Stability of Cubic and Quartic Functional Equations in Non-Archimedean Spaces

M. Eshaghi Gordji · M.B. Savadkouhi

Abstract We prove generalized Hyers-Ulam–Rassias stability of the cubic functional equation $f(kx + y) + f(kx - y) = k[f(x + y) + f(x - y)] + 2(k^3 - k)f(x)$ for all $k \in \mathbb{N}$ and the quartic functional equation $f(kx + y) + f(kx - y) = k^2[f(x + y) + f(x - y)] + 2k^2(k^2 - 1)f(x) - 2(k^2 - 1)f(y)$ for all $k \in \mathbb{N}$ in non-Archimedean normed spaces.

Keywords Generalized Hyers–Ulam–Rassias stability · Cubic functional equation · Quartic functional equation · Non-Archimedean space · p-adic

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1 Introduction

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation ϵ must be close to an exact solution of ϵ?"

If the problem accepts a solution, we say that the equation ϵ is stable. The first stability problem concerning group homomorphisms was raised by Ulam [28] in 1940.

We are given a group G and a metric group G' with metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $d(f(x), h(x)) < \epsilon$ for all $x \in G$?

Ulam's problem was partially solved by Hyers [13] in 1941. Let E_1 be a normed space, E_2 a Banach space and suppose that the mapping $f : E_1 \to E_2$ satisfies the inequality

$$\|f(x + y) - f(x) - f(y)\| \le \epsilon \quad (x, y \in E_1),$$

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where $\epsilon > 0$ is a constant. Then the limit $T(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ exists for each $x \in E_1$ and T is the unique additive mapping satisfying

$$\|f(x) - T(x)\| \le \epsilon \tag{1.1}$$

for all $x \in E_1$. Also, if for each x the function $t \longmapsto f(tx)$ from $\mathbb{R}$ to E_2 is continuous on $\mathbb{R}$, then T is linear. If f is continuous at a single point of E_1, then T is continuous everywhere in E_1. Moreover (1.1) is sharp.

In 1978, Th.M. Rassias [23] formulated and proved the following theorem, which implies Hyers' theorem as a special case. Suppose that E and F are real normed spaces with F a complete normed space, $f : E \to F$ is a mapping such that for each fixed $x \in E$ the mapping $t \longmapsto f(tx)$ is continuous on $\mathbb{R}$, and let there exist $\epsilon > 0$ and $p \in [0, 1)$ such that

$$\|f(x + y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.2}$$

for all $x, y \in E$. Then there exists a unique linear mapping $T : E \to F$ such that

$$\|f(x) - T(x)\| \le \frac{\epsilon \|x\|^p}{(1 - 2^{p-1})}$$

for all $x \in E$. The case of the existence of a unique additive mapping had been obtained by T. The terminology Hyers–Ulam stability originates from these historical backgrounds. The terminology can also be applied to the case of other functional equations. For more detailed definitions of such terminologies, we can refer to [8, 11, 14] and [15]. In 1994, P. Găvruta [9] provided a further generalization of Th.M. Rassias' theorem in which he replaced the bound $\epsilon(\|x\|^p + \|y\|^p)$ in (1.2) by a general control function $\varphi(x, y)$ for the existence of a unique linear mapping.

The functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping, see [24, 25]. A generalized Hyers–Ulam stability problem for the quadratic functional equation was proved by Skof [27] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. In [4], Czerwik proved the generalized Hyers–Ulam stability of the quadratic functional equation. Borelli and Forti [2] generalized the stability result as follows (cf. [20, 21]): Let G be an Abelian group, and X a Banach space. Assume that a mapping $f : G \to X$ satisfies the functional inequality

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \le \varphi(x, y)$$

for all $x, y \in G$, and $\varphi : G \times G \to [0, \infty)$ is a function such that

$$\Phi(x, y) := \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} \varphi(2^i x, 2^i y) < \infty$$

for all $x, y \in G$. Then there exists a unique quadratic mapping $Q : G \to X$ with the property $\|f(x) - Q(x)\| \le \Phi(x, x)$ for all $x \in G$.

Jun and Kim [16] introduced the following cubic functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x) \tag{1.3}$$

and they established the general solution and the generalized Hyers–Ulam–Rassias stability
for the functional equation (1.3). The function $f(x) = x^3$ satisfies the functional equation
(1.3), which is thus called a cubic functional equation. Every solution of the cubic functional
equation is said to be a cubic function. Now we introduce the following cubic functional
equation

$$f(kx + y) + f(kx - y) = k[f(x + y) + f(x - y)] + 2(k^3 - k)f(x) \qquad (1.4)$$

for all $k \in \mathbb{N}$.

In [22], Won-Gil Park and Jae-Hyeong Bae, introduced the following functional equation

$$f(x + 2y) + f(x - 2y) = 4[f(x + y) + f(x - y)] + 24f(y) - 6f(x) \qquad (1.5)$$

and they established the general solution of the functional equation (1.5). It is easy to see
that the function $f(x) = cx^4$ is a solution of the functional equation (1.5) for fixed $c \in \mathbb{R}$.
Thus, it is natural that (1.5) is called a quartic functional equation and every solution of the
quartic functional equation is said to be a quartic mapping. In this paper, we introduce the
following quartic functional equation

$$f(kx + y) + f(kx - y) = k^2[f(x + y) + f(x - y)] + 2k^2(k^2 - 1)f(x) - 2(k^2 - 1)f(y) \qquad (1.6)$$

for all $k \in \mathbb{N}$. For more detailed definitions of such terminologies, we can refer to [5–7, 19].
By a non-Archimedean field we mean a field K equipped with a function (valuation) $|.|$ from
K into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r + s| \leq \max\{|r|, |s|\}$
for all $r, s \in K$. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in N$.

Definition 1.1 Let X be a vector space over a scalar field K with a non-Archimedean non-
trivial valuation $|.|$. A function $\|.\| : X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it
satisfies the following conditions:

 (i) $\|x\| = 0$ if and only if $x = 0$;
 (ii) $\|rx\| = |r|\|x\|$ for all $r \in K$, $x \in X$;
(iii) the strong triangle inequality (ultrametric); namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}$$

for all $x, y \in X$. Then $(X, \|.\|)$ is called a non-Archimedean space.

Due to the fact that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n - 1\} \quad (n > m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean
space. By a complete non-Archimedean space we mean one in which every Cauchy se-
quence is convergent.

In 1897, Hensel [12] discovered the p-adic numbers as a number theoretical analogue of
power series in complex analysis. Fix a prime number p. For any nonzero rational number
x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b} p^{n_x}$, where a and b are integers not
divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion
of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$ which is called the p-adic

number field. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k \geq n_x}^{\infty} a_k p^k$, where $|a_k| \leq p - 1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $|\sum_{k \geq n_x}^{\infty} a_k p^k|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact field; see [10, 26].

During the last three decades p-adic numbers have gained the interest of physicists for their research in particular in problems coming from quantum physics, p-adic strings and superstrings (cf. [17]). A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: "for $x, y > 0$, there exists $n \in \mathbb{N}$ such that $x < ny$." It is very difficult to imagine a situation where this axiom does not hold, but in fact the very space and time we inhabit have both been shown by 20th century science to be unequivocally non-Archimedean: The Archimedean axiom breaks down at the Planck scale, that is, for distances less than 1.6×10^{-33} metres and durations less than 5.4×10^{-44} seconds. Despite our entrenched belief that space and time are continuous, homogeneous, infinitely divisible quantities, we are now confronted with the fact that below this scale, distances and durations cannot scaled up in order to produce macroscopic distances and durations. Equivalently, we cannot meaningfully measure distances or durations below this scale. So a suggestion emerges to abandon the Archimedean axiom at very small distances. This leads to a non-Euclidean and non-Riemannian geometry of space at small distances; cf. [29].

In [1], the authors investigated stability of approximate additive mappings $f : \mathbb{Q}_p \to \mathbb{R}$. They showed that if $f : \mathbb{Q}_p \to \mathbb{R}$ is a continuous mapping for which there exists a fixed ϵ such that $|f(x + y) - f(x) - f(y)| \leq \epsilon$ for all $x, y \in Q_p$, then there exists a unique additive mapping $T : \mathbb{Q}_p \to \mathbb{R}$ such that $|f(x) - T(x)| \leq \epsilon$ for all $x \in \mathbb{Q}_p$.

M.S. Moslehian and Th.M. Rassias [18] proved the generalized Hyers–Ulam stability of the Cauchy functional equation and the quadratic functional equation in non-Archimedean normed spaces.

In this paper, we solve the stability problem for cubic and quartic functional equations when the unknown function is one with values in a non-Archimedean space, in particular in the field of p-adic numbers.

2 Stability of the Cubic Functional Equation

In this section, we prove the generalized Hyers–Ulam–Rassias stability of the cubic functional equation (1.4). Throughout this section, we assume that G is an additive group and X is a complete non-Archimedean space. Now before taking up the main subject, given $f : G \to X$, we define the difference operator

$$(Df)(x, y) = f(kx + y) + f(kx - y) - k[f(x + y) + f(x - y)] - 2(k^3 - k)f(x)$$

for all $x, y \in G$ and all $k \in \mathbb{N}$. We consider the following functional inequality:

$$\|(Df)(x, y)\| \leq \varphi(x, y)$$

for an upper bound $\varphi : G \times G \to [0, \infty)$.

Theorem 2.1 *Let $\varphi : G \times G \to [0, \infty)$ be a function such that*

$$\lim_{n \to \infty} \frac{\varphi(k^n x, k^n y)}{|k|^{3n}} = 0 \tag{2.1}$$

for all $x, y \in G$, $k \in \mathbb{N}$ *and let for each* $x \in G$ *the limit*

$$\lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^{3j}} : 0 \le j < n\right\}, \tag{2.2}$$

denoted by $\tilde{\varphi}(x)$, *exists. Suppose that* $f : G \to X$ *is a mapping satisfying*

$$\|(Df)(x, y)\| \le \varphi(x, y) \tag{2.3}$$

for all $x, y \in G$. *Then there exists a cubic mapping* $C : G \to X$ *such that*

$$\|C(x) - f(x)\| \le \frac{1}{|2k^3|} \tilde{\varphi}(x) \tag{2.4}$$

for all $x \in G$, *if*

$$\lim_{i\to\infty} \lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^{3j}} : i \le j < n + i\right\} = 0$$

then C *is the unique cubic mapping satisfying* (2.4).

Proof Putting $y = 0$ in (2.3), we get

$$\left\|\frac{1}{k^3} f(kx) - f(x)\right\| \le \frac{1}{|2k^3|} \varphi(x, 0) \tag{2.5}$$

for all $x \in G$ and all $k \in \mathbb{N}$. Replacing x by $k^{n-1}x$ in (2.5), we get

$$\left\|\frac{1}{k^{3n}} f(k^n x) - \frac{1}{k^{3(n-1)}} f(k^{n-1}x)\right\| \le \frac{\varphi(k^{n-1}x, 0)}{|2k^{3n}|} \tag{2.6}$$

for all $x \in G$ and all $k \in \mathbb{N}$. It follows from (2.6) and (2.1) that the sequence $\{\frac{f(k^n x)}{k^{3n}}\}$ is Cauchy. Since X is complete, we conclude that $\{\frac{f(k^n x)}{k^{3n}}\}$ is convergent. Set $C(x) := \lim_{n\to\infty} \frac{f(k^n x)}{k^{3n}}$. Using induction one can show that

$$\left\|\frac{f(k^n x)}{k^{3n}} - f(x)\right\| \le \frac{1}{|2k^3|} \max\left\{\frac{\varphi(k^i x, 0)}{|k|^{3i}} : 0 \le i < n\right\} \tag{2.7}$$

for all $n \in \mathbb{N}$ and all $x \in G$. By taking n to approach infinity in (2.7) and using (2.2), we obtain (2.4). By (2.1) and (2.3), we get

$$\|(DC)(x, y)\| = \lim_{n\to\infty} \frac{1}{|k^{3n}|} \|(Df)(k^n x, k^n y)\| \le \lim_{n\to\infty} \frac{\varphi(k^n x, k^n y)}{|k|^{3n}} = 0$$

for all $x, y \in G$ and all $k \in \mathbb{N}$. Therefore the mapping $C : G \to X$ satisfies (1.4). If C' is another cubic mapping satisfying (2.4), then

$$\|C(x) - C'(x)\| = \lim_{i\to\infty} |k|^{-3i} \|C(k^i x) - C'(k^i x)\|$$

$$\le \lim_{i\to\infty} |k|^{-3i} \max\{\|C(k^i x) - f(k^i x)\|, \|f(k^i x) - C'(k^i x)\|\}$$

$$\leq \frac{1}{|2k^3|} \lim_{i\to\infty} \lim_{n\to\infty} \max\left\{ \frac{\varphi(k^j x, 0)}{|k|^{3j}} : i \leq j < n+i \right\}$$

$$= 0$$

for all $x \in G$ and all $k \in \mathbb{N}$. Therefore $C = C'$. This completes the proof of the uniqueness of C. □

Corollary 2.2 *Let $\alpha : [0, \infty) \to [0, \infty)$ be a function satisfying*

(i) $\alpha(|k|t) \leq \alpha(|k|)\alpha(t)$ *for all $t \geq 0$,*
(ii) $\alpha(|k|) < |k|^3$.

Let $\delta > 0$, let G be a normed space and let $f : G \to X$ fulfill the inequality

$$\|(Df)(x, y)\| \leq \delta[\alpha(\|x\|) + \alpha(\|y\|)]$$

for all $x, y \in G$. Then there exists a unique cubic mapping $C : G \to X$ such that

$$\|f(x) - C(x)\| \leq \frac{1}{|2k^3|} \delta\alpha(\|x\|) \tag{2.8}$$

for all $x \in G$ and all $k \in \mathbb{N}$.

Proof Defining $\varphi : G \times G \to [0, \infty)$ by $\varphi(x, y) := \delta[\alpha(\|x\|) + \alpha(\|y\|)]$ we have

$$\lim_{n\to\infty} \frac{\varphi(k^n x, k^n y)}{|k|^{3n}} \leq \lim_{n\to\infty} \left(\frac{\alpha(|k|)}{|k|^3} \right)^n \varphi(x, y) = 0$$

for all $x, y \in G$ and all $k \in \mathbb{N}$. We have

$$\tilde{\varphi}(x) = \lim_{n\to\infty} \max\left\{ \frac{\varphi(k^j x, 0)}{|k|^{3j}} : 0 \leq j < n \right\} = \varphi(x, 0)$$

and

$$\lim_{i\to\infty} \lim_{n\to\infty} \max\left\{ \frac{\varphi(k^j x, 0)}{|k|^{3j}} : i \leq j < n+i \right\} = \lim_{i\to\infty} \frac{\varphi(k^i x, 0)}{|k|^{3i}} = 0$$

for all $x \in G$ and all $k \in \mathbb{N}$. □

Remark 2.3 The classical example of the function α is the mapping $\alpha(t) = t^p$ for all $t \in [0, \infty)$, where $p > 3$ with the further assumption that $|k| < 1$.

Remark 2.4 We can formulate similar statements to Theorem 2.1 in which we can define the sequence $C(x) := \lim_{n\to\infty} k^{3n} f(\frac{x}{k^n})$ under suitable conditions on the function φ then obtain similar result to Corollary 2.2 for $p < 3$.

3 Stability of the Quartic Functional Equation

In this section, we prove the generalized Hyers–Ulam–Rassias stability of the quartic functional equation (1.6). Throughout this section, we assume that G is an additive group and

X is a complete non-Archimedean space. Now before taking up the main subject, given $f : G \to X$, we define the difference operator

$$(\Delta f)(x, y) = f(kx + y) + f(kx - y) - k^2[f(x + y) + f(x - y)]$$
$$- 2k^2(k^2 - 1)f(x) + 2(k^2 - 1)f(y)$$

for all $x, y \in G$ and all $k \in \mathbb{N}$. we consider the following functional inequality:

$$\|(\Delta f)(x, y)\| \leq \psi(x, y)$$

for an upper bound $\psi : G \times G \to [0, \infty)$.

Theorem 3.1 *Let $\psi : G \times G \to [0, \infty)$ be a function such that*

$$\lim_{n \to \infty} \frac{\psi(k^n x, k^n y)}{|k|^{4n}} = 0 \tag{3.1}$$

for all $x, y \in G$, $k \in \mathbb{N}$ and let for each $x \in G$ the limit

$$\lim_{n \to \infty} \max\left\{ \frac{\psi(k^j x, 0)}{|k|^{4j}} : 0 \leq j < n \right\}, \tag{3.2}$$

denoted by $\tilde{\psi}(x)$, exists. Suppose that $f : G \to X$ is a mapping satisfying $f(0) = 0$ and

$$\|(\Delta f)(x, y)\| \leq \psi(x, y) \tag{3.3}$$

for all $x, y \in G$. Then there exists a quartic mapping $Q : G \to X$ such that

$$\|Q(x) - f(x)\| \leq \frac{1}{|2k^4|} \tilde{\psi}(x) \tag{3.4}$$

for all $x \in G$, if

$$\lim_{i \to \infty} \lim_{n \to \infty} \max\left\{ \frac{\psi(k^j x, 0)}{|k|^{4j}} : i \leq j < n + i \right\} = 0$$

then Q is the unique quartic mapping satisfying (3.4).

Proof Putting $y = 0$ in (3.3) and knowing, by hypothesis, that $f(0) = 0$, we get

$$\left\| \frac{1}{k^4} f(kx) - f(x) \right\| \leq \frac{1}{|2k^4|} \psi(x, 0) \tag{3.5}$$

for all $x \in G$ and all $k \in \mathbb{N}$. Replacing x by $k^{n-1}x$ in (3.5), we get

$$\left\| \frac{1}{k^{4n}} f(k^n x) - \frac{1}{k^{4(n-1)}} f(k^{n-1}x) \right\| \leq \frac{\psi(k^{n-1}x, 0)}{|2k^{4n}|} \tag{3.6}$$

for all $x \in G$ and all $k \in \mathbb{N}$. It follows from (3.6) and (3.1) that the sequence $\{\frac{f(k^n x)}{k^{4n}}\}$ is Cauchy. Since X is complete, we conclude that $\{\frac{f(k^n x)}{k^{4n}}\}$ is convergent. Set $Q(x) :=$

$\lim_{n\to\infty} \frac{f(k^n x)}{k^{4n}}$. From the inequality (3.5) we use iterative methods and induction on n to prove our next relation:

$$\left\| \frac{f(k^n x)}{k^{4n}} - f(x) \right\| \leq \frac{1}{|2k^4|} \max\left\{ \frac{\psi(k^i x, 0)}{|k|^{4i}} : 0 \leq i < n \right\} \tag{3.7}$$

for all $n \in \mathbb{N}$ and all $x \in G$. By taking n to approach infinity in (3.7) and using (3.2), we obtain (3.4). By (3.1) and (3.3), we get

$$\|(\Delta Q)(x, y)\| = \lim_{n\to\infty} \frac{1}{|k^{4n}|} \|(\delta f)(k^n x, k^n y)\| \leq \lim_{n\to\infty} \frac{\psi(k^n x, k^n y)}{|k|^{4n}} = 0$$

for all $x, y \in G$ and all $k \in \mathbb{N}$. Therefore the mapping $Q : G \to X$ satisfies (1.6). The rest of proof is similar to the proof of Theorem 3.1. □

Corollary 3.2 *Let $\beta : [0, \infty) \to [0, \infty)$ be a function satisfying*

(i) $\beta(|k|t) \leq \beta(|k|)\beta(t)$ for all $t \geq 0$,
(ii) $\beta(|k|) < |k|^4$.

Let $\delta > 0$, let G be a normed space and let $f : G \to X$ fulfill the inequality

$$\|(\Delta f)(x, y)\| \leq \delta[\beta(\|x\|) + \beta(\|y\|)]$$

for all $x, y \in G$. Then there exists a unique quartic mapping $Q : G \to X$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{|2k^4|} \delta\beta(\|x\|) \tag{3.8}$$

for all $x \in G$ and all $k \in \mathbb{N}$.

Proof Putting $\psi(x, y) := \delta[\beta(\|x\|) + \beta(\|y\|)]$ in above theorem. We have

$$\lim_{n\to\infty} \frac{\psi(k^n x, k^n y)}{|k|^{4n}} \leq \lim_{n\to\infty} \left(\frac{\beta(|k|)}{|k|^4} \right)^n \psi(x, y) = 0$$

for all $x, y \in G$ and all $k \in \mathbb{N}$. We have

$$\tilde{\psi}(x) = \lim_{n\to\infty} \max\left\{ \frac{\psi(k^j x, 0)}{|k|^{4j}} : 0 \leq j < n \right\} = \psi(x, 0)$$

and

$$\lim_{i\to\infty} \lim_{n\to\infty} \max\left\{ \frac{\psi(k^j x, 0)}{|k|^{4j}} : i \leq j < n + i \right\} = \lim_{i\to\infty} \frac{\psi(k^i x, 0)}{|k|^{4i}} = 0$$

for all $x \in G$ and all $k \in \mathbb{N}$. □

Remark 3.3 The classical example of the function β is the mapping $\beta(t) = t^p$ for all $t \in [0, \infty)$, where $p > 4$ with the further assumption that $|k| < 1$.

Remark 3.4 We can formulate similar statements to Theorem 3.1 in which we can define the sequence $Q(x) := \lim_{n\to\infty} k^{4n} f(\frac{x}{k^n})$ under suitable conditions on the function ψ then obtain similar result to Corollary 3.2 for $\beta(t) = t^p$ with $p < 4$.

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