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Paths, Homotopy and Reduction in Digital Images

Loïc Mazo · Nicolas Passat · Michel Couprie · Christian Ronse

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Abstract The development of digital imaging (and its subsequent applications) has led to consideration and investigation of topological notions that are well-defined in continuous spaces, but not necessarily in discrete/digital ones. In this article, we focus on the classical notion of path. We establish in particular that the standard definition of path in algebraic topology is coherent w.r.t. the ones (often empirically) used in digital imaging. From this statement, we retrieve, and actually extend, an important result related to homotopy-type preservation, namely the equivalence between the fundamental group of a digital space and the group induced by digital paths. Based on this sound definition of paths, we also (re)explore various (and sometimes equivalent) ways to reduce a digital image in a homotopy-type preserving fashion.

Keywords Topology · Digital imaging · Paths · Fundamental group · Homotopy-type preservation

1 Introduction

Several different models have been proposed to deal with topological properties in finite sets. The first works dedicated to this issue have been developed by Alexandroff [1] in 1937.

L. Mazo (✉) · N. Passat · C. Ronse
LSIIT, UMR CNRS 7005, Université de Strasbourg, Parc d'Innovation, Bd S. Brant, BP 10413, 67412 Illkirch Cedex, France
e-mail: loic.mazo@unistra.fr

N. Passat
e-mail: passat@unistra.fr

C. Ronse
e-mail: cronse@unistra.fr

L. Mazo · M. Couprie
Laboratoire d'Informatique Gaspard-Monge, Université Paris-Est, Equipe A3SI, ESIEE Paris, France

M. Couprie
e-mail: m.couprie@esiee.fr

After this first attempt, no other works have been proposed for approximately 30 years, and we had to wait until the mid 60's to see (simultaneous) new contributions in the mathematics community [29, 36] and also in the computer science one [9, 34]. The rapid and important rise of digital imaging, and the associated need of efficient image analysis tools for 2-D, and later 3-D (and even 4-D) digital images have provided a strong motivation for the research related to the definition of sound discrete/digital topological models. Indeed, in order to be able to segment, process, or analyse digital images in various application fields, it is often fundamental to be able to preserve, retrieve or integrate topological information.

In the mathematics community, after the pioneering works of Alexandroff, McCord [29] firmly linked finite spaces with simplicial complexes, while Stong [36] undertook homeomorphism and homotopy type classifications. Many years later, at the end of the century, this subject yielded new developments whose main goal was to classify simplicial complexes via finite spaces [3, 10, 20, 31].

In the computer science community, works have essentially focused on specific—and pragmatic—questions related to topology, namely the definition of a notion of *adjacency* relation, and the induced notions of *connectivity* and *arcs*. These notions enable in particular to lead to high-level concepts of topology, such as homotopy, with natural applications to "homotopy type-preserving" transformations of topological spaces/digital images. The first—and very intuitive—solution to define an adjacency relation on $\mathbb{Z}^n$ is to consider that two points are adjacent if there are neighbours in the n-D cubic grid (possibly enriched by some well chosen sets of "diagonals"). This framework led—in order to avoid paradoxical intersections between objects—to the classical definition of *dual adjacencies* in digital images [9, 33, 34]. In this approach, known as *digital topology*, no topology is however actually defined and there are, in particular, no open/closed sets. To retrieve topological notions, a possible way is to define *continuous analogues* of n-D digital images, assuming that each point in such images physically corresponds to a unit n-cube of the Euclidean space. Following this analogy, it becomes possible to justify the use of dual adjacencies [32] and to define algebraic structures isomorphic to those used in topology [16, 22]. An alternative way to deal with connectivity in digital pictures is to find a topology in $\mathbb{Z}^n$, i.e., a family of subsets of $\mathbb{Z}^n$ (defined as open sets), leading to the desired adjacency relation (in this framework, two points x, y are adjacent if the set $\{x, y\}$ is connected). In [17], it is proved that there is only one convenient solution—the product of Khalimsky lines [13]—for defining such a framework, unfortunately this solution breaks the homogeneity of $\mathbb{Z}^n$. (To avoid this phenomenon, it is necessary to add points between those actually in the image, which is equivalent to identify the points of a digital image with some cells of abstract cellular complexes [15, 19].) All these topological models have found practical applications in the context of digital image analysis, especially for the definition of "topology-preserving" procedures (i.e., procedures enabling to modify a binary digital image without altering its homotopy type), including reduction ones (used for skeletonisation or segmentation), see e.g. [8].

The quite pragmatic motivations of the works on topological modelling of digital images can probably explain why most of the proposed definitions only aim at mimicking or adapting the definitions of the classical topology to retrieve intuitive notions such as connectivity and continuous deformation. Moreover, if the works of Alexandroff are relatively well known in both (mathematics and computer science) communities, those of McCord and Stong have visibly never been considered in the research related to topology in digital images. Consequently, it is generally believed that the classical definitions of topology cannot be "directly" embedded in the universe of digital images in a sound fashion[1] (i.e., while preserving their correct and intrinsic properties). In particular, it seems that *paths* in finite spaces have been quite systematically replaced by *ad hoc* definitions. This justifies to carefully explore the relations between continuous paths and digital paths of finite spaces.

The purpose of this article is to study the consequences of the use of the general topology standard definition of a path, namely a continuous function from [0, 1], when working in a digital space. We describe the images of such paths in a digital space and compare them with the regular digital paths defined in the framework of the Khalimsky topology [14] or in the equivalent framework of abstract cellular complexes [19]. We show that both definitions lead to very close geometrical objects: our first main result (Theorem 2) states that under each continuous path p, lies another continuous path which is a step function (for such a path, we say *a finite path*), whose image is included in the image of p and which is equal to p for at least one value of the parameter in each step interval. We also look at homotopy equivalence between paths and describe their discrete counterparts that we call *deformations*. We show that two finite paths with a same image are homotopically equivalent and our second main theorem (Theorem 3) establishes that two finite paths with distinct images are homotopically equivalent iff the image of one of them is a deformation of the image of the other one. Then, we retrieve (and in fact, extend, since we do not suppose the ambient space to be finite), without the need of high level preliminary results, the property recently proved in algebraic topology [3] that the fundamental group of a digital space is isomorphic to the group of digital paths equipped with the deformations. Since our model is based on classical definitions, we have the possibility of reinvest any external result in the field of image analysis and processing. In particular, we explore and compare some tools devoted to the reduction of finite, or countable, spaces and which have counterparts in continuous analogues embedded in the Euclidean space.

In order to do so, Sect. 2 first recalls background notions related to general topology and partially ordered sets. (These notions enable to make this article globally self-contained, and then more comprehensible for the reader.) In Sect. 3, we study in detail the paths in digital images, i.e., the continuous functions of $[0, 1] \to \mathbb{Z}^n$ (where $\mathbb{Z}^n$ is interpreted from the topological point of view mentioned above) and we justify why we can avoid to consider the "functional side" of paths. In particular, we prove that the fundamental group of a digital space is isomorphic to the "fundamental-like" group which is generally considered in digital image analysis. Then, topological algebraic structures being well defined, we can borrow any tool in the existing literature on countable/finite spaces for use in image analysis and processing. Thereby, in Sect. 4, we study and confront various ways to make minimal changes in a digital image while preserving, as far as possible, its topology. Concluding remarks will be found in Sect. 5.

2 Background Notions

2.1 General Topology

In this subsection, we recall (without proof) some basic definitions and classical properties of topology. The main purpose here is to introduce useful notations and to gather results needed in the sequel of the article. The reader interested in proofs, or details on a particular

[1] In [21], Latecki writes "topology is basically not a finite concept and reduces to triviality whenever applied to finite sets".

notion, can find them in any lecture book on general topology (for example, [30, 37]) or on algebraic topology (for example, [11, 24, 25].

A space X satisfies the *separation axiom* T_0 (or, shortly, is a T_0*-space*) if for every pair (x_1, x_2) $(x_1 \neq x_2)$ in X there is an open set of X which contains exactly one element of the pair. That is, one can distinguish them from a topological viewpoint. The pair (x_1, x_2) satisfies this condition exactly if x_1 does not belong to the closure of $\{x_2\}$ or x_2 does not belong to the closure of $\{x_1\}$. If for every pair (x_1, x_2) $(x_1 \neq x_2)$, x_1 does not belong to the closure of $\{x_2\}$ and x_2 does not belong to the closure of $\{x_1\}$, that is, if for each $x \in X$, $\{x\}$ is closed, then X is a T_1*-space*. *Hausdorff spaces*, or T_2*-spaces*, like $\mathbb{R}^n$ equipped with the usual topology, have a stronger property: any two distinct points have disjoint neighbourhoods. Note that a T_2-space is T_1 and a T_1-space is T_0.

If Y is a subset of X, Y is a *retract* of X if there exists a continuous map, called a *retraction*, $r : X \to Y$ such that $r(y) = y$ for all $y \in Y$. A continuous map $r : X \times [0, 1] \to X$ is a (*strong*) *deformation retraction* if, for every x in X and y in Y, we have $r(x, 0) = x$, $r(x, 1) \in Y$ and $r(y, 1) = y$ (and for every t in $[0, 1]$, $r(y, t) = y$). If such a map exists, Y is a (*strong*) *deformation retract* of X.

When Y is not a subspace of X, there exists however a notion similar to that of retraction. Two continuous maps $f, g : X \to Y$ are *homotopic* if there exists a continuous map, called a *homotopy*, $h : X \times [0, 1] \to Y$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in X$. The spaces X and Y are *homotopy equivalent* (or have the same *homotopy type*) if there exist two continuous maps $f : X \to Y$ and $g : Y \to X$, called *homotopy equivalences*, such that $g \circ f$ is homotopic to the identity map id_X and $f \circ g$ is homotopic to id_Y. If X and Y are homeomorphic, they are homotopy equivalent: given a homeomorphism φ between X and Y, φ and φ^{-1} are homotopy equivalences between X and Y. The converse is not true in general (for example, a ball is homotopy equivalent—but not homeomorphic—to a point). A topological space is *contractible* if it has the homotopy type of a single point.

Let X be a topological space. Two paths p, q in X are *equivalent* if they have the same extremities (i.e., $p(0) = q(0)$ and $p(1) = q(1)$) and are homotopic by an homotopy h such that $h(0, u) = p(0) = q(0)$ and $h(1, u) = p(1) = q(1)$ for all $u \in [0, 1]$. It is easy to check that this relation on paths is actually an equivalence relation. We write $[p]$ for the equivalence class of p. If p, q are two paths in X such that $p(1) = q(0)$ we can define the product $p \cdot q$ by:

$$(p \cdot q)(t) = \begin{cases} p(2t) & \text{if } t \in [0, \frac{1}{2}], \\ q(2t-1) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

This product is well defined on equivalence classes by $[p] \cdot [q] = [p \cdot q]$. Let x be a point of X. A *loop* at x is a path in X which starts and ends at x. The product of two loops at x is a loop at x and the set $\pi_1(X, x)$ of equivalence classes of loops at x is a group for this product. It is called the *fundamental group* of X (with *basepoint* x) or the *first homotopy group* of X. If X is path-connected, the group $\pi_1(X, x)$ does not depend on the basepoint (i.e., for any points $x, y \in X$, $\pi_1(X, x)$ and $\pi_1(X, y)$ are isomorphic). Higher homotopy groups are defined by replacing loops at x by continuous maps from $[0, 1]^n$ to X that associate the boundary of the n-cube to x. The product on such maps is then defined by gluing two faces of the n-cubes:

$$p \cdot q(t_1, \ldots, t_n) = \begin{cases} p(2t_1, t_2, \ldots, t_n) & \text{if } t_1 \in [0, \frac{1}{2}], \\ q(2t_1 - 1, t_2, \ldots, t_n) & \text{if } t_1 \in [\frac{1}{2}, 1]. \end{cases}$$

Conventionally, the set of path-connected components of X is denoted by $\pi_0(X, x)$, but it has no group structure.

Let X and Y be two topological spaces with base points x and y. A continuous map $f : X \to Y$ is a *weak homotopy equivalence* if the morphisms $f_n : \pi_n(X, x) \to \pi_n(Y, y)$ defined by $f_n([p]) = [f \circ p]$ are all bijective (f_0 is just a bijection, not a morphism). Two spaces X, Y are *weakly homotopy equivalent* if there is a sequence of spaces $X_0 = X, X_1, \ldots, X_r = Y$ ($r \geqslant 1$) such that there exist weak homotopy equivalences $X_{i-1} \to X_i$ or $X_i \to X_{i-1}$ for all $i \in [1, r]$. Hence, two weakly homotopy equivalent spaces X, Y have isomorphic homotopy groups.

Two homotopy equivalent spaces are weakly homotopy equivalent (the converse is not true in general but Whitehead's theorem [39, 40] implies that it is true for all spaces that are geometric realisations of simplicial or cubical complexes (see Sect. 4.1)).

2.2 Partially Ordered Sets

Because of their capacity to encompass all topological approaches on digital images, our work is presented in the framework of *posets* (Partially Ordered SETS). For this reason, but mainly to show how discrete spaces are concerned with continuity, this subsection on partially ordered sets is more detailed than the previous one. We give proofs, as far as possible, while we state properties with hypothesis close to our subject. Readers interested in more general hypothesis may refer to [1, 2, 26, 27, 29, 36].

Let X be a set. A binary relation on X is a *partial order* if it is reflexive, antisymmetric, and transitive. A *partially ordered set*, or *poset*, is a couple $(X, \trianglelefteqslant)$ where the relation $\trianglelefteqslant$ is a partial order on X. The relation $\trianglerighteqslant$ defined on X by $x \trianglerighteqslant y$ if $y \trianglelefteqslant x$ is a partial order on X called the *dual order*. We say that two points x, y in X are *comparable* if $x \trianglelefteqslant y$ or $y \trianglelefteqslant x$. We say that a poset is *locally finite* if for each point x in X, there are finitely many points comparable with x (note that for many authors, locally finite means that each point x has a finite neighbourhood). As an example, $\mathbb{N}$ equipped with the dual of the usual order (i.e., with $\geqslant$) is not locally finite with the definition we use though each point $n \in \mathbb{N}$ has a finite neighbourhood $[0, n]$ (see Theorem 1 for the definition of the topology). If, for all pairs (x, y) of elements of X, x and y are comparable, the relation $\trianglelefteqslant$ is a *total order* on X. A *chain* in X is a totally ordered subset of X. A poset is *finite-dimensional* if there is an integer n such that any chain in X has a cardinal less or equal than $n + 1$. The smallest integer n having this property is called the *dimension* of X and we write $n = \dim(X)$.

We write $x \lhd y$ when $x \trianglelefteqslant y$ and $x \neq y$ and we set:

- $x^{\uparrow} = \{y \in X \mid x \trianglelefteqslant y\}$ and $x^{\uparrow\star} = x^{\uparrow} \setminus \{x\} = \{y \in X \mid x \lhd y\}$;
- $x^{\downarrow} = \{y \in X \mid y \trianglelefteqslant x\}$ and $x^{\downarrow\star} = x^{\downarrow} \setminus \{x\} = \{y \in X \mid y \lhd x\}$.

If x and y are comparable, we write $x \asymp y$; otherwise, we write $x \not\asymp y$. The set of points comparable with a given point x is denoted $x^{\updownarrow}$ ($x^{\updownarrow} = x^{\downarrow} \cup x^{\uparrow}$) and $x^{\updownarrow\star} = x^{\updownarrow} \setminus \{x\} = x^{\downarrow\star} \cup x^{\uparrow\star}$. A point $x \in X$ is *minimal* if $x^{\downarrow} = \{x\}$ and *maximal* if $x^{\uparrow} = \{x\}$. A point $x \in X$ is the *minimum* of X if $x^{\uparrow} = X$ and is the *maximum* of X if $x^{\downarrow} = X$.

The Hasse diagram is the oriented graph of the *covering* relation defined by: y covers x ($x \prec y$) if $x \lhd y$ and there is no z such that $x \lhd z \lhd y$. Orienting all arcs from top to bottom, this diagram offers good visual representations of (small) posets (see Fig. 8).

2.2.1 Topology in Posets

Let us forget for a while posets in order to define Alexandroff spaces. A topological space X is an *Alexandroff space* if any intersection of open sets is an open set. In such a space, closed sets satisfy the definition properties of open sets, namely, $\emptyset$, X are closed sets, any union and any intersection of closed sets is a closed set, so one can exchange open and closed sets. The obtained topology is then called the *dual topology*. As any set has a closure, any element x of an Alexandroff space has a *smallest neighbourhood* (an open set included in any open set containing x), denoted by U_x, which is the closure of $\{x\}$ for the dual topology. Conversely, a topological space X in which each point has a smallest neighbourhood is an Alexandroff space. Moreover, since for any open set $V \subseteq X$, we have $V = \bigcup_{x \in V} U_x$, the set of smallest neighbourhoods is a basis for the topology. When a topological set X has the T_1-separation property, each singleton is closed; thus an Alexandroff space with the T_1-separation property is totally disconnected. It is the reason why the only Alexandroff spaces worthy of interest are non-T_1. We call the T_0-Alexandroff spaces *A-space*s. McCord has proved in [29] that if an Alexandroff space is not T_0, the identification of the points that share the same smallest neighbourhood give a homotopy equivalent quotient space which is T_0.

Now let us go back to posets with the next theorem known as Alexandroff specialisation theorem which establishes a canonical link between A-spaces and posets.

Theorem 1 [1] *Let $(X, \mathfrak{U})$ be an A-space. The relation $\trianglelefteqslant$ defined on X by $x \trianglelefteqslant y$ if $y \in U_x$ is a partial order on X. Conversely, let $(X, \trianglelefteqslant)$ be a poset. The set $\mathfrak{U}$ defined by $\mathfrak{U} = \{U \subseteq X \mid \forall x \in U, x^{\uparrow} \subseteq U\}$ is a topology on X, $(X, \mathfrak{U})$ is an A-space and, for all $x \in X$, $U_x = x^{\uparrow}$.*

If Y is a subset of X, the topology associated to the poset $(Y, \trianglelefteqslant)$ is the topology induced by the one associated to the poset $(X, \trianglelefteqslant)$. The dual topology of the topology associated to the poset $(X, \trianglelefteqslant)$ is the topology associated to the dual order $\trianglerighteqslant$.

From now on, posets will always be equipped with the topology $\mathfrak{U}$ described in Theorem 1.

The easy following property founds an interesting application when a continuous function, like a path, is defined from a compact subset of $\mathbb{R}^n$, that is a closed bounded subset, to a locally finite poset.

Property 1 *Any compact locally finite poset is finite.*

Proof Let X be a compact locally finite poset. Since X is compact, there exists a finite set $A \subseteq X$ such that $(U_x)_{x \in A}$ is a finite subcover of the open cover $(U_x)_{x \in X}$. As X is locally finite, each U_x is finite and, therefore, $X = \bigcup_{x \in A} U_x$ is finite. □

2.2.2 Continuity and Connectivity

Property 2 *Let X, Y be posets. A function $f : X \to Y$ is continuous iff it is non-decreasing.*

Proof [36] (Stong assumes X and Y to be finite, but he does not use it in his proof) Suppose that f is continuous. Let $x_1 \trianglelefteqslant x_2$ be two points in X. Since $f^{-1}(U_{f(x_1)})$ is an open set containing x_1, it includes $x_1^{\uparrow}$ so $x_2 \in f^{-1}(U_{f(x_1)})$ and $f(x_2) \in U_{f(x_1)}$, that is $f(x_1) \trianglelefteqslant f(x_2)$. Conversely, suppose that f is non-decreasing. For some $y \in Y$, take $x \in f^{-1}(U_y)$, which means $y \trianglelefteqslant f(x)$. For any $x' \in U_x$, $x \trianglelefteqslant x'$, so $y \trianglelefteqslant f(x) \trianglelefteqslant f(x')$ and $x' \in f^{-1}(U_y)$. Hence $U_x \subseteq f^{-1}(U_y)$ for any $x \in f^{-1}(U_y)$. That is, $f^{-1}(U_y)$ is open. □

Let $x, y \in X$. We say that x, y are *adjacent* if the set $\{x, y\}$ is connected. A sequence $(x_i)_{i=0}^r$ ($r \geqslant 0$) of points in X is an *arc* in X (*from x_0 to x_r*) if for all $i \in [1, r]$, x_{i-1} and x_i are distinct and adjacent. The integer r is the *length* of the arc $(x_i)_{i=0}^r$. If for all x_i,

$1 \leqslant i \leqslant r-1$, $x_{i-1} < x_i \Leftrightarrow x_i > x_{i+1}$, we say that the arc is *minimal*.[2] If for all $x, y \in X$ there exists an arc in X from x to y, we say that X is *arc-connected*.

Property 3 *Two points $x, y \in X$ are adjacent iff x and y are comparable.*

Proof Let $S = \{x, y\}$ and suppose x, y are not comparable, that is $x \notin U_y$ and $y \notin U_x$. Then, $U_x \cap S = \{x\}$ and $U_y \cap S = \{y\}$ are disjoint open sets of S. Therefore, S is not connected. If x, y are comparable, for example $x \trianglelefteq y$, every open set containing x contains y, so it is impossible to break S into two non-empty open sets. Thus S is connected. □

Lemma 1 *Let X be a poset. If x and y are comparable, then there is a path from x to y.*

Proof [36] (Stong assumes X and Y to be finite, but he does not use it in his proof) Suppose $x \trianglelefteq y$ and let $p : [0, 1] \to X$ be the function defined by $p(t) = x$ if $t \leqslant \frac{1}{2}$ and $p(t) = y$ if $t > \frac{1}{2}$. We claim that p is continuous, i.e., p is a path. To prove this assertion, it is sufficient to prove that for any U_z, $p^{-1}(U_z)$ is open in $[0, 1]$. If $x, y \notin U_z$, then $p^{-1}(U_z)$ is empty and thus is open. If $x \in U_z$, $z \trianglelefteq x \trianglelefteq y$ so $y \in U_z$ and $p^{-1}(U_z) = [0, 1]$ is open. If $x \notin U_z$ and $y \in U_z$, then $p^{-1}(U_z) =]\frac{1}{2}, 1]$ which is an open set of $[0, 1]$. □

The material for the next property, and for its proof, is also in [36].

Property 4 *Let X be a poset. The following statements are equivalent*:
1. *X is path-connected*;
2. *X is connected*;
3. *X is arc-connected.*

Proof $1 \Rightarrow 2$ is true in any topological space. To prove $2 \Rightarrow 3$, suppose X is connected and take a point $x \in X$. By Proposition 3, it is straightforward to prove that the sets A of points in X that are connected to x by an arc and its complement, $X = B \setminus A$, are open. As X is connected and $A \neq \emptyset$, B is empty and X is arc-connected. Finally to prove $3 \Rightarrow 1$, suppose X is arc-connected. From Lemma 1, we derive easily that X is path-connected. □

Observe that the above property means that the standard definition of paths and the digital one lead to the same path-connected components.

2.2.3 Homotopy

Let f, g be two continuous maps from a topological space Y to X. We write $f \trianglelefteq g$ when $f(a) \trianglelefteq g(a)$ for all $a \in Y$. It is straightforward that the relation $\trianglelefteq$ is a partial order on $\mathcal{C}(Y, X)$, the set of continuous maps from Y to X. For some given $x_1, x_2 \in X$, $y_1, y_2 \in Y$, we set $\mathcal{C}(Y, X)_\star = \{f \in \mathcal{C}(Y, X) \mid f(y_1) = x_1, f(y_2) = x_2\}$. Unlike others authors [2, 25, 36], we do not use here the compact-open topology on continuous functions but the Alexandroff topology associated to the poset $(\mathcal{C}(Y, X), \trianglelefteq)$.

[2] The definition of a path by Kovalevsky [19] in the framework of cellular complexes corresponds to the definition of an arc given above while the definition of a digital path by Kong et al. [18] in the framework of the Khalimsky topology corresponds to the definition of a minimal path above.

Property 5 [36] *Let X be a poset and Y any topological space. Let $p, p' \in \mathcal{C}(Y, X)$ be such that $p' \trianglelefteq p$. Then, there is a homotopy h between p and p' such that for all $y \in Y$, $p(y) = p'(y) \Rightarrow \forall u \in [0, 1], h(y, u) = p(y) = p'(y)$.*

Proof Define $h : Y \times [0, 1] \to X$ by $h(y, t) = p(y)$ if $t < 1$ and $h(y, 1) = p'(y)$. Let U_x be some smallest neighbourhood for some $x \in X$. Then, $h^{-1}(U_x) = p^{-1}(U_x) \times [0, 1[\cup p'^{-1}(U_x) \times \{1\}$. Now, $y \in p'^{-1}(U_x) \Rightarrow p'(y) \in U_x \Rightarrow p(y) \in U_x$ (for $p' \trianglelefteq p$) $\Rightarrow y \in p^{-1}(U_x)$. Thus, $p'^{-1}(U_x) \subseteq p^{-1}(U_x)$ and $h^{-1}(U_x) = p^{-1}(U_x) \times [0, 1[\cup p'^{-1}(U_x) \times [0, 1]$. As p, p' are continuous, $p^{-1}(U_x)$ and $p'^{-1}(U_x)$ are open and, therefore, $h^{-1}(U_x)$ is open which establishes the continuity of h. □

Corollary 1 *Let X be a poset. If X has a maximum, or a minimum, then X is contractible. In particular, for any $x \in X$, $x^{\downarrow}$ and $x^{\uparrow}$ are contractible.*

Proof Let x be the minimum of X and φ the constant map that takes X onto $\{x\}$. The function φ is non-decreasing and verifies $\varphi \trianglelefteq \mathrm{id}_X$. Hence, thanks to Property 5, we derive that $\{x\}$ is a strong deformation retract of X. □

The following corollary is a direct consequence of the Property 5 (taking $Y = [0, 1]$). It is of first importance for our study of paths in posets.

Corollary 2 *Let X be a poset and a, b be two points in X. Let p, p' be two paths in X from a to b such that $p' \trianglelefteq p$. Then, p and p' are equivalent.*[3]

Property 6 *Let X be a poset and Y a compact topological space. The connected components* of $\mathcal{C}(Y, X)$ *(resp. $\mathcal{C}(Y, X)_\star$), equipped with the binary relation $\trianglelefteq$, are the homotopy equivalence classes of* $\mathcal{C}(Y, X)$ *(resp.* $\mathcal{C}(Y, X)_\star$*).*

Proof Suppose that f and g are in the same connected component of $\mathcal{C}(Y, X)$ (resp. $\mathcal{C}(Y, X)_\star$). From Properties 4 and 3 (applied to the poset $\mathcal{C}(Y, X)$ or $\mathcal{C}(Y, X)_\star$), there exists a sequence $(q_i)_{i=0}^{r}$ ($r \geqslant 1$) of paths in $\mathcal{C}(Y, X)$ ($\mathcal{C}(Y, X)_\star$) such that $q_0 = f$, $q_r = g$ and, for all $i \in [1, r]$, q_{i-1}, q_i are comparable, and thus, thanks to Property 5, homotopic. Hence, f and g are homotopic (from Property 5, we easily derive that, if $f, g \in \mathcal{C}(Y, X)_\star$, there is a homotopy h such that for all $t \in [0, 1]$, $h(., t) \in \mathcal{C}(Y, X)_\star$). Conversely, let $h : Y \times [0, 1] \to X$ be a homotopy between some maps f and g of $\mathcal{C}(Y, X)$ (with $h(y_i, t) = x_i$ for all $t \in [0, 1]$ and $i \in \{1, 2\}$). Define $h_\star : [0, 1] \to \mathcal{C}(Y, X)$ by $(h_\star(t))(y) = h(y, t)$. It is clear that $h_\star(0) = f$ and $h_\star(1) = g$ (and $h_\star(t) \in \mathcal{C}(Y, X)_\star$). We want to prove that $h_\star$ is continuous and is therefore a path from f to g. Let t be a point in the preimage $h_\star^{-1}(U_p)$ of some smallest neighbourhood in $\mathcal{C}(Y, X)$. As h is continuous, for each $y \in Y$, there are open sets $V_y \subseteq Y$, $I_y \subseteq [0, 1]$ such that $y \in V_y$, $t \in I_y$ and $h(V_y \times I_y) \subseteq U_{p(y)}$. Thanks to the compactness of Y, there is a finite subset A of Y such that $\{V_y\}_{y \in A}$ is a finite cover of Y. Then $I = \bigcap_{y \in A} I_y$ is an open neighbourhood of t and for all $t' \in I$, $y \in Y$, $h(y, t') \in h(V_y, I) \subseteq h(V_y, I_y) \subseteq U_{p(y)}$ hence $t' \in h_\star^{-1}(U_p)$ and $I \subseteq h_\star^{-1}(U_p)$. We can now conclude that $h_\star$ is continuous and that f, g are in the same (path-)connected component of $\mathcal{C}(Y, X)$ ($\mathcal{C}(Y, X)_\star$). □

[3] See Sect. 2.

As a particular case of Property 6, we obtain that the connected components of $\Pi_{a,b}$, the set of paths in X from a to b equipped with the binary relation $\trianglelefteq$, are the equivalence classes of $\Pi_{a,b}$ and from Property 4 we derive immediately the following corollary.

Corollary 3 *Let X be a poset and a, b two points in X. Two paths p, p' in X from a to b are equivalent iff there exists a sequence $(p_i)_{i=0}^{r}$ $(r \geqslant 0)$ in $\Pi_{a,b}$ such that $p_0 = p$, $p_r = p'$ and, for all $i \in [1, r]$, p_{i-1} and p_i are comparable.*

3 Paths and Arcs

The aim of Sect. 3 is to understand precisely how paths behave in a poset and to study the link between their image and the arcs defined in Sect. 2.2.2. In the sequel of the article, $(X, \trianglelefteq)$ is a poset (X need not to be finite nor, even, locally finite).

3.1 Finite Paths

We say that a function $f : [0, 1] \to X$ is a *step function* if there exists finitely many intervals $(I_i)_{i=0}^{r}$ $(r \in \mathbb{N})$ such that f is constant on each interval I_i and $[0, 1] = \bigcup_{i=0}^{r} I_i$. If for all $i \in [1, r]$, $\sup(I_{i-1}) = \inf(I_i)$ and $f(I_{i-1}) \neq f(I_i)$, we write $f = \sum_{i=0}^{r} x_i \mathbf{1}_{I_i}$ where $\{x_i\} = f(I_i)$. Note that we use the notation $f = \sum_{i=0}^{r} x_i \mathbf{1}_{I_i}$ by analogy with mathematical analysis but it is purely formal and there is no meaning behind this summation.

As a path in X is a continuous map from $[0, 1]$ to X and $[0, 1]$ is compact, the image of a path p in a locally finite poset X is compact and therefore finite (Property 1). Nevertheless, this does not mean that p is a step function. For example, let $x \lhd y$ be points in X and consider the map $p : [0, 1] \to \{x, y\}$ defined by $p(0) = x$, $p(]\frac{1}{2r+1}, \frac{1}{2r}[) = \{y\}$ and $p([\frac{1}{2r}, \frac{1}{2r-1}]) = \{x\}$ for any positive integer r. The function p is a loop at x in X (continuity of p is obvious since $\emptyset, \{y\}, \{x, y\}$ are the only open sets in $\{x, y\}$) but this path goes through x and y countably many times and it is even impossible to tell which is the second point crossed by the path p. Observe that this path is greater than the constant path $p_0 : [0, 1] \to \{x\}$ and less than $p_1 : [0, 1] \to X$ defined by $p_1(0) = x$, $p(]0, \frac{1}{2}[) = \{y\}$, $p([\frac{1}{2}, 1]) = \{x\}$ and thus p is equivalent to p_0 and p_1 (Property 5).

Definition 1 (Finite path) A path p in X is a *finite path* if it is a step function $p = \sum_{i=0}^{r} x_i \mathbf{1}_{I_i}$. The sequence $(I_i)_{i=0}^{r}$ is called the *intervals sequence* of p and the sequence $(x_i)_{i=0}^{r}$ the *track* of p. A finite path is *regular* if there is no singleton in its intervals sequence. A finite path is *minimal* if for all x_i, $1 \leqslant i \leqslant r-1$, in the track of p, $x_{i-1} < x_i \Leftrightarrow x_i > x_{i+1}$.

Proposition 1 *The track of a finite path is an arc, and any arc is the track of a regular finite path.*

Proof Let $p = \sum_{i=0}^{r} x_i \mathbf{1}_{I_i}$, $(r \geqslant 0)$, be a finite path. If $r = 0$, it is obvious that χ is an arc. If $r \geqslant 1$, take $i \in [1, r]$. The set $\{x_{i-1}, x_i\} = p(I_{i-1} \cup I_i)$ is connected since $I_{i-1} \cup I_i$ is connected and p is continuous. Hence, χ is an arc.

Let $\chi = (x_i)_{i=0}^{r}$ $(r \geqslant 0)$ be an arc. If $r = 0$, the constant path p defined by $p([0, 1]) = \{x_0\}$ has track χ. If $r = 1$, from Lemma 1 and its proof, there exists a regular path from x_0 to x_1. If $r \geqslant 2$, the product $p_1 \cdots p_r$ of regular paths p_i from x_{i-1} to x_i $(1 \leqslant i \leqslant r)$ is a path with track χ and it can easily be seen, from the very definition of this product, that a product of regular paths is regular. □

Lemma 2 *A step function $p = \sum_{i=0}^{r} x_i \mathbf{1}_{I_i}$ is a finite path in X iff for all $i \in [0, r-1]$, $x_i \asymp x_{i+1}$ and $x_i \trianglelefteqslant x_{i+1} \Leftrightarrow \sup(I_i) \in I_i$.*

Proof Suppose p is continuous. Let $i \in [0, r-1]$. By Proposition 1, $x_i \asymp x_{i+1}$. If $x_i \trianglelefteqslant x_{i+1}$, then $x_i \notin U_{x_{i+1}}$ (since $x_i \neq x_{i+1}$ by convention when writing $p = \sum_{i=0}^{r} x_i \mathbf{1}_{I_i}$). So the open set $p^{-1}(U_{x_{i+1}})$ includes the interval I_{i+1} but not the interval I_i. Thus, $\inf(I_{i+1}) = \sup(I_i)$ is not in I_{i+1}, i.e., $\sup(I_i) \in I_i$. If the inequality $x_i \trianglelefteqslant x_{i+1}$ is false then $x_{i+1} \notin U_{x_i}$ and the open set $p^{-1}(U_{x_i})$ includes the interval I_i but not the interval I_{i+1}. So $\sup(I_i)$ is not in I_i. Hence, the equivalence $x_i \trianglelefteqslant x_{i+1} \Leftrightarrow \sup(I_i) \in I_i$ holds. Conversely, suppose that there is some $s \geqslant 0$ such that any step function $\sum_{i=0}^{r} x_i \mathbf{1}_{I_i}$ with $r \leqslant s$ is continuous when for all $i \in [0, r-1]$, $x_i \asymp x_{i+1}$ and $x_i \trianglelefteqslant x_{i+1} \Leftrightarrow \sup(I_i) \in I_i$. Let $p = \sum_{i=0}^{s+1} x_i \mathbf{1}_{I_i}$ be a step function such that for all $i \in [0, s]$, $x_i \asymp x_{i+1}$ and $x_i \trianglelefteqslant x_{i+1} \Leftrightarrow \sup(I_i) \in I_i$. Indeed, for all $i \in [0, s-1]$, $x_i \asymp x_{i+1}$ and $x_i \trianglelefteqslant x_{i+1} \Leftrightarrow \sup(I_i) \in I_i$ so the step function $p' = \sum_{i=0}^{s-1} x_i \mathbf{1}_{I_i} + x_s \mathbf{1}_{I_s \cup I_{s+1}}$ is continuous. Let U be an open set in X. If $x_s, x_{s+1} \notin U$, or $x_s, x_{s+1} \in U$, then $p^{-1}(U) = p'^{-1}(U)$ is open. If $x_s \in U$ and $x_{s+1} \notin U$ then necessarily the inequality $x_s \trianglelefteqslant x_{s+1}$ is false which implies that $\sup(I_s) \notin I_s$. Thus, I_{s+1} is closed and $p^{-1}(U) = p'^{-1}(U) \setminus I_{s+1}$ is open. If $x_s \notin U$ and $x_{s+1} \in U$ then, since x_s and x_{s+1} are comparable, $x_s \trianglelefteqslant x_{s+1}$ and, by hypothesis, $\sup(I_s) \in I_s$. Thus, I_{s+1} is open and $p^{-1}(U) = p'^{-1}(U) \cup I_{s+1}$ is open. As in each case the preimage of an open set is open, p is continuous. Observing that, if $s = 0$, the map p is constant and therefore, continuous, we may conclude by induction. $\square$

Theorem 2 is the main result of Sect. 3.1. It states that any path p in a poset is equivalent to a finite path, the track of which is “very close” to the image of p. Thus, it is a first link between the continuous notion of path and the discrete one of arc.

Theorem 2 *For all $x, y \in X$ and any path p from x to y, there exists a minimal regular finite path from x to y, $p' \trianglelefteq p$, the track of which is included in the image of p. Moreover, in any interval I in the interval sequence of p', there is an element t such that $p'(t) = p(t)$.*

Proof Let p be a path from x to y in X. For each $t \in [0, 1]$, $p^{-1}(U_{p(t)})$ is open and contains t. Let J_t be the connected component of $p^{-1}(U_{p(t)})$ containing t (J_t is an open interval). Since $[0, 1]$ is compact and the family $(J_t)_{t \in [0,1]}$ is an open cover of $[0, 1]$, there exists a finite subset A of $[0, 1]$, such that $(J_t)_{t \in A}$ covers $[0, 1]$. If, for some $t, t' \in A$, $J_t \cap J_{t'} \neq \emptyset$ and $p(t) \trianglelefteqslant p(t')$, we remove t' from A and we replace J_t by $J_t \cup J_{t'}$ so we can suppose that $J_t \cap J_{t'} \neq \emptyset \Rightarrow p(t) \not\trianglelefteqslant p(t')$ (observe that it implies that t cannot belong to $J_{t'}$). Let A' be a subset of A such that A' is a minimal cover of $[0, 1]$ (for any strict subset B of A', $(J_t)_{t \in B}$ does not cover $[0, 1]$). Let $(t_i)_{i=0}^{r}$ be the (strictly) ordered sequence of reals in A' (where r is the cardinal of A' minus one). From the hypothesis on A', we derive that the sequences $(\inf(J_{t_i}))_{i=0}^{r}$ and $(\sup(J_{t_i}))_{i=0}^{r}$ are strictly ordered, $J_{t_{i-1}} \cap J_{t_i} \neq \emptyset$ for all $i \in [1, r]$ and $J_{t_{i-1}} \cap J_{t_{i+1}} = \emptyset$ for all $i \in [1, r-1]$. Finally, for each $i = 1, \dots, r$, we choose a real w_i in $J_{t_{i-1}} \cap J_{t_i}$ and we set $w_0 = -\infty$, $w_{r+1} = +\infty$, $p(w_0) = x$, $p(w_{r+1}) = y$. Observe that for any $i = 1, \dots, r$, $p(t_{i-1}) \vartriangleleft p(w_i)$ and $p(t_i) \vartriangleleft p(w_i)$. We set $J_{w_0} = \{0\} \cup [0, \frac{t_0}{2}[$, $J_{w_{r+1}} = \{1\} \cup]\frac{t_r+1}{2}, 1]$ and, for $i \in [1, r]$, if $\overline{J_{w_i}} \not\subseteq J_{t_{i-1}} \cap J_{t_i}$, we change J_{w_i} to any open interval J such that $w_i \in J \subset \overline{J} \subset J_{w_i} \cap J_{t_{i-1}} \cap J_{t_i}$. We define $p' : [0, 1] \to p([0, 1])$ by:

$$p'(t) = \begin{cases} p(w_i) & \text{if } t \in J_{w_i} \ (0 \leqslant i \leqslant r+1), \\ p(t_i) & \text{if } t \in [\sup(J_{w_i}), \inf(J_{w_{i+1}})] \ (0 \leqslant i \leqslant r). \end{cases}$$

Since $[\sup(J_{w_i}), \inf(J_{w_{i+1}})] \subset J_{t_i}$ and, for all $t \in [0, 1]$ and $u \in J_t$, $p(t) \trianglelefteqslant p(u)$, we have straightforwardly $p' \trianglelefteq p$. Furthermore, p' is a step function. We have stated above that, for any $i \in [1, r]$, $p(t_{i-1}) \lhd p(w_i)$ and $p(t_i) \lhd p(w_i)$. So, in order to prove the minimality of p' and, thanks to Lemma 2, its continuity, we still need to look at the extremities, that is, to compare $p(w_0)$ with $p(t_0)$ and $p(t_r)$ with $p(w_{r+1})$. If $p(t_0) = x$, then $p'(t) = p(t_0) = x$ on $[0, \inf(J_1)]$, otherwise $0 \in J_{t_0}$ so $p(t_0) \lhd p(w_0)$. Similarly, if $p(t_r) = y$, then $p'(t) = p(t_r) = y$ on $[\sup(J_{t_r}), 1]$, otherwise $1 \in J_{t_r}$ so $p(t_r) \lhd p(w_{r+1})$. Now, we are able to conclude that p' is a minimal finite path from x to y. As for any $i \in [1, r]$, $\overline{J_{w_i}} \subset J_{t_{i-1}} \cap J_{t_i}$, we have $\emptyset \subset]\sup(J_{i-1}), \inf(J_{i+1})[\subset [\sup(J_{w_i}), \inf(J_{w_{i+1}})]$ and p' is regular. As $w_i \in J_{w_i}$ and $t_i \in [\sup(J_{w_i}), \inf(J_{w_{i+1}})]$ (for $t_i \notin J_{t_{i-1}} \cap J_{t_i}$ and $t_i \notin J_{t_i} \cap J_{t_{i+1}}$), in any interval I in the interval sequence of p', there is an element t such that $p'(t) = p(t)$. □

There is no hope to find in the general case finite paths greater than a given path. For instance, consider the poset $X = \{x, y, z\}$ where $x \trianglelefteq y, x \trianglelefteq z$. Let $p : [0, 1] \to X$ be the function defined by $p(t) = x$ if t belongs to the Cantor set C (i.e., t has a ternary numeral with no "1"), $p(t) = y$ if $t \notin C$ and the first "1" in a ternary numeral of t is in odd position (starting from point), $p(t) = z$ if $t \notin C$ and the first "1" in a ternary numeral of t is in even position. The map p is continuous because $p^{-1}(\{y\}) =]\frac{1}{3}, \frac{2}{3}[\cup]\frac{1}{27}, \frac{2}{27}[\cup]\frac{7}{27}, \frac{8}{27}[\cup]\frac{19}{27}, \frac{20}{27}[\cup]\frac{25}{27}, \frac{26}{27}[\cup \cdots$ is open and $p^{-1}(\{z\}) =]\frac{1}{9}, \frac{2}{9}[\cup]\frac{7}{9}, \frac{8}{9}[\cup \cdots$ is open. However any open set of $[0, 1]$ containing 0, contains real numbers with ternary numerals the first "1" of which is in even, or odd, position. Thus, a finite path greater than p should have a value near 0 greater than y and z. Such a value does not exist in X. Moreover, observe that, for any integer n, we can find a subset of $\mathbb{Z}^n$ isomorphic to X.

The two following technical results will be needed in the proof of Proposition 3 and Theorem 3.

Lemma 3 *For all $x, y \in X$ and any paths p_1, p_2, p_3 from x to y such that $p_1 \trianglelefteq p_2$ and $p_3 \trianglelefteq p_2$, there are three finite paths from x to y, $p'_1 \trianglelefteq p_1$, $p'_2 \trianglelefteq p_2$, $p'_3 \trianglelefteq p_3$, such that $p'_1 \trianglelefteq p'_2$ and $p'_3 \trianglelefteq p'_2$.*

Proof The proof of Lemma 3 is close to the proof of Theorem 2. However we need to make some changes in the proof of the theorem. For all $t \in [0, 1]$, we now define J_t as an interval containing t and included in $p_1^{-1}(U_{p_1(t)}) \cap p_2^{-1}(U_{p_2(t)}) \cap p_3^{-1}(U_{p_3(t)})$. The finite set A' is such that $(J_t)_{t \in A'}$ is a minimal cover of $[0, 1]$ and the sequences $(t_i)_{i=0}^r$, $(w_i)_{i=0}^r$ are defined as in the proof of Theorem 2. Observe that it is no longer possible to assume that $t_{i-1}, t_i \notin J_{t_{i-1}} \cap J_{t_i}$ and therefore, it may happen that $p(t_{i-1}) = p(w_i)$ or $p(t_i) = p(w_i)$. The maps p'_k, $k \in \{1, 2, 3\}$, are defined by:

$$p'_k(t) = \begin{cases} p_k(w_i) & \text{if } t \in J_{w_i} \ (0 \leqslant i \leqslant r+1), \\ p_k(t_i) & \text{if } t \in [\sup(J_{w_i}), \inf(J_{w_{i+1}})] \ (0 \leqslant i \leqslant r). \end{cases}$$

Of course, we still have $p'_k \trianglelefteq p_k$ for each $k \in \{1, 2, 3\}$ and the proof of continuity for the three maps need not to be changed (except that we replace $p(t_{i-1}) \lhd p(w_i)$ and $p(t_i) \lhd p(w_i)$ by $p(t_{i-1}) \trianglelefteq p(w_i)$ and $p(t_i) \trianglelefteq p(w_i)$). □

Lemma 4

– *Let p be a finite path. There is a regular path p' with same track as p such that $p' \trianglelefteq p$.*
– *Let $p_1 \trianglelefteq p_2$ be two finite paths. There are two regular paths $p'_1 \trianglelefteq p_1$, $p'_2 \trianglelefteq p_2$ with same tracks as p_1 and p_2 such that $p'_1 \trianglelefteq p'_2$.*

Proof Let p be a non-regular finite path. Let $u \in [0,1]$ such that $\{u\}$ is an interval of the intervals sequence of p and I, J be the intervals before and after $\{u\}$ in this sequence (if $u = 0$ or $u = 1$, we set $I = \emptyset$ or $J = \emptyset$). We denote by x the point in X such that $p(u) = x$. Since p is continuous, there is a real $\varepsilon > 0$ such that $p(]u-\varepsilon, u+\varepsilon[) \subseteq U_x$ and we can choose ε such that $]u-\varepsilon, u[\cap[0,1] \subseteq I$, $]u, u+\varepsilon[\cap[0,1]) \subseteq J$. Set $p_x^\varepsilon : [0,1] \to X$, the function defined by $p_x^\varepsilon(t) = x$ if $t \in [u - \frac{\varepsilon}{2}, u + \frac{\varepsilon}{2}]$ and $p_x^\varepsilon(t) = p(t)$ otherwise. Clearly, we have $p_x^\varepsilon \trianglelefteq p$ and, from Lemma 2, we derive that p_x^ε is a finite path (since p is itself a finite path) which has the same track as p. This way, we can remove all singletons from the intervals sequence of p, resulting in a regular path $p' \trianglelefteq p$ with same track than p.

Let $p_1 \trianglelefteq p_2$ be two finite paths. Thanks to the first part of the proof, we know there is a regular path $p_1' \trianglelefteq p_1 \trianglelefteq p_2$. We slightly modify the above construction of p' in order to get $p_1' \trianglelefteq p_2'$. Let $u \in [0,1]$ such that $\{u\}$ is an interval of the intervals sequence of p_2 and I_2, J_2 be the intervals before and after $\{u\}$ in this sequence (if $u = 0$, or $u = 1$, we set $I_2 = \emptyset$ or $J_2 = \emptyset$). Set $x = p(u)$. Take $\varepsilon > 0$ such that $p(]u-\varepsilon, u+\varepsilon[) \subseteq U_x$, $]u-\varepsilon, u[\cap[0,1] \subseteq I$, $]u, u+\varepsilon[\cap[0,1]) \subseteq J$ and either $]u-\varepsilon, u]$ or $[u, u+\varepsilon[$ is included in an interval of the intervals sequence of p_1' (such a choice is possible since p_1' is regular). Suppose, for example, that $[u, u+\varepsilon[$ is included in an interval of the intervals sequence of p_1' (the other case is similar) and, therefore, $u \neq 1$, $J \neq \emptyset$, and there is a point $y \trianglelefteq x$ (since $p_1' \trianglelefteq p_2$) in X such as $p_1'([u, u+\varepsilon[) = \{y\}$. Set $p_x^\varepsilon : [0,1] \to X$, the function defined by $p_x^\varepsilon(t) = x$ if $t \in [u, u + \frac{\varepsilon}{2}]$ and $p_x^\varepsilon(t) = p_2(t)$ otherwise. As above, we have p_2' continuous and $p_2' \trianglelefteq p_2$. Moreover, we have also $p_1' \trianglelefteq p_2'$. Doing successively this construction for all singletons in the intervals sequence of p_2, we obtain a regular path p_2' with same track as p_2 and such that $p_1' \trianglelefteq p_2' \trianglelefteq p_2$. □

3.2 Arcs

Theorem 2 means that every path in a poset is homotopic to a finite path, the image of which is an arc. Processing digital images, one usually either just look at images of paths, that is at arcs, and ignore the functional definition or link arcs with paths in continuous analogs. In this subsection we focus our attention on relations between arcs and paths in the poset itself.

We can think of a track (of a finite path) as a map from the set of finite paths onto arcs (Proposition 1). Obviously this map is not injective. The next proposition gives some light upon this point.

Proposition 2 *Two finite paths in X with same track are equivalent.*

Proof Let $p = \sum_{i=0}^r x_i \mathbf{1}_{I_i}$ and $p' = \sum_{i=0}^r x_i \mathbf{1}_{J_i}$ be two paths in X with same track (r is a non negative integer). For each $i = 0, \dots, r$, we denote α_i, β_i (α_i', β_i') the lower and upper bound of I_i (J_i). Thanks to Lemma 2, we know that, for each $i = 0, \dots, r$, intervals I_i and J_i have the same form: $\alpha_i \in I_i \Leftrightarrow \alpha_i' \in J_i$ and $\beta_i \in I_i \Leftrightarrow \beta_i' \in J_i$. For all $u \in [0,1]$, we denote $K_{i,u}$ the interval with the same form as I_i, J_i and the bounds of which are $(1-u)\alpha_i + u\alpha_i'$ and $(1-u)\beta_i + u\beta_i'$. It follows again from Lemma 2 that the step function $p_u = \sum_{i=0}^r x_i \mathbf{1}_{K_{i,u}}$ is a finite path. Let $h : [0,1] \times [0,1] \to X$ be the function defined by $h(t,u) = p_u(t)$. We have $h(t,0) = p(t)$ and $h(t,1) = p'(t)$ for all $t \in [0,1]$. It can be seen that for any open set U, $h^{-1}(U)$ is an union of open trapezoid in $[0,1] \times [0,1]$, the bases of which are $p^{-1}(U) \times \{0\}$ and $p'^{-1}(U) \times \{1\}$. Hence, h is continuous: p and p' are equivalent. □

Now a new question arises: it is not difficult to see that the converse of the previous proposition is false (i.e. unless X is a singleton, there are in X equivalent finite paths which

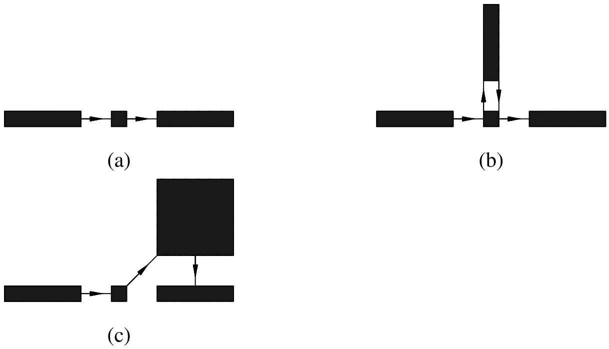


Fig. 1 (**a**) An arc χ. The *arrows* give the ordering of the sequence. (**b**), (**c**) Two elementary stretching of χ

have distinct tracks), but when two finite paths are homotopic, what about their tracks? To go further, we need to introduce the elementary modification on arcs that is illustrated on Fig. 1 (see also [7, 17]).

Definition 2 (Stretching) An arc $\chi = (x_i)_{i=0}^r$ ($r \geqslant 2$) is an *elementary stretching* of an arc χ' if for some $j \in [1, r-1]$, $\chi' = (x_i)_{i=0,i\neq j}^r$ or $x_{j-1} = x_{j+1}$ and $\chi' = (x_i)_{i=0,i\neq j-1,i\neq j}^r$. An arc χ is a *deformation* of an arc χ' if there is a sequence $(\chi_i)_{i=0}^s$ of arcs in X such that $\chi_0 = \chi$, $\chi_s = \chi'$ and for any $i \in [1, s]$, either χ_i is an elementary stretching of χ_{i-1} or χ_{i-1} is an elementary stretching of χ_i.

We will also call elementary stretching the transformation between an arc and an elementary stretching of this arc. Observe that if $\chi = (x_i)_{i=0}^r$ is an elementary stretching of $\chi' = (x_i)_{i=0,i\neq j}^r$, necessarily the three points x_{j-1}, x_j, x_{j+1} are mutually comparable. Barmak and Minian in [3] use a similar notion which leads to the same deformations: an arc $\chi = (x_i)_{i=0}^r$ ($r \geqslant 2$) is *close to* an arc χ' if for some $j \leqslant k \leqslant j'$ in $[1, r-1]$, $\chi' = (x_i)_{i=0,i\notin[j,j']}^r$ and $x_j \lhd \cdots \lhd x_k \rhd \cdots \rhd x_{j'}$ or $x_j \rhd \cdots \rhd x_k \lhd \cdots \lhd x_{j'}$.

Proposition 3 *Let p, p' be two finite paths with tracks χ, χ'. If χ' is a deformation of χ, then p and p' are equivalent.*

Proof Let p and p' be two finite paths in X with tracks χ, χ'. Since a deformation is a sequence of elementary stretchings and homotopy is an equivalence relation, it is sufficient to prove the result for an elementary stretching. So we assume that χ' is an elementary stretching of χ and, thanks to Lemma 4, we can also assume that p and p' are regular. We set $p = \sum_{i=0,i\neq j}^r x_i \mathbf{1}_{I_i}$ or $\sum_{i=0,i\neq j-1,i\neq j}^r x_i \mathbf{1}_{I_i}$ and $p' = \sum_{i=0}^r x_i \mathbf{1}_{J_i}$ ($2 \leqslant r$ and $1 \leqslant j \leqslant r-1$). If $x_{j-1} \lhd x_j \lhd x_{j+1}$ or $x_{j+1} \lhd x_j \lhd x_{j-1}$, we set $p_1(t) = p(t)$ if $t \in \bigcup_{i\neq j} J_i$ and $p_1(J_j) = \{x_{j-1}\}$. Otherwise ($x_j \lhd x_{j-1}$ and $x_j \lhd x_{j+1}$, or $x_{j-1} \lhd x_j$ and $x_{j+1} \lhd x_j$), let α and β be the lower bound and the upper bound of J_j ($\alpha \neq \beta$ since p' is regular) and $\gamma = \frac{\alpha+\beta}{2}$. We set $p_1(t) = p(t)$ if $t \in \bigcup_{i\neq j} J_i$, $p_1(t) = x_{j-1}$ if $t \in [\alpha, \gamma[$, $p_1(t) = x_{j+1}$ if $t \in]\gamma, \beta]$ and $p_1(\gamma) = x_{j-1}$ if $x_{j-1} \trianglelefteq x_{j+1}$, $p_1(\gamma) = x_{j+1}$ if $x_{j+1} \lhd x_{j-1}$ (see Fig. 2). In any case, we can derive from Lemma 2 that p_1 is a path. Since the tracks of p_1 and p are the same, p_1 and p are equivalent. Moreover, it can easily be seen that $p_1 \trianglelefteq p'$ or $p' \trianglelefteq p_1$. Thus p_1 and p' are equivalent and, by transitivity, p and p' are equivalent. □

We can now state that the notion of deformation is the discrete counterpart of the continuous notion of homotopy equivalence.

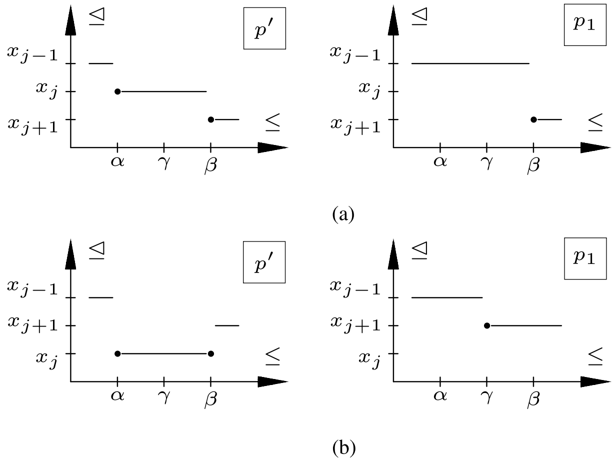


Fig. 2 (**a**) Case $x_{j+1} \lhd x_j \lhd x_{j-1}$ (case $x_{j-1} \lhd x_j \lhd x_{j+1}$ is similar). (**b**) Case $x_j \lhd x_{j-1}, x_j \lhd x_{j+1}$ (case $x_{j-1} \lhd x_j, x_{j+1} \lhd x_j$ is similar with an open interval J_j). Note that in this case, it could happen that $x_{j-1} = x_{j+1}$

Theorem 3 *Two finite paths p, p' in X with tracks $\chi \neq \chi'$ are equivalent iff χ is a deformation of χ'.*

Proof Let p and p' be two distinct finite equivalent paths in X from point a to point b and $\Pi_{a,b}$ be the poset of paths in X from a to b. Since p and p' are equivalent, there is a path from p to p' in $\Pi_{a,b}$ (Proposition 6) and, thus, there is an arc in $\Pi_{a,b}$ from p to p' (Property 4). Of course we can suppose that this arc is minimal (otherwise we delete the superfluous paths). Moreover, we claim that we can build a minimal arc in $\Pi_{a,b}$ from p to p', the elements of which are all finite. Suppose that $P = (p_i)_{i=0}^r$ ($r \geqslant 2$) is a minimal arc in $\Pi_{a,b}$ from p to p', the k first elements of which are finite ($1 \leqslant k \leqslant r-1$). Case 1: $p_k \trianglelefteq p_{k-1}$. Since P is minimal, we have $p_k \trianglelefteq p_{k+1}$. We derive from Theorem 2 that there is a finite path q in $\Pi_{a,b}$ such that $q \trianglelefteq p_k$. Thus, the sequence $P' = (q_i)_{i=0}^r$ where $q_k = q$ and $q_i = p_i$ otherwise, is a minimal arc in $\Pi_{a,b}$ from p to p', the $k+1$ first elements of which are all finite. Case 2: $p_{k-1} \trianglelefteq p_k$, and thus $p_{k+1} \trianglelefteq p_k$. Thanks to Lemma 3, we know there exist three finite paths q, q', q'' such that $q \trianglelefteq p_{k-1}, q' \trianglelefteq p_k, q'' \trianglelefteq p_{k+1}$ and $q \trianglelefteq q', q'' \trianglelefteq q'$. If $p_{k+1} \neq p'$ or $q'' = p'$ we set $P' = (q_i)_{i=0}^r$ where $q_{k-1} = q, q_k = q', q_{k+1} = q''$ and $q_i = p_i$ otherwise. Then, P' is a minimal arc in $\Pi_{a,b}$ from p to p', the $k+2$ first elements of which are finite. If $p_{k+1} = p'$ and $q'' \neq p'$ we set $P' = (q_i)_{i=0}^{r+1}$ where $q_i = p_i$ if $i \leqslant k-2$, $q_{k-1} = q, q_k = q', q_{k+1} = q''$ and $q_{r+1} = p'$. Then, P' is a minimal arc in $\Pi_{a,b}$ from p to p', the elements of which are all finite. This way, we build iteratively an arc in $\Pi_{a,b}$ from p to p', the elements of which are all finite.

Therefore, to prove that the track of p is a deformation of the track of p' it is sufficient to do so for two finite and comparable paths, say p_1 and p'_1. Moreover, thanks to Lemmas 4 and 2, we can easily build two comparable regular (finite) paths from a to b, $q = \sum_{i=0}^r x_i \mathbf{1}_{I_i} \trianglelefteq q' = \sum_{j=0}^s y_j \mathbf{1}_{J_j}$, with same tracks as p_1 and p'_1 and such that the intervals I_i ($0 \leqslant i \leqslant r$) have no common bounds with the intervals J_j ($0 \leqslant j \leqslant s$). Thus, we denote $(\alpha_i)_{i=0}^{r+s+1}$ the strictly increasing sequence the elements of which are the bounds of the intervals I_i and J_j: $\alpha_0 = 0, \alpha_{r+s+1} = 1$, for each $1 \leqslant i \leqslant r+s$ either q or q', but not both, change its value on α_i and no others changes occur. For each $i \in [1, r+s]$ and each $j \in [1, r+s]$ we define the step functions q_i and q'_j by $q_i(t) = q'(t)$ if $t < \alpha_i$, $q(t)$ otherwise and $q'_j(t) = q'(t)$ if $t < \frac{\alpha_j + \alpha_{j+1}}{2}$, $q(t)$ otherwise. In particular, $q_1 = q$ and $q'_{r+s} = q'$ (since $q'(0) = q(0)$ and $q'(1) = q(1)$). We denote by χ_i and χ'_j the tracks of q_i and q'_j

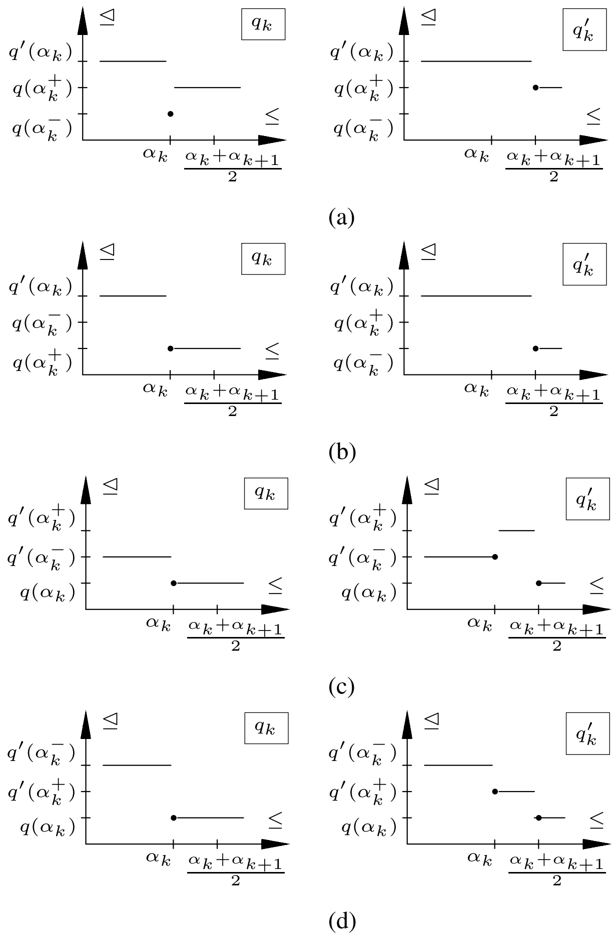


Fig. 3 (**a**) q_k is a stretching of q'_k (depending on $q(\alpha_k^+) = q'(\alpha_k)$ or not, we use one of the two cases in the definition of elementary stretchings). (**b**) $q_k = q'_k$. (**c**) q'_k is a stretching of q_k. (**d**) If $q'(\alpha_k^+) \neq q(\alpha_k)$, q'_k is a stretching of q_k, otherwise $q_k = q'_k$

($i, j \in [1, r+s]$). From Lemma 2, we easily derive that the step functions q_i and q'_j are finite paths from a to b. We want now to prove that, for all $i \in [1, r+s]$, either χ_k (χ_{k+1}) is equal to χ'_k or is a stretching of χ'_k or the converse. The proof consists in checking the 2×4 configurations relative to q_k and q'_k and to q'_k and q_{k+1}. These 8 configurations are depicted in Figs. 3 and 4 which clearly establish that in any case we have equality or stretching. Note that in Figs. 3 and 4 we denote by $f(t^-)$ and $f(t^+)$ the values taken by a finite path f on some intervals $]t-\varepsilon, t[$, $]t, t+\varepsilon[$ where $\varepsilon > 0$ is small enough to assume that f is constant on these intervals.

The converse part of the proof is given by Proposition 3. □

To go further in the parallelism between paths and arcs, homotopies and deformations, we will now study the arc product defined by $(x_0, \dots, x_r).(y_0, \dots, y_s) = (x_0, \dots, x_r, y_1, \dots, y_s)$. More formally:

Definition 3 (Arcs product) Let $\chi_1 = (x_i)_{i=0}^r$ and $\chi_2 = (y_i)_{i=0}^s$ ($r, s \geqslant 0$) be two arcs such that $x_r = y_0$. The *arc product* is defined by $\chi_1.\chi_2 = (z_i)_{i=0}^{r+s}$ where $z_i = x_i$ if $i \leqslant r$ and $z_i = y_{i-r}$ if $i \geqslant r$.

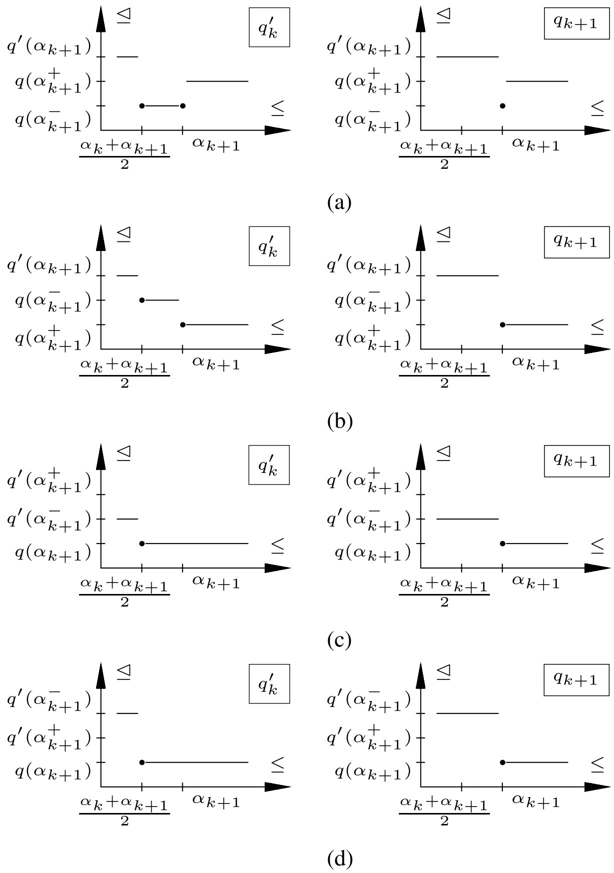


Fig. 4 (**a**) $q'_k = q_{k+1}$. (**b**) If $q'(\alpha_k) \neq q(\alpha^-_{k+1})$, q'_k is a stretching of q_{k+1}, otherwise $q'_k = q_{k+1}$. (**c**), (**d**) $q'_k = q_{k+1}$

Let x be a point in X. It is easy to check that being a deformation or equal is an equivalence relation in the set of arcs in X from x to x. We write $[\chi]$ for the equivalence class of an arc χ and we denote by $\rho(X, x)$ the set of equivalence classes. It is not more difficult to verify that the arc product is well defined on classes by $[\chi_1].[\chi_2] = [\chi_1.\chi_2]$ and $\rho(X, x)$ equipped with the arc product is a group (the identity element of which is $[(x)]$ and the inverse of $[(x_i)_{i=0}^r]$ is $[(x_i)_{i=r}^0]$).

Theorem 4 *Let $x \in X$. The fundamental group $\pi_1(X, x)$ of X with basepoint x is isomorphic to the group $\rho(X, x)$.*

Proof By Theorem 2 we know that there are finite paths in any class of $\pi_1(X, x)$ and by Theorem 3, we may define a map $\varphi : \pi_1(X, x) \to \rho(X, x)$ by $\varphi([p]) = [\chi]$ where χ is the track of any finite path equivalent to p. From Proposition 3, we derive that φ is injective and from Proposition 1, φ is surjective. Finally, φ is a morphism since we can easily see that the track of a product of two finite paths is the product of the tracks of these finite paths. □

Remark 1 Barmak and Minian in [3] have proved the same result in a different way and in the frame of finite spaces. They establish an isomorphism between $\rho(X, x)$ and a group of loops composed with edges of the simplicial complex $\mathcal{K}(X)$ associated to X (see Sect. 4.1), then invoke an isomorphism between the edge-paths group of $\mathcal{K}(X)$ and the fundamental group of its geometric realisation $|\mathcal{K}(X)|$ (see Sect. 4.1) described by Spanier [35] and conclude thanks to the weak homotopy equivalence between $|\mathcal{K}(X)|$ and X established by McCord (see Sect. 4.1).

4 Reduction

In this section, we are interested in retractions, or more general decreasing transformations, that minimally alter the topology of a poset and the topology of a continuous analogue. In particular, we will visit minimal modifications of such sets that do not change homotopy type. But before thinking at transformations, we present in Sect. 4.1 the way we embed a digital image in a poset and how the continuous analogue of the digital image is defined.

4.1 Complexes

Complexes are topological sets whose combinatorial organisation provide a way to link digital images, namely subspaces of $\mathbb{Z}^n$, with the continuous Euclidean space $\mathbb{R}^n$.

4.1.1 Simplicial Complexes

Simplicial complexes are among the simplest combinatorial structures. They are commonly used in the field of geometric modelling.

An *abstract simplicial complex* is a set K of non-empty subsets, called *simplices*, of a set V, such that each non-empty subset of a simplex is a simplex. The elements of V are called *vertices*. Each vertex must belong to at least one simplex. A non-empty (proper) subset of a simplex is a (*proper*) *face* of the given simplex. For in this section we focus on digital images, we assume that the simplices of a complex are finite and that their cardinalities are bounded. Thus, we can define the *dimension* of a simplex which is its number of vertices minus one and the dimension of a complex which is the maximum of the dimensions of its simplices.

In $\mathbb{R}^n$, a set of points are *geometrically independent* if any k-hyperplane ($k \leqslant n$) contains no more than $k+1$ of them. The (*geometric*) simplex spanned by a set of geometrically independent points is the convex hull of these points which are the *vertices* of the geometric simplex. A *k-face* of a simplex is a simplex spanned by k vertices of the simplex. A (*geometric*) *simplicial complex* K is a set of simplices in $\mathbb{R}^n$ such that any face of a simplex in K is a simplex in K and any intersection of two simplices in K is a simplex in K. The faces of the complex are the faces of its simplices. The vertices of the complex are the vertices of its simplices. Note that the vertices of a complex need not be geometrically independent. The *geometric realisation* $|K|$ of the complex K is the union of its simplices equipped with the topology the closed sets of which are the sets that intersect each simplex in a closed set of $\mathbb{R}^n$. Because a union of closed sets is not always a closed set, this topology could be different from the usual topology on $\mathbb{R}^n$. But here, as K is locally finite, i.e. any vertex belongs to finitely many simplices, this topology is the usual topology on $\mathbb{R}^n$. The *open simplices* of $|K|$ are the interiors of its k-faces ($k \geqslant 1$) and its 0-faces. Each point x in $|K|$ belongs to a unique open simplex spanned by some vertices $v_1, \dots, v_k$ ($k \geqslant 1$) and there exists a unique k-uple $(b_1, \dots, b_k)$ in $[0, 1]^k$ such that $x = \sum_{i=0}^{k} b_i v_i$. Let f be a function between the set of vertices of two complexes K and K', the function $|\mathcal{K}(f)|$ which associates to each point $x = \sum_{i=0}^{k} b_i v_i$ in $|K|$ the point y of $|K'|$ defined by $y = \sum_{i=0}^{k} b_i f(v_i)$ is *the simplicial map associated to* f. This map is continuous.

A *realisation* of an abstract simplicial complex K is a geometric simplicial complex whose vertices are in one to one correspondence with the vertices of K and whose simplices are spanned by the images of the simplices of K. Any abstract simplicial complex K of dimension n can be realised in $\mathbb{R}^{2n+1}$ [12].

There is a narrow link between posets and simplicial complexes discovered by Alexandroff [1]. Let X be a poset. The points in X are the vertices of a simplicial complex $\mathcal{K}(X)$ the simplices of which are the (finite) chains of X (see Fig. 6). Conversely, it is plain that the simplices of a given simplicial complex K, equipped with the inclusion relation, is a locally finite poset denoted $\mathcal{X}(K)$. Note that $\mathcal{K}(\mathcal{X}(K))$ is not equal to K but to a simplicial complex called the *barycentric subdivision* of the complex K. These correspondences are not only algebraic and the topologies on the poset and the geometric realisation of the complex are concerned as well. The following theorem due to McCord gives the key-properties of the map $\varphi_X : |\mathcal{K}(X)| \to X$ which associates to each point in the geometric realisation of $\mathcal{K}(X)$, the highest element of the unique open simplex it belongs to (remember that a simplex of $\mathcal{K}(X)$ is a chain).

Theorem 5 (McCord [29]) *Let X be a poset. There is a weak homotopy equivalence φ_X : $|\mathcal{K}(X)| \to X$. Furthermore, one can associate to each continuous map $f : X \to Y$ between two posets the simplicial map $|\mathcal{K}(f)|$ such that the following diagram is commutative*:

$$\begin{tikzcd} X \arrow[r, "f"'] & Y \\ {|\mathcal{K}(X)|} \arrow[u, "\varphi_X"] \arrow[r, "|\mathcal{K}(f)|"'] & {|\mathcal{K}(Y)|} \arrow[u, "\varphi_Y"'] \end{tikzcd}$$

Observe that, as we have proved that the fundamental group $\pi_1(X, x)$ of a poset X with basepoint x is isomorphic to the group $\rho(X, x)$ of its arcs from x to x (for any $x \in X$), Theorem 5 gives by transitivity an isomorphism between $\rho(X, x)$ and the fundamental group of the geometric realisation of $\mathcal{K}(X)$.

4.1.2 Cubical Complexes

In digital images, grids are often cubical ones, so it is interesting in image analysis to replace simplices in complexes by n-cubes.

We set $\mathbb{F}_0^1 = \{\{a\} \mid a \in \mathbb{Z}\}$ and $\mathbb{F}_1^1 = \{\{a, a+1\} \mid a \in \mathbb{Z}\}$. A subset f of $\mathbb{Z}^n$ which is the Cartesian product of m elements of $\mathbb{F}_1^1$ and $n-m$ elements of $\mathbb{F}_0^1$ is a *face* or an *m-face* (*of* $\mathbb{Z}^n$), m is the *dimension* of f, and we write $\dim(f) = m$. We denote by $\mathbb{F}_m^n$ the set composed of all m-faces of $\mathbb{Z}^n$and by $\mathbb{F}^n$ the set composed of all faces of $\mathbb{Z}^n$. Let $f \in \mathbb{F}^n$ be a face. The set $\{g \in \mathbb{F}^n \mid g \subseteq f\}$ is a *cell* and any union of cells is an *abstract cubical complex*. The *geometric cubical complexes* are defined in the same manner, except we change the definition of $\mathbb{F}_1^1$ by setting $\mathbb{F}_1^1 = \{[a, a+1] \mid a \in \mathbb{Z}\}$. The geometric realisation $|K|$ of a geometric cubical complex K is the union of its faces (see Fig. 5).

The points in a digital image are often a measure of a physical quantity on a piece of the Euclidean space. Then, the abstract cellular complex framework—and in particular the cubical complexes—enable to model the adjacency relations between these pieces of the Euclidean space in a topologically sound manner. Furthermore, as an abstract cellular complex (equipped with the inclusion) is a poset, Theorem 5 ensures that this complex is weakly

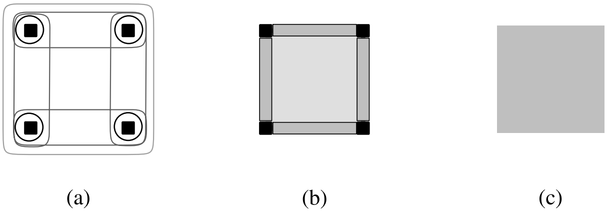
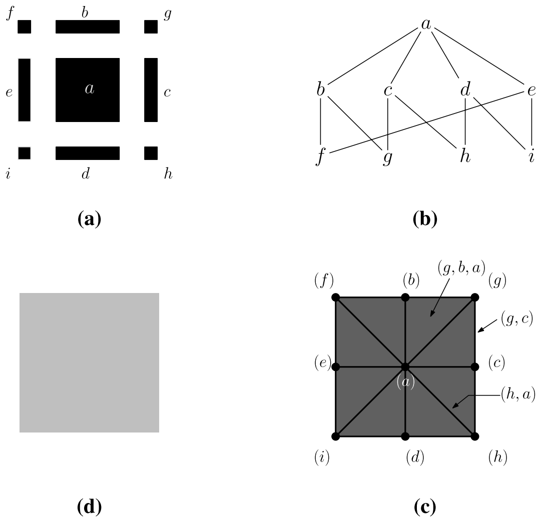


Fig. 5 (**a**) An abstract (cubical) cell C composed of one 2-face, four 1-faces and four 0-faces. The four small *black squares* represent four points in $\mathbb{Z}^n$ mutually 8-adjacent. (**b**) The geometric (cubical) cell gC which is the realisation of C. (**c**) The geometric realisation $|gC|$ of gC

Fig. 6 (**a**) An abstract cubical 2-cell f which models a digital point of $\mathbb{Z}^2$. (**b**) The Hasse diagram of $\mathcal{X}(f)$. (**c**) The simplicial complex $\mathcal{K}(\mathcal{X}(f))$. (**d**) The geometric realisation of $\mathcal{K}(\mathcal{X}(f))$



homotopy equivalent to its geometric realisation (more precisely, to the geometric realisation of the associated simplicial complex—see Fig. 6) which is a conceivable representation of the tessellation of the Euclidean space captured by the measure device. We say that this geometric realisation is the *continuous analogue* of the digital image. The second part of Theorem 5 says that any continuous transformations of the complex image has an equivalent on the continuous analogue compatible with the weak homotopy equivalence.

4.1.3 Collapses

Whitehead has defined elementary transformations on complexes as follows. Let X be a complex and (x, y) a pair of faces in X such that x is the only face of X including y. Then, (x, y) is a *free pair*, and the set $Y = X \setminus \{x, y\}$ is an *elementary collapse* of X, or X is an *elementary expansion* of Y. If a set Y is obtained from X by a sequence of elementary collapses (a sequence of elementary collapses and expansions), then Y is a *collapse* of X (X and Y are *simple-homotopy* equivalent) and one write $X \searrow Y$ ($X \nearrow\!\searrow Y$). A set is *collapsible* if it collapses onto a singleton.

If Y is a *collapse* of X then $|Y|$ is a strong deformation retract of $|X|$ (and thus, $|X|$ and $|Y|$ are homotopy equivalent) [38]. Figure 7 illustrates this property.

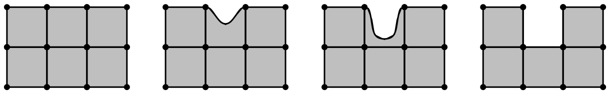


Fig. 7 (**a**) A complex X. (**d**) A complex Y which is an elementary collapse of X. (**b**)–(**c**) Two steps in a strong deformation retraction of $|X|$ onto $|Y|$

(a) (b) (c) (d)

4.2 Unipolar Points

In the 60's, Stong [36] introduced the notion of *(co)linear points* in order to classify finite spaces with respect to homotopy type. More recently, May [27] called them *beat points* and Bertrand [6] *unipolar points*. We keep this last designation. In the same article, and for the same goal, Stong also defined the *core* of a finite space (see Definition 5) which is the smallest subset of X homotopic to X. Most results in this subsection were first established in Stong's article for finite spaces. Most of his proofs can be easily adapted to posets so we do not recall them.

Definition 4 (Unipolar point) Let X be a poset.

- A point $x \in X$ is *down unipolar* if there is $y \lhd x$ such that $z \lhd x$ implies $z \trianglelefteqslant y$ (i.e. $x^{\downarrow\star} = y^{\downarrow}$).
- A point $x \in X$ is *up unipolar* if it is down unipolar for the dual order on X.
- A point is *unipolar* if it is either down unipolar or up unipolar.

Proposition 4 *Let X be a poset. A point $x \in X$ is unipolar iff $X \setminus \{x\}$ is a strong deformation retract of X.*

Proof The "only if" part of this proof is in [36]. The "if" part is original and rely on our Theorem 2.

Let us assume that $Y = X \setminus \{x\}$ is a strong deformation retract of X. Thus, there is an homotopy $h : X \times [0, 1] \to X$ such that $h(z, t) = z$ for all $(z, t) \in Y \times [0, 1]$ and $h(x, 0) = x, h(x, 1) \neq x$. The map $h(x, .) : [0, 1] \to X$ is a path in X from x to $h(x, 1)$ so, following Theorem 2, we denote $p = \sum_{i=0}^{r} x_i \mathbf{1}_{I_i}$ $(r \geqslant 1)$, with $p \trianglelefteq h(x, .)$, a regular finite path from x to $h(x, 1)$ with property that in any interval I of the interval sequence of p, there is an element t such that $p(t) = h(x, t)$. Let $t_1 \in I_1$ verifying $p(t_1) = h(x, t_1) = x_1$ which is an element of Y comparable to x (Proposition 1). The map $h(., t_1) : X \to X$ is continuous and, therefore, non-decreasing (Property 2) so for any $y \in Y$, $y \lhd x \Rightarrow y \trianglelefteqslant x_1$ and $x \lhd y \Rightarrow x_1 \trianglelefteqslant y$ (since $h(., t_1)$ is the identity map on Y). As x_1 is comparable to x, we derive that x is unipolar. □

Definition 5 (Core) Let $Y \subseteq X$. We say that Y is a core of X if it has no unipolar point and it is a strong deformation retract of X.

Property 7
1. *Any finite poset has a core and two cores of the same poset are homeomorphic.*
2. *Two finite posets are homotopy equivalent iff they have homeomorphic cores.*

Observe in particular that Property 7 implies that one can greedily remove the unipolar points of a finite poset in order to obtain a core which will be homeomorphic to any other core of the same poset. In particular, when the poset is contractible, we have the corollary below.

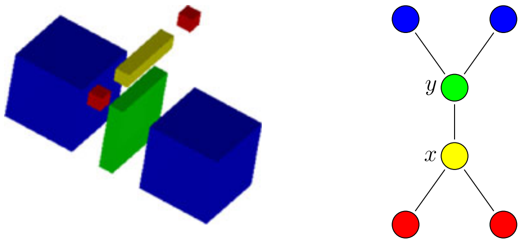


Fig. 8 *Left*: a subset X of $\mathbb{F}^n$. *Right*: its Hasse diagram. The 1-face x is up unipolar and the 2-face y is down unipolar. Neither x in $X \setminus \{y\}$ or y in $X \setminus \{x\}$ are unipolar

Corollary 4 *If X is finite and contractible, there is a sequence $(x_i)_{i=0}^r (r \geqslant 0)$ of points in X such that $X = \{x_j\}_{j=0}^r$ and, for all $i \in [1, r]$, x_i is unipolar in $\{x_j\}_{j=0}^i$. Furthermore, if $x \in X$ is unipolar, we can choose $x_r = x$.*

Proof The fact that X is contractible means that X is homotopy equivalent to a point. Since X is finite, X has a core and any core of X is a singleton (Property 7). It is not difficult to see that it implies that one can greedily construct a sequence $(x_i)_{i=0}^r$ $(r \geqslant 0)$ of points in X such that $X = \{x_j\}_{j=0}^r$ and, for all $i \in [1, r]$, x_i is unipolar in $\{x_j\}_{j=0}^i$. $\square$

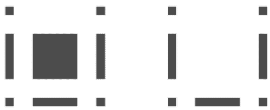
Bertrand [6] has established that down (or up) unipolar points can be deleted in parallel, that is, if $x \neq y$ are down unipolar points in X then y is down unipolar in $X \setminus \{x\}$. It is no longer true for unipolar points (forgetting "down") as shown by the example of Fig. 8. Nevertheless, we can state the next proposition.

Proposition 5 *If $x \neq y$ are unipolar points then either* (a) *y is unipolar in $X \setminus \{x\}$ or* (b), *for one order on X ($\trianglelefteq$ or $\trianglerighteq$), x is down-unipolar and covers y, for the other order y is down-unipolar and covers x and the map $\varphi : X \setminus \{x\} \to X \setminus \{y\}$ defined by $\varphi(z) = z$ if $z \neq y$ and $\varphi(y) = x$ is an homeomorphism.*

Proof Let $x \neq y$ be unipolar points in X. If x and y are not comparable, it is easy to see that y is unipolar in $X \setminus \{x\}$ since Definition 4 only involves comparable points. If x and y are comparable, we can set $x \lhd y$. If y is up-unipolar, y is unipolar in $X \setminus \{x\}$ since Definition 4 applied to y only involves points z such that $y \lhd z$. We suppose now that y is down unipolar and we denote z the maximum of $y^{\downarrow\star}$. Hence, for any $t \in X$, $t \lhd y \Leftrightarrow t \trianglelefteq z$ (1). If $x \neq z$, obviously this inference is true for any $t \in X \setminus \{x\}$ and y is unipolar in $X \setminus \{x\}$. If $x = z$ and x is down unipolar, we use the result established in [6]. If $x = z$ and x is up unipolar, necessarily y is the minimum of $x^{\uparrow\star}$: for any $t \in X$, $x \lhd t \Leftrightarrow y \trianglelefteq t$ (2). We define $\varphi : X \setminus \{x\} \to X \setminus \{y\}$ by $\varphi(t) = t$ if $t \neq y$ and $\varphi(y) = x$. Trivially, φ is a bijection and from (1) and (2) we derive that φ and φ^{-1} are non-decreasing, that is, continuous. $\square$

4.3 Simple Points

Simple points were first introduced by Bertrand in [6] in order to perform topologically sound thinning algorithms in posets. They have been used by Barmak and Minian [5] to define a collapse operation in posets which corresponds actually to the collapse operation in complexes associated to posets. The proofs of Property 8 and Theorem 6, which are out of scope of this paper, can be found in [5].



(a) (b)

Fig. 9 X is the subset of $\mathbb{F}^2$ depicted in (**a**) and x is the 2-face in X (note that $X = x^{\downarrow}$). The face x is simple since $x^{\downarrow\star}$, depicted in (**b**), is clearly contractible. But $X \setminus \{x\} = x^{\downarrow\star}$ is not a retraction of X, for a retraction, as any continuous function, preserves connectivity and it is impossible to find an image for x in $x^{\downarrow\star}$, while leaving unchanged the other points in X, without disconnecting some connected subset of X

Definition 6 (Simple point)

A point $x \in X$ is *down simple (in X)* if $x^{\downarrow\star}$ is contractible.
A point $x \in X$ is *up simple (in X)* if $x^{\uparrow\star}$ is contractible.
A point is *simple (in X)* if it is either down simple or up simple.

Observe that unipolar points are simple points since if $x \in X$ is a down (up) unipolar point, $x^{\downarrow\star}$ ($x^{\uparrow\star}$) has a maximum (minimum) and is therefore contractible (Corollary 1). We saw previously (Proposition 4) that the removal of a unipolar point is a strong deformation retraction. It is no longer true for simple points. See Fig. 9 for a counterexample where the removal of a simple point is not even a retraction. Nevertheless, Property 8 states that homotopy groups are not changed by such a deletion and, furthermore, Theorem 6 ensure that this deletion corresponds to a deformation retract on the continuous analogue.

Property 8 [5] *Let X be a finite poset. Let $x \in X$ be a simple point. Then, the inclusion $i : X \setminus \{x\} \to X$ is a weak homotopy equivalence.*

Theorem 6 (Barmak and Minian [5]) *Let X be a finite poset. Let $x \in X$ be a simple point and* $\mathrm{K}(X)$, $\mathrm{K}(X \setminus \{x\})$ *the simplicial complexes associated to X and $X \setminus \{x\}$. Then,* $\mathrm{K}(X) \searrow \mathrm{K}(X \setminus \{x\})$.

From an algorithmic point of view, simple points have good properties since they can be deleted in parallel. Obviously, if x, y are two points in X with $\dim(x) = \dim(y)$, there is no need to know whether x has been deleted from X or not to decide if $y^{\downarrow\star}$, or $x^{\uparrow\star}$ is contractible. Moreover, as we have seen above, the decision on the contractibility can be greedily performed. Thus, a topology-preserving thinning procedure consists of repeating until stability the removal of the k-dimensional simple points for $k = 0$ to n. Figure 10 gives an example of the result of such a procedure when applied to a 2D-picture. A detailed study of algorithms quite similar to the previous scheme can be found in [23].

4.4 Free Pairs and Unipolar/Simple Points

In order to perform thinning on a complex, it is usual to do collapses but, viewing this complex as a poset, it is possible to remove unipolar or simple points. So we want to compare these three ways to reduce a complex. Hence, in this subsection, X is a cubical complex (included in $\mathbb{F}^n$) and accordingly, for any face $x \in X$, $x^{\downarrow}$ is the cell $\{y \in \mathbb{F}^n \mid y \subseteq x\}$.

Lemma 5 *Let $0 \leqslant k \leqslant m \leqslant n$ and $x \in X$ such that* $\dim(x) = m$. *Let $y \in x^{\downarrow}$ be a k-face.*
1. *There exist exactly $m - k$ faces in $x^{\downarrow}$ of dimension $(k + 1)$ which include y.*

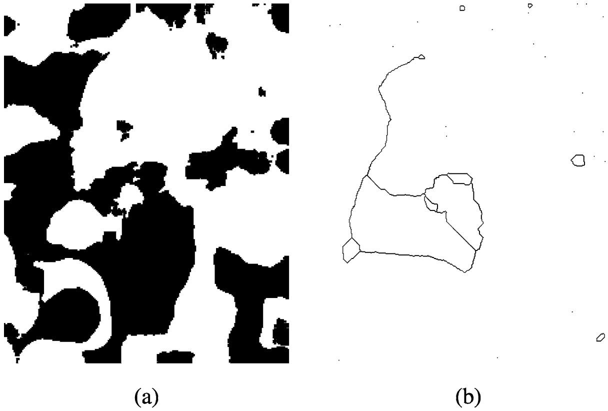


Fig. 10 *Left*: the original image. *Right*: a skeleton obtained by the parallel removal of simple points of same dimension until stability

2. *Let* x_1, x_2 *be two* $(m-1)$*-faces in* $x^{\downarrow}$ *such that* $x = x_1 \cup x_2$ *and* y *intersects both* x_1 *and* x_2. *If* $k \neq 0$, *there exists in* $y^{\downarrow}$ *exactly one* $(k-1)$*-face which is included in* (*or equal to*) x_1 *and one* $(k-1)$*-face which is included in* (*or equal to*) x_2.

Proof If $k = m$, Lemma 5 is trivial. Suppose now that $m > k$. Without loss of generality, we can assume that $x = \prod_{i=1}^{n} I_i$ where $I_i \in \mathbb{F}_1^1$ if $i \leqslant m$, $I_i \in \mathbb{F}_0^1$ otherwise (see Sect. 4.1.2) and $y = \prod_{i=1}^{n} J_i$ where $\emptyset \subset J_i \subset I_i$ if $i \leqslant m-k$ and $J_i = I_i$ otherwise.

1. It is plain that the only $(k+1)$-faces included in x and including y are the $m-k$ faces z_j, $1 \leqslant j \leqslant m-k$ defined by $z_j = \prod_{i=1}^{n} K_i$ with $K_i = J_i$ if $i \neq j$ and $K_j = I_j$.
2. Since y intersects both x_1 and x_2, there exists $j \in [m-k+1, m]$ such that $x_1 = \prod_{i=1}^{n} K_i^1$ and $x_2 = \prod_{i=1}^{n} K_i^2$ with $K_i^1 = K_i^2 = I_i$ if $i \neq j$, $\emptyset \subset K_j^1 \subset I_j$ and $K_j^2 = I_j \setminus K_j^1$. Therefore, the only $(k-1)$-face z included in y and in x_1 (resp. x_2) is $z = \prod_{i=1}^{n} L_i$ with $L_i = J_i$ if $i \neq j$ and $L_j = K_i^1$ (resp. $L_j = K_i^2$). □

An easy consequence of Lemma 5, is that the boundary $x^{\downarrow\star}$ of a cell $x^{\downarrow}$ in $\mathbb{F}^n$ is not contractible since for any k-face y in $x^{\downarrow\star}$, there exist at least two $(k+1)$-faces including y, except if y is maximal in $x^{\downarrow\star}$, and two $(k-1)$-faces included in y, except if y is minimal in $x^{\downarrow\star}$, and therefore $x^{\downarrow\star}$ has no unipolar point. So, $x^{\downarrow\star}$ is not contractible (Corollary 4).

Corollary 5 *The boundary* $x^{\downarrow\star}$ *of a cell* $x^{\downarrow}$ *in* $\mathbb{F}^n$ *is not contractible.*

Lemma 6 *Let* $x, y \in X$, $x \lhd y$, *be two faces with* $\dim(x) = \dim(y) - 1$. *Then,* $y^{\downarrow\star} \setminus \{x\}$ *is contractible.*

Proof We set $m = \dim(y)$ and $Y = y^{\downarrow\star} \setminus \{x\}$. If $m = 1$, Lemma 6 is trivial (Y is a singleton). Suppose now that $m \geqslant 2$. We denote x' the face opposite to x in $y^{\downarrow} : x' = y \setminus x$. We will shrink Y to $\{x'\}$, removing unipolar points from Y. First, we remove the faces in $x^{\downarrow\star}$ in decreasing order relatively to their dimension. For any $(m-2)$-face z in $x^{\downarrow}$ we derive from Lemma 5 that there are two $(m-1)$-faces in $y^{\downarrow}$ including z, one of which is x. Hence, z is up unipolar in Y and, thanks to Propositions 4 and 5, we deduce that the set $Y_1 = \{z \in Y \mid z \notin x^{\downarrow}$ or $\dim(z) < m-2\}$ is a strong deformation retract of Y. Since, according to Lemma 5, any $(m-k)$-face in $x^{\downarrow\star}$ is included in exactly one $(m-k+1)$-face in $y^{\downarrow} \setminus x^{\downarrow}$, we can inductively remove all faces of $x^{\downarrow}$ from Y with the same argumentation as above. Hence, $Z = Y \setminus x^{\downarrow}$ is a strong deformation retract of Y. In a second step, we are going to prove

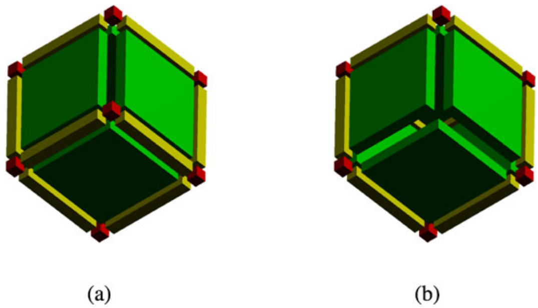


Fig. 11 Cores of the set $y^{\downarrow\star} \setminus \{x\}$ when y is a 3-face and x is (**a**): a 1-face; (**b**): a 0-face

that the faces in $Z \setminus x'^{\downarrow}$ are successively down unipolar if we remove them in an increasing order w.r.t. their dimension. Note that, since $x' = y \setminus x$, there is no 0-face in $Z \setminus x'^{\downarrow}$. So, suppose we have removed all faces in $Z \setminus x'^{\downarrow}$ of dimension less than k ($1 \leqslant k \leqslant m-1$) and let z be a k-face in $Z \setminus x'^{\downarrow}$. Lemma 5 ensures that there exists in $z^{\downarrow}$ exactly one $(k-1)$-face in $Z_1 = Z \setminus \{t \in Z \setminus x'^{\downarrow} \mid \dim(t) < k\}$ (which belongs to $x'^{\downarrow}$) so z is down unipolar in Z_1. Hence, we can inductively prove that $x'^{\downarrow}$ is a strong deformation retract of Y. As any cell is contractible (Corollary 1), we are done. $\square$

Remark 2 The previous lemma becomes false if we omit the hypothesis $\dim(x) = \dim(y) - 1$ and if $\dim(y) \geqslant 3$. Indeed, when the dimension m of y is greater than 2, for any face $x \in y^{\downarrow}$ with $0 < \dim(x) < m-1$, the set $Y = y^{\downarrow\star} \setminus \{x\}$ has no unipolar point (any k-face z in $y^{\downarrow\star}$ covers $2k$ faces and is covered by $m-k$ faces (Lemma 5), hence any k-face, $k \in [1, m-1]$ in Y covers at least 2 faces and any k-face, $k \in [0, m-2]$ in Y is covered by at least 2 faces). Therefore, Y is its proper core and is not contractible. Such a set Y is depicted in Fig. 11(a) for $m = 3$ and $\dim(x) = 1$. When $\dim(x) = 0$, it is easy to check that the removal of unipolar points in Y leads to pick off the 1-faces of $y^{\downarrow}$ including x and stops after these m steps showing a core of Y (see Fig. 11(b). Thus, Y is not contractible.

Proposition 6 *Let X be a cubical complex.*

(a) *If $x \in X$ is unipolar, then x is up simple and there exists $y \in X$ such as (y, x) is a free pair.*
(b) *If $x \in X$ is simple, there exist $y, z \in x^{\uparrow\star}$ such as (y, z) is a free pair.*
(c) *If (x, y) is a free pair, y is up unipolar and x is down simple in $X \setminus \{y\}$.*

Proof

(a) Let $x \in X$ be a unipolar point. Since X is a complex, $x^{\downarrow} \subseteq X$ and thus, x cannot be down unipolar (for a m-face in a cubical complex covers $2m$ faces). So, x is up unipolar, i.e. $x^{\uparrow\star}$ has a minimum (denoted y) and is therefore contractible (Corollary 1). Hence, x is up simple. Moreover, $\dim(y) = \dim(x) + 1$ (for X is a complex) and, y being the only face in $x^{\uparrow\star}$ with this dimension, we deduce from Lemma 5(a) that it does not exist any face $z \in x^{\uparrow\star}$ such that $\dim(z) > \dim(y)$. Thus, (y, x) is a free pair in X.
(b) Let $x \in X$ be a simple face. Then, $x^{\uparrow\star}$ is contractible (for $x^{\downarrow\star}$ is not contractible: Corollary 5). Hence, either $x^{\uparrow\star}$ is a singleton or there is a face y unipolar in $x^{\uparrow\star}$ (Corollary 4). If $x^{\uparrow\star}$ is a singleton $\{y\}$, (y, x) is a free pair. Otherwise, we derive from the previous part of this proof that there is a face z in $x^{\uparrow\star}$ such that (z, y) is a free pair in $x^{\uparrow\star}$ and thus in X.

(c) Let (x, y) be a free pair. The face x is the only face in $y^{\uparrow\star}$ so y is up unipolar and, since X is a complex, $\dim(y) = \dim(x) - 1$. Moreover, thanks to Lemma 6, we conclude that x is down simple in $X \setminus \{y\}$ (for $x^{\downarrow\star} \cap (X \setminus \{y\}) = x^{\downarrow\star} \setminus \{y\}$). □

4.5 w-simple Points

The example of Fig. 11 puts in evidence the need of a weaker condition on points to be deleted when processing the reduction of a digital image. The following definition of a w-simple point ("w" stands for "weak") and their properties are due to Barmak and Minian [4] who call them γ-points. Bertrand in [6] defines a quite similar notion.

Definition 7 A point x of a poset is a *w-simple point*, or simply a *w-point*, if the poset $x^{\updownarrow\star}$ is homotopically trivial, i.e. if it is connected and all its homotopy groups are trivial.

Property 9 gives several ways to prove that an element of a finite poset is a w-point and Property 10 ensures that the deletion of a w-point does not modify the homotopy groups.

Property 9 *Let X and Y be finite posets. Then $x^{\updownarrow\star}$ is homotopically trivial if $x^{\downarrow\star}$ or $x^{\uparrow\star}$ is homotopically trivial.*

Property 10 *Let X be a finite poset. Let $x \in X$ be a w-simple point. Then, the inclusion $i : X \setminus \{x\} \to X$ is a weak homotopy equivalence.*

Last, Theorem 7 states that, when deleting a w-point in a finite poset, the homotopy type of the continuous analogue keeps unchanged.

Theorem 7 *Let X be a finite poset and let $x \in X$ be a w-simple point. Then $|\mathcal{K}(X \setminus \{x\})|$ and $|\mathcal{K}(X)|$ are simple-homotopy equivalent.*

In a 3D-image X, the cost to decide whether the set $x^{\updownarrow\star}$ is homotopically trivial is not expensive. Indeed, $\mathcal{K}(x^{\updownarrow\star})$ is a 2-dimensional simplicial complex and it is enough to compute its connected components and its Euler characteristic. Moreover, the scheme proposed for the deletion of simple points is still valid (same dimensional w-simple points can be remove in parallel). An example of the use of this scheme on a 3-D image is given in Fig. 12.

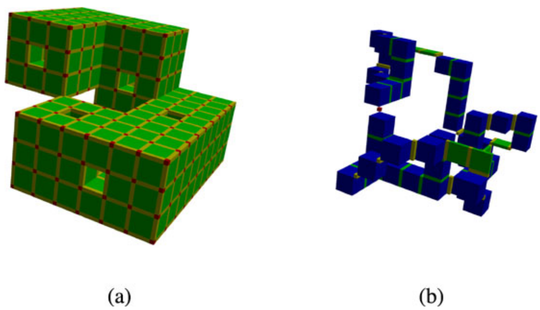


Fig. 12 Reduction by w-points removal in 3D-space. *Left*: a hollow pinched torus with five little holes. *Right*: The same torus after the removal of w-points until stability

5 Conclusion

We have studied the links between the standard notion of path in a topological space and the notion of path in a graph (here, the Hasse diagram) and showed that there are closer that it could be thought. In particular, they lead to the same fundamental group. It is a new validation of the use of posets, such as Khalimsky spaces or complexes, to analyse or process digital images. In a further work not yet published [28], we will study the relations between the digital paths, and the digital fundamental groups in $\mathbb{Z}^n$, as defined by [16], and the paths and fundamental groups in $\mathbb{F}^n$. Anyway, we hope we have succeeded to convince the reader that continuity is also a rich concept when applied to discrete or finite spaces. Though such notions as Jordan curves, surfaces, manifolds, which involve homeomorphisms, i.e. one-to-one correspondences with pieces of $\mathbb{R}^n$, cannot be used as-is in finite spaces and must be adapted, standard topology offers a set of tools usable in finite spaces and useful links between finite spaces and continuous analogues.

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