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Intersection Local Times of Independent Fractional Brownian Motions as Generalized White Noise Functionals

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Abstract In this work we present expansions of intersection local times of fractional Brownian motions in $\mathbb{R}^d$, for any dimension $d \geq 1$, with arbitrary Hurst coefficients in $(0,1)^d$. The expansions are in terms of Wick powers of white noises (corresponding to multiple Wiener integrals), being well-defined in the sense of generalized white noise functionals. As an application of our approach, a sufficient condition on d for the existence of intersection local times in L^2 is derived, extending the results in Nualart and Ortiz-Latorre (J. Theoret. Probab. 20(4):759–767, 2007) to different and more general Hurst coefficients.

Keywords Fractional Brownian motion · White noise analysis · Local time

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1 Introduction

In the recent years the fractional Brownian motion has become an object of intense study, namely, due to its special properties, such as short/long range dependence and self-similarity, yielding its proper and natural uses in several applications in different fields (e.g. mathematical finances [14], telecommunications engineering [16]).

Besides its own specific properties, the intersection properties of fractional Brownian motion paths have been studied by many authors as well, see e.g. the works done by Gradinaru et al. [6], Hu and Nualart [8, 9], Rosen [21], and the references therein.

One may consider intersections of sample paths with themselves, as in [5] and references therein, or with other independent fractional Brownian motions, as in [15].

This work concerns the latter standpoint. Within the white noise analysis framework (Sect. 2), a first purpose of this work is an extension of the results presented in [1] to two d-dimensional independent fractional Brownian motions $\mathbf{B}_{H_1}$ and $\mathbf{B}_{H_2}$ with different Hurst coefficients, H_1 and H_2. Technically, this approach has the advantage that the underlying probability space does not depend on any Hurst coefficient under consideration. As a consequence, one may analyze the intersection local time of any two independent fractional Brownian motions, without any restriction on the corresponding Hurst coefficients.

From the viewpoint of applications to physics, this absence of restrictions on the Hurst coefficients under consideration is meaningful to widen the modelling of polymers towards polymers molecules handling different types of polymers.

For low dimensions, that is, either for $d=1$ or for $d=2$, the white noise analysis framework allows the definition of the intersection local time of any two independent fractional Brownian motions $\mathbf{B}_{H_i}$ in terms of an integral over a Donsker's δ-function

$$L \equiv \int dt_1 dt_2\, \delta(\mathbf{B}_{H_1}(t_1) - \mathbf{B}_{H_2}(t_2)),$$

intended to sum up the contributions from each pair of moments of time t_1, t_2 for which the fractional Brownian motions $\mathbf{B}_{H_i}$ arrive at the same point.

A rigorous definition, such as, e.g., through a sequence of Gaussians approximating the δ-function,

$$(2\pi\varepsilon)^{-d/2} \exp\left(-\frac{|x|^2}{2\varepsilon}\right), \quad \varepsilon > 0,$$

will make L increasingly singular, and various "renormalizations" have to be done as the dimension d increases. Of course, besides the dimension of the space, the type of "renormalizations" needed depends as well on the Hurst coefficients $H_i \in (0,1)^d$ being considered. For $d>2$ with $1/\max_j H_{1,j} + 1/\max_j H_{2,j} \le d$, the expectation diverges in the limit and must be subtracted. Depending on the values of $\max_j H_{i,j}$, further kernel terms must be also subtracted (Theorem 10).

In this work we are particularly interested in the chaos decomposition of L. We expand L in terms of Wick powers [7] of white noise, an expansion which corresponds to that in terms of multiple Wiener integrals when one considers the Wiener process as the fundamental random variable. This allows us to derive the kernels for L. Due to the local structure of the Wick powers, the kernel functions are relatively simple and exhibit clearly the dimension dependence singularities of L (Proposition 9). For comparison, we also calculate the regularized kernel functions corresponding to the Gaussian δ-sequence mentioned above (Theorem 10).

As an application of this approach, in Theorem 12 we derive a sufficient condition for the existence of the intersection local times in L^2, extending the results obtained in [15] to different and more general Hurst coefficients.

2 Gaussian White Noise Calculus

In this section we briefly recall the concepts and results of white noise analysis used throughout this work (for a detailed explanation see e.g. [4, 7, 10, 12, 17]).

2.1 Fractional Brownian Motion

The starting point of white noise analysis for the construction of two independent d-dimensional, $d \geq 1$, fractional Brownian motions is the real Gelfand triple

$$S_{2d}(\mathbb{R}) \subset L^2_{2d}(\mathbb{R}) \subset S'_{2d}(\mathbb{R}),$$

where $L^2_{2d}(\mathbb{R}) := L^2(\mathbb{R}, \mathbb{R}^{2d})$ is the real Hilbert space of all vector valued square integrable functions with respect to the Lebesgue measure on $\mathbb{R}$, and $S_{2d}(\mathbb{R})$, $S'_{2d}(\mathbb{R})$ are the Schwartz spaces of the vector valued test functions and tempered distributions, respectively. We shall denote the $L^2_{2d}(\mathbb{R})$-norm by $|\cdot|_{2d}$ (or if there is no risk of confusion simply by $|\cdot|$) and the dual pairing between $S'_{2d}(\mathbb{R})$ and $S_{2d}(\mathbb{R})$ by $\langle \cdot, \cdot \rangle_{2d}$, or simply by $\langle \cdot, \cdot \rangle$, which is defined as the bilinear extension of the inner product on $L^2_{2d}(\mathbb{R})$, i.e.,

$$\langle \mathbf{g}, \mathbf{f} \rangle_{2d} = \sum_{i=1}^{2d} \int_{\mathbb{R}} dx \, g_i(x) f_i(x),$$

for all $\mathbf{g} = (g_1, \ldots, g_{2d}) \in L^2_{2d}(\mathbb{R})$ and all $\mathbf{f} = (f_1, \ldots, f_{2d}) \in S_{2d}(\mathbb{R})$. By the Minlos theorem, there is a unique probability measure μ on the σ-algebra $\mathcal{B}$ generated by the cylinder sets on $S'_{2d}(\mathbb{R})$ with characteristic function given by

$$C(\mathbf{f}) := \int_{S'_{2d}(\mathbb{R})} d\mu(\vec{\omega}) \, e^{i\langle \vec{\omega}, \mathbf{f} \rangle} = e^{-\frac{1}{2}|\mathbf{f}|^2}, \quad \mathbf{f} \in S_{2d}(\mathbb{R}).$$

In this way we have defined the white noise measure space $(S'_{2d}(\mathbb{R}), \mathcal{B}, \mu)$.

To construct two independent d-dimensional fractional Brownian motions we shall consider a $2d$-tuple of independent Gaussian white noises

$$\vec{\omega} := (\vec{\omega}_1, \vec{\omega}_2), \quad \vec{\omega}_i = (\omega_{i,1}, \ldots, \omega_{i,d}), \ i = 1, 2.$$

Within this formalism, a version of a d-dimensional Wiener Brownian motion is given by

$$\mathbf{B}(t) := \left(\langle \omega_1, \mathbb{1}_{[0,t]} \rangle, \ldots, \langle \omega_d, \mathbb{1}_{[0,t]} \rangle \right), \quad (\omega_1, \ldots, \omega_d) \in S'_d(\mathbb{R}),$$

where $\mathbb{1}_A$ denotes the indicator function of a set A and $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_1$. For an arbitrary d-dimensional Hurst parameter $H = (H_1, \ldots, H_d) \in (0, 1)^d$, a version of a d-dimensional fractional Brownian motion is given by

$$\mathbf{B}_H(t) := \left(\langle \omega_1, M_{H_1} \mathbb{1}_{[0,t]} \rangle, \ldots, \langle \omega_d, M_{H_d} \mathbb{1}_{[0,t]} \rangle \right), \quad (\omega_1, \ldots, \omega_d) \in S'_d(\mathbb{R}),$$

where, for a 1-dimensional Hurst parameter $H \in (0,1)$ and for a generic real valued function f,

$$(M_H f)(x) := \begin{cases} \frac{(\frac{1}{2}-H)K_H}{\Gamma(H+\frac{1}{2})} \lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty dy \, \frac{f(x)-f(x+y)}{y^{\frac{3}{2}-H}}, \quad H \in (0,1/2)\,, & (*) \\ f(x), \quad H = \frac{1}{2}, & \\ \frac{K_H}{\Gamma(H-\frac{1}{2})} \int_x^\infty dy \, f(y)(y-x)^{H-\frac{3}{2}}, \quad H \in (1/2,1) & (**) \end{cases}$$

provided the limit in $(*)$ exists for almost all $x \in \mathbb{R}$ and the integral in $(**)$ exists for all $x \in \mathbb{R}$ (for more details see e.g. [2] and [19] and the references therein). Independently of the case under consideration, the normalizing constant K_H is given by

$$K_H = \Gamma\left(H + \frac{1}{2}\right) \left(\frac{1}{2H} + \int_0^\infty ds \left((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\right)\right)^{-\frac{1}{2}}.$$

There are several examples of functions f for which $M_H f$ exists for any $H \in (0,1)$, namely, for $f = 1\!\!1_{[0,t]}$ with $t > 0$ or for $f \in S_1(\mathbb{R})$. For more details and proofs see e.g. [2, 3, 13, 20], and the references therein.

2.2 Hida Distributions and Characterization Results

Let us now consider the complex Hilbert space $(L^2) := L^2(S'_{2d}(\mathbb{R}), \mathcal{B}, \mu)$. For simplicity one introduces the notation

$$\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d, \quad n = \sum_{i=1}^d n_i, \quad \mathbf{n}! = \prod_{i=1}^d n_i!.$$

The space (L^2) is canonically isomorphic to the symmetric Fock space of symmetric square integrable functions,

$$(L^2) \simeq \left(\bigoplus_{k=0}^\infty \operatorname{Sym} L^2(\mathbb{R}^k, k! d^k x)\right)^{\otimes 2d},$$

which leads to the chaos expansion of the elements in (L^2),

$$\begin{aligned} F(\vec{\omega}_1, \vec{\omega}_2) &= \sum_{\mathbf{m}} \sum_{\mathbf{k}} \langle :\vec{\omega}_1^{\otimes \mathbf{m}}: \otimes :\vec{\omega}_2^{\otimes \mathbf{k}}:, \mathbf{f}_{\mathbf{m},\mathbf{k}} \rangle \\ &= \sum_{\mathbf{m}} \sum_{\mathbf{k}} \left\langle \bigotimes_{i=1}^d :\omega_{1,i}^{\otimes m_i}: \otimes \bigotimes_{j=1}^d :\omega_{2,j}^{\otimes k_j}:, \mathbf{f}_{\mathbf{m},\mathbf{k}} \right\rangle, \end{aligned}$$

with kernel functions $\mathbf{f}_{\mathbf{m},\mathbf{k}}$ in the Fock space, that is, square integrable functions of the $m + k$ arguments and symmetric in each m_i-, k_j-tuple.

To proceed further we have to consider a Gelfand triple around the space (L^2). We will use the space $(S)^*$ of Hida distributions (or generalized Brownian functionals) and the corresponding Gelfand triple $(S) \subset (L^2) \subset (S)^*$. Here (S) is the space of white noise test functions such that its dual space (with respect to (L^2)) is the space $(S)^*$. Instead of reproducing the explicit construction of $(S)^*$ (see e.g. [7]), in Theorem 2 below we characterize this space through its S-transform. We recall that given a $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in S_{2d}(\mathbb{R})$, and the Wick exponential

$$:\exp(\langle\vec{\omega}, \mathbf{f}\rangle) ::= \sum_{\mathbf{m}} \sum_{\mathbf{k}} \frac{1}{\mathbf{m}!\mathbf{k}!} \langle :\vec{\omega}_1^{\otimes \mathbf{m}} : \otimes : \vec{\omega}_2^{\otimes \mathbf{k}} :, \mathbf{f}_1^{\otimes \mathbf{m}} \otimes \mathbf{f}_2^{\otimes \mathbf{k}} \rangle = C(\mathbf{f}) e^{\langle \vec{\omega}, \mathbf{f}\rangle_{2d}},$$

we define the S-transform of a $\Phi \in (S)^*$ by

$$S\Phi(\mathbf{f}) := \langle\langle \Phi, :\exp(\langle \cdot, \mathbf{f}\rangle) :\rangle\rangle, \quad \forall \mathbf{f} \in S_{2d}(\mathbb{R}). \tag{1}$$

Here $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the dual pairing between $(S)^*$ and (S) which is defined as the bilinear extension of the sesquilinear inner product on (L^2). We observe that the multilinear expansion of (1),

$$S\Phi(\mathbf{f}) := \sum_{\mathbf{m}} \sum_{\mathbf{k}} \langle F_{\mathbf{m},\mathbf{k}}, \mathbf{f}_1^{\otimes \mathbf{m}} \otimes \mathbf{f}_2^{\otimes \mathbf{k}} \rangle,$$

extends the chaos expansion to $\Phi \in (S)^*$ with distribution valued kernels $F_{\mathbf{m},\mathbf{k}}$ such that

$$\langle\langle \Phi, \varphi \rangle\rangle = \sum_{\mathbf{m}} \sum_{\mathbf{k}} \mathbf{m}!\mathbf{k}! \langle F_{\mathbf{m},\mathbf{k}}, \varphi_{\mathbf{m},\mathbf{k}} \rangle, \tag{2}$$

for every generalized test function $\varphi \in (S)$ with kernel functions $\varphi_{\mathbf{m},\mathbf{k}}$.

In order to characterize the space $(S)^*$ through its S-transform we need the following definition.

Definition 1 A function $F : S_{2d}(\mathbb{R}) \to \mathbb{C}$ is called a U-functional whenever

1. for every $\mathbf{f}_1, \mathbf{f}_2 \in S_{2d}(\mathbb{R})$ the mapping $\mathbb{R} \ni \lambda \longmapsto F(\lambda \mathbf{f}_1 + \mathbf{f}_2)$ has an entire extension to $\lambda \in \mathbb{C}$,
2. there are constants $K_1, K_2 > 0$ such that

$$|F(z\mathbf{f})| \le K_1 e^{K_2 |z|^2 \|\mathbf{f}\|^2}, \quad \forall z \in \mathbb{C}, \mathbf{f} \in S_{2d}(\mathbb{R})$$

   for some continuous norm $\| \cdot \|$ on $S_{2d}(\mathbb{R})$.

We are now ready to state the aforementioned characterization result.

Theorem 2 [11, 18] *The S-transform defines a bijection between the space $(S)^*$ and the space of U-functionals.*

As a consequence of Theorem 2 one may derive the next two statements. The first one concerns the convergence of sequences of Hida distributions and the second one the Bochner integration of families of distributions of the same type (for more details and proofs see e.g. [7, 11, 18]).

Corollary 3 *Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence in $(S)^*$ such that*

(i) *for all $\mathbf{f} \in S_{2d}(\mathbb{R})$, $((S\Phi_n)(\mathbf{f}))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$,*

(ii) *there are constants* $K_1, K_2 > 0$ *such that for some continuous norm* $\|\cdot\|$ *on* $S_{2d}(\mathbb{R})$ *one has*

$$|(S\Phi_n)(z\mathbf{f})| \leq K_1 e^{K_2|z|^2\|\mathbf{f}\|^2}, \quad \forall z \in \mathbb{C}, \mathbf{f} \in S_{2d}(\mathbb{R}), n \in \mathbb{N}.$$

Then $(\Phi_n)_{n\in\mathbb{N}}$ *converges strongly in* $(S)^*$ *to a unique Hida distribution.*

Corollary 4 *Let* $(\Omega, \mathcal{B}, m)$ *be a measure space and* $\lambda \mapsto \Phi_\lambda$ *be a mapping from* Ω *to* $(S)^*$. *We assume that the* S*-transform of* Φ_λ *fulfills the following two properties*:

(i) *the mapping* $\lambda \mapsto (S\Phi_\lambda)(\mathbf{f})$ *is measurable for every* $\mathbf{f} \in S_{2d}(\mathbb{R})$,
(ii) *the* $S\Phi_\lambda$ *obeys a* U*-estimate*

$$|(S\Phi_\lambda)(z\mathbf{f})| \leq C_1(\lambda)e^{C_2(\lambda)|z|^2\|\mathbf{f}\|^2}, \quad z \in \mathbb{C}, \mathbf{f} \in S_{2d}(\mathbb{R}),$$

for some continuous norm $\|\cdot\|$ *on* $S_{2d}(\mathbb{R})$ *and for some* $C_1 \in L^1(\Omega, m)$, $C_2 \in L^\infty(\Omega, m)$.

Then

$$\int_\Omega dm(\lambda)\, \Phi_\lambda \in (S)^*$$

and

$$S\left(\int_\Omega dm(\lambda)\, \Phi_\lambda\right)(\mathbf{f}) = \int_\Omega dm(\lambda)\, (S\Phi_\lambda)\, (\mathbf{f}).$$

3 Chaos Expansions

Let us now consider two independent d-dimensional fractional Brownian motions $\mathbf{B}_{H_1}(t)$ and $\mathbf{B}_{H_2}(t)$ with Hurst multiparameters $H_1 = (H_{1,1}, \ldots, H_{1,d})$ and $H_2 = (H_{2,1}, \ldots, H_{2,d})$, respectively. That is, given a $2d$-tuple of independent white noises $(\omega_{1,1}, \ldots, \omega_{1,d}, \omega_{2,1}, \ldots, \omega_{2,d})$,

$$\mathbf{B}_{H_i}(t) := \left(\langle \omega_{i,1}, M_{H_{i,1}} \mathbb{1}_{[0,t]}\rangle, \ldots, \langle \omega_{i,d}, M_{H_{i,d}} \mathbb{1}_{[0,t]}\rangle\right), \quad i = 1, 2.$$

Proposition 5 *For each* t *and* s *strictly positive real numbers the Bochner integral*

$$\delta(\mathbf{B}_{H_1}(t) - \mathbf{B}_{H_2}(s)) := \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} d\lambda\, e^{i\lambda(\mathbf{B}_{H_1}(t) - \mathbf{B}_{H_2}(s))}$$

is a Hida distribution with S*-transform given by*

$$\begin{aligned} &S\delta(\mathbf{B}_{H_1}(t) - \mathbf{B}_{H_2}(s))(\mathbf{f}) \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^d \prod_{j=1}^d \frac{1}{\sqrt{t^{2H_{1,j}} + s^{2H_{2,j}}}} \\ &\quad \cdot e^{-\frac{1}{2}\sum_{j=1}^d \frac{1}{t^{2H_{1,j}} + s^{2H_{2,j}}} \left(\int_\mathbb{R} dx\, (f_{1,j}(x)(M_{H_{1,j}} \mathbb{1}_{[0,t]})(x) - f_{2,j}(x)(M_{H_{2,j}} \mathbb{1}_{[0,s]})(x))\right)^2}, \end{aligned} \tag{3}$$

for all $\mathbf{f} = (f_{1,1}, \ldots, f_{1,d}, f_{2,1}, \ldots, f_{2,d}) \in S_{2d}(\mathbb{R})$.

Remark 6 For motivation we note that

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^d} d\lambda\, e^{i\lambda(\mathbf{B}_{H_1}(t)-\mathbf{B}_{H_2}(s))} e^{-\frac{\varepsilon}{2}|\lambda|^2} = 0$$

whenever $(\mathbf{B}_{H_1}(t))(\vec{\omega}_1) \neq (\mathbf{B}_{H_2}(s))(\vec{\omega}_2)$, $\vec{\omega}_i = (\omega_{i,1}, \ldots, \omega_{i,d})$, $i = 1, 2$.

Proof The proof of this result follows from an application of Corollary 4 to the S-transform of the integrand function

$$\Phi(\vec{\omega}_1, \vec{\omega}_2) := e^{i\lambda(\mathbf{B}_{H_1}(t)-\mathbf{B}_{H_2}(s))}, \quad \vec{\omega}_i = (\omega_{i,1}, \ldots, \omega_{i,d}),\ i = 1, 2,$$

with respect to the Lebesgue measure on $\mathbb{R}^d$. For this purpose we begin by observing that since the fractional Brownian motions are independent one has

$$S\Phi(\mathbf{f}) = Se^{i\lambda \mathbf{B}_{H_1}(t)}(\mathbf{f}_1) \cdot Se^{-i\lambda \mathbf{B}_{H_2}(s)}(\mathbf{f}_2)$$

for every $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in S_{2d}(\mathbb{R})$, $\mathbf{f}_1 := (f_{1,1}, \ldots, f_{1,d})$, $\mathbf{f}_2 := (f_{2,1}, \ldots, f_{2,d})$. Hence, according e.g. to [7], for all $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d$ we obtain

$$S\Phi(\mathbf{f}) = \prod_{j=1}^{d} e^{i\lambda_j \int_{\mathbb{R}} dx\,(f_{1,j}(x)(M_{H_{1,j}}\mathbb{1}_{[0,t]})(x) - f_{2,j}(x)(M_{H_{2,j}}\mathbb{1}_{[0,s]})(x))} e^{-\frac{1}{2}\lambda_j^2 (t^{2H_{1,j}} + s^{2H_{2,j}})}, \tag{4}$$

which clearly fulfills the measurability condition. Moreover, for all $z \in \mathbb{C}$ we find

$$\begin{aligned}
|S\Phi(z\mathbf{f})| &= \prod_{j=1}^{d} e^{-\frac{1}{4}\lambda_j^2 (t^{2H_{1,j}} + s^{2H_{2,j}})} \\
&\quad \cdot \prod_{j=1}^{d} \left| e^{-\frac{1}{4}\lambda_j^2 (t^{2H_{1,j}} + s^{2H_{2,j}}) + iz\lambda_j \int_{\mathbb{R}} dx\,(f_{1,j}(x)(M_{H_{1,j}}\mathbb{1}_{[0,t]})(x) - f_{2,j}(x)(M_{H_{2,j}}\mathbb{1}_{[0,s]})(x))} \right| \\
&\leq \prod_{j=1}^{d} e^{-\frac{1}{4}\lambda_j^2 (t^{2H_{1,j}} + s^{2H_{2,j}})} \\
&\quad \cdot \prod_{j=1}^{d} e^{-\frac{1}{4}\lambda_j^2 (t^{2H_{1,j}} + s^{2H_{2,j}}) + |z||\lambda_j| \left| \int_{\mathbb{R}} dx\,(f_{1,j}(x)(M_{H_{1,j}}\mathbb{1}_{[0,t]})(x) - f_{2,j}(x)(M_{H_{2,j}}\mathbb{1}_{[0,s]})(x)) \right|},
\end{aligned}$$

where, for each $j = 1, \ldots, d$, the corresponding term in the second product is bounded by

$$\exp\left(\frac{|z|^2}{t^{2H_{1,j}} + s^{2H_{2,j}}} \left(\int_{\mathbb{R}} dx\, \left(f_{1,j}(x)(M_{H_{1,j}}\mathbb{1}_{[0,t]})(x) - f_{2,j}(x)(M_{H_{2,j}}\mathbb{1}_{[0,s]})(x)\right) \right)^2 \right),$$

because

$$\begin{aligned}
&-\frac{1}{4}\lambda_j^2 (t^{2H_{1,j}} + s^{2H_{2,j}}) \\
&\qquad + |z||\lambda_j| \left| \int_{\mathbb{R}} dx\, \left(f_{1,j}(x)(M_{H_{1,j}}\mathbb{1}_{[0,t]})(x) - f_{2,j}(x)(M_{H_{2,j}}\mathbb{1}_{[0,s]})(x)\right) \right|
\end{aligned}$$

$$= -\left(\frac{|z|}{\sqrt{t^{2H_{1,j}} + s^{2H_{2,j}}}} \left| \int_{\mathbb{R}} dx \, \left(f_{1,j}(x)(M_{H_{1,j}} \mathbb{1}_{[0,t]})(x) - f_{2,j}(x)(M_{H_{2,j}} \mathbb{1}_{[0,s]})(x)\right) \right| \right.$$
$$\left. - \frac{|\lambda_j|}{2} \sqrt{t^{2H_{1,j}} + s^{2H_{2,j}}} \right)^2$$
$$+ \frac{|z|^2}{t^{2H_{1,j}} + s^{2H_{2,j}}} \left(\int_{\mathbb{R}} dx \, \left(f_{1,j}(x)(M_{H_{1,j}} \mathbb{1}_{[0,t]})(x) - f_{2,j}(x)(M_{H_{2,j}} \mathbb{1}_{[0,s]})(x)\right) \right)^2 .$$

As a result,

$$|S\Phi(z\mathbf{f})| \leq e^{-\frac{1}{4}\sum_{j=1}^{d} \lambda_j^2 (t^{2H_{1,j}} + s^{2H_{2,j}})}$$
$$\cdot e^{|z|^2 \sum_{j=1}^{d} \frac{1}{t^{2H_{1,j}} + s^{2H_{2,j}}} (\int_{\mathbb{R}} dx \, (f_{1,j}(x)(M_{H_{1,j}} \mathbb{1}_{[0,t]})(x) - f_{2,j}(x)(M_{H_{2,j}} \mathbb{1}_{[0,s]})(x)))^2},$$

where, as a function of λ, the first exponential is integrable on $\mathbb{R}^d$ and the second exponential is constant.

An application of the result mentioned above completes the proof. In particular, it yields (3) by integrating (4) over λ. □

In order to proceed further the next result shows to be very useful. It improves the estimate obtained in [2, Theorem 2.3] towards the characterization results stated in Corollaries 3 and 4.

Lemma 7 [5] *Let $H \in (0, 1)$ and $f \in S_1(\mathbb{R})$ be given. There is a non-negative constant C_H independent of f and t such that*

$$\left| \int_{\mathbb{R}} dx \, f(x)(M_H \mathbb{1}_{[0,t]})(x) \right| \leq C_H t \left(\sup_{x\in\mathbb{R}} |f(x)| + \sup_{x\in\mathbb{R}} |f'(x)| + |f| \right)$$

for all $t > 0$.

In particular, the use of Lemma 7 allows to state the next result on intersection local times L_{H_1,H_2} as well as on their subtracted counterparts $L^{(N)}_{H_1,H_2}$. There, and throughout the rest of this work as well, given a $H = (H_1, \ldots, H_d) \in (0, 1)^d$ we shall use the notation

$$\bar{H} := \max_{j=1,\ldots,d} H_j.$$

Theorem 8 *Let $T > 0$ be given. For any pair of integer numbers $d \geq 1$, $N \geq 0$ and for any pair of Hurst multiparameters $H_1, H_2 \in (0, 1)^d$ such that*

$$\max\{\bar{H}_1, \bar{H}_2\} \left(N + \frac{d}{2} - \frac{1}{2\min\{\bar{H}_1, \bar{H}_2\}} \right) < N + \frac{1}{2},$$

the Bochner integral

$$L^{(N)}_{H_1,H_2} := \int_0^T dt \int_0^T ds \, \delta^{(N)}(\mathbf{B}_{H_1}(t) - \mathbf{B}_{H_2}(s))$$

is a Hida distribution, where

$$\delta^{(N)}(\mathbf{B}_{H_1}(t)-\mathbf{B}_{H_2}(s)) := \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} d\lambda \, \exp_N\left(i\lambda(\mathbf{B}_{H_1}(t)-\mathbf{B}_{H_2}(s))\right),$$

with $\exp_N(x) := \sum_{n=N}^{\infty} \frac{x^n}{n!}$.

Proof To prove this result we shall again use Corollary 4 with respect to the Lebesgue measure on $[0, T]^2$.

By proceeding as in the proof of (3), one shows that for every $t, s > 0$ the S-transform of $\delta^{(N)}(\mathbf{B}_{H_1}(t)-\mathbf{B}_{H_2}(s))$ is given by

$$\begin{aligned} &S\delta^{(N)}(\mathbf{B}_{H_1}(t)-\mathbf{B}_{H_2}(s))(\mathbf{f}) \\ &\quad= \left(\frac{1}{\sqrt{2\pi}}\right)^d \prod_{j=1}^d \frac{1}{\sqrt{t^{2H_{1,j}}+s^{2H_{2,j}}}} \exp_N\left(-\frac{1}{2}\sum_{j=1}^d \frac{1}{t^{2H_{1,j}}+s^{2H_{2,j}}}\right. \\ &\qquad \left.\cdot \left(\int_{\mathbb{R}} dx\, \left(f_{1,j}(x)(M_{H_{1,j}}\mathbb{1}_{[0,t]})(x) - f_{2,j}(x)(M_{H_{2,j}}\mathbb{1}_{[0,s]})(x)\right)\right)^2\right), \end{aligned} \tag{5}$$

which is a measurable function.

In order to check the boundedness condition, on $S_{2d}(\mathbb{R})$ let us consider the norm $\|\cdot\|$ defined for all $\mathbf{f} = (f_1, \ldots, f_{2d}) \in S_{2d}(\mathbb{R})$ by

$$\|\mathbf{f}\| := \left(\sum_{i=1}^{2d}\left(\sup_{x\in\mathbb{R}}|f_i(x)| + \sup_{x\in\mathbb{R}}|f_i'(x)| + |f_i|\right)^2\right)^{\frac{1}{2}}. \tag{6}$$

We observe that on $S_1(\mathbb{R})$ this norm reduces to the continuous norm

$$\|f\| = \sup_{x\in\mathbb{R}}|f(x)| + \sup_{x\in\mathbb{R}}|f'(x)| + |f|, \quad f \in S_1(\mathbb{R}),$$

which implies the continuity of the norm (6) for higher dimensions.

By Lemma 7, for each $j = 1, \ldots, d$, we obtain

$$\begin{aligned} &\left(\int_{\mathbb{R}} dx\, \left(f_{1,j}(x)(M_{H_{1,j}}\mathbb{1}_{[0,t]})(x) - f_{2,j}(x)(M_{H_{2,j}}\mathbb{1}_{[0,s]})(x)\right)\right)^2 \\ &\quad\leq 2\left(\int_{\mathbb{R}} dx\, f_{1,j}(x)(M_{H_{1,j}}\mathbb{1}_{[0,t]})(x)\right)^2 + 2\left(\int_{\mathbb{R}} dx\, f_{2,j}(x)(M_{H_{2,j}}\mathbb{1}_{[0,s]})(x)\right)^2 \\ &\quad\leq 2t^2 C_{H_{1,j}}^2 \|f_{1,j}\|^2 + 2s^2 C_{H_{2,j}}^2 \|f_{2,j}\|^2, \end{aligned}$$

and thus, for all $z \in \mathbb{C}$ and all $\mathbf{f} \in S_{2d}(\mathbb{R})$,

$$\begin{aligned} &\left|S(\delta^{(N)}(\mathbf{B}_{H_1}(t)-\mathbf{B}_{H_2}(s)))(z\mathbf{f})\right| \\ &\quad\leq \left(\frac{1}{\sqrt{2\pi}}\right)^d \prod_{j=1}^d \frac{1}{\sqrt{t^{2H_{1,j}}+s^{2H_{2,j}}}} \exp_N\left(|z|^2 C_{H_1,H_2}^2 \frac{t^2+s^2}{t^{2H_{(1)}}+s^{2H_{(2)}}}\|\mathbf{f}\|^2\right) \end{aligned}$$

with $C_{H_1,H_2} := \max\{C_{H_{1,j}}, C_{H_{2,j}} : j = 1, \ldots, d\}$ and

$$H_{(1)} := \begin{cases} \bar{H}_1 = \max_{j=1,\ldots,d} H_{1,j}, & 0 < t \leq 1, \\ \min_{j=1,\ldots,d} H_{1,j}, & t > 1, \end{cases}$$

$$H_{(2)} := \begin{cases} \bar{H}_2 = \max_{j=1,\ldots,d} H_{2,j}, & 0 < s \leq 1, \\ \min_{j=1,\ldots,d} H_{2,j}, & s > 1. \end{cases}$$

Therefore, for $0 < t, s \leq 1$ one has

$$\frac{t^2 + s^2}{t^{2H_{(1)}} + s^{2H_{(2)}}} = \frac{t^2 + s^2}{t^{2\bar{H}_1} + s^{2\bar{H}_2}} \leq 1,$$

and thus

$$\exp_N\left(|z|^2 C^2_{H_1,H_2} \frac{t^2 + s^2}{t^{2\bar{H}_1} + s^{2\bar{H}_2}} \|\mathbf{f}\|^2\right) \leq \left(\frac{t^2 + s^2}{t^{2\bar{H}_1} + s^{2\bar{H}_2}}\right)^N e^{|z|^2 C^2_{H_1,H_2} \|\mathbf{f}\|^2};$$

while either for $t > 1$ or for $s > 1$ one finds

$$\exp_N\left(|z|^2 C^2_{H_1,H_2} \frac{t^2 + s^2}{t^{2H_{(1)}} + s^{2H_{(2)}}} \|\mathbf{f}\|^2\right)$$
$$\leq \left(\frac{t^2 + s^2}{t^{2H_{(1)}} + s^{2H_{(2)}}}\right)^N e^{|z|^2 C^2_{H_1,H_2} \left(\frac{t^2+s^2}{t^{2H_{(1)}}+s^{2H_{(2)}}} + N\right) \|\mathbf{f}\|^2}.$$

As a consequence, independently of T being smaller or greater than 1 there is always a function $C = C(t, s) > 0$ bounded on $[0, T]^2$ such that

$$\left|S(\delta^{(N)}(\mathbf{B}_{H_1}(t) - \mathbf{B}_{H_2}(s)))(z\mathbf{f})\right|$$
$$\leq \left(\frac{1}{\sqrt{2\pi}}\right)^d \prod_{j=1}^d \frac{1}{\sqrt{t^{2H_{1,j}} + s^{2H_{2,j}}}} \left(\frac{t^2 + s^2}{t^{2H_{(1)}} + s^{2H_{(2)}}}\right)^N e^{|z|^2 C^2_{H_1,H_2} C(t,s) \|\mathbf{f}\|^2}. \tag{7}$$

The proof then amounts to prove the integrability on $[0, T]^2$ of the expression

$$\prod_{j=1}^d \frac{1}{\sqrt{t^{2H_{1,j}} + s^{2H_{2,j}}}} \left(\frac{t^2 + s^2}{t^{2H_{(1)}} + s^{2H_{(2)}}}\right)^N$$

appearing in (7). For this purpose one observes that due to the singular point at the origin this expression is integrable on $[0, T]^2$ if and only if it is integrable on $[0, 1]^2$. As shown in the Appendix A (Lemma 15), this occurs whenever

$$2\max\{\bar{H}_1, \bar{H}_2\}\left(N + \frac{d}{2} - \frac{1}{2\min\{\bar{H}_1, \bar{H}_2\}}\right) - 2N < 1.$$

The proof is then completed by an application of Corollary 4. □

As a consequence, one may derive the chaos expansion for the (truncated) local times $L^{(N)}_{H_1,H_2}$.

Proposition 9 *Under the conditions of Theorem 8, $L^{(N)}_{H_1,H_2}$ has the chaos expansion*

$$L^{(N)}_{H_1,H_2}(\vec{\omega}_1,\vec{\omega}_2)=\sum_{\mathbf{m}}\sum_{\mathbf{k}}\langle :\vec{\omega}_1^{\otimes\mathbf{m}}:\otimes:\vec{\omega}_2^{\otimes\mathbf{k}}:, F_{H_1,H_2,\mathbf{m},\mathbf{k}}\rangle$$

where the kernel functions $F_{H_1,H_2,\mathbf{m},\mathbf{k}}$ are given by

$$\begin{aligned}F_{H_1,H_2,\mathbf{m},\mathbf{k}}&=\left(\frac{1}{\pi}\right)^{\frac{d}{2}}\frac{(-1)^{\frac{m+3k}{2}}}{(\frac{\mathbf{m}+\mathbf{k}}{2})!}\left(\frac{1}{2}\right)^{\frac{m+k+d}{2}}\binom{\mathbf{m}+\mathbf{k}}{\mathbf{m}}\\&\quad\cdot\int_0^T dt\int_0^T ds\prod_{j=1}^d\left(\frac{1}{t^{2H_{1,j}}+s^{2H_{2,j}}}\right)^{\frac{m_j+k_j+1}{2}}\\&\quad\cdot\bigotimes_{j=1}^d\left((M_{H_{1,j}}1\!\!1_{[0,t]})^{\otimes m_j}\otimes(M_{H_{2,j}}1\!\!1_{[0,s]})^{\otimes k_j}\right)\end{aligned}$$

for each $\mathbf{m}=(m_1,\ldots,m_d)$ and each $\mathbf{k}=(k_1,\ldots,k_d)$ such that $m+k\geq 2N$ and all sums m_j+k_j, $j=1,\ldots,d$, are even numbers. All other kernel functions $F_{H_1,H_2,\mathbf{m},\mathbf{k}}$ are identically equal to zero.

Proof According to Corollary 4, the S-transform of the (truncated) local time $L^{(N)}_{H_1,H_2}$ is obtained by integrating (5) over $[0,T]^2$. Hence, given a $\mathbf{f}=(f_{1,1},\ldots,f_{1,d},f_{2,1},\ldots,f_{2,d})\in S_{2d}(\mathbb{R})$ one has

$$\begin{aligned}SL^{(N)}_{H_1,H_2}(\mathbf{f})&=\left(\frac{1}{\sqrt{2\pi}}\right)^d\int_0^T dt\int_0^T ds\prod_{j=1}^d\frac{1}{\sqrt{t^{2H_{1,j}}+s^{2H_{2,j}}}}\\&\quad\cdot\sum_{n=N}^\infty\frac{(-1)^n}{2^n n!}\sum_{\substack{n_1,\ldots,n_d\\ n_1+\cdots+n_d=n}}\frac{n!}{n_1!\cdots n_d!}\prod_{j=1}^d\left(\frac{1}{t^{2H_{1,j}}+s^{2H_{2,j}}}\right)^{n_j}\\&\quad\cdot\left(\int_{\mathbb{R}}dx\,\left(f_{1,j}(x)(M_{H_{1,j}}1\!\!1_{[0,t]})(x)-f_{2,j}(x)(M_{H_{2,j}}1\!\!1_{[0,s]})(x)\right)\right)^{2n_j}\end{aligned}\tag{8}$$

with (8) being equal to

$$\begin{aligned}&\sum_{\substack{m_j,k_j\\ m_j+k_j=2n_j}}(-1)^{k_j}\binom{2n_j}{m_j}\left(\int_{\mathbb{R}}dx\,f_{1,j}(x)(M_{H_{1,j}}1\!\!1_{[0,t]})(x)\right)^{m_j}\\&\quad\cdot\left(\int_{\mathbb{R}}dx\,f_{2,j}(x)(M_{H_{2,j}}1\!\!1_{[0,s]})(x)\right)^{k_j}.\end{aligned}$$

From these calculations follow the equality

$$SL^{(N)}_{H_1,H_2}(\mathbf{f}) = \left(\frac{1}{\pi}\right)^{\frac{d}{2}} \int_0^T dt \int_0^T ds \sum_{n=N}^{\infty} \sum_{\substack{n_1,\ldots,n_d\\ n_1+\cdots+n_d=n}} \sum_{\substack{m_1,\ldots,m_d,k_1,\ldots,k_d\\ m_j+k_j=2n_j, j=1,\ldots,d}}$$

$$\left\{\prod_{j=1}^{d} \frac{(-1)^{\frac{m_j+3k_j}{2}}}{(\frac{m_j+k_j}{2})!} \left(\frac{1}{2(t^{2H_{1,j}}+s^{2H_{2,j}})}\right)^{\frac{m_j+k_j+1}{2}}\right\}$$

$$\cdot \left\{\prod_{j=1}^{d} \binom{m_j+k_j}{m_j} \left(\int_{\mathbb{R}} dx\, f_{1,j}(x)(M_{H_{1,j}}1\!\!1_{[0,t]})(x)\right)^{m_j}\right.$$

$$\left.\cdot \left(\int_{\mathbb{R}} dx\, f_{2,j}(x)(M_{H_{2,j}}1\!\!1_{[0,s]})(x)\right)^{k_j}\right\},$$

which is equivalent to

$$\left(\frac{1}{\pi}\right)^{\frac{d}{2}} \int_0^T dt \int_0^T ds \sum_{\substack{\mathbf{m},\mathbf{k}\\ m+k\geq 2N\\ m_j+k_j\ even, j=1,\ldots,d}} \frac{(-1)^{\frac{m+3k}{2}}}{(\frac{\mathbf{m}+\mathbf{k}}{2})!} \left(\frac{1}{2}\right)^{\frac{m+k+d}{2}} \binom{\mathbf{m}+\mathbf{k}}{\mathbf{m}}$$

$$\cdot \prod_{j=1}^{d} \left(\frac{1}{t^{2H_{1,j}}+s^{2H_{2,j}}}\right)^{\frac{m_j+k_j+1}{2}} \left(\int_{\mathbb{R}} dx\, f_{1,j}(x)(M_{H_{1,j}}1\!\!1_{[0,t]})(x)\right)^{m_j}$$

$$\cdot \left(\int_{\mathbb{R}} dx\, f_{2,j}(x)(M_{H_{2,j}}1\!\!1_{[0,s]})(x)\right)^{k_j}.$$

Comparing with the general form of the chaos expansion

$$\sum_{\mathbf{m}} \sum_{\mathbf{k}} \langle : \vec{\omega}_1^{\otimes \mathbf{m}} : \otimes : \vec{\omega}_2^{\otimes \mathbf{k}} :, F_{H_1,H_2,\mathbf{m},\mathbf{k}} \rangle,$$

one concludes that the kernels $F_{H_1,H_2,\mathbf{m},\mathbf{k}}$ vanish whenever either there is a $j = 1, \ldots, d$ such that $m_j + k_j$ is an odd number or $m + k < 2N$, while for all other cases

$$F_{H_1,H_2,\mathbf{m},\mathbf{k}} = \left(\frac{1}{\pi}\right)^{\frac{d}{2}} \frac{(-1)^{\frac{m+3k}{2}}}{(\frac{\mathbf{m}+\mathbf{k}}{2})!} \left(\frac{1}{2}\right)^{\frac{m+k+d}{2}} \binom{\mathbf{m}+\mathbf{k}}{\mathbf{m}}$$

$$\cdot \int_0^T dt \int_0^T ds \prod_{j=1}^{d} \left(\frac{1}{t^{2H_{1,j}}+s^{2H_{2,j}}}\right)^{\frac{m_j+k_j+1}{2}}$$

$$\cdot \bigotimes_{j=1}^{d} \left((M_{H_{1,j}}1\!\!1_{[0,t]})^{\otimes m_j} \otimes (M_{H_{2,j}}1\!\!1_{[0,s]})^{\otimes k_j}\right).$$

□

Theorem 8 shows that for $d = 1$ or $d = 2$ all intersection local times L_{H_1,H_2} are well-defined for all possible Hurst multiparameters $H_1,\ H_2$ in $(0, 1)^d$. For $d > 2$, intersection local times are well-defined only for $1/\bar{H}_1 + 1/\bar{H}_2 > d$. Under these conditions, Proposition 9 in addition yields

$$\mathbb{E}_\mu(L_{H_1,H_2}) = F_{H_1,H_2,0,0} = \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_0^T dt \int_0^T ds \prod_{j=1}^d \frac{1}{\sqrt{t^{2H_{1,j}} + s^{2H_{2,j}}}}.$$

Informally speaking, for $1/\bar{H}_1 + 1/\bar{H}_2 \le d$ with $d > 2$, the local times only become well-defined once subtracted the divergent terms. This "renormalization" procedure motivates the study of a regularization. As a computationally simple regularization we discuss

$$L_{H_1,H_2,\varepsilon} := \int_0^T dt \int_0^T ds\, \delta_\varepsilon(\mathbf{B}_{H_1}(t) - \mathbf{B}_{H_2}(s)), \quad \varepsilon > 0,$$

where

$$\delta_\varepsilon(\mathbf{B}_{H_1}(t) - \mathbf{B}_{H_2}(s)) := \left(\frac{1}{\sqrt{2\pi\varepsilon}}\right)^d e^{-\frac{(\mathbf{B}_{H_1}(t) - \mathbf{B}_{H_2}(s))^2}{2\varepsilon}}.$$

Theorem 10 *Let $\varepsilon > 0$ be given. For all $H_1, H_2 \in (0,1)^d$ and all dimensions $d \ge 1$ the intersection local time $L_{H_1,H_2,\varepsilon}$ is a Hida distribution with kernel functions given by*

$$\begin{aligned} F_{H_1,H_2,\varepsilon,\mathbf{m},\mathbf{k}} &= \left(\frac{1}{\pi}\right)^{\frac{d}{2}} \frac{(-1)^{\frac{m+3k}{2}}}{(\frac{\mathbf{m}+\mathbf{k}}{2})!} \left(\frac{1}{2}\right)^{\frac{m+k+d}{2}} \binom{\mathbf{m}+\mathbf{k}}{\mathbf{m}} \\ &\cdot \int_0^T dt \int_0^T ds \prod_{j=1}^d \left(\frac{1}{\varepsilon + t^{2H_{1,j}} + s^{2H_{2,j}}}\right)^{\frac{m_j+k_j+1}{2}} \\ &\cdot \bigotimes_{j=1}^d \left((M_{H_{1,j}} \mathbb{1}_{[0,t]})^{\otimes m_j} \otimes (M_{H_{2,j}} \mathbb{1}_{[0,s]})^{\otimes k_j}\right) \end{aligned}$$

for all $\mathbf{m} = (m_1, \ldots, m_d), \mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}_0^d$ such that all sums $m_i + k_j$, $j = 1, \ldots, d$, are even numbers, and $F_{H_1,H_2,\varepsilon,\mathbf{m},\mathbf{k}} \equiv 0$ if at least one of the sums $m_i + k_i$ is an odd number. Moreover, if $\max\{\bar{H}_1, \bar{H}_2\}(N + \frac{d}{2} - \frac{1}{2\min\{\bar{H}_1,\bar{H}_2\}}) < N + \frac{1}{2}$, then when ε tends to zero the (truncated) intersection local time $L^{(N)}_{H_1,H_2,\varepsilon}$ converges strongly in $(S)^*$ to the (truncated) local time $L^{(N)}_{H_1,H_2}$.

Proof As before, the first part of the proof follows from the Corollary 4 with respect to the Lebesgue measure on $[0,T]^2$. By the definition of the S-transform, for all $\mathbf{f} = (f_{1,1}, \ldots, f_{1,d}, f_{2,1}, \ldots, f_{2,d}) \in S_{2d}(\mathbb{R})$ one finds

$$\begin{aligned} &S\delta_\varepsilon(\mathbf{B}_{H_1}(t) - \mathbf{B}_{H_2}(s))(\mathbf{f}) \\ &= \prod_{j=1}^d \frac{1}{\sqrt{2\pi(\varepsilon + t^{2H_{1,j}} + s^{2H_{2,j}})}} \\ &\quad \cdot e^{-\frac{1}{2}\sum_{j=1}^d \frac{1}{\varepsilon + t^{2H_{1,j}} + s^{2H_{2,j}}} \left(\int_{\mathbb{R}} dx\, (f_{1,j}(x)(M_{H_{1,j}}\mathbb{1}_{[0,t]})(x) - f_{2,j}(x)(M_{H_{2,j}}\mathbb{1}_{[0,s]})(x))\right)^2}, \end{aligned}$$

which is measurable. Hence, similarly to the proof of Theorem 8, an application of Lemma 7 yields for all $z \in \mathbb{C}$ and all $\mathbf{f} \in S_{2d}(\mathbb{R})$

$$\begin{aligned}
&\left|S(\delta_\varepsilon(\mathbf{B}_{H_1}(t) - \mathbf{B}_{H_2}(s)))(z\mathbf{f})\right| \\
&\quad \leq \prod_{j=1}^{d} \frac{1}{\sqrt{2\pi(\varepsilon + t^{2H_{1,j}} + s^{2H_{2,j}})}} e^{|z|^2 C^2_{H_1,H_2} \frac{t^2+s^2}{\varepsilon + t^{2H(1)} + s^{2H(2)}} \|\mathbf{f}\|^2}
\end{aligned}$$

with $\frac{t^2+s^2}{\varepsilon+t^{2H(1)}+s^{2H(2)}}$ bounded on $[0,T]^2$ and $\prod_{j=1}^{d} \frac{1}{\sqrt{2\pi(\varepsilon+t^{2H_{1,j}}+s^{2H_{2,j}})}}$ integrable on $[0,T]^2$. By Corollary 4, one may then conclude that $L_{H_1,H_2,\varepsilon} \in (S)^*$ and, moreover, for every $\mathbf{f} = (f_{1,1}, \dots, f_{1,d}, f_{2,1}, \dots, f_{2,d}) \in S_{2d}(\mathbb{R})$,

$$\begin{aligned}
SL_{H_1,H_2,\varepsilon}(\mathbf{f}) &= \int_0^T dt \int_0^T ds\, S\delta_\varepsilon(\mathbf{B}_{H_1}(t) - \mathbf{B}_{H_2}(s))(\mathbf{f}) \\
&= \left(\frac{1}{\pi}\right)^{\frac{d}{2}} \int_0^T dt \int_0^T ds \sum_{\substack{\mathbf{m},\mathbf{k} \\ m_j+k_j\ even,\, j=1,\dots,d}} \frac{(-1)^{\frac{m+3k}{2}}}{(\frac{\mathbf{m}+\mathbf{k}}{2})!} \left(\frac{1}{2}\right)^{\frac{m+k+d}{2}} \binom{\mathbf{m}+\mathbf{k}}{\mathbf{m}} \\
&\quad \cdot \prod_{j=1}^{d} \left(\frac{1}{\varepsilon + t^{2H_{1,j}} + s^{2H_{2,j}}}\right)^{\frac{m_j+k_j+1}{2}} \left(\int_{\mathbb{R}} dx\, f_{1,j}(x)(M_{H_{1,j}} \mathbb{1}_{[0,t]})(x)\right)^{m_j} \\
&\quad \cdot \left(\int_{\mathbb{R}} dx\, f_{2,j}(x)(M_{H_{2,j}} \mathbb{1}_{[0,s]})(x)\right)^{k_j}.
\end{aligned}$$

As in the proof of Proposition 9, it follows from the latter expression that the kernels $F_{H_1,H_2,\varepsilon,\mathbf{m},\mathbf{k}}$ appearing in the chaos expansion of $L_{H_1,H_2,\varepsilon}$ vanish if at least one of the $m_i + k_i$ in $\mathbf{m} + \mathbf{k} = (m_1 + k_1, \dots, m_d + k_d)$ is an odd number, otherwise they are given by

$$\begin{aligned}
F_{H_1,H_2,\varepsilon,\mathbf{m},\mathbf{k}} &= \left(\frac{1}{\pi}\right)^{\frac{d}{2}} \frac{(-1)^{\frac{m+3k}{2}}}{(\frac{\mathbf{m}+\mathbf{k}}{2})!} \left(\frac{1}{2}\right)^{\frac{m+k+d}{2}} \binom{\mathbf{m}+\mathbf{k}}{\mathbf{m}} \\
&\quad \cdot \int_0^T dt \int_0^T ds \prod_{j=1}^{d} \left(\frac{1}{\varepsilon + t^{2H_{1,j}} + s^{2H_{2,j}}}\right)^{\frac{m_j+k_j+1}{2}} \\
&\quad \cdot \bigotimes_{j=1}^{d} \left((M_{H_{1,j}} \mathbb{1}_{[0,t]})^{\otimes m_j} \otimes (M_{H_{2,j}} \mathbb{1}_{[0,s]})^{\otimes k_j}\right).
\end{aligned}$$

To complete the proof amounts to check the convergence. For this purpose we shall use Corollary 3. Since

$$SL^{(N)}_{H_1,H_2,\varepsilon}(\mathbf{f}) = \int_0^T dt \int_0^T ds\, S\delta^{(N)}_\varepsilon(\mathbf{B}_{H_1}(t) - \mathbf{B}_{H_2}(s))(\mathbf{f}),$$

for every $z \in \mathbb{C}$ and every $\mathbf{f} \in S_{2d}(\mathbb{R})$, a similar procedure used to prove Theorem 8 yields

$$\begin{aligned}\left|SL^{(N)}_{H_1,H_2,\varepsilon}(z\mathbf{f})\right| &\leq \int_0^T dt \int_0^T ds \,\left|S\delta_\varepsilon^{(N)}(\mathbf{B}_{H_1}(t)-\mathbf{B}_{H_2}(s))(z\mathbf{f})\right| \\ &\leq \left(\frac{1}{\sqrt{2\pi}}\right)^d e^{|z|^2\|\mathbf{f}\|^2 C^2_{H_1,H_2}\sup_{t,s\in[0,T]}C(t,s)} \\ &\quad \cdot \int_0^T dt \int_0^T ds \prod_{j=1}^d \frac{1}{\sqrt{t^{2H_{1,j}}+s^{2H_{2,j}}}}\left(\frac{t^2+s^2}{t^{2H_{(1)}}+s^{2H_{(2)}}}\right)^N,\end{aligned}$$

showing the boundedness condition. Furthermore, we have

$$\begin{aligned}&\left|S\delta_\varepsilon^{(N)}(\mathbf{B}_{H_1}(t)-\mathbf{B}_{H_2}(s))(\mathbf{f})\right| \\ &\leq \left(\frac{1}{\sqrt{2\pi}}\right)^d \prod_{j=1}^d \frac{1}{\sqrt{t^{2H_{1,j}}+s^{2H_{2,j}}}}\left(\frac{t^2+s^2}{t^{2H_{(1)}}+s^{2H_{(2)}}}\right)^N e^{C^2_{H_1,H_2}\|\mathbf{f}\|^2\sup_{t,s\in[0,T]}C(t,s)},\end{aligned}$$

which allows the use of the Lebesgue dominated convergence theorem to infer the other condition needed for the application of Corollary 3. □

Given any pair of Hurst multiparameters $H_1, H_2 \in (0,1)^d$, $d \geq 1$, such that $d < 1/\bar{H}_1 + 1/\bar{H}_2$, according to the convergence result stated in Theorem 10, for any $\mathbf{f} \in S_{2d}(\mathbb{R})$ fixed, $SL_{H_1,H_2,\varepsilon}(\mathbf{f})$ converges to $SL_{H_1,H_2}(\mathbf{f})$. This fact combined with the characterization result of the convergence in (L^2) in terms of the S-transform, recalled in Proposition 11, allows to improve the previous statements concerning the intersection local times (Theorem 12 below). In particular, this theorem extends the results obtained in [15] to different and more general Hurst multiparameters.

Proposition 11 *Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence in (L^2) and $\Phi \in (L^2)$. The following two assertions are equivalent*:

(i) $(\Phi_n)_{n\in\mathbb{N}}$ *converges in* (L^2) *to* Φ;
(ii) *the sequence* $(\|\Phi_n\|)_{n\in\mathbb{N}}$ *converges to* $\|\Phi\|$ *and, for all* $\mathbf{f} \in S_{2d}(\mathbb{R})$, $(S\Phi_n(\mathbf{f}))_{n\in\mathbb{N}}$ *converges to* $S\Phi(\mathbf{f})$.

Here $\|\cdot\|$ *denotes the norm defined on* (L^2).

Theorem 12 *For any pair of Hurst multiparameters* $H_1, H_2 \in (0,1)^d$, $d \geq 1$, *such that* $d < 1/\bar{H}_1 + 1/\bar{H}_2$, *the intersection local times* L_{H_1,H_2} *as well as all* $L_{H_1,H_2,\varepsilon}$, $\varepsilon > 0$, *exist in* (L^2), *and the sequence of* $L_{H_1,H_2,\varepsilon}$ *converges in* (L^2) *to* L_{H_1,H_2} *as* ε *tends to zero.*

Proof According to the previous considerations, the proof amounts to show that L_{H_1,H_2}, $L_{H_1,H_2,\varepsilon} \in (L^2)$, for all $\varepsilon > 0$, and that the convergence (in ε) of their (L^2)-norms holds. For this purpose we begin by showing that the sums

$$\sum_{\mathbf{m}}\sum_{\mathbf{k}} \mathbf{m}!\mathbf{k}!\left|F_{H_1,H_2,\varepsilon,\mathbf{m},\mathbf{k}}\right|^2_{(L^2_{2d}(\mathbb{R}))^{\otimes(m+k)}}, \qquad \sum_{\mathbf{m}}\sum_{\mathbf{k}} \mathbf{m}!\mathbf{k}!\left|F_{H_1,H_2,\mathbf{m},\mathbf{k}}\right|^2_{(L^2_{2d}(\mathbb{R}))^{\otimes(m+k)}}, \tag{9}$$

converge, where $F_{H_1,H_2,\varepsilon,\mathbf{m},\mathbf{k}}$ and $F_{H_1,H_2,\mathbf{m},\mathbf{k}}$ are the kernels given by Theorem 10 and Proposition 9, respectively. By (2), this will prove that $L_{H_1,H_2,\varepsilon}, L_{H_1,H_2} \in (L^2)$ with $\|L_{H_1,H_2,\varepsilon}\|^2$ given by the first sum appearing in (9) and $\|L_{H_1,H_2}\|^2$ by the second one.

Similar calculations done to prove Theorem 10 yield

$$\begin{aligned}
&\sum_{\mathbf{m}}\sum_{\mathbf{k}} \mathbf{m}!\mathbf{k}! \left| F_{H_1,H_2,\varepsilon,\mathbf{m},\mathbf{k}} \right|^2_{(L^2_{2d}(\mathbb{R}))^{\otimes(m+k)}} \\
&\quad = \sum_{\mathbf{m}}\sum_{\mathbf{k}} \mathbf{m}!\mathbf{k}! \left(\frac{1}{2\pi}\right)^d \frac{(-1)^{m+3k}}{((\frac{\mathbf{m}+\mathbf{k}}{2})!)^2} \left(\frac{1}{2}\right)^{m+k} \binom{\mathbf{m}+\mathbf{k}}{\mathbf{m}}^2 \\
&\quad\quad \cdot \int_0^T dt \int_0^T ds \int_0^T dt' \int_0^T ds' \\
&\quad\quad \cdot \prod_{j=1}^d \left(\frac{1}{\sqrt{(\varepsilon + t^{2H_{1,j}} + s^{2H_{2,j}})(\varepsilon + t'^{2H_{1,j}} + s'^{2H_{2,j}})}} \right)^{m_j+k_j+1} \\
&\quad\quad \cdot \langle M_{H_{1,j}} \mathbb{1}_{[0,t]}, M_{H_{1,j}} \mathbb{1}_{[0,t']} \rangle^{m_j} \langle M_{H_{2,j}} \mathbb{1}_{[0,s]}, M_{H_{2,j}} \mathbb{1}_{[0,s']} \rangle^{k_j},
\end{aligned}$$

with the inner products being equal to

$$\begin{aligned}
\langle M_{H_{1,j}} \mathbb{1}_{[0,t]}, M_{H_{1,j}} \mathbb{1}_{[0,t']} \rangle &= \frac{1}{2}\left(t^{2H_{1,j}} + t'^{2H_{1,j}} - |t-t'|^{2H_{1,j}} \right), \\
\langle M_{H_{2,j}} \mathbb{1}_{[0,s]}, M_{H_{2,j}} \mathbb{1}_{[0,s']} \rangle &= \frac{1}{2}\left(s^{2H_{2,j}} + s'^{2H_{2,j}} - |s-s'|^{2H_{2,j}} \right),
\end{aligned}$$

for each $j = 1, \ldots, d$,

$$\begin{aligned}
&= \left(\frac{1}{2\pi}\right)^d \int_0^T dt \int_0^T ds \int_0^T dt' \int_0^T ds' \prod_{j=1}^d \frac{1}{\sqrt{(\varepsilon + t^{2H_{1,j}} + s^{2H_{2,j}})(\varepsilon + t'^{2H_{1,j}} + s'^{2H_{2,j}})}} \\
&\quad \cdot \sum_{n=0}^\infty \frac{1}{4^n n!} \sum_{\substack{n_1,\ldots,n_d \\ n_1+\cdots+n_d=n}} \frac{n!}{n_1!\ldots n_d!} \\
&\quad \cdot \prod_{j=1}^d \frac{(2n_j)!}{n_j!} \left(\frac{1}{(\varepsilon + t^{2H_{1,j}} + s^{2H_{2,j}})(\varepsilon + t'^{2H_{1,j}} + s'^{2H_{2,j}})} \right)^{n_j} \\
&\quad \cdot \left(\frac{1}{2}\left(t^{2H_{1,j}} + t'^{2H_{1,j}} - |t-t'|^{2H_{1,j}} + s^{2H_{2,j}} + s'^{2H_{2,j}} - |s-s'|^{2H_{2,j}} \right) \right)^{2n_j}.
\end{aligned}$$

Concerning the integrand function, observe that using the following equalities for the Gamma function,

$$\begin{aligned}
\frac{(2n!)}{n!} &= \frac{2^{2n}}{\sqrt{\pi}} \Gamma\left(n + \frac{1}{2}\right), \\
\Gamma\left(n + \frac{1}{2}\right) &= \Gamma\left(\frac{1}{2}\right) \prod_{i=0}^{n-1} \left(\frac{1}{2} + i\right) = \sqrt{\pi} \prod_{i=0}^{n-1} \left(\frac{1}{2} + i\right),
\end{aligned}$$

one may rewrite it as

$$\prod_{j=1}^{d} \frac{1}{\sqrt{(\varepsilon + t^{2H_{1,j}} + s^{2H_{2,j}})(\varepsilon + t'^{2H_{1,j}} + s'^{2H_{2,j}})}}$$
$$\cdot \sum_{n=0}^{\infty} \sum_{\substack{n_1,\dots,n_d \\ n_1+\cdots+n_d=n}} \prod_{j=1}^{d} \frac{\Gamma\left(n_j + \frac{1}{2}\right)}{\sqrt{\pi} n_j!} \left(\frac{1}{(\varepsilon + t^{2H_{1,j}} + s^{2H_{2,j}})(\varepsilon + t'^{2H_{1,j}} + s'^{2H_{2,j}})} \right)^{n_j}$$
$$\cdot \left(\frac{1}{2} \left(t^{2H_{1,j}} + t'^{2H_{1,j}} - |t - t'|^{2H_{1,j}} + s^{2H_{2,j}} + s'^{2H_{2,j}} - |s - s'|^{2H_{2,j}} \right) \right)^{2n_j}$$
$$= \prod_{j=1}^{d} \frac{1}{\sqrt{(\varepsilon + t^{2H_{1,j}} + s^{2H_{2,j}})(\varepsilon + t'^{2H_{1,j}} + s'^{2H_{2,j}})}}$$
$$\cdot \sum_{n=0}^{\infty} \sum_{\substack{n_1,\dots,n_d \\ n_1+\cdots+n_d=n}} \prod_{j=1}^{d} \frac{1}{n_j!} \left(\prod_{i=0}^{n_j-1} \left(\frac{1}{2} + i \right) \right)$$
$$\cdot \prod_{j=1}^{d} \frac{(t^{2H_{1,j}} + t'^{2H_{1,j}} - |t - t'|^{2H_{1,j}} + s^{2H_{2,j}} + s'^{2H_{2,j}} - |s - s'|^{2H_{2,j}})^{2n_j}}{4^{n_j} (\varepsilon + t^{2H_{1,j}} + s^{2H_{2,j}})^{n_j} (\varepsilon + t'^{2H_{1,j}} + s'^{2H_{2,j}})^{n_j}}. \tag{10}$$

Concerning the sum in (10), observe that it is equal to

$$\sum_{n=0}^{\infty} \sum_{\substack{n_1,\dots,n_d \\ n_1+\cdots+n_d=n}} \prod_{j=1}^{d} \frac{1}{n_j!} \left(\prod_{i=0}^{n_j-1} \left(-\frac{1}{2} - i \right) \right)$$
$$\cdot \prod_{j=1}^{d} (-1)^{n_j} \frac{(t^{2H_{1,j}} + t'^{2H_{1,j}} - |t - t'|^{2H_{1,j}} + s^{2H_{2,j}} + s'^{2H_{2,j}} - |s - s'|^{2H_{2,j}})^{2n_j}}{4^{n_j} (\varepsilon + t^{2H_{1,j}} + s^{2H_{2,j}})^{n_j} (\varepsilon + t'^{2H_{1,j}} + s'^{2H_{2,j}})^{n_j}}$$
$$= \prod_{j=1}^{d} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \prod_{i=0}^{n-1} \left(-\frac{1}{2} - i \right) \right.$$
$$\left. \cdot \frac{(t^{2H_{1,j}} + t'^{2H_{1,j}} - |t - t'|^{2H_{1,j}} + s^{2H_{2,j}} + s'^{2H_{2,j}} - |s - s'|^{2H_{2,j}})^{2n}}{4^{n} (\varepsilon + t^{2H_{1,j}} + s^{2H_{2,j}})^{n} (\varepsilon + t'^{2H_{1,j}} + s'^{2H_{2,j}})^{n}} \right\}. \tag{11}$$

Hence, taking into account that for any $0 \leq t, t', s, s' \leq T$ and for any $j = 1, \dots, d$

$$0 \leq \frac{(t^{2H_{1,j}} + t'^{2H_{1,j}} - |t - t'|^{2H_{1,j}} + s^{2H_{2,j}} + s'^{2H_{2,j}} - |s - s'|^{2H_{2,j}})^2}{4(\varepsilon + t^{2H_{1,j}} + s^{2H_{2,j}})(\varepsilon + t'^{2H_{1,j}} + s'^{2H_{2,j}})} < 1,$$

one recognizes that each sum in (11) is the Taylor expansion of the function

$$\left(\sqrt{1 - \frac{(t^{2H_{1,j}} + t'^{2H_{1,j}} - |t - t'|^{2H_{1,j}} + s^{2H_{2,j}} + s'^{2H_{2,j}} - |s - s'|^{2H_{2,j}})^2}{4(\varepsilon + t^{2H_{1,j}} + s^{2H_{2,j}})(\varepsilon + t'^{2H_{1,j}} + s'^{2H_{2,j}})}} \right)^{-1},$$

and thus the sum in (10) is given by

$$\prod_{j=1}^{d}\left(\sqrt{1-\frac{(t^{2H_{1,j}}+t'^{2H_{1,j}}-|t-t'|^{2H_{1,j}}+s^{2H_{2,j}}+s'^{2H_{2,j}}-|s-s'|^{2H_{2,j}})^2}{4(\varepsilon+t^{2H_{1,j}}+s^{2H_{2,j}})(\varepsilon+t'^{2H_{1,j}}+s'^{2H_{2,j}})}}\right)^{-1}.$$

As a result,

$$\begin{aligned}&\sum_{\mathbf{m}}\sum_{\mathbf{k}}\mathbf{m}!\mathbf{k}!\left|F_{H_1,H_2,\varepsilon,\mathbf{m},\mathbf{k}}\right|^2_{(L^2_{2d}(\mathbb{R}))^{\otimes(m+k)}}\\&=\left(\frac{1}{2\pi}\right)^d\int_0^T dt\int_0^T ds\int_0^T dt'\int_0^T ds'\prod_{j=1}^{d}\Big((\varepsilon+t^{2H_{1,j}}+s^{2H_{2,j}})(\varepsilon+t'^{2H_{1,j}}+s'^{2H_{2,j}})\\&\quad-\frac{1}{4}\left(t^{2H_{1,j}}+t'^{2H_{1,j}}-|t-t'|^{2H_{1,j}}+s^{2H_{2,j}}+s'^{2H_{2,j}}-|s-s'|^{2H_{2,j}}\right)^2\Big)^{-\frac{1}{2}}.\end{aligned}$$

Independently of the dimension d and the Hurst multiparameters under consideration, clearly such a multiple integral is always finite.

Concerning the second sum in (9), first we note that the expression of the kernels $F_{H_1,H_2,\mathbf{m},\mathbf{k}}$ coincides with $F_{H_1,H_2,\varepsilon,\mathbf{m},\mathbf{k}}$ for $\varepsilon=0$. Thus, one may apply the previous scheme, just replacing ε by zero, with the slight difference that in this case one has

$$0<\frac{(t^{2H_{1,j}}+t'^{2H_{1,j}}-|t-t'|^{2H_{1,j}}+s^{2H_{2,j}}+s'^{2H_{2,j}}-|s-s'|^{2H_{2,j}})^2}{4(t^{2H_{1,j}}+s^{2H_{2,j}})(t'^{2H_{1,j}}+s'^{2H_{2,j}})}<1,$$

only for $0<t,t',s,s'\leq T$ such that $t\neq t'$ and $s\neq s'$. Thus, only for $0<t,t',s,s'\leq T$ such that $t\neq t'$ and $s\neq s'$ the sum corresponding to the sum in (10) converges. As a result, in this case we obtain

$$\begin{aligned}&\sum_{\mathbf{m}}\sum_{\mathbf{k}}\mathbf{m}!\mathbf{k}!\left|F_{H_1,H_2,\mathbf{m},\mathbf{k}}\right|^2_{(L^2_{2d}(\mathbb{R}))^{\otimes(m+k)}}\\&=4\left(\frac{1}{2\pi}\right)^d\int_0^T dt\int_0^t dt'\int_0^T ds\int_0^s ds'\prod_{j=1}^{d}\Big((t^{2H_{1,j}}+s^{2H_{2,j}})(t'^{2H_{1,j}}+s'^{2H_{2,j}})\\&\quad-\frac{1}{4}\left(t^{2H_{1,j}}+t'^{2H_{1,j}}-(t-t')^{2H_{1,j}}+s^{2H_{2,j}}+s'^{2H_{2,j}}-(s-s')^{2H_{2,j}}\right)^2\Big)^{-\frac{1}{2}}.\end{aligned}$$

Due to the existence of singular points, an additional analysis is now needed in order to show the convergence of this multiple integral. As before, it is enough to consider the case $T=1$.

As a first step we use the fact that for each $j=1,\dots,d$ fixed one has

$$\begin{aligned}&(t^{2H_{1,j}}+s^{2H_{2,j}})(t'^{2H_{1,j}}+s'^{2H_{2,j}}) &(12)\\&\quad-\frac{1}{4}\left(t^{2H_{1,j}}+t'^{2H_{1,j}}-|t-t'|^{2H_{1,j}}+s^{2H_{2,j}}+s'^{2H_{2,j}}-|s-s'|^{2H_{2,j}}\right)^2\\&\geq t^{2H_{1,j}}t'^{2H_{1,j}}-\frac{1}{4}\left(t^{2H_{1,j}}+t'^{2H_{1,j}}-|t-t'|^{2H_{1,j}}\right)^2 &(13)\end{aligned}$$

$$+ s^{2H_{2,j}} s'^{2H_{2,j}} - \frac{1}{4}\left(s^{2H_{2,j}} + s'^{2H_{2,j}} - |s-s'|^{2H_{2,j}}\right)^2, \tag{14}$$

with the advantage that, in contrast to (12), (13) as well as (14) only depend of a unique Hurst parameter. Moreover, (13) and (14) are both of the type

$$\varphi_H(u,v) := u^{2H} v^{2H} - \frac{1}{4}\left(u^{2H} + v^{2H} - |u-v|^{2H}\right)^2,$$

which, as a function of u and v, is an homogeneous function of order $4H$. Therefore, for every $0 < v < u < 1$ one has

$$\begin{aligned} &u^{2H} v^{2H} - \frac{1}{4}\left(u^{2H} + v^{2H} - (u-v)^{2H}\right)^2 \\ &= u^{4H}\left[\left(\frac{v}{u}\right)^{2H} - \frac{1}{4}\left(1 + \left(\frac{v}{u}\right)^{2H} - \left(1 - \frac{v}{u}\right)^{2H}\right)^2\right], \end{aligned}$$

where, for $v/u \in (0,1)$ fixed, the expression between the square brackets is a decreasing function of $H \in (0,1)$ (Appendix B).

As a consequence,

$$\begin{aligned} &\sum_{\mathbf{m}} \sum_{\mathbf{k}} \mathbf{m}!\mathbf{k}! \left|F_{H_1,H_2,\mathbf{m},\mathbf{k}}\right|^2_{(L^2_{2d}(\mathbb{R}))^{\otimes(m+k)}} \\ &\leq 4\left(\frac{1}{2\pi}\right)^d \int_0^1 dt \int_0^t dt' \int_0^1 ds \int_0^s ds' \left(t^{2\bar{H}_1} t'^{2\bar{H}_1} - \frac{1}{4}\left(t^{2\bar{H}_1} + t'^{2\bar{H}_1} - (t-t')^{2\bar{H}_1}\right)^2\right. \\ &\quad \left. + s^{2\bar{H}_2} s'^{2\bar{H}_2} - \frac{1}{4}\left(s^{2\bar{H}_2} + s'^{2\bar{H}_2} - (s-s')^{2\bar{H}_2}\right)^2\right)^{-\frac{d}{2}}. \end{aligned} \tag{15}$$

Now the proof follows closely the one in [15, Proof of Lemma 4], based on the fact that

$$\lambda^{-\frac{d}{2}} = \frac{1}{\Gamma(\frac{d}{2})} \int_0^{+\infty} dz\, e^{-\lambda z} z^{\frac{d}{2}-1},$$

which allows to rewrite the multiple integral in (15) as

$$\frac{1}{\Gamma(\frac{d}{2})} \int_0^{+\infty} dz\, z^{\frac{d}{2}-1} \left(\int_0^1 dt \int_0^t ds\, e^{-z\varphi_{\bar{H}_1}(t,s)}\right)\left(\int_0^1 dt \int_0^t ds\, e^{-z\varphi_{\bar{H}_2}(t,s)}\right). \tag{16}$$

Since

$$\forall z \in [0,1], \qquad \int_0^1 dt \int_0^t ds\, e^{-z\varphi_{\bar{H}_i}(t,s)} < +\infty, \quad i = 1,2,$$

the convergence of the integral (16) then will follow from the convergence of the integral

$$\int_1^{+\infty} dz\, z^{\frac{d}{2}-1} \left(\int_0^1 dt \int_0^t ds\, e^{-z\varphi_{\bar{H}_1}(t,s)}\right)\left(\int_0^1 dt \int_0^t ds\, e^{-z\varphi_{\bar{H}_2}(t,s)}\right).$$

As in [15, Proof of Lemma 4], the homogeneity property of $\varphi_{\bar{H}_1}$ and $\varphi_{\bar{H}_2}$ yields

$$\int_1^{+\infty} dz\, z^{\frac{d}{2}-1} \left(\int_0^1 dt \int_0^t ds\, e^{-z\varphi_{\bar{H}_1}(t,s)}\right)\left(\int_0^1 dt \int_0^t ds\, e^{-z\varphi_{\bar{H}_2}(t,s)}\right)$$
$$= \int_1^{+\infty} dz\, z^{\frac{d}{2}-1-\frac{1}{2\bar{H}_1}-\frac{1}{2\bar{H}_2}} \left(\int_0^{z^{\frac{1}{4\bar{H}_1}}} dx \int_0^x dy\, e^{-\varphi_{\bar{H}_1}(x,y)}\right)$$
$$\times \left(\int_0^{z^{\frac{1}{4\bar{H}_2}}} dx \int_0^x dy\, e^{-\varphi_{\bar{H}_2}(x,y)}\right),$$

where a double change of coordinates leads for each $i = 1, 2$ to

$$\int_0^{z^{\frac{1}{4\bar{H}_i}}} dx \int_0^x dy\, e^{-\varphi_{\bar{H}_i}(x,y)}$$
$$\le \frac{1}{4\bar{H}_i} \int_0^{\pi/4} d\theta\, (\varphi_{\bar{H}_i}(\cos\theta, \sin\theta))^{-\frac{1}{2\bar{H}_i}} \gamma\left(\frac{1}{2\bar{H}_i}, 2^{2\bar{H}_i} z\varphi_{\bar{H}_i}(\cos\theta, \sin\theta)\right).$$

Here γ is the lower incomplete gamma function, that is,

$$\gamma(\alpha, x) := \int_0^x dy\, e^{-y} y^{\alpha-1}, \quad \alpha > 0,$$

which, as shown in [15, Lemma 2], is bounded by

$$\gamma(\alpha, x) \le K(\alpha)x^\epsilon, \quad K(\alpha) := \max\left\{\frac{1}{\alpha}, \Gamma(\alpha)\right\},$$

for all $x > 0$ and for every $0 < \epsilon < \alpha$.

Hence, for all $0 < \epsilon < \frac{1}{2\max\{\bar{H}_1,\bar{H}_2\}}$ one finally obtains

$$\int_1^{+\infty} dz\, z^{\frac{d}{2}-1} \left(\int_0^1 dt \int_0^t ds\, e^{-z\varphi_{\bar{H}_1}(t,s)}\right)\left(\int_0^1 dt \int_0^t ds\, e^{-z\varphi_{\bar{H}_2}(t,s)}\right)$$
$$\le \frac{2^{2\epsilon(\bar{H}_1+\bar{H}_2)}}{16\bar{H}_1\bar{H}_2} K\left(\frac{1}{2\bar{H}_1}\right) K\left(\frac{1}{2\bar{H}_2}\right) \int_1^{+\infty} dz\, z^{\frac{d}{2}-1-\frac{1}{2\bar{H}_1}-\frac{1}{2\bar{H}_2}+2\epsilon}$$
$$\cdot \left(\int_0^{\pi/4} d\theta\, (\varphi_{\bar{H}_1}(\cos\theta, \sin\theta))^{\epsilon-\frac{1}{2\bar{H}_1}}\right)\left(\int_0^{\pi/4} d\theta\, (\varphi_{\bar{H}_2}(\cos\theta, \sin\theta))^{\epsilon-\frac{1}{2\bar{H}_2}}\right). \quad (17)$$

Concerning the integral in z, clearly it converges provided $\epsilon < \frac{\bar{H}_1+\bar{H}_2-d\bar{H}_1\bar{H}_2}{4\bar{H}_1\bar{H}_2}$, being $\frac{\bar{H}_1+\bar{H}_2-d\bar{H}_1\bar{H}_2}{4\bar{H}_1\bar{H}_2}$ always a positive number, because $d < 1/\bar{H}_1 + 1/\bar{H}_2$. These facts combined mean that in (17) one shall fix a

$$0 < \epsilon < \min\left\{\frac{1}{2\max\{\bar{H}_1, \bar{H}_2\}}, \frac{\bar{H}_1 + \bar{H}_2 - d\bar{H}_1\bar{H}_2}{4\bar{H}_1\bar{H}_2}\right\}.$$

For such a ϵ fixed, also both integrals in θ converge, cf. [15, Proof of Lemma 4], and thus (17) converges.

In this way we have shown that, on the one hand, $L_{H_1,H_2} \in (L^2)$, and, on the other hand, that one may apply a Lebesgue dominated convergence argument to infer the convergence in ε of $\|L_{H_1,H_2,\varepsilon}\|^2$ to $\|L_{H_1,H_2}\|^2$. □

Remark 13 Under the conditions of Theorem 12, one has

$$\min\left\{\frac{1}{2\max\{\bar{H}_1,\bar{H}_2\}}, \frac{\bar{H}_1+\bar{H}_2-d\bar{H}_1\bar{H}_2}{4\bar{H}_1\bar{H}_2}\right\} = \frac{\bar{H}_1+\bar{H}_2-d\bar{H}_1\bar{H}_2}{4\bar{H}_1\bar{H}_2}$$

whenever $d \geq \frac{1}{\min\{\bar{H}_1,\bar{H}_2\}} - \frac{1}{\max\{\bar{H}_1,\bar{H}_2\}}$, while for $d < \frac{1}{\min\{\bar{H}_1,\bar{H}_2\}} - \frac{1}{\max\{\bar{H}_1,\bar{H}_2\}}$,

$$\min\left\{\frac{1}{2\max\{\bar{H}_1,\bar{H}_2\}}, \frac{\bar{H}_1+\bar{H}_2-d\bar{H}_1\bar{H}_2}{4\bar{H}_1\bar{H}_2}\right\} = \frac{1}{2\max\{\bar{H}_1,\bar{H}_2\}}.$$

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Appendix A

Lemma 14 *Let* $0 < H_1 < H_2 < 1$ *be given. The integral*

$$\int_0^1 dt \int_0^1 ds\, \frac{(t^2+s^2)^N}{(t^{2H_1}+s^{2H_2})^{N+\frac{d}{2}}}$$

is finite if and only if $2H_2(N+\frac{d}{2}) < 1 + 2N + \frac{H_2}{H_1}$.

Proof Through the change of variables $t = u^{\frac{H_2}{H_1}}$ one obtains

$$\begin{aligned}
&\int_0^1 dt \int_0^1 ds\, \frac{(t^2+s^2)^N}{(t^{2H_1}+s^{2H_2})^{N+\frac{d}{2}}}\\
&\quad = \frac{H_2}{H_1}\sum_{n=0}^{N}\binom{N}{n}\int_0^1 du \int_0^1 ds\, \frac{u^{2n\frac{H_2}{H_1}} s^{2(N-n)}}{(u^{2H_2}+s^{2H_2})^{N+\frac{d}{2}}} u^{\frac{H_2}{H_1}-1},
\end{aligned}\tag{18}$$

where each double integral appearing in (18) is finite if and only if the integrand function is integrable on the unit ball $B_1(0) \subset \mathbb{R}^2$. For each $n = 0, 1, \ldots, N$ a polar change of coordinates yields

$$\begin{aligned}
&\int_{B_1(0)} duds\, \frac{u^{2n\frac{H_2}{H_1}} s^{2(N-n)}}{(u^{2H_2}+s^{2H_2})^{N+\frac{d}{2}}} u^{\frac{H_2}{H_1}-1}\\
&\quad = \int_0^{2\pi} d\theta\, \frac{\cos^{(2n+1)\frac{H_2}{H_1}-1}\theta \cdot \sin^{2(N-n)}\theta}{(\cos^{2H_2}\theta+\sin^{2H_2}\theta)^{N+\frac{d}{2}}} \int_0^1 dr\, \frac{1}{r^{2H_2(N+\frac{d}{2})-(2n+1)\frac{H_2}{H_1}-2(N-n)}},
\end{aligned}$$

which is finite if and only if the integral in r converges, that is, if and only if $2H_2(N+\frac{d}{2}) - (2n+1)\frac{H_2}{H_1} - 2(N-n) < 1$. This shows that a necessary and sufficient condition for the convergence of the sum in (18) is given by $2H_2(N+\frac{d}{2}) - 2N - \frac{H_2}{H_1} < 1$. □

Lemma 15 *Given $H_i = (H_{i,1}, \dots, H_{i,d}) \in (0,1)^d$, $i = 1, 2$, assume that $\bar{H}_1 < \bar{H}_2$. Then*

$$\int_0^1 dt \int_0^1 ds \prod_{j=1}^{d} \frac{1}{\sqrt{t^{2H_{1,j}} + s^{2H_{2,j}}}} \left(\frac{t^2 + s^2}{t^{2\bar{H}_1} + s^{2\bar{H}_2}} \right)^N < \infty$$

whenever $2\bar{H}_2(N+\frac{d}{2}) < 1 + 2N + \frac{\bar{H}_2}{\bar{H}_1}$.

Proof Since in $[0,1]^2$ the following inequality holds

$$\prod_{j=1}^{d} \frac{1}{\sqrt{t^{2H_{1,j}} + s^{2H_{2,j}}}} \left(\frac{t^2 + s^2}{t^{2\bar{H}_1} + s^{2\bar{H}_2}} \right)^N \le \frac{(t^2+s^2)^N}{(t^{2\bar{H}_1} + s^{2\bar{H}_2})^{N+\frac{d}{2}}},$$

the proof reduces to an application of the previous lemma. □

Appendix B

For $v/u \in (0,1)$ fixed, let $a := v/u$ and

$$f(H) = f_a(H) := a^{2H} - \frac{1}{4}\left(1 + a^{2H} - (1-a)^{2H}\right)^2, \quad H \in (0,1) \tag{19}$$

with derivative function given by

$$f'(H) = 2a^{2H}\ln a - \left(a^{2H} + 1 - (1-a)^{2H}\right)\left(a^{2H}\ln a - (1-a)^{2H}\ln(1-a)\right).$$

Since $0 < (1-a)^{2H} < 1$, one has $a^{2H} + 1 - (1-a)^{2H} > 0$, $(1-a)^{2H}\ln(1-a) < 0$, and thus

$$\begin{aligned} f'(H) &\le 2a^{2H}\ln a - a^{2H}\ln a\left(a^{2H} + 1 - (1-a)^{2H}\right) \\ &= \left(2a^{2H} - a^{2H}\left(a^{2H} + 1 - (1-a)^{2H}\right)\right)\ln a, \end{aligned}$$

where $\ln a < 0$ and the expression between brackets is positive:

$$\begin{aligned} 2a^{2H} - a^{2H}\left(a^{2H} + 1 - (1-a)^{2H}\right) &= \left(1 + (1-a)^{2H}\right)a^{2H} - a^{4H} \\ &> \left(1 + (1-a)^{2H}\right)a^{4H} - a^{4H} \\ &= (1-a)^{2H}a^{4H} > 0. \end{aligned}$$

In other words, (19) defines a decreasing function of $H \in (0,1)$.

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