Acta Appl Math (2011) 116:119–132
DOI 10.1007/s10440-011-9633-7

Comparison Theorem for Stochastic Differential Delay Equations with Jumps

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Received: 5 October 2010 / Accepted: 7 July 2011 / Published online: 23 July 2011

Abstract In this paper we establish a comparison theorem for stochastic differential delay equations with jumps. An example is constructed to demonstrate that the comparison theorem need not hold whenever the diffusion term contains a delay function although the jump-diffusion coefficient could contain a delay function. Moreover, another example is established to show that the comparison theorem is not necessary to be true provided that the jump-diffusion term is non-increasing with respect to the delay variable.

Keywords Comparison theorem · Stochastic differential delay equation with jumps

Mathematics Subject Classification (2000) 39A11 · 37H10

1 Introduction

For most of the practical cases, the dynamical systems will be disturbed by some stochastic perturbation [13]. One type of stochastic perturbation is continuous and can be modeled by stochastic integral w.r.t. the continuous martingale, e.g., Brownian motion. Non-Gaussian random processes also play an important role in modelling stochastic dynamical systems (see, e.g., Applebaum [2], Situ [13], Peszat and Zabczyk [12]). Typical examples of non-Gaussian stochastic processes are Lévy processes and processes arising by Poisson random measures. In [14], Woyczyński describes a number of phenomena from fluid mechanics, solid state physics, polymer chemistry, economic science, etc., for which non-Gaussian Lévy processes can be used as their mathematical model in describing the related probability behaviour. On the other hand, control engineering intuition suggests that time-delays are common in practical systems and are often the cause of instability and/or poor performance [17]. Moreover, it is usually difficult to obtain accurate values for the delay and conservative estimates often have to be used. The importance of time delay has already motivated by several studies on the stability of stochastic diffusion with time delay (see, e.g., [3] and [9]).

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In the past few years, comparison theorems for two Stochastic Differential Equations (SDEs) have received a lot of attention, for example, Anderson [1], Gal'cuk and Davis [4], Ikeda and Watanable [5], Mao [6], O'Brien [8], Yamada [15], Yan [16] and references therein. Recently, comparison theorem has made a great development that Peng and Zhu [11] obtain a necessary and sufficient condition for comparison theorem of SDEs with jumps by applying a criteria of "viability condition", Peng and Yang [10] give a comparison theorem for anticipated backward SDEs, and, for a class of Stochastic Differential Delay Equations (SDDEs), Yang, Mao and Yuan [17] also establish a comparison theorem.

In this paper we shall establish a comparison theorem for SDDEs with jumps. It should be pointed out that the approach of this paper is inspired by Peng and Yang [10], Peng and Zhu [11] and Yang, Mao and Yuan [17]. We construct an example, which demonstrates that the comparison theorem need not hold whenever the diffusion term contains a delay function although the jump-diffusion coefficient could contain a delay function just as Example 2.1 below shows. Moreover, another example, Example 2.3, is established to show that the comparison theorem is not necessary to be true provided that the jump-diffusion term is non-increasing w.r.t. the delay variable.

The organization of this paper goes as follows: In Sect. 2 we establish a comparison theorem for two one-dimensional SDDEs with pure jumps; Similar comparison results are given for SDDEs with compensation jump processes in Sect. 3.

2 Comparison Theorem for SDDEs with Pure Jumps

Let $W(t), t \geq 0$, be a real-valued Wiener process defined on a certain probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), and $N(\cdot,\cdot)$ is a Poisson counting process with characteristic measure λ on a measurable subset $\mathbb{Y}$ of $[0,\infty)$ with $\lambda(\mathbb{Y}) < \infty$, $\tilde{N}(dt, du) := N(dt, du) - \lambda(du)dt$ is a compensation martingale process. For given $\tau > 0$ denote $D([-\tau, 0]; \mathbb{R})$ the space of all càdlàg paths from $[-\tau, 0]$ into $\mathbb{R}$ with the norm $\|u\| := \sup_{-\tau\leq\theta\leq 0} |u(\theta)|$, and $L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R})$ the family of $\mathcal{F}_0$-measurable $D([-\tau, 0]; \mathbb{R})$-valued random variables $\xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$ such that $\mathbb{E}\|\xi\|^2 < \infty$. Throughout this paper, we assume that $W(t)$ and $N(dt, du)$ are independent.

Fix $T > 0$ and consider SDDE with jumps for $t \in [0, T]$

$$\begin{aligned} dX(t) &= f(X(t), X(t-\tau), t)dt + g(X(t), X(t-\tau), t)dW(t) \\ &\quad + \int_{\mathbb{Y}} \gamma(X(t-), X((t-\tau)-), t, u)\tilde{N}(dt, du) \end{aligned} \tag{2.1}$$

with an initial condition $X(\theta) = \xi(\theta)$ for any $\theta \in [-\tau, 0]$, where $\xi \in D([-\tau, 0]; \mathbb{R})$, and $X(t-) := \lim_{s\uparrow t} X(s)$. Let the maps $f = f(x, y, t, \omega), g = g(x, y, t, \omega)$ and $\gamma = \gamma(x, y, t, u, \omega)$ be given by

$$f, g : \mathbb{R} \times \mathbb{R} \times [0, T] \times \Omega \to \mathbb{R} \quad \text{and} \quad \gamma : \mathbb{R} \times \mathbb{R} \times [0, T] \times \mathbb{Y} \times \Omega \to \mathbb{R},$$

and be predictable. As usual by writing $f(x, y, t)$ we mean the map $\omega \mapsto f(x, y, t, \omega)$. Analogously for $g(x, y, t)$ and $\gamma(x, y, t, u)$. We assume that f, g and γ satisfy local Lipschitz condition and linear growth condition. That is, for all $n \in \mathbb{N}$ there exists some positive constant L_n such that on $[0, T] \times \Omega$

$$\begin{aligned}&|f(x_1, y_1, t) - f(x_2, y_2, t)|^2 + |g(x_1, y_1, t) - g(x_2, y_2, t)|^2 \\ &\quad + \int_{\mathbb{Y}} |\gamma(x_1, y_1, t, u) - \gamma(x_2, y_2, t, u)|^2 \lambda(du) \\ &\le L_n(|x_1 - x_2|^2 + |y_1 - y_2|^2)\end{aligned} \tag{2.2}$$

for those $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $|x_1| \vee |x_2| \vee |y_1| \vee |y_2| \le n$; and there exists an $L > 0$ such that on $[0, T] \times \Omega$

$$|f(x, y, t)|^2 + |g(x, y, t)|^2 + \int_{\mathbb{Y}} |\gamma(x, y, t, u)|^2 \lambda(du) \le L(1 + |x|^2 + |y|^2) \tag{2.3}$$

for any $x, y \in \mathbb{R}$.

By the standard Banach fixed point theorem and truncation approach (see, e.g., [7, p. 57]), the following existence and uniqueness result can be found.

Lemma 2.1 *Under conditions (2.2) and (2.3), for an initial condition $\xi \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R})$, (2.1) has a unique solution $X(t)$ with property $\mathbb{E} \sup_{-\tau \le t \le T} |X(t)|^2 < \infty$.*

In order to state our main results, we need the following lemma.

Lemma 2.2 *Consider two one-dimensional SDEs with jumps for any $t \in [0, T]$*

$$\begin{aligned}X_1(t) = x_1 &+ \int_0^t f_1(X_1(s), s) ds + \int_0^t g(X_1(s), s) dW(s) \\ &+ \int_0^t \int_{\mathbb{Y}} \gamma_1(X_1(s-), s, u) N(ds, du)\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}X_2(t) = x_2 &+ \int_0^t f_2(X_2(s), s) ds + \int_0^t g(X_2(s), s) dW(s) \\ &+ \int_0^t \int_{\mathbb{Y}} \gamma_2(X_2(s-), s, u) N(ds, du).\end{aligned} \tag{2.5}$$

Let $f_i, g, \gamma_i, i = 1, 2$, be predictable and assume that there exists an $L > 0$ such that on $[0, T] \times \Omega$

$$\begin{aligned}&|f_i(x, t) - f_i(y, t)|^2 + |g(x, t) - g(y, t)|^2 + \int_{\mathbb{Y}} |\gamma_i(x, t, u) - \gamma_i(y, t, u)|^2 \lambda(du) \\ &\le L|x - y|^2\end{aligned} \tag{2.6}$$

for any $x, y \in \mathbb{R}$ with $\mathbb{E} \sup_{0 \le t \le T} (|f_i(0, t)|^2 + |g(0, t)|^2 + \int_{\mathbb{Y}} |\gamma_i(0, t, u)|^2 \lambda(du)) < \infty$ and on $[0, T] \times \Omega$

$$f_1(x, t) \ge f_2(x, t) \quad \text{and} \quad \gamma_1(x, t, u) \ge \gamma_2(x, t, u), \quad \lambda(du) - a.e., \quad \text{for } x \in \mathbb{R}. \tag{2.7}$$

Assume further that on $[0, T] \times \Omega$

$$x + \gamma_1(x, t, u) \le y + \gamma_1(y, t, u), \quad \lambda(du) - a.e., \text{ whenever } x \le y \tag{2.8}$$

for any $x, y \in \mathbb{R}$. *Then we have*

$$X_1(t) \ge X_2(t), \quad \forall t \in [0, T], \ \mathbb{P} - a.s. \text{ provided that } x_1 \ge x_2.$$

Proof By Lemma 2.1, both (2.4) and (2.5) have unique solutions, respectively. Applying the Tanaka-type formula [13, Theorem 152, p. 120], we have for any $t \in [0, T]$

$$\begin{aligned}
&(X_2(t) - X_1(t))^+ \\
&= (x_2 - x_1)^+ + \int_0^t I_A[f_2(X_2(s), s) - f_1(X_1(s), s)]ds \\
&\quad + \int_0^t I_A[g(X_2(s), s) - g(X_1(s), s)]dW(s) \\
&\quad + \int_0^t \int_{\mathbb{Y}} [(X_2(s-) - X_1(s-) + \gamma_2(X_2(s-), s, u) - \gamma_1(X_1(s-), s, u))^+ \\
&\qquad - (X_2(s-) - X_1(s-))^+]N(ds, du) \\
&\le \int_0^t I_A[(f_1(X_2(s), s) - f_1(X_1(s), s)) + (f_2(X_2(s), s) - f_1(X_2(s), s))]ds \\
&\quad + \int_0^t I_A[g(X_2(s), s) - g(X_1(s), s)]dW(s) \\
&\quad + \int_0^t \int_{\mathbb{Y}} I_A(\gamma_1(X_2(s-), s, u) - \gamma_1(X_1(s-), s, u))N(ds, du) \\
&\quad + \int_0^t \int_{\mathbb{Y}} [(X_2(s-) - X_1(s-) + \gamma_1(X_2(s-), s, u) - \gamma_1(X_1(s-), s, u) \\
&\qquad + \gamma_2(X_2(s-), s, u) - \gamma_1(X_2(s-), s, u))^+ - (X_2(s-) - X_1(s-))^+ \\
&\qquad - I_A(\gamma_1(X_2(s-), s, u) - \gamma_1(X_1(s-), s, u))]N(ds, du),
\end{aligned}$$

in which $A := \{X_2(s) - X_1(s) > 0\}$ and the second inequality is due to $x_1 \ge x_2$. Taking expectations, together with (2.7), we obtain

$$\begin{aligned}
&\mathbb{E}(X_2(t) - X_1(t))^+ \\
&\le \mathbb{E} \int_0^t I_A[f_1(X_2(s), s) - f_1(X_1(s), s)]ds \\
&\quad + \mathbb{E} \int_0^t \int_{\mathbb{Y}} I_A(\gamma_1(X_2(s), s, u) - \gamma_1(X_1(s), s, u))\lambda(du)ds \\
&\quad + \mathbb{E} \int_0^t \int_{\mathbb{Y}} [(X_2(s) - X_1(s) + \gamma_1(X_2(s), s, u) - \gamma_1(X_1(s), s, u))^+ \\
&\qquad - (X_2(s) - X_1(s))^+ - I_A(\gamma_1(X_2(s), s, u) \\
&\qquad - \gamma_1(X_1(s), s, u))]\lambda(du)ds,
\end{aligned}$$

where we have replaced $X_1(s-)$ with $X_1(s)$ because this will not have any effect on the Lebesgue integrals involved. On the other hand, thanks to (2.8), it follows that

$$\mathbb{E}\int_0^t \int_{\mathbb{Y}} [(X_2(s) - X_1(s) + \gamma_1(X_2(s), s, u) - \gamma_1(X_1(s), s, u))^+ - (X_2(s) - X_1(s))^+ \\ - I_A(\gamma_1(X_2(s), s, u) - \gamma_1(X_1(s), s, u))]\lambda(du)ds \le 0.$$

Hence, taking into account (2.6) yields

$$\begin{aligned}\mathbb{E}(X_2(t) - X_1(t))^+ &\le (1 + \lambda^{\frac{1}{2}}(\mathbb{Y}))L^{\frac{1}{2}}\mathbb{E}\int_0^t I_{\{X_2(s)-X_1(s)>0\}}|X_2(s) - X_1(s)|ds \\ &= (1 + \lambda^{\frac{1}{2}}(\mathbb{Y}))L^{\frac{1}{2}}\mathbb{E}\int_0^t (X_2(s) - X_1(s))^+ds.\end{aligned}$$

This, in addition to Gronwall's inequality, implies $\mathbb{E}(X_2(t) - X_1(t))^+ = 0$ and then yields $X_2(t) \le X_1(t), t \in [0, T]$, $\mathbb{P}$ − a.s., due to the fact that $(X_2(t) - X_1(t))^+$ is a nonnegative random variable for fixed t, as required. □

Remark 2.1 Peng and Zhu [11, Theorem 3.1] obtain a necessary and sufficient condition of comparison theorem for two one-dimensional SDEs driven by compensation jump processes such that

$$\begin{aligned}X_1(t) = x_1 &+ \int_0^t f_1(X_1(s), s)ds + \int_0^t g_1(X_1(s), s)dW(s) \\ &+ \int_0^t \int_{\mathbb{Y}} \gamma_1(X_1(s-), s, u)\tilde{N}(ds, du)\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}X_2(t) = x_2 &+ \int_0^t f_2(X_2(s), s)ds + \int_0^t g_2(X_2(s), s)dW(s) \\ &+ \int_0^t \int_{\mathbb{Y}} \gamma_2(X_2(s-), s, u)\tilde{N}(ds, du).\end{aligned} \tag{2.10}$$

The main result is as follows: $x_1 \ge x_2 \Rightarrow X_1(t) \ge X_2(t)$, $\mathbb{P}$ − a.s., if and only if

$$f_1(x, t) \ge f_2(x, t), g_1(x, t) = g_2(x, t), \gamma_1(x, t, u) = \gamma_2(x, t, u), \lambda(du) - \text{a.e.}$$

as well as (2.8) holds. For (2.4) and (2.5) which are driven by pure jump processes, we consider a comparison result in Lemma 2.2, where it is not necessary to impose $\gamma_1 = \gamma_2$. Clearly, (2.4) and (2.5) can be easily transformed to (2.9) and (2.10), respectively. However, we shall use this lemma to establish a comparison theorem for SDDEs driven by jump processes.

In the work [17], where comparison theorem of one-dimensional stochastic hybrid delay systems is studied, a very suggestive example (Example 3.3 in [17]) shows that comparison theorem need not hold whenever diffusion terms contain a delay function. While for stochastic delay systems with jumps, the following example demonstrates that the jump-diffusion terms could contain a delay function.

Example 2.1 Consider the following two one-dimensional SDDEs with jumps

$$\begin{cases} X(t) = c + \int_0^t \int_0^\infty \gamma(u) X((s-\tau)-)\tilde{N}(ds, du), & t \in [0, T]; \\ X(\theta) = c, & \theta \in [-\tau, 0) \end{cases} \tag{2.11}$$

and

$$\begin{cases} Y(t) = \int_0^t \int_0^\infty \gamma(u) Y((s-\tau)-)\tilde{N}(ds, du), & t \in [0, T]; \\ Y(\theta) = 0, & \theta \in [-\tau, 0) \end{cases} \tag{2.12}$$

where $c < 0$ is a constant. We further assume that

$$\gamma(u) > 0, \quad u \in (0, \infty) \tag{2.13}$$

and

$$\tau \int_0^\infty \gamma(u)\lambda(du) < 1. \tag{2.14}$$

For any $t \in [0, \tau]$

$$\begin{aligned} X(t) &= c\left(1 + \int_0^t \int_0^\infty \gamma(u)\tilde{N}(ds, du)\right) \\ &= c\left(1 + \int_0^t \int_0^\infty \gamma(u)N(ds, du) - \int_0^t \int_0^\infty \gamma(u)\lambda(du)ds\right). \end{aligned} \tag{2.15}$$

By (2.13), combining the definition of Poisson stochastic calculus, it follows that for $t \in [0, \tau]$

$$\int_0^t \int_0^\infty \gamma(u)N(ds, du) \geq 0$$

and

$$-\int_0^t \int_0^\infty \gamma(u)\lambda(du)ds \geq -\int_0^\tau \int_0^\infty \gamma(u)\lambda(du)ds = -\tau \int_0^\infty \gamma(u)\lambda(du).$$

Hence, together with (2.14), in (2.15) $X(t) \leq 0$ while $Y(t) \equiv 0$ for $t \in [0, \tau]$. As a consequence, we could derive the following comparison result: the solutions $X(t)$ of (2.11) and $Y(t)$ of (2.12) obey the property that for $t \in [0, \tau]$

$$X(t) \leq Y(t), \quad \mathbb{P}-\text{a.s.}$$

Motivated by [17, Example 3.3] we can also establish an example to show that, for stochastic delay systems with jumps, comparison theorem need not hold if diffusion term contains a delay function.

Example 2.2 Consider the following two one-dimensional equations

$$\begin{cases} X(t) = c + \int_0^t X(s-\tau)dW(s) - \int_0^t X((s-\tau)-)dN(s), & t \in [0, T]; \\ X(\theta) = c, & \theta \in [-\tau, 0) \end{cases}$$

and

$$\begin{cases} Y(t) = \int_0^t Y(s-\tau)dW(s) - \int_0^t Y((s-\tau)-)dN(s), & t \in [0,T]; \\ Y(\theta) = 0, & \theta \in [-\tau, 0) \end{cases}$$

where $c < 0$ is a constant, N is a Poisson process, independent of Brownian motion W. Clearly, for any $t \in [0, \tau]$, $Y(t) \equiv 0$ while

$$X(t) = c(1 + W(t) - N(t)).$$

Noting that $N(t) \geq 0$ and the relation

$$\{\omega \in \Omega : W(t) < -1\} \subseteq \{\omega \in \Omega : 1 + W(t) - N(t) < 0\},$$

hence

$$\mathbb{P}\{\omega \in \Omega : 1 + W(t) - N(t) < 0\} \geq \mathbb{P}\{\omega \in \Omega : W(t) < -1\} > 0,$$

since W obeys the normal distribution. This, together with $c < 0$, yields

$$\mathbb{P}\{\omega \in \Omega : X(t, \omega) > 0\} > 0.$$

Consequently, we could conclude that comparison theorem need not hold if diffusion coefficient contains a delay function. What's more, the following example shows that if the jump-diffusion coefficients are not increasing w.r.t. the delay variables, comparison theorem also need not hold.

Example 2.3 Consider the following two one-dimensional equations

$$\begin{cases} X(t) = c - 2\int_0^t X((s-\tau)-)dN(s), & t \in [0,T]; \\ X(\theta) = c, & \theta \in [-\tau, 0) \end{cases} \tag{2.16}$$

and

$$\begin{cases} Y(t) = -2\int_0^t I_{\{Y((s-\tau)-)<0\}} Y((s-\tau)-)dN(s), & t \in [0,T]; \\ Y(\theta) = 0, & \theta \in [-\tau, 0), \end{cases}$$

where $c < 0$ is a constant and N is a Poisson process with intensity λ.

By (2.16) it is easy to see that for any $t \in [0, \tau]$

$$X(t) = c(1 - 2N(t)).$$

In what follows we intend to show

$$\mathbb{P}\{\omega \in \Omega : X(t, \omega) > 0\} > 0. \tag{2.17}$$

Indeed, noting that

$$\{1 - 2N(t) < 0\} = \{N(t) \geq 1\},$$

we have

$$\mathbb{P}\{1 - 2N(t) < 0\} = 1 - e^{-\lambda t} > 0 \quad \text{whenever } 0 < t < \tau,$$

which further gives (2.17). Although

$$-2y \le -2yI_{\{y<0\}} \quad \text{and} \quad c < 0,$$

we cannot deduce that

$$X(t) \le Y(t), \quad \mathbb{P} - \text{a.s.}$$

due to (2.17) and $Y(t) \equiv 0$, $t \in [0, T]$.

Based on the previous discussion, now we state a comparison theorem for SDDEs driven by pure jump processes. In the proof, Lemma 2.2 is used.

Theorem 2.1 *Consider two one-dimensional SDDEs with pure jumps for any* $t \in [0, T]$

$$\begin{cases} dX_1(t) = f_1(X_1(t), X_1(t-\tau), t)dt + g(X_1(t), t)dW(t) \\ \qquad\qquad + \int_{\mathbb{Y}} \gamma(X_1(t-), X_1((t-\tau)-), t, u)N(dt, du) \\ X_1(t) = \xi_1(t), \quad t \in [-\tau, 0], \end{cases} \tag{2.18}$$

and

$$\begin{cases} dX_2(t) = f_2(X_2(t), X_2(t-\tau), t)dt + g(X_2(t), t)dW(t) \\ \qquad\qquad + \int_{\mathbb{Y}} \gamma(X_2(t-), X_2((t-\tau)-), t, u)N(dt, du) \\ X_2(t) = \xi_2(t), \quad t \in [-\tau, 0]. \end{cases} \tag{2.19}$$

Let $f_i, g, \gamma, i = 1, 2$, *be predictable and, for* $x_1, x_2, y_1, y_2, x, y \in \mathbb{R}$, *assume that there exists an* $L > 0$ *such that on* $[0, T] \times \Omega$

$$\begin{aligned} &|f_i(x_1, y_1, t) - f_i(x_2, y_2, t)|^2 + \int_{\mathbb{Y}} |\gamma(x_1, y_1, t, u) - \gamma(x_2, y_2, t, u)|^2 \lambda(du) \\ &\quad \le L(|x_1 - x_2|^2 + |y_1 - y_2|^2) \end{aligned} \tag{2.20}$$

and

$$|g(x, t) - g(y, t)|^2 \le L|x - y|^2 \tag{2.21}$$

with the property that $\mathbb{E}\sup_{0 \le t \le T}(|f_i(0, 0, t)|^2 + |g(0, t)|^2 + \int_{\mathbb{Y}} |\gamma(0, 0, t, u)|^2 \lambda(du)) < \infty$. *For* $x, y, z \in \mathbb{R}$, *assume further that on* $[0, T] \times \Omega$

$$f_1(x, y, t) \ge f_2(x, y, t) \tag{2.22}$$

and

$$x + \gamma(x, z, t, u) \le y + \gamma(y, z, t, u), \quad \lambda(du) - a.e. \text{ whenever } x \le y. \tag{2.23}$$

Moreover, we suppose that f_2 *and* γ *is nondecreasing with respect to the second variable, that is, on* $[0, T] \times \Omega$, *for* $x, y, z \in \mathbb{R}$

$$f_2(x, y, t) \ge f_2(x, z, t) \quad \text{and} \quad \gamma(x, y, t, u) \ge \gamma(x, z, t, u), \ \ \lambda(du) - a.e., \text{ whenever } y \ge z. \tag{2.24}$$

Then we have that for $\xi_1, \xi_2 \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R})$

$$X_1(t) \ge X_2(t), \quad t \in [0, T], \quad \mathbb{P} - a.s., \text{ provided that } \xi_1(t) \ge \xi_2(t), \ \mathbb{P} - a.s.$$

Proof The proof is motivated by that of [17, Theorem 3.4]. Under conditions (2.20) and (2.21), both (2.18) and (2.19) have unique solutions $X_1(t), t \in [0, T]$ and $X_2(t), t \in [0, T]$, respectively. Now consider SDDE with pure jumps for any $t \in [-\tau, T]$

$$\begin{cases} dX_3(t) = f_2(X_3(t), X_1(t-\tau), t)dt + g(X_3(t), t)dW(t) \\ \qquad\qquad + \int_{\mathbb{Y}} \gamma(X_3(t-), X_1((t-\tau)-), t, u)N(dt, du) \\ X_3(t) = \xi_2(t), \quad t \in [-\tau, 0]. \end{cases}$$

Noting by (2.22) that $f_1(x, X_1(t-\tau), t) \geq f_2(x, X_1(t-\tau), t)$, together with $\xi_1(t) \geq \xi_2(t), \mathbb{P}-$a.s., for $t \in [-\tau, 0]$, we conclude by Lemma 2.2 that $X_1(t) \geq X_3(t), t \in [-\tau, T]$, $\mathbb{P}-$a.s. Next consider SDDE with pure jumps

$$\begin{cases} dX_4(t) = f_2(X_4(t), X_3(t-\tau), t)dt + g(X_4(t), t)dW(t) \\ \qquad\qquad + \int_{\mathbb{Y}} \gamma(X_4(t-), X_3((t-\tau)-), t, u)N(dt, du) \\ X_4(t) = \xi_2(t), \quad t \in [-\tau, 0], \end{cases}$$

which could be rewritten as

$$\begin{cases} dX_4(t) = [f_2(X_4(t), X_1(t-\tau), t) + (f_2(X_4(t), X_3(t-\tau), t) \\ \qquad\qquad - f_2(X_4(t), X_1(t-\tau), t))]dt \\ \qquad\qquad + g(X_4(t), t)dW(t) + \int_{\mathbb{Y}}[\gamma(X_4(t-), X_1((t-\tau)-), t, u) \\ \qquad\qquad + (\gamma(X_4(t-), X_3((t-\tau)-), t, u) \\ \qquad\qquad - \gamma(X_4(t-), X_1((t-\tau)-), t, u))]N(dt, du) \\ X_4(t) = \xi_2(t), \quad t \in [-\tau, 0]. \end{cases}$$

Recalling $X_1(t) \geq X_3(t), t \in [-\tau, T], \mathbb{P}-$a.s., by (2.24) and Lemma 2.2, $X_3(t) \geq X_4(t), t \in [-\tau, T], \mathbb{P}-$a.s. In what follows, repeating the previous procedure we can get the sequence

$$X_1(t) \geq X_3(t) \geq X_4(t) \geq X_5(t) \geq \cdots \geq X_n(t) \geq \cdots, \quad \mathbb{P}-\text{a.s.}, \tag{2.25}$$

where $X_n(t)$ satisfies the following equation

$$\begin{cases} dX_n(t) = f_2(X_n(t), X_{n-1}(t-\tau), t)dt + g(X_n(t), t)dW(t) \\ \qquad\qquad + \int_{\mathbb{Y}} \gamma(X_n(t-), X_{n-1}((t-\tau)-), t, u)N(dt, du) \\ X_n(t) = \xi_2(t), \quad t \in [-\tau, 0]. \end{cases}$$

In what follows we intend to show that $X_n(t)$ is a Cauchy sequence, which has a unique limit $X(t)$, and $X(t) = X_2(t), t \in [0, T]$. Denote by $L^2_{\mathcal{F}_t}([0, T]; \mathbb{R})$ the space of $\mathbb{R}$-valued $\mathcal{F}_t$-adapted stochastic processes with $\mathbb{E}\int_0^T |v(t)|^2 dt < \infty$, equipped with the norm

$$\|v\|_{-\beta} := \left(\mathbb{E}\int_0^T |v(s)|^2 e^{-\beta s} ds\right)^{\frac{1}{2}},$$

where β is a positive constant to be determined. Obviously, the norm $\|v\|_{-\beta}$ is equivalent to the original one $\|v\| := (\mathbb{E}\int_0^T |v(t)|^2 dt)^{\frac{1}{2}}$ for $v \in L^2_{\mathcal{F}_t}([0, T]; \mathbb{R})$. For simplicity, set

$\bar{X}_n(t) := X_n(t) - X_{n-1}(t), n \geq 4$. Applying Itô's formula we find for any $t \in [0, T]$

$$\begin{aligned}
&\mathbb{E}(e^{-\beta t}|\bar{X}_n(t)|^2)\\
&= \mathbb{E}\int_0^t -\beta e^{-\beta s}|\bar{X}_n(s)|^2 ds + \mathbb{E}\int_0^t e^{-\beta s}[2\bar{X}_n(s)(f_2(X_n(s), X_{n-1}(s-\tau), s)\\
&\quad - f_2(X_{n-1}(s), X_{n-2}(s-\tau), s)) + |g(X_n(s), s) - g(X_{n-1}(s), s)|^2]ds\\
&\quad + \mathbb{E}\int_0^t \int_{\mathbb{Y}} e^{-\beta s}\Big[2\bar{X}_n(s-)\Big(\gamma(X_n(s-), X_{n-1}((s-\tau)-), s, u)\\
&\quad - \gamma(X_{n-1}(s-), X_{n-2}((s-\tau)-), s, u)\Big)\\
&\quad + \Big|\gamma(X_n(s-), X_{n-1}((s-\tau)-), s, u) - \gamma(X_{n-1}(s-), X_{n-2}((s-\tau)-), s, u)\Big|^2\Big]\\
&\quad \times N(ds, du).
\end{aligned}$$

This, together with (2.20) and (2.21), yields that

$$\begin{aligned}
&\mathbb{E}(e^{-\beta t}|\bar{X}_n(t)|^2)\\
&\leq \mathbb{E}\int_0^t -\beta e^{-\beta s}|\bar{X}_n(s)|^2 ds + \mathbb{E}\int_0^t e^{-\beta s}[2L|\bar{X}_n(s)|^2 + L|\bar{X}_{n-1}(s)|^2\\
&\quad + 2L^{\frac{1}{2}}(1 + (\lambda(\mathbb{Y}))^{\frac{1}{2}})|\bar{X}_n(s)|(|\bar{X}_n(s)| + |\bar{X}_{n-1}(s)|)]ds\\
&\leq (-\beta + 2L + 3L^{\frac{1}{2}}(1 + (\lambda(\mathbb{Y}))^{\frac{1}{2}}))\mathbb{E}\int_0^t e^{-\beta s}|\bar{X}_n(s)|^2 ds\\
&\quad + (L + L^{\frac{1}{2}}(1 + (\lambda(\mathbb{Y}))^{\frac{1}{2}}))\mathbb{E}\int_0^t e^{-\beta s}|\bar{X}_{n-1}(s)|^2 ds.
\end{aligned}$$

Letting

$$\beta = 4L + 5L^{\frac{1}{2}}(1 + (\lambda(\mathbb{Y}))^{\frac{1}{2}}),$$

we then have $-\beta + 2L + 3L^{\frac{1}{2}}(1 + (\lambda(\mathbb{Y}))^{\frac{1}{2}}) < 0$ and

$$(L + L^{\frac{1}{2}}(1 + (\lambda(\mathbb{Y}))^{\frac{1}{2}}))/(\beta - (2L + 3L^{\frac{1}{2}}(1 + (\lambda(\mathbb{Y}))^{\frac{1}{2}}))) = \frac{1}{2}.$$

Hence

$$\mathbb{E}\int_0^t e^{-\beta s}|\bar{X}_n(s)|^2 ds \leq \frac{1}{2}\mathbb{E}\int_0^t e^{-\beta s}|\bar{X}_{n-1}(s)|^2 ds,$$

which implies by induction arguments that

$$\mathbb{E}\int_0^t e^{-\beta s}|\bar{X}_n(s)|^2 ds \leq \frac{1}{2^{n-4}}\mathbb{E}\int_0^t e^{-\beta s}|\bar{X}_4(s)|^2 ds.$$

This gives that $\bar{X}_n(t)$ is a Cauchy sequence in $L^2_{\mathcal{F}_t}([0, T]; \mathbb{R})$ with the norm $\|\cdot\|_{-\beta}$. Thus $X_n(t)$ is also a Cauchy sequence and has a unique limit denoted by $X(t) \in L^2_{\mathcal{F}_t}([0, T]; \mathbb{R})$, which is a complete norm space under the norm $\|\cdot\|_{-\beta}$. Next we show $X_2(t) = X(t)$ by the uniqueness. In fact, by (2.20)

$$
\begin{aligned}
&\mathbb{E}\int_0^T e^{-\beta t}\left|\int_0^t [f_2(X_n(s), X_{n-1}(s-\tau), s) - f_2(X(s), X(s-\tau), s)]ds\right|^2 dt \\
&\le LT\mathbb{E}\int_0^T \int_0^t e^{-\beta(t-s)}e^{-\beta s}(|X_n(s) - X(s)|^2 + |X_{n-1}(s) - X(s)|^2)dsdt \\
&\le LT^2\mathbb{E}\int_0^T e^{-\beta s}(|X_n(s) - X(s)|^2 + |X_{n-1}(s) - X(s)|^2)ds \\
&\to 0 \text{ as } n \to \infty,
\end{aligned}
$$

and, according to Itô's isometry

$$
\begin{aligned}
&\mathbb{E}\int_0^T e^{-\beta t}\bigg|\int_0^t \int_{\mathbb{Y}} [\gamma(X_n(s-), X_{n-1}((s-\tau)-), s, u) \\
&\qquad - \gamma(X(s-), X((s-\tau)-), s, u)]N(ds, du)\bigg|^2 dt \\
&\le C\mathbb{E}\int_0^T e^{-\beta t}\int_0^t \int_{\mathbb{Y}} |\gamma(X_n(s), X_{n-1}(s-\tau), s, u) \\
&\qquad - \gamma(X(s), X(s-\tau), s, u)|^2 \lambda(du)dsdt \\
&\le LCT\mathbb{E}\int_0^T e^{-\beta s}(|X_n(s) - X(s)|^2 + |X_{n-1}(s) - X(s)|^2)ds \\
&\to 0 \text{ as } n \to \infty,
\end{aligned}
$$

where $C := 2(1 + T\lambda(\mathbb{Y}))$, and, carrying out similar arguments,

$$
\mathbb{E}\int_0^T e^{-\beta t}\left|\int_0^t [g(X_n(s), s) - g(X(s), s)]dW(s)\right|^2 dt \to 0 \quad \text{as} \quad n \to \infty.
$$

As a consequence, we can conclude that X satisfies

$$
\begin{cases}
dX(t) = f_2(X(t), X(t-\tau), t)dt + g(X(t), t)dW(t) \\
\qquad\qquad + \int_{\mathbb{Y}} \gamma(X(t-), X((t-\tau)-), t, u)N(dt, du) \\
X(t) = \xi_2(t), \quad t \in [-\tau, 0].
\end{cases}
$$

By the uniqueness of solution of (2.19), we conclude that $X(t) = X_2(t)$ and the desired assertion is complete by recalling (2.25). □

Remark 2.2 [17] established an example to show that condition (2.22) is vital for the comparison theorem for SDDEs. By Example 2.3, we could conclude that, if the jump diffusion γ is nonincreasing in second variable, namely, delay term, comparison theorem might not be available. Therefore condition (2.24) is natural. For (2.23), we can refer to Situ [13] and Peng and Zhu [11] for more details.

Remark 2.3 By carrying out the techniques of stopping times, the derived comparison theorem can be extend to the case in which Lipschitz condition is replaced by the Carathéodory-type condition [13, p. 292].

3 Comparison Theorem for SDDEs with Compensation Jump Processes

In the last section we established a comparison theorem for SDDEs with pure jump processes. To make the content more comprehensive, in this part we aim to discuss the comparison problems for SDDEs with compensation jump process.

Consider two one-dimensional SDDEs with jumps for any $t \in [0, T]$

$$\begin{cases} dX_1(t) = f_1(X_1(t), X_1(t-\tau), t)dt + g(X_1(t), t)dW(t) \\ \qquad\qquad + \int_{\mathbb{Y}} \gamma(X_1(t-), X_1((t-\tau)-), t, u)\tilde{N}(dt, du) \\ X_1(t) = \xi_1(t), \quad t \in [-\tau, 0], \end{cases} \tag{3.1}$$

and

$$\begin{cases} dX_2(t) = f_2(X_2(t), X_2(t-\tau), t)dt + g(X_2(t), t)dW(t) \\ \qquad\qquad + \int_{\mathbb{Y}} \gamma(X_2(t-), X_2((t-\tau)-), t, u)\tilde{N}(dt, du) \\ X_2(t) = \xi_2(t), \quad t \in [-\tau, 0]. \end{cases} \tag{3.2}$$

Noting that $\tilde{N}(dt, du) = N(dt, du) - \lambda(du)dt$, (3.1) and (3.2) are equivalent to

$$\begin{cases} dX_1(t) = [f_1(X_1(t), X_1(t-\tau), t) - \int_{\mathbb{Y}} \gamma(X_1(t), X_1(t-\tau), t, u)\lambda(du)]dt \\ \qquad\qquad + g(X_1(t), t)dW(t) + \int_{\mathbb{Y}} \gamma(X_1(t-), X_1((t-\tau)-), t, u)N(dt, du) \\ X_1(t) = \xi_1(t), \quad t \in [-\tau, 0], \end{cases}$$

and

$$\begin{cases} dX_2(t) = [f_2(X_2(t), X_2(t-\tau), t) - \int_{\mathbb{Y}} \gamma(X_2(t), X_2(t-\tau), t, u)\lambda(du)]dt \\ \qquad\qquad + g(X_2(t), t)dW(t) + \int_{\mathbb{Y}} \gamma(X_2(t-), X_2((t-\tau)-), t, u)N(dt, du) \\ X_2(t) = \xi_2(t), \quad t \in [-\tau, 0], \end{cases}$$

respectively.

Applying comparison theorem, Theorem 2.1, we can derive the following comparison results for stochastic delay systems with compensation jump processes.

Theorem 3.1 *Let conditions* (2.20)–(2.23) *hold. Moreover, we suppose that* $f_2 - \gamma$ *and* γ *is non-decreasing with respect to the second variable, that is, on* $[0, T] \times \Omega$, *for* $x, y, z \in \mathbb{R}$

$$f_2(x, y, t) - \int_{\mathbb{Y}} \gamma(x, y, t, u)\lambda(du) \geq f_2(x, z, t) - \int_{\mathbb{Y}} \gamma(x, z, t, u)\lambda(du) \tag{3.3}$$

and

$$\gamma(x, y, t, u) \geq \gamma(x, z, t, u), \quad \lambda(du) - a.e. \tag{3.4}$$

whenever $y \geq z$. *Then we have for* $\xi_1, \xi_2 \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R})$

$$X_1(t) \geq X_2(t), t \in [0, T], \quad \mathbb{P} - a.s., \text{ provided that } \xi_1(t) \geq \xi_2(t), \quad \mathbb{P} - a.s.$$

Example 3.1 Consider two one-dimensional SDDEs with jumps

$$\begin{cases} dX_1(t) = f_1(X_1(t), X_1(t-\tau), t)dt + g(X_1(t), t)dW(t) \\ \qquad\qquad + \int_{\mathbb{Y}} \rho(u) f_2(X_1(t-), X_1((t-\tau)-), t)\tilde{N}(dt, du) \\ X_1(t) = \xi_1(t), \quad t \in [-\tau, 0], \end{cases}$$

and

$$\begin{cases} dX_2(t) = f_2(X_2(t), X_2(t-\tau), t)dt + g(X_2(t), t)dW(t) \\ \qquad\qquad + \int_{\mathbb{Y}} \rho(u) f_2(X_2(t-), X_2((t-\tau)-), t)\tilde{N}(dt, du) \\ X_2(t) = \xi_2(t), \quad t \in [-\tau, 0], \end{cases}$$

where f_1, f_2, g satisfy conditions (2.20)–(2.22) and $\xi_1(t) \geq \xi_2(t)$, $\mathbb{P} -$ a.s., for $\xi_1, \xi_2 \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R})$.

In what follows, we further assume that $\rho > 0$, $\int_{\mathbb{Y}} \rho(u)\lambda(du) < 1$ and

$$f_2(x, y, t) \geq f_2(x, z, t) \quad \text{whenever } y \geq z. \tag{3.5}$$

By Theorem 3.1, to show that $X_1(t) \geq X_2(t), t \in [-\tau, T]$, $\mathbb{P} -$ a.s., it is sufficient to check conditions (2.23), (3.3) and (3.4). By (3.5) and $\rho > 0$, it is easy to see that conditions (2.23) and (3.4) hold. On the other hand, recalling $\int_{\mathbb{Y}} \rho(u)\lambda(du) < 1$, we have

$$f_2(x, y, t) - \int_{\mathbb{Y}} \rho(u)\lambda(du) f_2(x, y, t) = \left(1 - \int_{\mathbb{Y}} \rho(u)\right) f_2(x, y, t),$$

and, combining (3.5), condition (3.3) is also true.

Acknowledgements The authors wish to express their sincere thanks to the anonymous referees for their careful comments and valuable suggestions, which greatly improved the quality of the paper.

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