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On Exponentially Shaped Josephson Junctions

Monica De Angelis

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Abstract The paper deals with a third order semilinear equation which characterizes exponentially shaped Josephson junctions in superconductivity. The initial-boundary problem with Dirichlet conditions is analyzed. When the source term F is a linear function, the problem is explicitly solved by means of a Fourier series with properties of rapid convergence. When F is nonlinear, appropriate estimates of this series allow to deduce a priori estimates, continuous dependence and asymptotic behaviour of the solution.

Keywords Superconductivity · Josephson junction · Partial differential equations · Fundamental solution

Mathematics Subject Classification 35K35 · 35E05

1 Introduction

We refer to the semilinear equation

$$\mathcal{L}_\varepsilon u = F(x, t, u) \tag{1}$$

where $\mathcal{L}_\varepsilon$ is the third-order parabolic operator:

$$\mathcal{L}_\varepsilon = (\partial_{xx} - \lambda\partial_x)(\varepsilon\partial_t + 1) - \partial_t(\partial_t + \alpha). \tag{2}$$

Equation (1) characterizes the evolution of several dissipative models such as the motions of viscoelastic fluids or solids [1–4]; the sound propagation in viscous gases [5]; the heat conduction at low temperature [6, 7] and the propagation of localized magnetohydrodinamic models in plasma physics [8]. Moreover, it can also be referred to reaction diffusion systems [9].

M. De Angelis (✉)
Faculty of Engineering. Dept. of Math. and Appl., University of Naples Federico II, Via Claudio 21, 80125 Naples, Italy
e-mail: modeange@unina.it

As example of perturbed model of the phase evolution, we will consider the non linear phenomenon concerning the Josephson effects in superconductivity.

More precisely, if $\varphi = \varphi(x, t)$ is the phase difference in a rectangular junction and γ is the normalized current bias; when $\lambda = 0$ and $F = \sin\varphi - \gamma$, Eq. (1) gives the well-known perturbed Sine-Gordon equation (PSGE) [10]:

$$\varepsilon\varphi_{xxt} + \varphi_{xx} - \varphi_{tt} - \alpha\varphi_t = \sin\varphi - \gamma. \tag{3}$$

The terms $\varepsilon\varphi_{xxt}$ and $\alpha\varphi_t$ characterize the dissipative normal electron current flow respectively along and across the junction. They represent the *perturbations* with respect to the classic Sine Gordon equation [10, 11]. When the surface resistance is negligible, then $\varepsilon(< 1)$ is vanishing and a singular perturbation problem for Eq. (3) could appear [12]. As for the coefficient α of (3), it depends on the shunt conductance [13] and generally one has $a < 1$ [14–16]. However, if the resistance of the junction is so small as to short completely the capacitance, the case $a > 1$ arises [10, 17].

More recently, the case of the exponentially shaped Josephson junction (ESJJ) has been considered. The evolution of the phase inside this junction is described by the third order equation:

$$\varepsilon\varphi_{xxt} + \varphi_{xx} - \varphi_{tt} - \varepsilon\lambda\varphi_{xt} - \lambda\varphi_x - a\varphi_t = \sin\varphi - \gamma \tag{4}$$

where λ is a positive constant generally less than one [18, 19] while the terms $\lambda\varphi_x$ and $\lambda\varepsilon\varphi_{xt}$ represent the current due to the tapering. In particular $\lambda\varphi_x$ correspond to a geometrical force driving the fluxons from the wide edge to the narrow edge [19, 20].

According to recent literature [14, 18, 20–23], an exponentially shaped Josephson junction provides several advantages with respect to a rectangular junction. For instance in [18] it has been proved that in an ESJJ it is possible to obtain a voltage which is not chaotic anymore, but rather periodic excluding, in this way, some among the possible causes of large spectral width. It is also proved that the problem of trapped flux can be avoided. Moreover, some devices as SQUIDs were built with exponentially tapered loop areas [24].

The analysis of many initial—boundary problems related to the PSGE (3) has been discussed in a lot of papers. In particular in[25], to deduce an exhaustive asymptotic analysis, the Green function of the linear operator

$$\mathcal{L} = \partial_{xx}\left(\varepsilon\partial_t + c^2\right) - \partial_t(\partial_t + \alpha) \tag{5}$$

has been determined by Fourier series. By means of its properties an exponential decrease of both linear and non linear solutions is deduced.

The aim of this paper is the analysis of the Dirichlet boundary value problem related to Eq. (4).

The Green function G of the linear strip problem is determined by Fourier series and properties of rapid convergence are established. So, when the source term F is a linear function, then the explicit solution is obtained and an exponential decrease of the solution is deduced.

When F is nonlinear, the problem is reduced to an integral equation with kernel G and an appropriate analysis implies results on the existence and uniqueness of the solution. Moreover, by means of suitable properties of G, a priori estimates, continuous dependence upon the data and asymptotic behaviour of the solution are achieved, too.

2 Statement of the Problem and Properties of the Green Function

Let l, T be arbitrary positive constants and let

$$\Omega_T = \left\{(x, t) : 0 < x < l, 0 < t \le T\right\}.$$

The boundary value problem related to equation (1) is the following:

$$\begin{cases} (\partial_{xx} - \lambda\partial_x)(\varepsilon u_t + u) - \partial_t(u_t + \alpha u) = F(x,t,u), & (x,t) \in \Omega_T, \\ u(x,0) = h_0(x), \qquad u_t(x,0) = h_1(x), & x \in [0,l], \\ u(0,t) = 0, \qquad u(l,t) = 0, \quad 0 < t \le T. \end{cases} \tag{6}$$

By Fourier method it is possible to determine the Green function of the linear operator $\mathcal{L}_\varepsilon$. So, let

$$\gamma_n = \frac{n\pi}{l}, \qquad b_n = \left(\gamma_n^2 + \lambda^2/4\right), \qquad g_n = \frac{1}{2}(\alpha + \varepsilon b_n), \qquad \omega_n = \sqrt{g_n^2 - b_n} \tag{7}$$

and

$$G_n(t) = \frac{1}{\omega_n} e^{-g_n t} \sinh(\omega_n t), \tag{8}$$

by standard techniques, the Green function can be given the form:

$$G(x,t,\xi) = \frac{2}{l} e^{\frac{\lambda}{2}x} \sum_{n=1}^{\infty} G_n(t) \sin\gamma_n\xi \sin\gamma_n x. \tag{9}$$

This series is endowed of rapid convergence and it is exponentially vanishing as t tends to infinity. In fact, if we denote by

$$a_\lambda = \alpha + \varepsilon\lambda^2/4 \tag{10}$$

and

$$p_\lambda = \frac{\pi^2}{\varepsilon\pi^2 + a_\lambda l^2}, \qquad q_\lambda = \frac{a_\lambda + \varepsilon(\pi/l)^2}{2}, \qquad \delta \equiv \min(p_\lambda, q_\lambda), \tag{11}$$

the following theorem holds:

Theorem 1 *Whatever the constants $\alpha, \varepsilon, \lambda$ may be in $\Re^+$, the function $G(x,\xi,t)$ defined in* (9) *and all its time derivatives are continuous functions in Ω_T and it results*:

$$\left|G(x,\xi,t)\right| \le M e^{-\delta t}, \qquad \left|\frac{\partial^j G}{\partial t^j}\right| \le N_j e^{-\delta t}, \quad j \in \mathbb{N} \tag{12}$$

where M, N_j are constants depending on $\alpha, \lambda, \varepsilon$.

Proof Physical problems lead to consider $\alpha\varepsilon < 1$ and denoting by

$$N_{1,2}^{\lambda} = \frac{l}{2\varepsilon^2\pi}\left[4(1 \mp \sqrt{1-\alpha\varepsilon})^2 - \varepsilon^2\lambda^2\right]^{1/2}, \tag{13}$$

let us assume that $N_{1,2}^{\lambda} > 1$. So, let k be a positive constant less than one and let $\bar{N}_{1,2}, N_k^{\lambda}$ be the lowest integers such that

$$\begin{cases} \bar{N}_1 < N_1^{\lambda}, \qquad \bar{N}_2 > N_2^{\lambda}; \\ N_k^{\lambda} > \dfrac{l}{2\varepsilon^2\pi k}[4(1 \mp \sqrt{1-\alpha k\varepsilon})^2 - \varepsilon^2 k\lambda^2]^{1/2}. \end{cases} \tag{14}$$

We start analysing the hyperbolic terms when $n \ge \bar{N}_2$. Letting

$$X_n = \frac{b_n}{g_n^2} < 1, \qquad \varphi_n = g_n(-1 + \sqrt{1 - X_n}), \tag{15}$$

it is possible to prove that $\varphi_n \le -\frac{\gamma_n^2}{2g_n}$. So it results:

$$e^{-t(g_n-\omega_n)} \le e^{-p_\lambda t}. \tag{16}$$

Furthermore, it is easily verified that for all $n \ge N_k^\lambda (\ge \bar{N}_2)$ it results $\frac{b_n}{g_n^2} \le k$ and hence one has:

$$\omega_n \ge g_n(1-k)^{1/2} \ge n^2 \frac{2l^2}{\varepsilon\pi^2}(1-k)^{1/2}. \tag{17}$$

Other terms can be treated similarly. For instance, as for circular terms, it can be proved that $e^{-g_n t} \le e^{-q_\lambda t}$.

In consequence estimate $(12)_1$ holds $\forall n \ge 1$.

As for $(12)_2$, one has:

$$g_n - \omega_n = \frac{b_n}{g_n+\omega_n} \le \frac{2}{\varepsilon} + \frac{\lambda^2}{4\varepsilon q_\lambda}, \quad \forall n \ge 1 \tag{18}$$

and by means of standard computations, $(12)_2$ can be deduced, too.

It may be similarly proved that the theorem holds also when $\alpha\varepsilon \ge 1$ or when the conditions $N_{1,2}^\lambda > 1$ do not hold.

Finally we notice that when $N_{1,2}^\lambda$ are integers, the constants M and N_j in (12) could depend on t. □

As for the x-derivatives of Fourier series like (9), attention is needed towards convergence problems. For this, we will consider x-differentiations of the operator $(\varepsilon\partial_t + 1)G$ instead of G and G_t.

Theorem 2 *Whatever $\alpha, \varepsilon, \lambda$ may be, the function $G(x,\xi,t)$ defined in (9) is such that*:

$$\left|\partial_x^{(i)}(\varepsilon G_t + G)\right| \le A_i e^{-\delta t} \quad (i = 0, 1, 2) \tag{19}$$

where δ is defined in (11) and A_i $(i = 0, 1, 2)$ are constants depending on a, ε, λ.

Proof As for the hyperbolic terms in G, it results:

$$\varepsilon G_t + G = e^{\frac{\lambda x}{2}} \sum_{n=1}^{\infty} \frac{e^{-g_n t}}{l\omega_n}\left\{\left[1-\varepsilon(g_n-\omega_n)\right]e^{\omega_n t} - \left[1-\varepsilon(g_n+\omega_n)\right]e^{-\omega_n t}\right\}. \tag{20}$$

where according to (15), it results:

$$1-\varepsilon(g_n-\omega_n) = 1+\varepsilon\varphi_n.$$

So, by means of Taylor's formula, one has:

$$1-\varepsilon(g_n-\omega_n) = 1 - \frac{\varepsilon}{2}g_n X_n - \frac{\varepsilon}{8}g_n X_n^2 - \frac{3}{16}\varepsilon g_n \int_0^{X_n} \frac{(X_n-y)^2}{(1-y)^{5/2}}dy. \tag{21}$$

Besides, it is possible to prove that $\forall n \ge 1$ one has:

$$X_n < \frac{c^2}{n^2} \quad \left(c = l\sqrt{4+\lambda^2}/\varepsilon\pi\right) \tag{22}$$

and for all $n \ge c(1+c_1)(c_1 > 0)$ it results:

$$\int_0^{X_n} \frac{(X_n-y)^2}{(1-y)^{5/2}}dy \le \frac{2}{3}X_n^2\left[\frac{(c_1+1)^3}{c_1(c_1+2)]^{3/2}} - 1\right]. \tag{23}$$

So, taking into account that

$$\frac{\varepsilon}{2} g_n X_n = 1 - \frac{\alpha}{a_\lambda + \varepsilon \gamma_n^2},$$
$$\frac{\varepsilon}{8} g_n X_n^2 = \frac{\varepsilon b_n^2}{(a_\lambda + \varepsilon \gamma_n^2)^3};$$

there exists a positive constant k_1 such that:

$$|1 + \varepsilon \varphi_n| \le \frac{1}{n^2}\left(\frac{\alpha l^2}{\varepsilon \pi^2} + \frac{k_1}{n^2}\right). \tag{24}$$

Estimates of Theorem 1 together with (24) show that the series terms related to the operator $\varepsilon G_t + G$ have order at least of n^{-4}. So it can be differentiated term by term with respect to x and the estimate (19) can be deduced. □

As solution of the equation $\mathcal{L}_\varepsilon v = 0$ we will mean a continuous function $v(x,t)$ which has continuous the derivatives v_t, v_{tt}, $\partial_x(\varepsilon v_t + v)$, $\partial_{xx}(\varepsilon v_t + v)$ and these derivatives verify the equation.

So, we are able to prove the following theorem:

Theorem 3 *The function $G(x,t)$ defined in (9) is a solution of the equation*

$$\mathcal{L}_\varepsilon G = (\partial_{xx} - \lambda \partial_x)(\varepsilon G_t + G) - \partial_t(G_t + \alpha G) = 0. \tag{25}$$

Proof The uniform convergence proved in Theorems 1–2 allows to deduce that:

$$(\partial_t + \alpha)\frac{\partial G}{\partial t} = \frac{2e^{\frac{\lambda x}{2}}}{l} \sum_{n=1}^{\infty} \left\{\left[b_n(\varepsilon g_n - 1)G_n - \varepsilon b_n e^{-g_n t} \cosh \omega_n t\right] \sin \gamma_n \xi \sin \gamma_n x\right\}, \tag{26}$$

$$\begin{aligned}\partial_x(\varepsilon \partial_t + 1)G = \frac{2}{l} e^{\frac{\lambda x}{2}} \sum_{n=1}^{\infty} \Bigg\{ & \left[\frac{\lambda}{2}(1 - \varepsilon g_n)G_n + \frac{\lambda \varepsilon}{2} e^{-g_n t} \cosh \omega_n t\right] \sin \gamma_n \xi \sin \gamma_n x \\ & + \left[G_n \gamma_n (1 - \varepsilon g_n) + \varepsilon \gamma_n e^{-g_n t} \cosh \omega_n t\right] \sin \gamma_n \xi \cos \gamma_n x \Bigg\}. \end{aligned} \tag{27}$$

Moreover, being

$$\begin{aligned}\partial_{xx}(\varepsilon \partial_t + 1)G = \frac{2}{l} e^{\frac{\lambda x}{2}} \sum_{n=1}^{\infty} \Bigg\{ & \Bigg[(\varepsilon g_n - 1)\left(b_n - \frac{\lambda^2}{2}\right) G_n \\ & + \left(-\frac{\lambda^2 \varepsilon}{4} + \varepsilon \gamma_n^2\right) e^{-g_n t} \cosh \omega_n t\Bigg] \sin \gamma_n \xi \sin \gamma_n x \\ & + \left[G_n \lambda \gamma_n (1 - \varepsilon g_n) + \varepsilon \lambda \gamma_n e^{-g_n t} \cosh \omega_n t\right] \sin \gamma_n \xi \cos \gamma_n x \Bigg\}, \end{aligned} \tag{28}$$

(25) can be deduced. □

3 Properties of the Convolution

To achieve the solution of the strip problem (6), the convolution of the function G with the data must be analysed. For this, let $h(x)$ be a continuous function on $(0,l)$ and let:

$$u_h(x,t) = \int_0^l h(\xi)G(x,\xi,t)d\xi \tag{29}$$

$$u_h^*(x,t) = (\partial_t + \alpha + \varepsilon\lambda\partial_x - \varepsilon\partial_{xx})u_h(x,t). \tag{30}$$

The following theorems hold:

Theorem 4 *If the data $h(x)$ is a $C^1(0,l)$ function, then u_h defined by (29) is a solution of the equation $\mathcal{L}_\varepsilon = 0$ and it results*:

$$\lim_{t\to 0} u_h(x,t) = 0, \qquad \lim_{t\to 0} \partial_t u_h(x,t) = h(x), \tag{31}$$

uniformly for all $x \in [0,l]$.

Proof The absolute convergence of u_h with its partial derivatives is proved by means of Theorems 1 and 2 and continuity of function $h(x)$. So, since (25) $\mathcal{L}_\varepsilon u_h = 0$ is verified, while Theorem 1 and hypotheses on $h(x)$ imply $(31)_1$.

More being:

$$\frac{\partial G}{\partial t} = -\frac{2}{\pi}\frac{\partial}{\partial \xi}\sum_{n=1}^{\infty}\frac{\partial G_n}{\partial t}\frac{\cos\gamma_n\xi}{n}\sin\gamma_n x \tag{32}$$

and

$$\begin{aligned}\partial_t u_h = &-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\partial G_n}{\partial t}\big[h(\xi)\cos\gamma_n\xi\big]_0^l\frac{\sin\gamma_n x}{n}\\ &+\frac{2}{\pi}\int_0^l\sum_{n=1}^{\infty}\frac{\partial G_n}{\partial t}h'(\xi)\frac{\cos\gamma_n\xi}{n}d\xi\sin\gamma_n x,\end{aligned} \tag{33}$$

denoting by $\eta(x)$ the Heaviside function, it results:

$$\lim_{t\to 0}\partial_t u_h = \frac{x}{l}\big[h(l)-h(0)\big] + h(0) - \int_0^l h'(\xi)\left[\eta(\xi - x) + \frac{x}{l} - 1\right]d\xi = h(x). \tag{34}$$

□

Theorem 5 *Let $h(x)$ be a $C^3(0,l)$ function such that $h^{(i)}(0) = h^{(i)}(l) = 0 (i = 1,2,3)$. Then u_h^* defined in (30) is a solution of the equation $\mathcal{L}_\varepsilon = 0$ and it results*:

$$\lim_{t\to 0} u_h^*(x,t) = h, \qquad \lim_{t\to 0}\partial_t u_h^*(x,t) = 0 \tag{35}$$

uniformly for all $x \in (0,l)$.

Proof Properties of $h(x)$ assure that:

$$\begin{aligned}&(\lambda\partial_x - \partial_{xx})u_h(x,t)\\ &= -\frac{2}{l}(\lambda\partial_x - \partial_{xx})e^{\frac{\lambda x}{2}}\sum_{n=1}^{\infty}G_n(t)\int_0^l h''(\xi)\frac{\sin\gamma_n\xi}{\gamma_n^2}d\xi\sin\gamma_n x\\ &= -\frac{2}{l}e^{\frac{\lambda x}{2}}\sum_{n=1}^{\infty}\left[G_n(t)\left(\gamma_n^2 + \frac{\lambda^2}{4}\right)\int_0^l h''(\xi)\frac{\sin\gamma_n\xi}{\gamma_n^2}d\xi\sin\gamma_n x\right]\\ &= -u_{h''}(x,t) + \frac{\lambda^2}{4}u_h(x,t).\end{aligned} \tag{36}$$

So, since Theorem 3, $\mathcal{L}_\varepsilon u_h^* = 0$ is verified. Moreover, being:

$$\partial_t u_h^* = (\partial_{xx} - \lambda \partial_x) u_h \tag{37}$$

(36) implies $(35)_2$, too. Finally, owing to (31) and (36) , one obtains:

$$\lim_{t \to 0} u_h^* = \lim_{t \to 0} \left[\partial_t u_h + \varepsilon \left(\frac{\lambda^2}{4} u_h - u_{h''} \right) \right] = h(x). \tag{38}$$

□

4 Solution of the Linear Problem

Let us consider the homogeneous case. From Theorems 4, 5 the following result is obtained:

Theorem 6 *When $F = 0$ and the initial data $h_1(x)$, and $h_0(x)$ verify the hypotheses of Theorems 4–5, then the function*:

$$u(x,t) = u_{h_1} + (\partial_t + \alpha + \varepsilon\lambda\partial_x - \varepsilon\partial_{xx}) u_{h_0} \tag{39}$$

represents a solution of the homogeneous strip problem (6).

Otherwise, when $F = f(x,t)$, let consider

$$u_f(x,t) = -\int_0^t d\tau \int_0^l f(\xi,\tau) G(x,\xi,t-\tau) d\xi. \tag{40}$$

Standard computations lead to consider at first the problem (6) with $g_0 = g_1 = 0$. For this the following theorem is proved:

Theorem 7 *If the function $f(x,t)$ is a continuous function in Ω_T with continuous derivative with respect to x, then the function u_f represents a solution of the nonhomogeneous strip problem.*

Proof Since $(31)_1$ it results:

$$\partial_t u_f(x,t) = \int_0^t d\tau \int_0^l f(\xi,\tau) G_t(x,\xi,t-\tau) d\xi \tag{41}$$

and as proved in Theorem 4, one obtains:

$$\lim_{\tau \to t} \partial_t u_f(x,t) = f(x,t). \tag{42}$$

Hence, one has:

$$\partial_t^2 u_f = f(x,t) + \int_0^t d\tau \int_0^l f(\xi,\tau) G_{tt}(x,\xi,t-\tau) d\xi \tag{43}$$

and Theorem 3 assures that $\mathcal{L}_\varepsilon u_f = f(x,t)$.

Furthermore, owing to (40)–(41) and estimates (12), if B_i $(i = 1,2)$ are two positive constants, it results:

$$|u_f| \le B_1 \left(1 - e^{-\delta t}\right); \qquad |\partial_t u_f| \le B_2 \left(1 - e^{-\delta t}\right) \tag{44}$$

from which initial homogeneous conditions follow. □

The uniqueness is a consequence of the energy-method and we have:

Theorem 8 *When the source term $f(x,t)$ satisfies Theorem 7 and the initial data (h_0, h_1) satisfy Theorem 6, then the function*

$$u(x,t) = u_{h_1} + (\partial_t + \alpha + \varepsilon\lambda\partial_x - \varepsilon\partial_{xx})u_{h_0} + u_f \tag{45}$$

is the unique solution of the linear non-homogeneous strip problem (6).

5 Solution of the Non-linear Problem

Let us consider now the non-linear problem:

$$\begin{cases} (\partial_{xx} - \lambda\partial_x)(\varepsilon u_t + u) - \partial_t(u_t + \alpha u) = F(x,t,u), & (x,t) \in \Omega_T, \\ u(x,0) = h_0(x), \qquad u_t(x,0) = h_1(x), & x \in [0,l], \\ u(0,t) = 0, \qquad u(l,t) = 0, & 0 < t \le T. \end{cases} \tag{46}$$

As for the data F and $h_i(x)$ $(i = 0, 1)$ we shall admit:

Assumption 9 *The functions $h_i(x)$ $(i = 0, 1)$ are continuously differentiable and bounded together with $h_1'(x)$ and $h_0^{(k)}$ $(k = 1, 2)$. The function $F(x,t,u)$ is defined and continuous on the set*

$$D_T \equiv \big\{(x,t,u) : (x,t) \in \Omega_T, -\infty < u < \infty\big\} \tag{47}$$

and more it is uniformly Lipschitz continuous in (x,t,u) for each compact subset of Ω_T. Besides, F is bounded for bounded u and there exists a constant C_F such that the estimate

$$\big|F(x,t,u_1) - F(x,t,u_2)\big| \le C_F |u_1 - u_2| \tag{48}$$

holds for all (u_1, u_2).

When the problem (46) admits a solution u then, properties of G and the Assumption 9 ensure that u must satisfy the integral equation

$$\begin{aligned} u(x,t) = {} & \int_0^l h_1(\xi)G(x,\xi,t)d\xi + (\partial_t + \alpha + \varepsilon\lambda\partial_x - \varepsilon\partial_{xx})\int_0^l h_0(\xi)G(x,\xi,t)d\xi \\ & + \int_0^t d\tau \int_0^l G(x,\xi,t-\tau)F\big(\xi,\tau,u(\xi,\tau)\big)d\xi, \end{aligned} \tag{49}$$

and it is possible to prove that [26–28]

Theorem 10 *The non linear problem (46) admits a unique solution if and only if the integral equation (49) has a unique solution which is continuous on Ω_T.*

Moreover, let $\|v\|_T = \sup_{\Omega_T} |v(x,t)|$ and let $\mathcal{B}_T$ denote the Banach space

$$\mathcal{B}_T \equiv \big\{v(x,t) : v \in C(\Omega_T),\ \|v\|_T < \infty\big\}. \tag{50}$$

By means of standard methods related to integral equations it is possible to prove that the mapping ψ defined by (49) is a contraction of $\mathcal{B}_T$ in $\mathcal{B}_T$ and so it admits a unique fixed point $u(x,t)$. In consequence the following theorem holds:

Theorem 11 *When the initial data $h_i (i = 0, 1)$ and the source term F verify the Assumption 9, then the problem (46) admits a unique regular solution.*

6 Applications

All these results allow us to obtain continuous dependence upon the data, a priori estimates of the solution and asymptotic properties.

According to Assumption 9, let

$$\|h_i\| = \sup_{(0,l)}\big|h_i(x)\big| \quad (i = 0, 1), \qquad \big\|h_0''\big\| = \sup_{(0,l)}\big|h_0''(x)\big|,$$
$$\|u\|_T = \sup_{\Omega_T}\big|u(x,t)\big|, \qquad \|F\| = \sup_{D_T}\big|F(x,t,u)\big|.$$

So, by means of the following theorem the dependence upon the data can be proved:

Theorem 12 *Let u_1, u_2 be two solutions of the problem related to the data (h_0, h_1, F_1) and $(\gamma_0, \gamma_1 F_2)$ which satisfy the Assumption 9. Then, there exists a positive constant C such that*

$$\|u_1 - u_2\|_T \le C \sup_{\Omega_T}|h_0 - \gamma_0| + C \sup_{\Omega_T}|h_1 - \gamma_1| + C \sup_{D_T}\big|F_1(x,t,u) - F_2(x,t,u)\big|,$$

where C depends on C_F, T and on the parameters $\alpha, \varepsilon, \lambda$.

The integral equation and the properties proved for Green Function G imply a priori estimates, too.

Theorem 13 *When the data (h_0, h_1, F) of the problem (46) verify the Assumption 9, then the following estimate holds*:

$$\big\|u(x,t)\big\|_T \le \frac{1}{\delta}\big(1 - e^{-\delta t}\big)\|F\| + K\big[\|h_1\| + \|h_0\| + \|h_0''\|\big]e^{-\delta t} \tag{51}$$

where the constants δ- defined in (11) and K depend on $\alpha, \varepsilon, \lambda$.

As for the asymptotic properties, obviously the behaviour of the solution depends upon the shape of the source term.

For instance, in the linear case one has:

Theorem 14 *When the source term $f(x,t)$ satisfies the condition*:

$$\big|f(x,t)\big| \le Ce^{-mt} \quad (C, m = \text{const} > 0), \tag{52}$$

one has:

$$\big|u(x,t)\big| \le ke^{-m^* t}, \quad m^* = \min\{\delta, m\},\ k = \text{const}. \tag{53}$$

An exponentially decreasing behaviour is also possible in the non linear case. In fact, according to [29], let us consider a normed space where

$$\big\|u(x,t)\big\| = \max_{x\in(0,l)}\big|u(x,t)\big| \tag{54}$$

is such that

$$\big\|u(x,t)\big\| \le \beta e^{-\delta t} \tag{55}$$

being β a positive constant and δ is defined in (11). Furthermore, let us introduce the following definition [29]:

Definition 15 When the function F is such that $|F(x,t,u)| \leq \gamma \|u\| e^{-\delta t}$, then F is an exponential Lipschitz function.

So the following theorem can be proved:

Theorem 16 *If the non linear source F is an exponential Lipschitz function, then the solution of the semilinear problem* (46) *vanishes as follows*:

$$\left|u(x,t)\right| \leq K_1 e^{-\delta t} \tag{56}$$

where K_1 is a positive constant depending on $\alpha, \varepsilon, \lambda$.

Since $|\sin u| \leq |u|$, a similar behaviour is also verified for the model of superconductivity when $F(x,t,u) = \sin u$.

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