0n}"'] \arrow[d, ".F_{0n}"] & D^{1\times p'_{1n}} \arrow[l, ".R'_{1n}"'] & D^{1\times p'_{2n}} \arrow[l, ".R'_{2n}"'] \\
D^{1\times p_{-1(n-1)}} \arrow[d, ".F_{-1(n-1)}"] & D^{1\times p_{0(n-1)}} \arrow[l, ".R_{0(n-1)}"'] \arrow[d, ".F_{0(n-1)}"] & D^{1\times p'_{1(n-1)}} \arrow[l, ".R'_{1(n-1)}"'] & D^{1\times p'_{2(n-1)}} \arrow[l, ".R'_{2(n-1)}"'] \\
\vdots \arrow[d, ".F_{-14}"] & \vdots \arrow[d, ".F_{04}"] & \vdots & \vdots \\
D^{1\times p_{-13}} \arrow[d, ".F_{-13}"] & D^{1\times p_{03}} \arrow[l, ".R_{03}"'] \arrow[d, ".F_{03}"] & D^{1\times p'_{13}} \arrow[l, ".R'_{13}"'] & D^{1\times p'_{23}} \arrow[l, ".R'_{23}"'] \\
D^{1\times p_{-12}} \arrow[d, ".F_{-12}"] & D^{1\times p_{02}} \arrow[l, ".R_{02}"'] \arrow[d, ".F_{02}"] & D^{1\times p'_{12}} \arrow[l, ".R'_{12}"'] & D^{1\times p'_{22}} \arrow[l, ".R'_{22}"'] \\
D^{1\times p_{-11}} & D^{1\times p_{01}} \arrow[l, ".R_{01}"'] & D^{1\times p'_{11}} \arrow[l, ".R'_{11}"'] & D^{1\times p'_{21}}. \arrow[l, ".R'_{21}"']
\end{tikzcd}
\tag{69}
$$

$$
\begin{tikzcd}
D^{1\times p_{-1n}} \arrow[d, ".F_{-1n}"] & {\color{blue}D^{1\times p_{0n}}} \arrow[l, ".R_{0n}"'] \arrow[d, ".F_{0n}"] & {\color{green}D^{1\times p'_{1n}}} \arrow[l, ".R'_{1n}"'] \arrow[d, ".F'_{1n}"] & D^{1\times p'_{2n}} \arrow[l, ".R'_{2n}"'] \arrow[d, ".F'_{2n}"] \\
D^{1\times p_{-1(n-1)}} \arrow[d, ".F_{-1(n-1)}"] & {\color{blue}D^{1\times p_{0(n-1)}}} \arrow[l, ".R_{0(n-1)}"'] \arrow[d, ".F_{0(n-1)}"] & {\color{green}D^{1\times p'_{1(n-1)}}} \arrow[l, ".R'_{1(n-1)}"'] \arrow[d, ".F'_{1(n-1)}"] & D^{1\times p'_{2(n-1)}} \arrow[l, ".R'_{2(n-1)}"'] \arrow[d, ".F'_{2(n-1)}"] \\
\vdots \arrow[d, ".F_{-14}"] & \vdots \arrow[d, ".F_{04}"] & \vdots \arrow[d, ".F'_{14}"] & \vdots \arrow[d, ".F'_{24}"] \\
D^{1\times p_{-13}} \arrow[d, ".F_{-13}"] & {\color{blue}D^{1\times p_{03}}} \arrow[l, ".R_{03}"'] \arrow[d, ".F_{03}"] & {\color{green}D^{1\times p'_{13}}} \arrow[l, ".R'_{13}"'] \arrow[d, ".F'_{13}"] & D^{1\times p'_{23}} \arrow[l, ".R'_{23}"'] \arrow[d, ".F'_{23}"] \\
D^{1\times p_{-12}} \arrow[d, ".F_{-12}"] & {\color{blue}D^{1\times p_{02}}} \arrow[l, ".R_{02}"'] \arrow[d, ".F_{02}"] & {\color{green}D^{1\times p'_{12}}} \arrow[l, ".R'_{12}"'] \arrow[d, ".F'_{12}"] & D^{1\times p'_{22}} \arrow[l, ".R'_{22}"'] \arrow[d, ".F'_{22}"] \\
D^{1\times p_{-11}} & {\color{blue}D^{1\times p_{01}}} \arrow[l, ".R_{01}"'] & {\color{green}D^{1\times p'_{11}}} \arrow[l, ".R'_{11}"'] & D^{1\times p'_{21}}. \arrow[l, ".R'_{21}"']
\end{tikzcd}
\tag{70}
$$

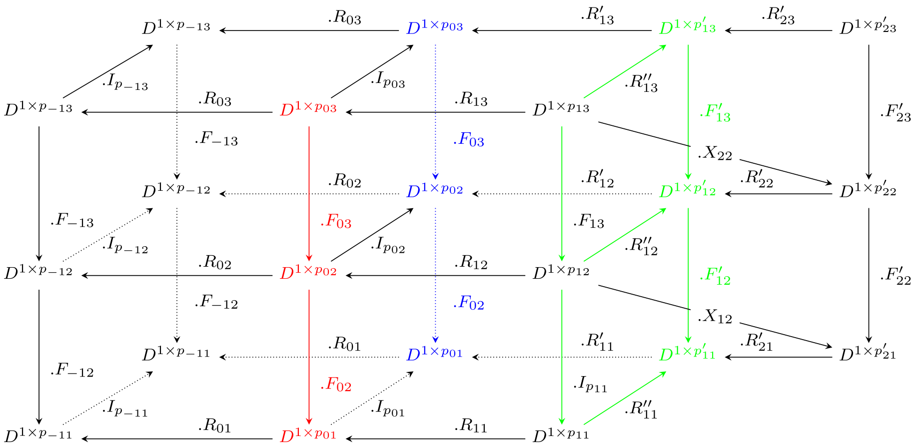

Fig. 1 Bottom part of the main diagram defining the grade filtration of M

Let us now deduce identities which will be used in what follows. Combining (28), (29), (30), (64), and (67), for $i = 1, \ldots, n$, we get

$$\begin{aligned} F_{1(i+1)}\big(R''_{1i} R'_{1i}\big) &= F_{1(i+1)} R_{1i} = R_{1(i+1)} F_{0(i+1)} = \big(R''_{1(i+1)} R'_{1(i+1)}\big) F_{0(i+1)} \\ &= R''_{1(i+1)} F'_{1(i+1)} R'_{1i}, \end{aligned}$$

and thus $(F_{1(i+1)} R''_{1i} - R''_{1(i+1)} F'_{1(i+1)}) R'_{1i} = 0$, i.e.,

$$D^{1\times p_{1(i+1)}}\big(F_{1(i+1)} R''_{1i} - R''_{1(i+1)} F'_{1(i+1)}\big) \subseteq \ker_D\big(.R'_{1i}\big) = D^{1\times p'_{2i}} R'_{2i},$$

which proves the existence of a matrix $X_{i2} \in D^{p_{1(i+1)}\times p'_{2i}}$ such that:

$$\forall i = 1, \ldots, n-1, \quad F_{1(i+1)} R''_{1i} - R''_{1(i+1)} F'_{1(i+1)} = X_{i2} R'_{2i}. \tag{71}$$

Now, $\gamma_{(i+1)i} \in \hom_D(T_{i+1}, T_i)$ then yields $\overline{\gamma}_{(i+1)i} \in \hom_D(L_{i+1}, L_i)$ defined by

$$\forall i = 1, \ldots, n-1, \quad \overline{\gamma}_{(i+1)i} = \chi_i^{-1} \circ \gamma_{(i+1)i} \circ \chi_{i+1}, \tag{72}$$

where the χ_i's are defined by (66) and $\gamma_{(i+1)i}$ by (49). Using (67), we get

$$\begin{aligned} \overline{\gamma}_{(i+1)i}\big(\rho'_{(i+1)}(\lambda)\big) &= \big(\chi_i^{-1} \circ \gamma_{(i+1)i}\big)\big(\rho_{i+1}\big(\lambda R'_{1(i+1)}\big)\big) = \chi_i^{-1}\big(\rho_i\big(\lambda R'_{1(i+1)} F_{0(i+1)}\big)\big) \\ &= \chi_i^{-1}\big(\rho_i\big(\lambda F'_{1(i+1)} R'_{1i}\big)\big) = \rho'_i\big(\lambda F'_{1(i+1)}\big), \end{aligned} \tag{73}$$

for all $\lambda \in D^{1\times p'_{1(i+1)}}$. Moreover, using (68) and (71), for $i = 1, \ldots, n-1$, we obtain

$$\begin{pmatrix} R''_{1(i+1)} \\ R'_{2(i+1)} \end{pmatrix} F'_{1(i+1)} = \begin{pmatrix} F_{1(i+1)} R''_{1i} - X_{i2} R'_{2i} \\ F'_{2(i+1)} R'_{2i} \end{pmatrix} = \begin{pmatrix} F_{1(i+1)} & -X_{i2} \\ 0 & F'_{2(i+1)} \end{pmatrix} \begin{pmatrix} R''_{1i} \\ R'_{2i} \end{pmatrix}, \tag{74}$$

which yields the following commutative exact diagram

$$\begin{tikzcd} D^{1\times(p_{1(i+1)}+p'_{2(i+1)})} \arrow[r, ".(R''^T_{1(i+1)} \quad R'^T_{2(i+1)})^T"] \arrow[d, ".G'_{1(i+1)}"] & D^{1\times p'_{1(i+1)}} \arrow[r, "\rho'_{i+1}"] \arrow[d, ".F'_{1(i+1)}"] & L_{i+1} \arrow[r] \arrow[d, "\overline{\gamma}_{(i+1)i}"] & 0 \\ D^{1\times(p_{1i}+p'_{2i})} \arrow[r, ".(R''^T_{1i} \quad R'^T_{2i})^T"] & D^{1\times p'_{1i}} \arrow[r, "\rho'_i"] & L_i \arrow[r] & 0, \end{tikzcd} \tag{75}$$

where $G'_{1(i+1)} \in D^{(p_{1(i+1)}+p'_{2(i+1)})\times(p_{1i}+p'_{2i})}$ is the first matrix appearing in the last equality of (74).

The identities $R_{11} = R''_{11} R'_{11}$ (see (64)) and $R'_{21} R'_{11} = 0$ yield the following commutative exact diagram

$$\begin{tikzcd} D^{1\times(p_{11}+p'_{21})} \arrow[r, ".(R''^T_{11} \quad R'^T_{21})^T"] \arrow[d, ".\begin{pmatrix} I_{p_{11}} \\ 0 \end{pmatrix}"] & D^{1\times p'_{11}} \arrow[r, "\rho'_1"] \arrow[d, ".R'_{11}"] & L_1 \arrow[r] \arrow[d, "\overline{\gamma}_{10}=\gamma_{10}\circ\chi_1"] & 0 \\ D^{1\times p_{11}} \arrow[r, ".R_{11}"] & D^{1\times p_{01}} \arrow[r, "\pi"] & M \arrow[r] & 0, \end{tikzcd} \tag{76}$$

where $\overline{\gamma}_{10} = \gamma_{10} \circ \chi_1 \in \hom_D(L_1, M)$ is defined by:

$$\forall \lambda \in D^{1\times p'_{11}}, \quad \overline{\gamma}_{10}\big(\rho'_1(\lambda)\big) = \pi\left(\lambda R'_{11}\right). \tag{77}$$

The matrices previously introduced can be rearranged into the three dimensional diagram whose bottom part is shown in Fig. 1. Each two dimensional diagram of Fig. 1 commutes except for the two diagrams marked in green ("faces in the depth direction") (see (71)). The horizontal sequences are in the foreground complexes and are exact in the background sequences. The vertical sequences are not complexes. The defect of exactness $T_i = \mathrm{ext}^i_D(N_{ii}, D)$ of the ith horizontal complex at $D^{1\times p_{0i}}$ (marked in red (color version online)) is isomorphic to the cokernel L_i of the left D-homomorphism $D^{1\times(p_{1i}+p'_{2i})} \longrightarrow D^{1\times p'_{1i}}$ defined by the two left D-homomorphisms $.R''_{1i} \colon D^{1\times p_{1i}} \longrightarrow D^{1\times p'_{1i}}$ and $.R'_{2i} \colon D^{1\times p'_{2i}} \longrightarrow D^{1\times p'_{1i}}$ arriving at $D^{1\times p'_{1i}}$ (marked in green (color version online)), i.e., $L_i = D^{1\times p'_{1i}}/(D^{1\times(p_{1i}+p'_{2i})}(R''^T_{1i} \quad R'^T_{2i})^T)$. The left D-homomorphism $\gamma_{i(i-1)} \colon T_i \longrightarrow T_{i-1}$ defined by (49), i.e., by means of the left D-homomorphism $.F_{0i}$ (marked in red (color version online)), then induces $\overline{\gamma}_{i(i-1)} \in \hom_D(L_i, L_{i-1})$ defined by (73), i.e., by means of the left D-homomorphism $.F'_{1i}$ (marked in green (color version online)).

Algorithm 3

Input: A regular ring D satisfying (38), $\mathrm{gld}(D) = n$, and a matrix $R \in D^{q\times p}$.
Output: A sequence $\{L_i\}_{i=1,\dots,n}$ of finitely presented left D-modules and a sequence $\{\overline{\gamma}_{10} \in \hom_D(L_1, M)\} \cup \{\overline{\gamma}_{(i+1)i} \in \hom_D(L_{i+1}, L_i)\}_{i=1,\dots,n-1}$ of left D-homomorphisms defined by (66).

1. Apply Algorithm 2 to D and $R \in D^{q\times p}$ to get matrices $R_{0i} \in D^{p_{0i}\times p_{-1i}}$ for $i = 1,\dots,n$, $F_{0i} \in D^{p_{0i}\times p_{0(i-1)}}$ for $i = 2,\dots,n$, and $R'_{1i} \in D^{p'_{1i}\times p_{0i}}$ such that $\ker_D(.R_{0i}) = D^{1\times p'_{1i}} R'_{1i}$ for $i = 1,\dots,n$.

2. Compute $R'_{2i} \in D^{p'_{2i} \times p'_{1i}}$ such that $\ker_D(.R'_{1i}) = D^{1\times p'_{2i}} R'_{2i}$ for $i = 1, \dots, n$.
3. Left factorize R_{1i} by R'_{1i} to get $R''_{1i} \in D^{p_{1i} \times p'_{1i}}$ such that $R_{1i} = R''_{1i} R'_{1i}$ for $i = 1, \dots, n$.
4. Compute $F'_{1i} \in D^{p'_{1i} \times p'_{1(i-1)}}$ such that $R'_{1i} F_{0i} = F'_{1i} R'_{1(i-1)}$ for $i = 2, \dots, n$.
5. Return $L_i = D^{1\times p'_{1i}}/(D^{1\times(p_{1i}+p'_{2i})}(R''^T_{1i} \quad R'^T_{2i})^T)$ for $i = 1, \dots, n$, the matrix R'_{11} which defines $\overline{\gamma}_{10} \in \hom_D(L_1, M)$ defined by (77), and the matrices $F'_{1(i+1)}$ which define $\overline{\gamma}_{(i+1)i} \in \hom_D(L_{i+1}, L_i)$ by (73) for $i = 1, \dots, n-1$.

Algorithm 3 is implemented in the PURITYFILTRATION package [45].

Using 3 of Proposition 5, we get the following finite presentation of $\operatorname{coker} \overline{\gamma}_{(i+1)i}$:

$$\operatorname{coker} \overline{\gamma}_{(i+1)i} = D^{1\times p'_{1i}} / \big(D^{1\times p'_{1i}} F'_{1i} + D^{1\times p_{1i}} R''_{1i} + D^{1\times p'_{2i}} R'_{2i}\big), \quad i = 1, \dots, n-1. \tag{78}$$

We denote by $\sigma_i : D^{1\times p'_{1i}} \longrightarrow \operatorname{coker} \overline{\gamma}_{(i+1)i}$ the canonical projection onto $\operatorname{coker} \overline{\gamma}_{(i+1)i}$.

Up to isomorphism, the short exact sequences

$$0 \longrightarrow T_{i+1} \xrightarrow{\gamma_{(i+1)i}} T_i \longrightarrow \operatorname{coker} \gamma_{(i+1)i} \longrightarrow 0, \quad i = 1, \dots, n-1,$$

defined in (54) (see also (46)) give rise to the following exact sequences:

$$0 \longrightarrow L_{i+1} \xrightarrow{\overline{\gamma}_{(i+1)i}} L_i \xrightarrow{\theta_i} \operatorname{coker} \overline{\gamma}_{(i+1)i} \longrightarrow 0, \quad i = 1, \dots, n-1. \tag{79}$$

Since both γ_{10} and χ_1 are injective so is $\overline{\gamma}_{10}$, and (76) yields the following short exact sequence

$$0 \longrightarrow L_1 \xrightarrow{\overline{\gamma}_{10}} M \xrightarrow{\rho} M/M_1 \longrightarrow 0, \tag{80}$$

where $M/M_1 \cong D^{1\times p_{01}} / \ker_D(.R_{01}) = D^{1\times p_{01}}/(D^{1\times p'_{11}} R'_{11})$ (see (51)).

We recall that $\operatorname{coker} \overline{\gamma}_{(i+1)i} \cong \operatorname{coker} \gamma_{(i+1)i} \subseteq \operatorname{ext}^i_D(\operatorname{ext}^i_D(M, D), D)$ (see (55)), and thus $\operatorname{coker} \overline{\gamma}_{(i+1)i}$ is either zero or an i-pure left D-module when D is an Auslander regular ring (see 3 of Theorem 8 and Remark 6).

Exact sequences (79) and (80) will be used in Sect. 4.

Remark 15 Let us point out that the left D-modules M_i's can also be characterized by means of the left D-homomorphisms $\overline{\gamma}_{i(i-1)}$'s. Combining (75) with (76), we obtain the following commutative exact diagram:

$$\begin{tikzcd}
D^{1\times(p_{1i}+p'_{2i})} \arrow[r, ".(R''^T_{1i} \quad R'^T_{2i})^T"] \arrow[d, ".\left(G'_{1i}\dots G'_{12}\begin{pmatrix} I_{p_{11}} \\ 0\end{pmatrix}\right)"] & D^{1\times p'_{1i}} \arrow[r, "\rho'_i"] \arrow[d, ".(F'_{1i}\dots F'_{12}R'_{11})"] & L_i \arrow[r] \arrow[d, "\overline{\gamma}_{10}\circ\dots\circ\overline{\gamma}_{i(i-1)}"] & 0 \\
D^{1\times p_{11}} \arrow[r, ".R_{11}"] & D^{1\times p_{01}} \arrow[r, "\pi"] & M \arrow[r] & 0.
\end{tikzcd}$$

By construction (see (67)), the identity $R'_{1i} F_{1i} \dots F_{12} = F'_{1i} \dots F'_{12} R'_{11}$ holds. Hence, using (63) and 2 of Proposition 5, we obtain:

$$M_i = \operatorname{im}(\overline{\gamma}_{10} \circ \dots \circ \overline{\gamma}_{i(i-1)}) = \big(D^{1\times p'_{1i}}\big(F'_{1i} \dots F'_{12} R'_{11}\big) + D^{1\times p_{11}} R_{11}\big)/\big(D^{1\times p_{11}} R_{11}\big).$$

The residue classes of the rows of the matrix $R'_{1i} F_{1i} \dots F_{12} = F'_{1i} \dots F'_{12} R'_{11}$ in the left D-module $M = D^{1\times p_{01}}/(D^{1\times p_{11}} R_{11})$ generate the left D-module M_i for $i = 1, \dots, n$.

Finally, we explain an efficient way to obtain the grade filtration of a nontrivial $\mathrm{ext}^i_D(N, D)$ for $i \geq 1$. We consider the case of a right D-module N (the case of a left D-module is similar). Let us first study the case of $\mathrm{ext}^1_D(N, D)$, where $N = D^q/(R D^p)$. If we introduce the Auslander transpose $M = D^{1\times p}/(D^{1\times q} R)$ of N, then the above results shows that $t_1(M) = \mathrm{ext}^1_D(N, D)$, and thus the grade filtration of $\mathrm{ext}^1_D(N, D)$ can be obtained by computing the grade filtration of M. Let us now study the case $i \geq 2$. Considering a free resolution (4) of N and introducing the right D-module $P = D^{q_{i-1}}/(S_i D^{q_i}) \cong \mathrm{im}_D(S_{i-1}.)$, then applying Proposition 1 to the following long exact sequence

$$0 \longleftarrow N \xleftarrow{\kappa} D^{q_0} \xleftarrow{S_1.} D^{q_1} \xleftarrow{S_2.} \cdots \xleftarrow{S_{i-2}.} D^{q_{i-2}} \longleftarrow P \longleftarrow 0,$$

we get $\mathrm{ext}^i_D(N, D) \cong \mathrm{ext}^1_D(P, D) = t_1(L)$, where $L = D^{1\times q_i}/(D^{1\times p_{i-1}} S_i)$ is the Auslander transpose of P, which shows that the grade filtration of L gives the grade filtration of $\mathrm{ext}^i_D(N, D)$. The corresponding algorithm is implemented in the PurityFiltration package [45].

4 Equidimensional Triangularization of Linear Systems

The purpose of this section is to apply Theorem 7 on Baer's extensions to the short exact sequences (79) and (80) to get a block-triangular matrix which presents the left D-module M, and whose block-diagonal matrices are presentation matrices of the pure left D-modules M_i/M_{i+1}, where $\{M_i\}_{i=0,\ldots,n}$ is the filtration (58) of M.

To simplify the exposition, we shall only consider the first three terms of the filtration (58) of M, i.e., $M_3 \subseteq M_2 \subseteq M_1 \subseteq M$, to obtain a presentation matrix P of M based on the presentation matrices of the left D-modules M_3, M_2/M_3, M_1/M_2, and M/M_1. If D is an Auslander regular ring, then M/M_1 (resp., M_1/M_2, M_2/M_3) is 0-pure (resp., 1-pure, 2-pure). The left D-module M_3 satisfies $j_D(M_3) \geq 3$ but it is generally not 3-pure (it is the case if $\mathrm{gld}(D) = 3$). From the clear pattern of the presentation matrix P, we can then easily obtain the result for the general case.

The approach we use here also emphasizes another advantage of our approach over the ones based on more sophisticated techniques of homological algebra. If we do no want to separate the elements of M of grade number greater than or equal to j, then we only need to compute the first j terms of the filtration (58) of M.

By (79) and (80), the following short exact sequences hold

$$\begin{aligned} &0 \longrightarrow L_3 \xrightarrow{\overline{\gamma}_{32}} L_2 \xrightarrow{\theta_2} \mathrm{coker}\,\overline{\gamma}_{32} \longrightarrow 0,\\ &0 \longrightarrow L_2 \xrightarrow{\overline{\gamma}_{21}} L_1 \xrightarrow{\theta_1} \mathrm{coker}\,\overline{\gamma}_{21} \longrightarrow 0,\\ &0 \longrightarrow L_1 \xrightarrow{\overline{\gamma}_{10}} M \xrightarrow{\rho} M/M_1 \longrightarrow 0, \end{aligned} \tag{81}$$

where L_i (resp., $\mathrm{coker}\,\overline{\gamma}_{(i+1)i}$) is defined by (65) (resp., (78)) and:

$$M/M_1 \cong D^{1\times p_{01}}/\big(D^{1\times p'_{11}} R'_{11}\big).$$

Using the definitions of L_2, L_3, and $\operatorname{coker}\overline{\gamma}_{32}$ (see (66) and (78)), the following commutative exact diagram holds

$$\begin{tikzcd}
& & 0 \arrow[d] & & \\
& & D^{1\times p_{12}} R''_{12} + D^{1\times p'_{22}} R'_{22} \arrow[d] & & \\
& D^{1\times(p'_{13}+p_{12}+p'_{22})} \arrow[r, ".(F'^T_{13} \quad R''^T_{12} \quad R'^T_{22})^T"] \arrow[d, "\psi_2"] & D^{1\times p'_{12}} \arrow[r, "\sigma_2"] \arrow[d, "\rho'_2"] & \operatorname{coker}\overline{\gamma}_{32} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L_3 \arrow[r, "\overline{\gamma}_{32}"] & L_2 \arrow[r, "\theta_2"] \arrow[d] & \operatorname{coker}\overline{\gamma}_{32} \arrow[r] & 0, \\
& & 0 & &
\end{tikzcd}$$

where $\psi_2 \colon D^{1\times(p'_{13}+p_{12}+p'_{22})} \longrightarrow L_3$ is the left D-homomorphism defined by:

$$\psi_2(e_i) = \begin{cases} \rho'_3(e_i), & i = 1, \ldots, p'_{13}, \\ 0, & i = p'_{13}+1, \ldots, p'_{13}+p_{12}+p'_{22}. \end{cases}$$

Applying Theorem 7 to the first short exact sequence of (81) with the matrix

$$A = \begin{pmatrix} I^T_{p'_{13}} & 0^T & 0^T \end{pmatrix}^T \in D^{(p'_{13}+p_{12}+p'_{22})\times p'_{13}}$$

(see Corollary 2), we obtain the following characterization of the left D-module L_2 in terms of the presentations of the left D-modules L_3 and $\operatorname{coker}\overline{\gamma}_{32}$.

Proposition 9 *With the previous notations, let us consider*

$$P_2 = \begin{pmatrix} F'_{13} & -I_{p'_{13}} \\ R''_{12} & 0 \\ R'_{22} & 0 \\ 0 & R''_{13} \\ 0 & R'_{23} \end{pmatrix} \in D^{(p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})\times(p'_{12}+p'_{13})},$$

$$Q_2 = \begin{pmatrix} R''_{12} \\ R'_{22} \end{pmatrix} \in D^{(p_{12}+p'_{22})\times p'_{12}},$$

and the following two finitely presented left D-modules:

$$\begin{cases} L_2 = D^{1\times p'_{12}} / \big(D^{1\times(p_{12}+p'_{22})}\, Q_2\big), \\ E_2 = D^{1\times(p'_{12}+p'_{13})} / \big(D^{1\times(p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})}\, P_2\big). \end{cases}$$

If $\varrho_2 : D^{1\times(p'_{12}+p'_{13})} \longrightarrow E_2$ is the canonical projection onto E_2, then we have $E_2 \cong L_2$, where the left D-isomorphism is defined by:

$$\begin{aligned} \phi_2 \colon L_2 &\longrightarrow E_2 \\ \rho'_2(\mu) &\longmapsto \varrho_2\big(\mu(I_{p'_{12}} \quad 0)\big), \\ \phi_2^{-1} \colon E_2 &\longrightarrow L_2 \\ \varrho_2(\nu) &\longmapsto \rho'_2\big(\nu\big(I^T_{p'_{12}} \quad F'^T_{13}\big)^T\big). \end{aligned} \tag{82}$$

Proof Let us consider the following matrices:

$$\begin{aligned} V_2 &= (I_{p'_{12}} \quad 0) \in D^{p'_{12}\times(p'_{12}+p'_{13})}, \\ W_2 &= \begin{pmatrix} 0 & I_{p_{12}} & 0 & 0 & 0 \\ 0 & 0 & I_{p'_{22}} & 0 & 0 \end{pmatrix} \in D^{(p_{12}+p'_{22})\times(p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})}, \\ X_2 &= \begin{pmatrix} I_{p'_{12}} \\ F'_{13} \end{pmatrix} \in D^{(p'_{12}+p'_{13})\times p'_{12}}, \\ Y_2 &= \begin{pmatrix} 0 & 0 \\ I_{p_{12}} & 0 \\ 0 & I_{p'_{22}} \\ F_{13} & -X_{22} \\ 0 & F'_{23} \end{pmatrix} \in D^{(p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})\times(p_{12}+p'_{22})}. \end{aligned}$$

Using (68) and (71) (see also Fig. 1), we can check that $Q_2 V_2 = W_2 P_2$ (resp., $P_2 X_2 = Y_2 Q_2$), which by Proposition 5 induces $\phi_2 \in \hom_D(L_2, E_2)$ (resp., $\psi_2 \in \hom_D(E_2, L_2)$) defined by (82). Since $V_2 X_2 = I_{p'_{12}}$, we get $\psi_2 \circ \phi_2 = \mathrm{id}_{L_2}$, which shows that ϕ_2 is injective. Using 3 of Proposition 5, $\operatorname{coker}\phi_2$ is finitely presented by the matrix $(V_2^T \quad P_2^T)^T \in D^{(p'_{12}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})\times(p'_{12}+p'_{13})}$, which admits the following left inverse

$$\begin{pmatrix} I_{p'_{12}} & 0 & 0 & 0 & 0 \\ F'_{13} & -I_{p'_{13}} & 0 & 0 & 0 \end{pmatrix},$$

which yields $\operatorname{coker}\phi_2 = 0$, i.e., ϕ_2 is surjective, and finally shows that ϕ_2 is an isomorphism, $E_2 \cong L_2$, and $\phi_2^{-1} = \psi_2$. □

Using the left D-isomorphism $\phi_2^{-1} \colon E_2 \longrightarrow L_2$ defined by (82), the second short exact sequence of (81) yields the following short exact sequence

$$0 \longrightarrow E_2 \xrightarrow{\overline{\gamma}_{21}\circ\phi_2^{-1}} L_1 \xrightarrow{\theta_1} \operatorname{coker}\overline{\gamma}_{21} \longrightarrow 0, \tag{83}$$

where using (73), the left D-homomorphism $\overline{\gamma}_{21} \circ \phi_2^{-1} : E_2 \longrightarrow L_1$ is defined by

$$\big(\overline{\gamma}_{21} \circ \phi_2^{-1}\big)\big(\varrho_2(\nu)\big) = \overline{\gamma}_{21}\left(\rho'_2\left(\nu\begin{pmatrix} I_{p'_{12}} \\ F'_{13} \end{pmatrix}\right)\right) = \rho'_1\left(\nu\begin{pmatrix} F'_{12} \\ F'_{13}F'_{12} \end{pmatrix}\right),$$

for all $\nu \in D^{1\times(p'_{12}+p'_{13})}$. Using the definitions of L_1, E_2, and $\operatorname{coker}\overline{\gamma}_{21}$ (see (66), Proposition 9 and (78)), we get the commutative exact diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & D^{1\times p_{11}}R''_{11}+D^{1\times p'_{21}}R'_{21} \arrow[d] & & \\
 & D^{1\times(p'_{12}+p_{11}+p'_{21})} \arrow[r, ".(F'^{T}_{12}\quad R''^{T}_{11}\quad R'^{T}_{21})^{T}"] \arrow[d, "\psi_1"] & D^{1\times p'_{11}} \arrow[r, "\sigma_1"] \arrow[d, "\rho'_1"] & \operatorname{coker}\overline{\gamma}_{21} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E_2 \arrow[r, "\overline{\gamma}_{21}\circ\phi_2^{-1}"] & L_1 \arrow[r, "\theta_1"] \arrow[d] & \operatorname{coker}\overline{\gamma}_{21} \arrow[r] & 0, \\
 & & 0 & &
\end{tikzcd}$$

where $\psi_1 : D^{1\times(p'_{12}+p_{11}+p'_{21})} \longrightarrow E_2$ is the left D-homomorphism defined by

$$\psi_1(f_j) = \begin{cases} \varrho_2(f_j F), & j = 1, \ldots, p'_{12}, \\ 0, & j = p'_{12}+1, \ldots, p'_{12}+p_{11}+p'_{21}, \end{cases}$$

$\{f_j\}_{j=1,\ldots,p'_{12}+p_{11}+p'_{21}}$ is the standard basis of $D^{1\times(p'_{12}+p_{11}+p'_{21})}$ and:

$$F = \begin{pmatrix} I_{p'_{12}} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in D^{(p'_{12}+p_{11}+p'_{21})\times(p'_{12}+p'_{13})}.$$

Applying Theorem 7 to the short exact sequence (83) with the matrix $A = F$ (see Corollary 2), we obtain the following proposition.

Proposition 10 *With the previous notations, let us consider the following matrices*

$$P_1 = \begin{pmatrix} F'_{12} & -I_{p'_{12}} & 0 \\ R''_{11} & 0 & 0 \\ R'_{21} & 0 & 0 \\ 0 & F'_{13} & -I_{p'_{13}} \\ 0 & R''_{12} & 0 \\ 0 & R'_{22} & 0 \\ 0 & 0 & R''_{13} \\ 0 & 0 & R'_{23} \end{pmatrix} \in D^{(p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})\times(p'_{11}+p'_{12}+p'_{13})},$$

$$Q_1 = \begin{pmatrix} R''_{11} \\ R'_{21} \end{pmatrix} \in D^{(p_{11}+p'_{21})\times p'_{11}},$$

and the following two finitely presented left D-modules:

$$\begin{cases} L_1 = D^{1\times p'_{11}}/(D^{1\times(p_{11}+p'_{21})}\,Q_1), \\ E_1 = D^{1\times(p'_{11}+p'_{12}+p'_{13})}/(D^{1\times(p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})}\,P_1). \end{cases}$$

If $\varrho_1 : D^{1\times(p'_{11}+p'_{12}+p'_{13})} \longrightarrow E_1$ *is the canonical projection onto* E_1, *then we have* $E_1 \cong L_1$, *where the left* D-*isomorphism is defined by*:

$$\begin{aligned} \phi_1 : L_1 &\longrightarrow E_1 \\ \rho'_1(\nu) &\longmapsto \varrho_1\big(\nu(I_{p'_{11}} \quad 0 \quad 0)\big), \\ \phi_1^{-1} : E_1 &\longrightarrow L_1 \\ \varrho_1(\lambda) &\longmapsto \rho'_1\left(\lambda \begin{pmatrix} I_{p'_{11}} \\ F'_{12} \\ F'_{13}F'_{12} \end{pmatrix}\right). \end{aligned} \tag{84}$$

Finally, we have $L_1 \cong M_1$, *with the following left* D-*isomorphisms*:

$$\begin{aligned} \chi_1 : L_1 &\longrightarrow M_1 & \qquad \chi_1^{-1} : M_1 &\longrightarrow L_1 \\ \rho'_1(\nu) &\longmapsto \pi\big(\nu R'_{11}\big), & \pi\big(\nu R'_{11}\big) &\longmapsto \rho'_1(\nu). \end{aligned}$$

Proof Let us consider the following matrices:

$$V_1 = (I_{p'_{11}} \quad 0 \quad 0) \in D^{p'_{11}\times(p'_{11}+p'_{12}+p'_{13})},$$

$$X_1 = \big(I_{p'_{11}}^T \quad F_{12}'^T \quad (F'_{13}F'_{12})^T\big)^T \in D^{(p'_{11}+p'_{12}+p'_{13})\times p'_{11}},$$

$$W_1 = \begin{pmatrix} 0 & I_{p_{11}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{p'_{21}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in D^{(p_{11}+p'_{21})\times(p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})},$$

$$Y_1 = \begin{pmatrix} 0 & 0 \\ I_{p_{11}} & 0 \\ 0 & I_{p'_{21}} \\ 0 & 0 \\ I_{p_{11}} & -X_{12} \\ 0 & F'_{22} \\ F_{13} & -F_{13}X_{12} - X_{22}F'_{22} \\ 0 & F'_{23}F'_{22} \end{pmatrix} \in D^{(p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})\times(p_{11}+p'_{21})}.$$

Using the identities (68) and (71) (see also Fig. 1), and $F_{12} = I_{p_{11}}$, we can check that $Q_1 V_1 = W_1 P_1$ (resp., $P_1 X_1 = Y_1 Q_1$), which by Proposition 5 induces $\phi_1 \in \hom_D(L_1, E_1)$ (resp., $\psi_1 \in \hom_D(E_1, L_1)$) defined by (84). Since $V_1 X_1 = I_{p'_{11}}$, we get $\psi_1 \circ \phi_1 = \mathrm{id}_{L_1}$, which shows that ϕ_1 is injective. Using 3 of Proposition 5, the left D-module $\operatorname{coker}\phi_1$ is finitely presented by the following matrix

$$\big(V_1^T \quad P_1^T\big)^T \in D^{(p'_{11}+p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})\times(p'_{11}+p'_{12}+p'_{13})},$$

which admits the following left inverse:

$$\begin{pmatrix} I_{p'_{11}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ F'_{12} & -I_{p'_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ F'_{13}F'_{12} & -F'_{13} & 0 & 0 & -I_{p'_{13}} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, $\operatorname{coker}\phi_1 = 0$, i.e., ϕ_1 is surjective, and thus ϕ_1 is an isomorphism, $E_1 \cong L_1$, and $\phi_1^{-1} = \psi_1$. Finally, the last result of Proposition 10 was proved in Remark 15. □

Using Proposition 10 and Remark 15, $\overline{\gamma}_{10} \circ \phi_1^{-1} : E_1 \longrightarrow M_1$ is then defined by:

$$\left(\chi_1 \circ \phi_1^{-1}\right)\left(\varrho_1(\lambda)\right) = \pi\left(\lambda \begin{pmatrix} R'_{11} \\ F'_{12}R'_{11} \\ F'_{13}F'_{12}R'_{11} \end{pmatrix}\right).$$

Then, the third short exact sequence (81) yields the following one:

$$0 \longrightarrow E_1 \xrightarrow{\overline{\gamma}_{10}\circ\phi_1^{-1}} M \xrightarrow{\rho} M/M_1 \longrightarrow 0. \tag{85}$$

Now, we can easily check that the following commutative exact diagram holds

$$\begin{tikzcd} & D^{1\times p'_{11}} \arrow[r, ".R'_{11}"] \arrow[d, "\psi"] & D^{1\times p_{01}} \arrow[r, "\pi'"] \arrow[d, "\pi"] & M/M_1 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & E_1 \arrow[r, "\overline{\gamma}_{10}\circ\phi_1^{-1}"] & M \arrow[r, "\rho"] & M/M_1 \arrow[r] & 0, \end{tikzcd}$$

where $\psi : D^{1\times p'_{11}} \longrightarrow E_1$ is defined by $\psi(g_k) = \varrho_1(g_k(I_{p'_{11}}\ 0\ 0))$, and $\{g_k\}_{k=1,\ldots,p'_{11}}$ is the standard basis of $D^{1\times p'_{11}}$. Then, we can apply Theorem 7 to the short exact sequence (85) with $A = (I_{p'_{11}}\ 0\ 0) \in D^{p'_{11}\times(p'_{11}+p'_{12}+p'_{13})}$ (see Corollary 2) to get the following theorem.

Theorem 12 *Let D be a regular ring which satisfies* (38). *With the previous notations, let* $P \in D^{(p'_{11}+p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})\times(p_{01}+p'_{11}+p'_{12}+p'_{13})}$ *be defined by*

$$P = \begin{pmatrix} R'_{11} & -I_{p'_{11}} & 0 & 0 \\ 0 & F'_{12} & -I_{p'_{12}} & 0 \\ 0 & R''_{11} & 0 & 0 \\ 0 & R'_{21} & 0 & 0 \\ 0 & 0 & F'_{13} & -I_{p'_{13}} \\ 0 & 0 & R''_{12} & 0 \\ 0 & 0 & R'_{22} & 0 \\ 0 & 0 & 0 & R''_{13} \\ 0 & 0 & 0 & R'_{23} \end{pmatrix},$$

and the following two finitely presented left D-modules:

$$\begin{cases} M = D^{1\times p_{01}}/\big(D^{1\times p_{11}} R_{11}\big), \\ E = D^{1\times (p_{01}+p'_{11}+p'_{12}+p'_{13})}/\big(D^{1\times (p'_{11}+p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})} P\big). \end{cases}$$

If $\varrho\colon D^{1\times (p_{01}+p'_{11}+p'_{12}+p'_{13})} \longrightarrow E$ *is the canonical projection onto* E, *then we have* $E \cong M$, *where the left* D*-isomorphism is defined by*:

$$\begin{aligned} \phi\colon M &\longrightarrow E \\ \pi(\lambda) &\longmapsto \varrho\big(\lambda (I_{p_{01}} \quad 0 \quad 0 \quad 0)\big), \\ \phi^{-1}\colon E &\longrightarrow M \\ \varrho(\epsilon) &\longmapsto \pi\left(\epsilon \begin{pmatrix} I_{p_{01}} \\ R'_{11} \\ F'_{12} R'_{11} \\ F'_{13} F'_{12} R'_{11} \end{pmatrix}\right). \end{aligned} \tag{86}$$

Proof Let us consider the following matrices:

$$\begin{aligned} &V = (I_{p_{01}} \quad 0 \quad 0 \quad 0) \in D^{p_{01}\times (p_{01}+p'_{11}+p'_{12}+p'_{13})}, \\ &W = \big(R''_{11} \quad 0 \quad I_{p'_{11}} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\big) \in D^{p_{11}\times (p'_{11}+p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})}, \\ &X = \begin{pmatrix} I_{p_{01}} \\ R'_{11} \\ F'_{12} R'_{11} \\ F'_{13} F'_{12} R'_{11} \end{pmatrix} \in D^{(p_{01}+p'_{11}+p'_{12}+p'_{13})\times p_{01}}, \\ &Y = \begin{pmatrix} 0 \\ 0 \\ I_{p_{11}} \\ 0 \\ 0 \\ I_{p_{11}} \\ 0 \\ F_{13} \\ 0 \end{pmatrix} \in D^{(p'_{11}+p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})\times p_{11}}. \end{aligned}$$

Now, using (68) and (71) (see also Fig. 1), we can check that $R_{11} V = W P$ (resp., $P X = Y R_{11}$), which by Proposition 5 induces $\phi \in \hom_D(M, E)$ (resp., $\psi \in \hom_D(E, M)$) defined by (86). Moreover, since $V X = I_{p_{01}}$, we get $\psi \circ \phi = \mathrm{id}_M$, which shows that ϕ is injective. Using 3 of Proposition 5, the left D-module $\operatorname{coker}\phi$ is finitely presented by the matrix

$$\big(V^T \quad P^T\big)^T \in D^{(p_{01}+p'_{11}+p'_{12}+p'_{13})\times (p_{01}+p'_{11}+p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})},$$

which admits the following left inverse

$$\begin{pmatrix} I_{p_{01}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ R'_{11} & -I_{p'_{11}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ F'_{12} R'_{11} & -F'_{12} & -I_{p'_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ F'_{13} F'_{12} R'_{11} & -F'_{13} F'_{12} & -F'_{13} & 0 & 0 & -I_{p'_{13}} & 0 & 0 & 0 & 0 \end{pmatrix},$$

which yields $\operatorname{coker}\phi = 0$, i.e., ϕ is surjective, and finally shows that ϕ is an isomorphism, $E \cong M$, and $\phi^{-1} = \psi$. □

We note that (71) for $i = 1$ and $F_{12} = I_{p_{11}}$ yield the following identity:

$$R''_{11} = R''_{12} F'_{12} + X_{12} R'_{21}. \tag{87}$$

Hence, we can check that

$$\begin{pmatrix} 0 & R''_{11} & 0 & 0 \end{pmatrix} = R''_{12} \begin{pmatrix} 0 & F'_{12} & -I_{p'_{12}} & 0 \end{pmatrix} + X_{12} \begin{pmatrix} 0 & R'_{21} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & R''_{12} & 0 \end{pmatrix},$$

which shows that the third row of P containing the matrix R''_{11} is just a left D-linear combination of the others. We obtain the following corollary of Theorem 12.

Corollary 4 *With the hypotheses and the notations of Theorem 12, if*

$$Q = \begin{pmatrix} R'_{11} & -I_{p'_{11}} & 0 & 0 \\ 0 & F'_{12} & -I_{p'_{12}} & 0 \\ 0 & R'_{21} & 0 & 0 \\ 0 & 0 & F'_{13} & -I_{p'_{13}} \\ 0 & 0 & R''_{12} & 0 \\ 0 & 0 & R'_{22} & 0 \\ 0 & 0 & 0 & R''_{13} \\ 0 & 0 & 0 & R'_{23} \end{pmatrix}$$
$$\in D^{(p'_{11}+p'_{12}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})\times(p_{01}+p'_{11}+p'_{12}+p'_{13})},$$

then we have

$$\begin{aligned} M &= D^{1\times p_{01}} / \big(D^{1\times p_{11}} R_{11}\big) \\ &\cong E = D^{1\times(p_{01}+p'_{11}+p'_{12}+p'_{13})} / \big(D^{1\times(p'_{11}+p'_{12}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})} Q\big), \end{aligned}$$

where the isomorphism is defined by (86).

If $\mathcal{F}$ is a left D-module, then $M \cong E$ yields $\ker_{\mathcal{F}}(R_{11}.) \cong \ker_{\mathcal{F}}(P.) = \ker_{\mathcal{F}}(Q.)$ (see Theorem 1). Applying the contravariant $\hom_D(\cdot, \mathcal{F})$ to the diagram defined in Fig. 1, we obtain the diagram defined in Fig. 2 formed by horizontal complexes of abelian groups. Using (86) and $R = R_{11}$, we get the following corollary.

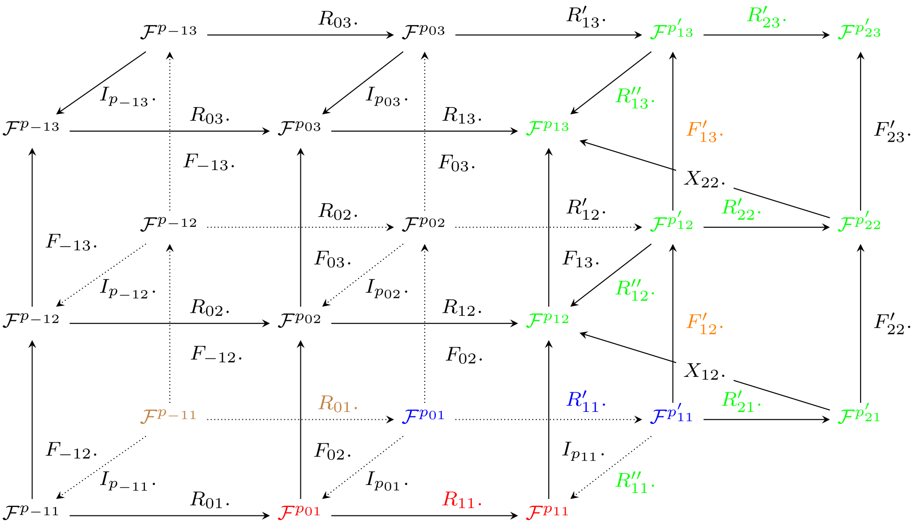

Fig. 2 Dual of Fig. 1

Corollary 5 *If D is a regular ring which satisfies (38), $R \in D^{q\times p}$, and $\mathcal{F}$ a left D-module, then* $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(Q.)$, *i.e., the following system equivalence holds*

$$R\eta = 0 \quad \Leftrightarrow \quad \begin{cases} R'_{11}\zeta - \tau_1 = 0, \\ F'_{12}\tau_1 - \tau_2 = 0, \\ R'_{21}\tau_1 = 0, \\ F'_{13}\tau_2 - \tau_3 = 0, \\ R''_{12}\tau_2 = 0, \\ R'_{22}\tau_2 = 0, \\ R''_{13}\tau_3 = 0, \\ R'_{23}\tau_3 = 0, \end{cases} \tag{88}$$

under the following invertible transformations:

$$\begin{array}{ll} \gamma: \ \ker_{\mathcal{F}}(Q.) \longrightarrow \ker_{\mathcal{F}}(R.) & \gamma^{-1}: \ \ker_{\mathcal{F}}(R.) \longrightarrow \ker_{\mathcal{F}}(Q.) \\ \begin{pmatrix} \zeta \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} \longmapsto \eta = \zeta, & \eta \longmapsto \begin{pmatrix} \zeta \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} I_{p_01} \\ R'_{11} \\ F'_{12}R'_{11} \\ F'_{13}F'_{12}R'_{11} \end{pmatrix} \eta. \end{array} \tag{89}$$

Remark 16 If D is an Auslander regular ring and a Cohen-Macaulay ring and

$$S_0 = R'_{11}, \qquad S_1 = \begin{pmatrix} F'_{12} \\ R''_{11} \\ R'_{21} \end{pmatrix}, \qquad S'_1 = \begin{pmatrix} F'_{12} \\ R'_{21} \end{pmatrix},$$

$$S_2 = \begin{pmatrix} F'_{13} \\ R''_{12} \\ R'_{22} \end{pmatrix}, \qquad S_3 = \begin{pmatrix} R''_{13} \\ R'_{23} \end{pmatrix},$$

then, within mathematical systems theory, we have:

1. $\ker_{\mathcal{F}}(S_3.) \cong \hom_D(L_3, \mathcal{F}) \cong \hom_D(\mathrm{ext}^3_D(N_{33}, D), \mathcal{F})$ is either 0 or has dimension less than or equal to $\dim(D) - 3$,
2. $\ker_{\mathcal{F}}(S_2.) \cong \hom_D(\mathrm{coker}\,\overline{\gamma}_{32}, \mathcal{F}) \cong \hom_D(\mathrm{coker}\,\gamma_{32}, \mathcal{F})$ is either 0 or has dimension $\dim(D) - 2$,
3. $\ker_{\mathcal{F}}(S_1.) = \ker_{\mathcal{F}}(S'_1.) \cong \hom_D(\mathrm{coker}\,\overline{\gamma}_{21}, \mathcal{F}) \cong \hom_D(\mathrm{coker}\,\gamma_{21}, \mathcal{F})$ is either 0 or has dimension $\dim(D) - 1$,
4. $\ker_{\mathcal{F}}(S_0.) \cong \hom_D(M/M_1, \mathcal{F})$ has dimension $\dim(D)$ when it is nonzero.

If R_3 has full row rank, i.e., $\ker_D(.R_3) = 0$, then $N_{33} \cong \mathrm{ext}^3_D(M, D)$, and thus $\mathrm{ext}^3_D(N_{33}, D) \cong \mathrm{ext}^3_D(\mathrm{ext}^3_D(M, D), D)$ is either zero or a 3-pure left D-module, which shows that $\ker_{\mathcal{F}}(S_3.)$ has $\dim(D) - 3$ when it is nonzero.

Hence, the linear system $\ker_{\mathcal{F}}(R.)$ can be obtained by integrating the linear system $\ker_{\mathcal{F}}(Q.)$, i.e., by integrating in cascade the linear system $\ker_{\mathcal{F}}(S_3.)$ of dimension less than or equal to $\dim(D) - 3$, then the inhomogeneous linear systems of dimension respectively $\dim(D) - 2$, $\dim(D) - 1$, and $\dim(D)$. Finally, if $\mathcal{F}$ is an injective left D-module, then the linear system $\ker_{\mathcal{F}}(R'_{11}.)$ of dimension $\dim(D)$ is parametrized by R_{01}, i.e., $\ker_{\mathcal{F}}(R'_{11}.) = R_{01}\mathcal{F}^{p-l_1}$.

Example 6 Let us consider an example, first studied by Janet (see [26] and the references therein) and considered again in [37], defined by the $D = \mathbb{Q}[\partial_1, \partial_2, \partial_3]$-module $M = D^{1\times 4}/(D^{1\times 6} R)$ finitely presented by the following matrix:

$$R = \begin{pmatrix} 0 & -2\partial_1 & \partial_3 - 2\partial_2 - \partial_1 & -1 \\ 0 & \partial_3 - 2\partial_1 & 2\partial_2 - 3\partial_1 & 1 \\ \partial_3 & -6\partial_1 & -2\partial_2 - 5\partial_1 & -1 \\ 0 & \partial_2 - \partial_1 & \partial_2 - \partial_1 & 0 \\ \partial_2 & -\partial_1 & -\partial_2 - \partial_1 & 0 \\ \partial_1 & -\partial_1 & -2\partial_1 & 0 \end{pmatrix}.$$

The D-module M admits the following finite free resolution:

$$0 \longleftarrow M \xleftarrow{\pi} D^{1\times 4} \xleftarrow{.R} D^{1\times 6} \xleftarrow{.R_2} D^{1\times 4} \xleftarrow{.R_3} D \longleftarrow 0,$$

$$R_2 = \begin{pmatrix} 2\partial_2 & \partial_2 & -\partial_2 & -\partial_3 & \partial_3 & 0 \\ 2\partial_1 & \partial_2 & -2\partial_1 + \partial_2 & -\partial_3 & 8\partial_1 - \partial_3 & -8\partial_2 + 2\partial_3 \\ 0 & \partial_1 - \partial_2 & \partial_1 - \partial_2 & \partial_3 & -8\partial_1 + \partial_3 & 8\partial_2 - \partial_3 \\ 0 & 0 & 0 & \partial_1 & -\partial_1 & \partial_2 \end{pmatrix},$$

$$R_3 = (\partial_1 \quad \partial_2 \quad -\partial_2 \quad \partial_3).$$

Using the notations $R_{11} = R$, $R_{22} = R_2$, and $R_{33} = R_3$, the commutative diagram (32) becomes the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & D \arrow[r, "R_{13}."] & D^4 \arrow[r, "R_{23}."] & D^4 \arrow[r, "R_{33}."] & D \arrow[r, "\kappa_{33}"] & N_{33} \arrow[r] & 0 \\
 & 0 \arrow[r] & D^3 \arrow[u, "F_{03}."] \arrow[r, "R_{12}."] & D^6 \arrow[u, "F_{13}."] \arrow[r, "R_{22}."] & D^4 \arrow[u, equal] \arrow[r, "\kappa_{22}"] & N_{22} \arrow[r] & 0 & \\
0 \arrow[r] & D \arrow[r, "R_{01}."] & D^4 \arrow[u, "F_{02}."] \arrow[r, "R_{11}."] & D^6 \arrow[u, equal] \arrow[r, "\kappa_{11}"] & N_{11} \arrow[r] & 0 & & \\
 & 0 \arrow[r] & D^4 \arrow[u, equal] \arrow[r, "\kappa_{00}"] & N_{00} \arrow[r] & 0, & & &
\end{tikzcd}$$

which horizontal sequences are exact and with the following notations:

$$R_{01} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ \partial_1 - 2\partial_2 + \partial_3 \end{pmatrix}, \qquad R_{12} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 4\partial_1 - \partial_3 & 0 \\ 1 & 4\partial_1 - \partial_3 & \partial_3 \\ 0 & \partial_1 - \partial_2 & 0 \\ 0 & \partial_1 - \partial_2 & 0 \\ 0 & 0 & \partial_1 \end{pmatrix},$$

$$R_{23} = \begin{pmatrix} -\partial_3 & \partial_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \partial_1 & -1 & \partial_3 \\ \partial_1 & 0 & 0 & \partial_2 \end{pmatrix}, \qquad R_{13} = \begin{pmatrix} -\partial_2 \\ -\partial_3 \\ 0 \\ \partial_1 \end{pmatrix},$$

$$F_{02} = \begin{pmatrix} 0 & -2\partial_1 & -\partial_1 - 2\partial_2 + \partial_3 & -1 \\ 0 & -1 & -1 & 0 \\ 1 & -1 & -2 & 0 \end{pmatrix}, \qquad F_{03} = (0 \quad 0 \quad 1),$$

$$F_{13} = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 0 \\ 2\partial_1 & \partial_2 & -2\partial_1 + \partial_2 & -\partial_3 & 8\partial_1 - \partial_3 & -8\partial_2 + 2\partial_3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$R_{03} = 0$, and $R_{02} = 0$. Using Remark 14 with $p_{03} = 1$ and $p_{02} = 3$, we get $R'_{13} = 1$, $R'_{12} = I_3$, $R'_{21} = 0$, $R'_{22} = 0$, and $R'_{23} = 0$. Then, (70) becomes the following the commutative diagram

$$\begin{array}{ccccccc}
0 & \longleftarrow & D & \xleftarrow{.R'_{13}} & D & \longleftarrow & 0 \\
 & & \downarrow .F_{03} & & \downarrow .F'_{13} & & \\
0 & \longleftarrow & D^{1\times 3} & \xleftarrow{.R'_{12}} & D^{1\times 3} & \longleftarrow & 0 \\
 & & \downarrow .F_{02} & & \downarrow .F'_{12} & & \\
D & \xleftarrow{.R_{01}} & D^{1\times 4} & \xleftarrow{.R'_{11}} & D^{1\times 3} & \longleftarrow & 0,
\end{array}$$

with the following notations:

$$R'_{11} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \partial_1 - 2\partial_2 + \partial_3 & -1 \end{pmatrix},$$

$$F'_{13} = F_{03}, \qquad F'_{12} = \begin{pmatrix} 0 & -2\partial_1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

Moreover, using (64), we have $R''_{13} = R_{13}$, $R''_{12} = R_{12}$, and:

$$R''_{11} = \begin{pmatrix} 0 & -2\partial_1 & 1 \\ 0 & -2\partial_1 + \partial_3 & -1 \\ \partial_3 & -6\partial_1 & 1 \\ 0 & -\partial_1 + \partial_2 & 0 \\ \partial_2 & -\partial_1 & 0 \\ \partial_1 & -\partial_1 & 0 \end{pmatrix}.$$

Since $\ker_D(.R_3) = 0$, we have $N_{33} \cong \mathrm{ext}_D^3(M, D)$, and thus $\mathrm{ext}_D^3(N_{33}, D) \cong \mathrm{ext}_D^3(\mathrm{ext}_D^3(M, D), D)$, which shows that $\{M_i\}_{i=0,\ldots,3}$ defined by (58) is the grade filtration of M.

Using (45) and (65), where $N_{11} = D^6/(R_{11}D^4)$, $N_{22} = D^4/(R_{22}D^6)$, and $N_{33} = D/(R_{33}D^4)$, we obtain the following finitely left D-modules:

$$\begin{cases} L_1 = D^{1\times 3}/\big(D^{1\times 6}R''_{11}\big) \cong \mathrm{ext}_D^1(N_{11}, D) \cong t(M), \\ L_2 = D^{1\times 3}/\big(D^{1\times 6}R_{12}\big) \cong \mathrm{ext}_D^2(N_{22}, D), \\ L_3 = D/\big(D^{1\times 4}R_{13}\big) \cong \mathrm{ext}_D^3(N_{33}, D). \end{cases}$$

Corollary 4 yields $M \cong E = D^{1\times 11}/(D^{1\times 17} Q)$, where Q is defined by:

$$Q = \begin{pmatrix} 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_1 - 2\partial_2 + \partial_3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\partial_1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4\partial_1 - \partial_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4\partial_1 - \partial_3 & \partial_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 - \partial_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 - \partial_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 \end{pmatrix}.$$

Let us compute $\ker_{\mathcal{F}}(Q.)$, where $\mathcal{F} = C^\infty(\mathbb{R}^3)$. We first integrate the last diagonal block of Q, i.e., the 0-dimensional (holonomic) linear system $\ker_{\mathcal{F}}(R_{13}.)$:

$$\begin{cases} -\partial_2 \tau_3 = 0, \\ -\partial_3 \tau_3 = 0, \\ \partial_1 \tau_3 = 0, \end{cases} \quad \Leftrightarrow \quad \tau_3 = c_1 \in \mathbb{R}.$$

Then, we integrate the inhomogeneous linear system in $\tau_2 = (\tau_{21}\ \tau_{22}\ \tau_{23})^T$ and τ_3 formed by the third triangular block of Q (which homogeneous part is purely subholonomic), namely:

$$\begin{cases} \tau_{23} - \tau_3 = 0, \\ \tau_{21} = 0, \\ -\tau_{21} + (4\partial_1 - \partial_3)\tau_{22} = 0, \\ \tau_{21} + (4\partial_1 - \partial_3)\tau_{22} + \partial_3 \tau_{23} = 0, \\ (\partial_1 - \partial_2)\tau_{22} = 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} \tau_{23} = \tau_3 = c_1, \\ \tau_{21} = 0, \\ (4\partial_1 - \partial_3)\tau_{22} = 0, \\ (\partial_1 - \partial_2)\tau_{22} = 0. \end{cases}$$

We obtain $\tau_{21} = 0$, $\tau_{22} = f_1(x_3 + \frac{1}{4}(x_1 + x_2))$, where f_1 is an arbitrary smooth function, and $\tau_{23} = c_1$, where c_1 is an arbitrary constant. Then, we integrate the inhomogeneous linear system in $\tau_1 = (\tau_{11} \quad \tau_{12} \quad \tau_{13})^T$ and τ_2 formed by the second triangular block of Q, namely:

$$\begin{cases} -2\partial_1 \tau_{12} + \tau_{13} - \tau_{21} = 0, \\ -\tau_{12} - \tau_{22} = 0, \\ \tau_{11} - \tau_{12} - \tau_{23} = 0, \end{cases} \Leftrightarrow \begin{cases} \tau_{12} = -\tau_{22} = -f_1\left(x_3 + \frac{1}{4}(x_1 + x_2)\right), \\ \tau_{11} = -\tau_{22} + \tau_{23} = -f_1\left(x_3 + \frac{1}{4}(x_1 + x_2)\right) + c_1, \\ \tau_{13} = -2\partial_1 \tau_{22} + \tau_{21} = -\frac{1}{2}\dot{f}_1\left(x_3 + \frac{1}{4}(x_1 + x_2)\right). \end{cases}$$

The entries of τ_1 are 1-dimensional and not 2-dimensional. This result comes from the fact that the matrix S'_1 defined in Remark 16 admits a left inverse. Thus, we have $M_1/M_2 = 0$, i.e., $M_1 = M_2$, which yields $\ker_{\mathcal{F}}(S'_1.) \cong \hom_D(\operatorname{coker} \overline{\gamma}_{21}, \mathcal{F}) \cong \hom_D(\operatorname{coker} \gamma_{21}, \mathcal{F}) = 0$. Finally, we integrate the inhomogeneous linear system in $\zeta = (\zeta_1 \ldots \zeta_4)^T$ and τ_1 formed by the first triangular block of P, namely:

$$\begin{cases} \zeta_1 - \zeta_3 - \tau_{11} = 0, \\ \zeta_2 + \zeta_3 - \tau_{12} = 0, \\ (\partial_1 - 2\partial_2 + \partial_3)\zeta_3 - \zeta_4 - \tau_{13} = 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} \zeta_1 - \zeta_2 = -f_1\left(x_3 + \frac{1}{4}(x_1 + x_2)\right) + c_1, \\ \zeta_2 + \zeta_3 = -f_1\left(x_3 + \frac{1}{4}(x_1 + x_2)\right), \\ (\partial_1 - 2\partial_2 + \partial_3)\zeta_3 - \zeta_4 = -\frac{1}{2}\dot{f}_1\left(x_3 + \frac{1}{4}(x_1 + x_2)\right). \end{cases} \tag{90}$$

The torsion-free D-module $M/t(M) = D^{1\times 4}/(D^{1\times 3} R'_{11})$ can be parametrized by means of R_{01}, i.e., $M/t(M) \cong D^{1\times 4} R_{01}$. Since $\mathcal{F}$ is an injective D-module, the linear system $\ker_{\mathcal{F}}(R'_{11}.)$ is parametrized by R_{01}, i.e., $\ker_{\mathcal{F}}(R'_{11}.) = R_{01}\mathcal{F}$. Moreover, R'_{11} admits the following right inverse over D

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and the *Quillen-Suslin theorem* (see, e.g., [20, 50]) implies that $M/t(M)$ is a free D-module of rank 1. The general $\mathcal{F}$-solution of (90) is then defined by $\zeta = R_{01}\xi + X\tau_1$

$$\begin{cases} \zeta_1 = \xi - f_1\left(x_3 + \frac{1}{4}(x_1 + x_2)\right) + c_1, \\ \zeta_2 = -\xi - f_1\left(x_3 + \frac{1}{4}(x_1 + x_2)\right), \\ \zeta_3 = \xi, \\ \zeta_4 = (\partial_1 - 2\partial_2 + \partial_3)\xi + \frac{1}{2}\dot{f}_1\left(x_3 + \frac{1}{4}(x_1 + x_2)\right), \end{cases}$$

for all $\xi \in C^\infty(\mathbb{R}^3)$, for all $f_1 \in C^\infty(\mathbb{R})$, and for all $c_1 \in \mathbb{R}$. For more details, see [46]. Using the D-isomorphism γ defined by (89), we obtain

$$\begin{cases} -2\partial_1\eta_2 + \partial_3\eta_3 - 2\partial_2\eta_3 - \partial_1\eta_3 - \eta_4 = 0, \\ \partial_3\eta_2 - 2\partial_1\eta_2 + 2\partial_2\eta_3 - 3\partial_1\eta_3 + \eta_4 = 0, \\ \partial_3\eta_1 - 6\partial_1\eta_2 - 2\partial_2\eta_3 - 5\partial_1\eta_3 - \eta_4 = 0, \\ \partial_2\eta_2 - \partial_1\eta_2 + \partial_2\eta_3 - \partial_1\eta_3 = 0, \\ \partial_2\eta_1 - \partial_1\eta_2 - \partial_2\eta_3 - \partial_1\eta_3 = 0, \\ \partial_1\eta_1 - \partial_1\eta_2 - 2\partial_1\eta_3 = 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} \eta_1 = \xi - f_1\left(x_3 + \dfrac{1}{4}(x_1 + x_2)\right) + c_1, \\ \eta_2 = -\xi - f_1\left(x_3 + \dfrac{1}{4}(x_1 + x_2)\right), \\ \eta_3 = \xi, \\ \eta_4 = (\partial_1 - 2\partial_2 + \partial_3)\xi + \dfrac{1}{2}\dot{f}_1\left(x_3 + \dfrac{1}{4}(x_1 + x_2)\right), \end{cases}$$

where ξ (resp., f_1, c_1) is an arbitrary function of $C^\infty(\mathbb{R}^3)$ (resp., $C^\infty(\mathbb{R})$, constant).

We note that the presentation matrix Q of E can be simplied by elementary operations. In particular, we can check $M \cong E \cong \overline{E} = D^{1\times 3}/(D^{1\times 5}\overline{Q})$, where

$$\overline{Q} = \begin{pmatrix} \partial_3 - 4\partial_1 & 0 & 0 \\ \partial_3 - 4\partial_2 & 0 & 0 \\ 0 & 0 & \partial_1 \\ 0 & 0 & \partial_2 \\ 0 & 0 & \partial_3 \end{pmatrix},$$

and the isomorphism $f : M \longrightarrow \overline{E}$ is defined by $f(\pi(\lambda)) = \overline{\varrho}(\lambda U)$ for all $\lambda \in D^{1\times 4}$, where $\overline{\varrho} : D^{1\times 3} \longrightarrow \overline{E}$ is the canonical projection onto $\overline{E}$ and U is defined by:

$$U = \begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \\ 2\partial_1 & -\partial_1 + 2\partial_2 - \partial_3 & 0 \end{pmatrix}.$$

Finally, we have $M \cong D \oplus D/(\partial_3 - 4\partial_1, \partial_3 - 4\partial_2) \oplus D/(\partial_1, \partial_2, \partial_3)$, i.e., M is the direct sum of the 0-pure D-module D, of the 1-pure D-module $D/(\partial_3 - 4\partial_1, \partial_3 - 4\partial_2)$, and of the 3-pure D-module $D/(\partial_1, \partial_2, \partial_3)$.

For more examples coming from mathematical physics, mathematical systems theory, and algebraic geometry, see [43, 45]. For instance, using the PURITYFILTRATION package, we can show that the torsion submodule of the differential module M defined by the linearized Einstein equations in the vacuum (see, e.g., [14]) is 1-pure (see [45]), and thus every nontrivial torsion element m of M defines a pure differential module of dimension 3. For more details, see [43].

Using the regular patterns of the matrix P and (86), we can easily generalize Theorem 12, Corollary 5, and Remark 16 as follows.

Theorem 13 *Let D be a regular ring D satisfying* (38), $\mathrm{gld}(D) = n$, *and* $R \in D^{q\times p}$. *Then, there exists a matrix $\overline{R} \in D^{\overline{q}\times \overline{p}}$ of the form*

$$\overline{R} = \begin{pmatrix} R'_{11} & -I_{p'_{11}} & 0 & 0 & 0 & 0 \\ 0 & F'_{12} & -I_{p'_{12}} & 0 & 0 & 0 \\ 0 & R'_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & \vdots & \vdots & 0 & 0 \\ 0 & 0 & \vdots & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & F'_{1n} & -I_{p'_{1n}} \\ 0 & 0 & 0 & 0 & R''_{1(n-1)} & 0 \\ 0 & 0 & 0 & 0 & R'_{2(n-1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & R''_{1n} \\ 0 & 0 & 0 & 0 & 0 & R'_{2n} \end{pmatrix} \tag{91}$$

such that $M = D^{1\times p}/(D^{1\times q} R) \cong \overline{M} = D^{1\times \overline{p}}/(D^{1\times \overline{q}}\,\overline{R})$.

Moreover, if $R'_{11} \in D^{p'_{11}\times p_{01}}$ *and* $\overline{\pi} : D^{1\times \overline{p}} \longrightarrow \overline{M}$ *is the canonical projection onto* $\overline{M}$, *then there exist matrices* F'_{1i} *for* $i = 2, \ldots, n$ *such that*:

$$\begin{aligned} \varphi : M &\longrightarrow \overline{M} \\ \pi(\lambda) &\longmapsto \overline{\pi}\big(\lambda(I_{p_{01}} \quad 0 \quad \cdots \quad 0)\big), \\ \varphi^{-1} : \overline{M} &\longrightarrow M \\ \overline{\pi}(\mu) &\longmapsto \pi\left(\mu\begin{pmatrix} I_{p_{01}} \\ R'_{11} \\ F'_{12}R'_{11} \\ \vdots \\ F'_{1n}\cdots F'_{12}R'_{11}\end{pmatrix}\right). \end{aligned}$$

If $\mathcal{F}$ is a left D-module, then $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(\overline{R}.)$, *where*:

$$\overline{\gamma} : \ker_{\mathcal{F}}(\overline{R}.) \longrightarrow \ker_{\mathcal{F}}(R.) \qquad \overline{\gamma}^{-1} : \ker_{\mathcal{F}}(R.) \longrightarrow \ker_{\mathcal{F}}(\overline{R}.)$$

$$\begin{pmatrix} \zeta \\ \tau_1 \\ \vdots \\ \tau_n \end{pmatrix} \longmapsto \eta = \zeta, \qquad \eta \longmapsto \begin{pmatrix} \zeta \\ \tau_1 \\ \vdots \\ \tau_n \end{pmatrix} = \begin{pmatrix} I_{p_{01}} \\ R'_{11} \\ \vdots \\ F'_{1n} \cdots F'_{12}R'_{11} \end{pmatrix}\eta.$$

Finally, if D is an Auslander regular ring, then the grade filtration $\{M_i\}_{i=0,\ldots,n}$ of M is defined by the left D-module M_i finitely presented by $(R''^T_{1i} \quad R'^T_{2i})^T$, and M_i/M_{i+1} is the i-pure left D-module finitely presented by R'_{11} for i=0, by $(F'^T_{1(i+1)} \; R''^T_{1i} \; R'^T_{2i})^T$ for $i = 1, \ldots, n-1$, and by $(R''^T_{1n} \; R'^T_{2n})^T$ for $i = n$.

We note that the presentation matrix $\overline{R}$ has a remarkably simple form (block-diagonal and single off-diagonal). It does not seem that it can easily be obtained from the classical black-box spectral sequence approach [2, 9, 10, 22, 31, 49].

Remark 17 We note that $M_i = M_{i+1}$ iff $S_i = (F'^T_{1(i+1)}\ R''^T_{1i}\ R'^T_{2i})^T$ admits a left inverse. It shows that the matrix $\overline{R}$ can sometimes be simplified especially if Gröbner/Janet bases can be computed over D. Moreover, elementary operations can also be applied to simplify the matrix S_i (see, e.g., Example 6). Using inductively Proposition 6, we can then obtain a simple presentation matrix of M with a triangular-block form and whose diagonal blocks present the nontrivial left D-modules M_i/M_{i+1}'s. Such a procedure is implemented in the PURITYFILTRATION package. For related results, see Appendix A of [2]. Finally, if D is a commutative polynomial ring, then Remark 4 can also be used to check whether or not $M_i \cong M_i/M_{i+1} \oplus M_{i+1}$, i.e., whether or not the corresponding matrix $(I^T_{p'_i}\ 0^T\ 0^T)^T$ can be replaced by the trivial matrix $(0^T\ 0^T\ 0^T)^T$ (which generally helps the integration of the corresponding linear functional system).

Even if the size of the matrix $\overline{R}$ is larger than the one of R, this presentation matrix is more tractable than R for a fine study of the module properties of the left D-module $M \cong \overline{M}$, for the study of the structural properties of $\ker_{\mathcal{F}}(R.)$, as well as for computing closed-form solutions of $\ker_{\mathcal{F}}(R.)$ (when they exist). For instance, overdetermined/underdetermined linear PD systems $\ker_{\mathcal{F}}(R.)$, which cannot be directly integrated by means of standard computer algebra systems such as Maple, can be done using their equivalent forms $\ker_{\mathcal{F}}(\overline{R}.)$. For more details, see [43, 45].

5 An Embedding Theorem

If D is a domain, then a torsion-free left D-module M can be embedded into a free left D-module (see the comment after Proposition 4), and thus into a projective left D-module. Using Example 4, a 0-pure left D-module M can then be embedded into a left D-module of left projective dimension 0. This result is a particular case of the following theorem for which we give a construct proof.

Theorem 14 ([10]) *Let D be an Auslander regular ring and M an i-pure left D-module. Then, M can be embedded into a left D-module P_i of left projective dimension i, i.e., there exist a left D-module P_i with $\mathrm{lpd}_D(P_i) = i$ and an injective homomorphism $\epsilon_i \in \hom_D(M, P_i)$.*

Proof Let us first suppose that $M = D^{1\times p}/(D^{1\times q} R)$ is 0-pure, i.e., $t_0(M) = M$ and $t_1(M) = 0$. Since $j_D(M) = 0$, $\ker_D(R.) \cong \hom_D(M, D) \neq 0$ (see Theorem 1), which shows that the Auslander transpose $N_{11} = D^{p_{11}}/(R_{11} D^{p_{01}})$ of $M = D^{1\times p_{01}}/(D^{1\times p_{11}} R_{11})$ ($R_{11} = R$, $p_{01} = p$, $p_{11} = q$) admits a free resolution of the form

$$\cdots \xrightarrow{R_{-11}.} D^{p_{-11}} \xrightarrow{R_{01}.} D^{p_{01}} \xrightarrow{R_{11}.} D^{p_{11}} \xrightarrow{\kappa_{11}} N_{11} \longrightarrow 0,$$

where $R_{01} \neq 0$. Since $T_1 = \mathrm{ext}^1_D(N_{11}, D) \cong M_1 = t_1(M) = 0$ (see Theorem 11), then we get the exact sequence $D^{1\times p_{-11}} \xleftarrow{.R_{01}} D^{1\times p_{01}} \xleftarrow{.R_{11}} D^{1\times p_{11}}$, which yields $M = \mathrm{coker}_D(.R_{11}) \cong \mathrm{im}_D(.R_{01}) \subseteq D^{1\times p_{-11}}$ and $\mathrm{lpd}(D^{1\times p_{-11}}) = 0$.

Let us now suppose that $i \geq 1$. Since M is i-pure and $j_D(M) = i$. If (24) is a free resolution of M, then $N_{ii} = D^{p_{ii}}/(R_{ii} D^{p_{(i-1)i}})$ admits the free resolution (62), where $R_{ii} = R_i$, $p_{ii} = p_i$, and $p_{i(i+1)} = p_{ii}$ (see the notations of Sect. 3). Now, $\mathrm{ext}^i_D(M, D) \cong \ker_D(R_{(i+1)(i+1)}.)/\mathrm{im}_D(R_{ii}.) = (R_{i(i+1)} D^{p_{(i-1)(i+1)}})/(R_{ii} D^{p_{(i-1)i}})$ is a left D-submodule of N_{ii}. Using Proposition 4 for right D-modules, we get

$$\mathrm{ext}^i_D(M, D) \cong D^{p_{(i-1)(i+1)}} / \big((F_{(i-1)(i+1)} \quad R_{(i-1)(i+1)}) D^{p_{(i-1)i}+p_{(i-2)(i+1)}}\big)$$

and the following commutative exact diagram holds

$$\begin{tikzcd}
D^{p_{(i-1)i}+p_{(i-2)(i+1)}} \arrow[r, "(F_{(i-1)(i+1)} \quad R_{(i-1)(i+1)})."] \arrow[d, "(I_{p_{(i-1)i}} \quad 0)."] & D^{p_{(i-1)(i+1)}} \arrow[r] \arrow[d, "R_{i(i+1)}."] & \mathrm{ext}^i_D(M, D) \arrow[r] \arrow[d, "u"] & 0 \\
D^{p_{(i-1)i}} \arrow[r, "R_{ii}."] & D^{p_{ii}} \arrow[r, "\kappa_{ii}"] & N_{ii} \arrow[r] & 0.
\end{tikzcd}$$

where u is an injective right D-homomorphism.

Let $q_0 = p_{(i-1)(i+1)}$, $q_1 = p_{(i-1)i} + p_{(i-2)(i+1)}$, $Q_1 = (F_{(i-1)(i+1)} \quad R_{(i-1)(i+1)})$, $L_0 = R_{i(i+1)}$, and $L_1 = (I_{p_{(i-1)i}} \quad 0)$. Extending the presentation of $\mathrm{ext}^i_D(M, D)$ to get a free resolution of $\mathrm{ext}^i_D(M, D)$ and using Remark 13, $u \in \hom_D(\mathrm{ext}^i_D(M, D), N_{ii})$ induces the following commutative exact diagram:

$$\begin{tikzcd}
D^{q_{i+1}} \arrow[r, "Q_{i+1}."] \arrow[d, "L_{i+1}."] & D^{q_i} \arrow[r, "Q_i."] \arrow[d, "L_i."] & \dots \arrow[r, "Q_2."] & D^{q_1} \arrow[r, "Q_1."] \arrow[d, "L_1."] & D^{q_0} \arrow[r] \arrow[d, "L_0."] & \mathrm{ext}^i_D(M, D) \arrow[r] \arrow[d, "u"] & 0 \\
D^{p_{-11}} \arrow[r, "R_{01}."] & D^{p_{01}} \arrow[r, "R_{11}."] & \dots \arrow[r, "R_{(i-1)(i-1)}."] & D^{p_{(i-1)(i-1)}} \arrow[r, "R_{ii}."] & D^{p_{ii}} \arrow[r, "\kappa_{ii}"] & N_{ii} \arrow[r] & 0.
\end{tikzcd} \tag{92}$$

Since $j_D(M) = i \geq 1$, Theorem 1 shows that $\ker_D(R_{11}.) \cong \hom_D(M, D) = 0$, i.e., $R_{01} = 0$. Dualizing (92), we get the following commutative diagram:

$$\begin{tikzcd}
D^{1\times q_{i+1}} & D^{1\times q_i} \arrow[l, ".Q_{i+1}"'] & \dots \arrow[l, ".Q_i"'] & D^{1\times q_1} \arrow[l, ".Q_2"'] & D^{1\times q_0} \arrow[l, ".Q_1"'] & 0 \arrow[l] \\
0 \arrow[u] & D^{1\times p_{01}} \arrow[l] \arrow[u, ".L_i"'] & \dots \arrow[l, ".R_{11}"'] & D^{1\times p_{(i-1)(i-1)}} \arrow[l, ".R_{(i-1)(i-1)}"'] \arrow[u, ".L_1"'] & D^{1\times p_{ii}} \arrow[l, ".R_{ii}"'] \arrow[u, ".L_0"'] & 0. \arrow[l]
\end{tikzcd} \tag{93}$$

Since D is Auslander regular, Remark 7 shows that $\mathrm{ext}^j_D(\mathrm{ext}^i_D(M, D), D) = 0$ for $j = 1, \dots, i-1$, which shows that the top horizontal complex of (93) is exact at $D^{1\times q_j}$ for $j = 0, \dots, i-1$. The defect of exactness of the top horizontal complex at $D^{1\times q_i}$ is $\mathrm{ext}^i_D(\mathrm{ext}^i_D(M, D), D) \cong \ker_D(.Q_{i+1})/\mathrm{im}_D(.Q_i)$, and the defect of exactness of the bottom horizontal complex at $D^{1\times p_{01}}$ is $\mathrm{ext}^i_D(N_{ii}, D) \cong D^{1\times p_{01}}/(D^{1\times p_{11}} R_{11}) = M$. Hence, $.L_i$ induces the following left D-homomorphism

$$\begin{aligned} \varepsilon_i : M &\longrightarrow \ker_D(.Q_{i+1})/\mathrm{im}_D(.Q_i) \\ \pi(\lambda) &\longmapsto o(\lambda L_i), \end{aligned}$$

where $o\colon \ker_D(.Q_{i+1}) \longrightarrow \ker_D(.Q_{i+1})/\operatorname{im}_D(.Q_i)$ is the projection and $\lambda \in D^{1\times p_{01}}$. Since M is i-pure, 1 of Theorem 8 then implies that ε_i is injective.

The exactness of the top horizontal complex of (93) at $D^{1\times q_j}$ for $j = 0, \ldots, i-1$ shows that the left D-module $P_i = D^{1\times q_i}/(D^{1\times q_{i-1}} Q_i)$ admits a free resolution of length i defined by $.Q_j$ with $j = i, \ldots, 1$, which implies that $\operatorname{ext}_D^j(P_i, D) = 0$ for all $j > i$. Using this free resolution of P_i, we obtain $\operatorname{ext}_D^i(P_i, D) \cong D^{q_0}/(Q_1 D^{q_1}) = \operatorname{ext}_D^i(M, D) \neq 0$, which proves that $\operatorname{lpd}_D(P_i) = i$ by Proposition 2.

Finally, $\ker_D(.Q_{i+1})/\operatorname{im}_D(.Q_i)$ is a left D-submodule of $P_i = D^{1\times q_i}/(D^{1\times q_{i-1}} Q_i)$, and thus ε_i induces an injective left D-homomorphism $\epsilon_i\colon M \longrightarrow P_i$ defined by $\epsilon_i(\pi(\lambda)) = \sigma_i(\lambda L_i)$ for all $\lambda \in D^{1\times p_{01}}$, where $\sigma_i : D^{1\times q_i} \longrightarrow P_i$ is the canonical projection onto P_i, which proves the result. □

Theorem 14 is implemented in the PURITYFILTRATION package.

A proof of Theorem 14 based on Spencer cohomology [52] was obtained in [38].

Example 7 Let D be an Auslander regular ring with $\operatorname{gld}(D) = n$ and M a nonzero holonomic left D-module. Then, $\operatorname{gld}(D) = n$ yields $\operatorname{ext}_D^i(M, D) = 0$ for $i > n$ by Proposition 3. By definition of a holonomic module, $j_D(M) = n$, and thus $\operatorname{ext}_D^n(M, D) \neq 0$, which proves that $\operatorname{lpd}_D(M) = n$ by Proposition 2. Finally, since M is n-pure, we can take $P_n = M$ and $\epsilon_n = \operatorname{id}_M$ in Theorem 14.

Example 8 Let D be an Auslander regular ring and M a nonzero left D-module defined by the free resolution $0 \longrightarrow D^{1\times p} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0$. Since $M \cong \operatorname{ext}_D^1(\operatorname{ext}_D^1(M, D), D)$, i.e., M is 1-pure, and $\operatorname{lpd}_D(M) = 1$, we can then take $P_1 = M$ and $\epsilon_1 = \operatorname{id}_M$ in Theorem 14. If D is also a Cohen-Macaulay ring, then $\dim_D(M) = \dim(D) - 1$. If D is the ring of PD operators with coefficients in a differential field K of characteristic 0, then this result proves *Janet's conjecture* [25], which was first proved by Johnson in [27] (see also [39, 40]).

Corollary 6 *Let D be an Auslander regular ring, $M = D^{1\times p}/(D^{1\times q} R)$ an i-pure left D-module, and $\mathcal{F}$ an injective left D-module. Then, there exist $Q \in D^{s\times r}$ and $L \in D^{p\times r}$ such that* $\operatorname{lpd}_D(P) = i$, *where $P = D^{1\times r}/(D^{1\times s} Q)$, and*

$$\ker_{\mathcal{F}}(R.) = L \ker_{\mathcal{F}}(Q.),$$

i.e., an i-pure linear system is the image of a linear system of projective dimension i.

Proof The proof of Theorem 14 shows that the commutative exact diagram holds

$$\begin{array}{ccccccc} 0 \longleftarrow & P_i & \xleftarrow{\sigma_i} & D^{1\times q_i} & \xleftarrow{.Q_i} & D^{1\times q_{i-1}} \\ & \uparrow \epsilon_i & & \uparrow .L_i & & \uparrow .L_{i-1} \\ 0 \longleftarrow & M & \xleftarrow{\pi} & D^{1\times p_{01}} & \xleftarrow{.R_{11}} & D^{1\times p_{11}}, \end{array} \tag{94}$$

where $\epsilon_i \in \hom_D(M, P_i)$ is injective, $R_{11} = R$, $p_{01} = p$, and $p_{11} = q$. Applying the contravariant exact functor $\hom_D(\,\cdot\,, \mathcal{F})$ to (94), we obtain the following commutative exact diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker_{\mathcal{F}}(Q_i.) & \longrightarrow & \mathcal{F}^{q_i} & \xrightarrow{Q_i.} & \mathcal{F}^{q_{i-1}} \\ & & \downarrow \epsilon_i^\star & & \downarrow L_i. & & \downarrow L_{i-1}. \\ 0 & \longrightarrow & \ker_{\mathcal{F}}(R_{11}.) & \longrightarrow & \mathcal{F}^{p_{01}} & \xrightarrow{R_{11}.} & \mathcal{F}^{p_{11}}, \end{array}$$

where $\epsilon_i^\star : \ker_{\mathcal{F}}(Q_i.) \longrightarrow \ker_{\mathcal{F}}(R.)$ is defined by $\epsilon_i^\star(\xi) = L_i \xi$ for all $\xi \in \ker_{\mathcal{F}}(Q_i.)$. By Theorem 3, the short exact sequence $0 \longrightarrow M \xrightarrow{\epsilon_i} P_i \longrightarrow \operatorname{coker} \epsilon_i \longrightarrow 0$ yields the following long exact sequence:

$$0 \longrightarrow \hom_D(\operatorname{coker} \epsilon_i, \mathcal{F}) \longrightarrow \hom_D(P_i, \mathcal{F}) \xrightarrow{\epsilon_i^*} \hom_D(M, \mathcal{F}) \longrightarrow \operatorname{ext}_D^1(\operatorname{coker} \epsilon_i, \mathcal{F}).$$

Since $\mathcal{F}$ is an injective left D-module, $\operatorname{ext}_D^1(\operatorname{coker} \epsilon_i, \mathcal{F}) = 0$ (see Definition 3), which shows that $\epsilon_i^\star$ is surjective, i.e., using Theorem 1, for every $\eta \in \ker_{\mathcal{F}}(R.)$, there exists $\xi \in \ker_{\mathcal{F}}(Q_i.)$ such that $\eta = L_i \xi$. Finally, we note that $\epsilon_i^\star$ is injective iff $\hom_D(\operatorname{coker} \epsilon_i, \mathcal{F}) \cong \ker_{\mathcal{F}}((L_i^T \; Q_i^T)^T.) = 0$. □

Example 9 Let M be the $D = \mathbb{Q}[\partial_1, \partial_2, \partial_3]$-module finitely presented by:

$$R = \begin{pmatrix} \partial_1 & 0 \\ 0 & \partial_1 \\ \partial_2 & -\partial_3 \end{pmatrix} \in D^{3 \times 2}.$$

Then, the D-module M admits the following free resolution:

$$0 \longleftarrow M \xleftarrow{\pi} D^{1\times 2} \xleftarrow{.R} D^{1\times 3} \xleftarrow{.R_2} D \longleftarrow 0, \quad R_2 = (-\partial_2 \quad \partial_3 \quad \partial_1).$$

Clearly, $\operatorname{ext}_D^2(M, D) = D/(\partial_1, \partial_2, \partial_3) \neq 0$, which shows that $\operatorname{pd}_D(M) = 2$ by Proposition 2. Using Algorithm 1, we can check that $M = M_1 = t(M)$ and $M_2 \cong \operatorname{ext}_D^2(N_{22}, D) = 0$, where $N_{22} = D/(\partial_1, \partial_2, \partial_3)$, which shows that M is a 1-pure D-module. With the notations of Sect. 3 and of the proof of Theorem 14, i.e., $R_{11} = R$, $R_{22} = R_2$, $\ker_D(R_{22}.) = R_{12} D^3$, $\ker_D(R_{12}.) = R_{02} D$, $R_{12} F_{02} = R_{11}$, $Q_1 = (F_{02} \; R_{02})$, $L_0 = R_{12}$, and $L_1 = (I_2 \; 0)$, where

$$R_{12} = \begin{pmatrix} \partial_3 & \partial_1 & 0 \\ \partial_2 & 0 & \partial_1 \\ 0 & \partial_2 & -\partial_3 \end{pmatrix}, \qquad F_{02} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad R_{02} = \begin{pmatrix} -\partial_1 \\ \partial_3 \\ \partial_2 \end{pmatrix},$$

we obtain $\operatorname{ext}_D^1(M, D) = \ker_D(R_{22}.)/(R_{11} D^2) = (R_{12} D^3)/(R_{11} D^2) \cong D^3/(Q_1 D^3)$. By Theorem 14, the D-homomorphism $\epsilon : M \longrightarrow P_1 = D^{1\times 3}/(D^{1\times 3} Q_1)$ defined by $\epsilon_1(\pi(\lambda)) = \sigma_1(\lambda L_1)$ is injective. Since the matrix Q_1 has full row rank and $P_1 \neq 0$, $\operatorname{pd}_D(P_1) = 1$, which shows that the 1-pure D-module M can be embedded into the D-module P_1 of projective dimension 1. If $\mathcal{F} = C^\infty(\mathbb{R}^3)$ (see Example 2), then

$$\ker_{\mathcal{F}}(Q_1.) = \left\{ \left(\partial_3 \phi(x_2, x_3) \quad \partial_2 \phi(x_2, x_3) \quad -\phi(x_2, x_3) \right)^T \mid \forall \phi \in C^\infty\left(\mathbb{R}^2\right) \right\},$$

which finally yields:

$$\ker_{\mathcal{F}}(R.) = L_1 \ker_{\mathcal{F}}(Q_1.) = \left\{ \left(\partial_3 \phi(x_2, x_3) \quad \partial_2 \phi(x_2, x_3) \right)^T \mid \forall \phi \in C^\infty\left(\mathbb{R}^2\right) \right\}.$$

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