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On Contact Equivalence of Monge-Ampère Equations to Linear Equations with Constant Coefficients

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Abstract We solve a problem of local contact equivalence of hyperbolic and elliptic Monge-Ampère equations to linear equations with constant coefficients. We find normal forms for such equations: the telegraph equation and the Helmholtz equation.

Keywords Monge-Ampère equations · The Laplace and the Cotton invariants · Contact geometry

Mathematics Subject Classification (2000) 35J15 · 35J60 · 35L10 · 35L70 · 53D10

1 Introduction

The class of Monge-Ampère equations is distinguished from the variety of second-order partial differential equations by the property that this class is closed under contact transformations and contains quasilinear equations. This fact was known already to Sophus Lie who studied the Monge-Ampère equations using methods of contact geometry [19, 20].

The Monge-Ampère equation has the following form:

$$Av_{xx} + 2Bv_{xy} + Cv_{yy} + D(v_{xx}v_{yy} - v_{xy}^2) + E = 0, \tag{1}$$

where A, B, C, D, and E are functions of independent variables x, y, an unknown function $v = v(x, y)$, and its first partial derivatives v_x and v_y.

In 1978 Lychagin noted that the classical Monge-Ampère equation and its multidimensional analogues admit a description in terms of effective differential forms on the

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space of 1-jets of smooth functions [21, 22]. His idea was fruitful, and it generated new approach to Monge-Ampère equations. The first advantage of this approach is a reduction of the order of the jet space: we use a simpler space of 1-jets J^1M instead of the space of 2-jets J^2M where Monge-Ampère equations should be *ad hoc*, being second-order partial differential equations [27]. In this paper we follow his approach.

A problem of reducibility of hyperbolic and elliptic Monge-Ampère equations (1), where coefficients A, B, C, D, and E do not depend on the variable v (they call such equations *symplectic*), to the equations with constant coefficients was solved by Lychagin and Rubtsov in 1983 [23]. Later on Tunitskii took off this restriction and solved the problem for general equations [26]. The problem of classification of Monge-Ampère equations was also considered by Morimoto [25].

The special class of parabolic Monge-Ampère equations were considered by Blanco, Manno and Pugliese [1].

The general classification problems for symplectic Monge-Ampère equations were studied by Kruglikov [6–8] and the author [10–13].

A contact linearization problem for general hyperbolic and elliptic Monge-Ampère equations was solved by author in the series of papers [14–16].

Problems of the equivalence and classification of second-order differential equations which admit infinitesimal group of symmetries were studied in the series of papers by Ibragimov [4]. A complete system of differential invariants for linear hyperbolic equations was constructed in his paper [5].

One can find review old and new results in the theory of Monge-Ampère equations in [17].

The present paper can be considered as continuation of the paper [15]. Here we give a complete solution of the following problem for hyperbolic and elliptic Monge-Ampère equations:

Find conditions under which Monge-Ampère equations are locally contact equivalent to linear equations with constant coefficients.

In the paper we find conditions when Monge-Ampère equations can be transformed to one of the following:

$v_{xy} = k(x, y)v$;
the telegraph equation $v_{xy} = v$;
$v_{xx} + v_{xx} = k(x, y)v + f(x, y)$;
the Helmholtz equation $v_{xx} + v_{yy} = \pm v + f(x, y)$.

Note that proofs of all theorems of the paper are constructive and contain algorithms for transformation of Monge-Ampère equations to corresponding normal forms.

2 Lychagin's Approach to Monge-Ampère Equations and Laplace Forms

2.1 Nonlinear Differential Operator and Effective Differential Forms

Let J^1M be the manifold of 1-jets of functions on a 2-dimensional smooth manifold M. We denote by $\Omega^s(J^1M)$ the module of differential s-forms on J^1M.

The manifold J^1M is endowed with the natural contact structure (Cartan's distribution)

$$C : a \in J^1M \mapsto C(a) \subset T_a(J^1M)$$

given by the universal differential 1-form $U \in \Omega^1(J^1M)$ (Cartan's form), where $C(a) = \ker U_a$.

In the local canonical coordinates $(q, u, p) = (q_1, q_2, u, p_1, p_2)$ on J^1M the Cartan form can be written as follows:

$$U = du - p_1 dq_1 - p_2 dq_2.$$

With any differential 2-form ω on J^1M, we associate a differential operator $\Delta_\omega : C^\infty(M) \to \Omega^2(M)$, which acts as

$$\Delta_\omega(v) = \omega|_{j_1(v)(M)}.$$

Here $v \in C^\infty(M)$ and $j_1(v)(M) \subset J^1M$ is the graph of the 1-jet of v (see [21]).

The equation $E_\omega = \{\Delta_\omega(v) = 0\} \subset J^2M$ has the form of (1).

The constructed map "differential 2-forms" → "differential operators" is not a 1-to-1 map. In order to make it 1-to-1 we should restrict a class of differential 2-forms and consider only effective differential forms.

Differential 2-forms that vanish on any integral manifold of the Cartan distribution form a submodule I^2 of the module $\Omega^2(J^1M)$. Such forms produce zero differential operators.

Elements of the quotient module

$$\Omega^2_\varepsilon = \Omega^2(J^1M)/I^2$$

are called *effective* 2-forms.

Denote the equivalence class corresponding to the form $\omega \in \Omega^2(J^1M)$ by $\omega_\varepsilon = \omega + I^2$.

Remark that for any element of the factor module Ω^2_ε, one can choose a unique representative $\omega \in \Omega^2(J^1M)$ such that $X_1 \rfloor \omega = 0$ and $\omega \wedge dU = 0$. Here X_1 is a contact vector field with generating function 1.

In the local canonical coordinates such representatives have the form

$$\begin{aligned} \omega = {} & E dq_1 \wedge dq_2 + B(dq_1 \wedge dp_1 - dq_2 \wedge dp_2) \\ & + C dq_1 \wedge dp_2 - A dq_2 \wedge dp_1 + D dp_1 \wedge dp_2, \end{aligned} \qquad (2)$$

where A, B, C, D and E are smooth functions on J^1M.

Remark also, that form (2) generates equation (1).

We identify effective forms as elements of the factor module Ω^2_ε with differential forms of type (2) and also call such differential forms effective.

Note that, generally speaking, a contact diffeomorphisms do not preserves the Cartan form and, because of it, do not act directly to the chosen representatives of effective forms.

Let ω and θ be two effective differential 2-forms. Two Monge-Ampère equations E_ω and E_θ are called *locally contact equivalent at a point* $a_0 \in J^1M$ if there exists a local contact diffeomorphism ϕ, which preserves this point and such that $\phi^*(\omega)_\varepsilon = h\theta$ in a neighborhood of a_0. Here h is a function.

2.2 Geometric Structure on J^1M

At each point $a \in J^1M$, the restriction of the differential of the Cartan form to the 4-dimensional space $C(a)$ defines the symplectic structure

$$\Omega_a = dU|_{C(a)}.$$

With any effective differential 2-form ω one can associate a smooth function $\mathrm{Pf}(\omega)$ on J^1M by means of the following equality:

$$\mathrm{Pf}(\omega)\Omega \wedge \Omega = \omega \wedge \omega. \tag{3}$$

We say that the Monge-Ampère equation E_ω is *hyperbolic, elliptic* or *parabolic* at a domain $\mathcal{D} \subset J^1M$ if the function $\mathrm{Pf}(\omega)$ is negative, positive or zero at each point of $\mathcal{D}$, respectively. The hyperbolic and elliptic equations are called *nondegenerate*.

An effective 2-form ω generates the "non-holonomic"[1] field of endomorphisms A_ω on the Cartan distribution by means of the formula

$$X \rfloor \omega = A_\omega X \rfloor \Omega, \tag{4}$$

where X is a vector field from the Cartan distribution.

The square of operator A_ω is scalar and

$$A_\omega^2 + \mathrm{Pf}(\omega) = 0. \tag{5}$$

Effective forms ω and $h\omega$, where h is a nonvanishing function, define the same equation. Therefore, for a non-degenerate equation E_ω the form ω can be normed in such a way that $|\,\mathrm{Pf}(\omega)| = 1$. It is sufficient to replace ω by $\frac{\omega}{\sqrt{|\mathrm{Pf}(\omega)|}}$.

By (5), the hyperbolic and elliptic equations generate a product structure $A_{\omega,a} = 1$ and complex structure $A_{\omega,a} = -1$ on $C(a)$ respectively.

Eigenspaces of the operator A_ω generate two distributions C_+ and C_- on J^1M. These distributions are real for hyperbolic equations and complex for elliptic ones.[2] We call them *characteristic*.

The first derivatives of the characteristic distributions $C_\pm^{(1)} = C_\pm + [C_\pm, C_\pm]$ are 3-dimensional. Their intersection $l = C_+^{(1)} \cap C_-^{(1)}$ is a one-dimensional distribution, which is transversal to the Cartan distribution [24].

For the hyperbolic equations the tangent space T_aJ^1M splits into the direct sum

$$T_aJ^1M = C_+(a) \oplus l(a) \oplus C_-(a) \tag{6}$$

at each point $a \in J^1M$.

For elliptic equations we get similar decomposition of the complexification of T_aJ^1M. In this case the distributions l is real also.

A non-degenerate equation is called *regular* if the derivatives $C_\pm^{(k)}$ $(k = 1, 2, 3)$ of the characteristic distributions are distributions too.

2.3 The Laplace Forms

Define four tensor fields $q^1_{2,3}, q^2_{1,1}, q^2_{3,3}$, and $q^3_{1,2}$ on J^1M:

$$q^s_{j,k}(X, Y) = -\mathbf{P}_s[\mathbf{P}_j X, \mathbf{P}_k Y],$$

where $j, k, s = 1, 2, 3; s \neq j, k$. Here $\mathbf{P}_j : D(J^1M) \to D_j$ is the projector from the module of vector fields $D(J^1M)$ on J^1M to the module of vector fields D_j on the distribution P_j $(j = 1, 2, 3)$ (see [15]).

[1] The endomorphism $A_{\omega,a}$ is defined on $C(a)$ only.

[2] In elliptic case they are complex conjugate.

Then two differential 2-forms λ_- and λ_+, which are "wedge contractions"

$$\lambda_- = \langle q^2_{1,1}, q^1_{2,3}\rangle \quad \text{and} \quad \lambda_+ = \langle q^2_{3,3}, q^3_{1,2}\rangle, \tag{7}$$

of these tensor fields, we call *Laplace forms* (see [15]).

Here a contraction $\langle\cdot,\cdot\rangle$ is defined by the formula

$$\langle \alpha\otimes X, \beta\otimes Y\rangle = (Y\rfloor\alpha)\wedge(X\rfloor\beta)$$

for tensors $\alpha\otimes X$ and $\beta\otimes Y$. For tensors $\sum\alpha\otimes X$ we define this contraction by linearity. It is not hard to check that this definition is correct.

Coefficients of the Laplace forms for linear hyperbolic equations are the classical Laplace invariants [18]. This observation justifies our definition.

By the construction the Laplace forms are contact invariants of equations.

For elliptic equations they are complex conjugate. In case of linear elliptic equations coefficients of the Laplace forms are also known as the Cotton invariants [2].

2.4 The Classes $H_{k,l}$

Let E_ω be a non-degenerate regular equation and let C_j be one of the characteristic distributions. Then for the distribution we expect one of the following four cases:

(1) $C_j \neq C_j^{(1)} = C_j^{(2)}$ and $\dim C_j^{(1)} = 3$;
(2) $C_j \neq C_j^{(1)} \neq C_j^{(2)} = C_j^{(3)}$ and $\dim C_j^{(1)} = 3$, $\dim C_j^{(2)} = 4$;
(3) $C_j \neq C_j^{(1)} \neq C_j^{(2)} \neq C_j^{(3)} = TJ^1M$ and $\dim C_j^{(1)} = 3$, $\dim C_j^{(2)} = 4$;
(4) $C_j \neq C_j^{(1)} \neq C_j^{(2)} = TJ^1M$ and $\dim C_j^{(1)} = 3$.

For complex distributions, "dim" means the complex dimension.

We say that a Monge-Ampère equation *belongs to the class* $H_{k,l}$ $(k,l = 0,\dots,4;\, k\le l)$ if case (k) holds for one of the characteristic distributions and case (l) holds for the another one [25].

Classes $H_{k,l}$ are invariant under contact transformations.

3 Hyperbolic Equations

3.1 Normal Form $v_{xy} = k(x,y)v$

First of all, we describe a class of Monge-Ampère equations which are locally contact equivalent to the equations of the form

$$v_{xy} = k(x,y)v, \tag{8}$$

where k is a smooth function.

Let (q^0, u^0, p^0) be local coordinates of a point $a_0 \in J^1M$. Suppose that the function k does not vanish at the point $q^0\in M$, then (8) belongs to the class $H_{2,2}$.

Theorem 1 *A hyperbolic Monge-Ampère equation of the class* $H_{2,2}$ *is locally contact equivalent to* (8) *at a point* $a_0\in J^1M$ *if and only if the following conditions for its Laplace forms*:

1. $\lambda_+\wedge\lambda_- = 0$,
2. $d\lambda_+ = d\lambda_- = 0$,
3. $\lambda_+ + \lambda_- = 0$

hold.

Proof Necessity we check by straightforward computations. Indeed, for (8) the Laplace forms are

$$\lambda_{\pm} = \pm k(x, y) dx \wedge dy,$$

and, therefore, conditions 1–3 hold. Now we prove the sufficiency.

Due to the theorem of contact linearization [15], the equation E is locally contact equivalent to linear equation of the form:

$$v_{xy} = a(x, y)v_x + b(x, y)v_y + c(x, y)v + g(x, y). \tag{9}$$

Here a, b, c and g are smooth functions. Third condition of the theorem implies the equality $b_{q_2} = a_{q_1}$, i.e., $a = \varphi_{q_2}$ and $b = \varphi_{q_1}$ for some smooth function $\varphi = \varphi(q_1, q_2)$. This function is defined up to an additive constant. We choose this constant in such a way that $\varphi(q^0) = 0$.

The effective form corresponding to (9) is

$$\omega = dq_1 \wedge dp_1 - dq_2 \wedge dp_2 - 2(\varphi_{q_2} p_1 + \varphi_{q_1} p_2 + cu + g) dq_1 \wedge dq_2.$$

Applying the contact transformation

$$\phi : \begin{cases} q_1 \mapsto q_1, \\ q_2 \mapsto q_2, \\ u \mapsto e^{\varphi}(u + \alpha(q_1^0 - q_1) + \beta(q_2^0 - q_2)), \\ p_1 \mapsto e^{\varphi}(p_1 - \alpha + (u + \alpha(q_1^0 - q_1) + \beta(q_2^0 - q_2))\varphi_{q_1}), \\ p_2 \mapsto e^{\varphi}(p_2 - \beta + (u + \alpha(q_1^0 - q_1) + \beta(q_2^0 - q_2))\varphi_{q_2}), \end{cases}$$

where $\alpha = u^0 \varphi_{q_1}(q^0)$ and $\beta = u^0 \varphi_{q_2}(q^0)$, to the 2-form ω and taking the effective part of the 2-form $\phi^*(\omega)$, we get the form

$$\begin{aligned} \phi^*(\omega)_\varepsilon = e^{\varphi}(dq_1 \wedge dp_1 - dq_2 \wedge dp_2) - 2(g + e^{\varphi}(u + \alpha(q_1^0 - q_1) \\ + \beta(q_2^0 - q_2))(c + \varphi_{q_1}\varphi_{q_2} - \varphi_{q_1 q_2})) dq_1 \wedge dq_2. \end{aligned}$$

The Monge-Ampère equation corresponding to this form is

$$v_{q_1 q_2} = kv + \tilde{g}, \tag{10}$$

where

$$k = c + \varphi_{q_1}\varphi_{q_2} - \varphi_{q_1 q_2} = ab + c - a_{q_1}$$

is one of the Laplace invariants of (9) and

$$\tilde{g} = g e^{-\varphi} + (\alpha(q_1^0 - q_1) + \beta(q_2^0 - q_2))\tilde{c}.$$

Let $v = \psi(q_1, q_2)$ be a solution of (10) with the following Cauchy data:

$$\psi|_{q_2 = q_2^0} = \gamma_0(q_1) \quad \text{and} \quad \left.\frac{\partial \psi}{\partial q_2}\right|_{q_2 = q_2^0} = \gamma_1(q_1),$$

where γ_0 and γ_1 are smooth functions such that $\gamma_0(q_1^0) = \gamma_0'(q_1^0) = \gamma_1(q_1^0) = 0$. Due to the existence theorem, in some neighborhood of the point q^0 there exists a solution of the Cauchy problem and its partial derivative by q_1 vanishes at the point q^0, i.e., $\frac{\partial\psi}{\partial q_1}|_{q^0} = 0$.

The contact transformation

$$(q_1, q_2, u, p_1, p_2) \mapsto (q_1, q_2, u + \psi, p_1 + \psi_{q_1}, p_2 + \psi_{q_2}),$$

takes (10) to (8). □

3.2 Normal Form $v_{xy} = v$

Consider the linear equation

$$v_{xy} = a(x, y)v_x + b(x, y)v_y + c(x, y)v + g(x, y). \tag{11}$$

To formulate conditions of equivalence of (11) to equation of the form

$$v_{xy} = \alpha v_x + \beta v_y + \gamma v + f(x, y) \tag{12}$$

with constant coefficients α, β, and γ we note that the Laplace invariants of (11) are $k = ab + c - b_y$ and $h = ab + c - a_x$.

Remark also that any equation which is equivalent to (12) belongs to the class $H_{1,1}$ (when $k \equiv h \equiv 0$) or $H_{2,2}$ (when $k \neq 0$ and $h \neq 0$). All equations of the class $H_{1,1}$ are locally equivalent to the wave equation $v_{xy} = 0$ [26]. Therefore, we shall consider only equations of the class $H_{2,2}$.

Equation (12) belongs to the class $H_{2,2}$ if and only if $\alpha\beta + \gamma \neq 0$. In this case it can be reduced to the telegraph equation

$$v_{xy} = v.$$

It worth to note, that this fact was known to Euler [3].

Theorem 2 *Differential equation* (11) *is locally equivalent to the telegraph equation* $v_{xy} = v$ *at point* $q^0 \in M$ *if and only if its Laplace invariants coincide in some neighborhood of* q^0, *do not vanish at this point and satisfy the following differential equation*:

$$kk_{xy} - k_x k_y = 0. \tag{13}$$

Proof Necessity we check by straightforward computations. Indeed, for the telegraph equation the Laplace invariant $k(x, y) = 1$ and therefore, the conditions of the Theorem hold. Let's prove the sufficiency.

Since the Laplace forms of (11) satisfy Theorem 1, this equation is locally equivalent to (8).

Equation (13) is equivalent to the equation

$$\frac{\partial^2 \ln|k|}{\partial q_1 \partial q_2} = 0.$$

Therefore $k(q) = \tilde{X}(q_1)\tilde{Y}(q_2)$ for some functions $\tilde{X}$ and $\tilde{Y}$. Let X and Y be integrals of $\tilde{X}$ and $\tilde{Y}$ respectively, such that $X(q_1^0) = q_1^0$ and $Y(q_2^0) = q_2^0$. Therefore, the effective form corresponding to (13) is

$$\omega = dq_1 \wedge dp_1 - dq_2 \wedge dp_2 - 2X'(q_1)Y'(q_2)u\,dq_1 \wedge dq_2.$$

Since the function k does not vanish at the point q^0, there exist the inverse functions for X and Y. We denote them by χ and ψ respectively. Note that $\chi(q_1^0) = q_1^0$, $\psi(q_2^0) = q_2^0$, $\chi'(q_1^0) \neq 0$ and $\psi'(q_2^0) \neq 0$.

The contact transformation

$$(q_1, q_2, u, p_1, p_2) \mapsto \left(\chi(q_1), \psi(q_2), u - \xi q_1 - \eta q_2, \frac{p_1 - \xi}{\chi'(q_1)}, \frac{p_2 - \eta}{\psi'(q_2)}\right),$$

where

$$\xi = p_1^0\left(\frac{1}{\chi'(q_1^0)} - 1\right) \quad \text{and} \quad \eta = p_2^0\left(\frac{1}{\psi'(q_2^0)} - 1\right),$$

transforms the form ω to the form

$$\begin{aligned} \widetilde{\omega} = {} & dq_1 \wedge dp_1 - dq_2 \wedge dp_2 \\ & - 2X'(\chi(q_1))Y'(\psi(q_2))(u - \xi q_1 - \eta q_2)\chi'(q_1)\psi'(q_2) dq_1 \wedge dq_2. \end{aligned}$$

Since

$$\chi'(q_1) = \frac{1}{X'(\chi(q_1))} \quad \text{and} \quad \psi'(q_2) = \frac{1}{Y'(\psi(q_2))},$$

we get

$$\widetilde{\omega} = dq_1 \wedge dp_1 - dq_2 \wedge dp_2 - 2(u + \gamma(q)) dq_1 \wedge dq_2,$$

where

$$\gamma(q) = -X'(\chi(q_1))Y'(\psi(q_2))(\xi q_1 + \eta q_2).$$

The equation corresponding to the form $\widetilde{\omega}$ is

$$v_{q_1 q_2} = v + \gamma(q), \tag{14}$$

and this equation is locally contact equivalent to the telegraph equation. □

The similar theorem holds for Monge-Ampère equations.

Theorem 3 *A hyperbolic Monge-Ampère equation is locally equivalent to the telegraph equation at a point $a_0 \in J^1 M$ if and only if its Laplace forms are*

$$\lambda_+ = \Phi(g, h) dg \wedge dh \quad \textit{and} \quad \lambda_- = -\Phi(g, h) dg \wedge dh, \tag{15}$$

where g and h are first integrals of the distributions $C_+^{(2)}$ and $C_-^{(2)}$ respectively, and the function $\Phi(g, h)$ does not vanish at this point and satisfies the following differential equation:

$$\Phi \Phi_{gh} - \Phi_g \Phi_h = 0. \tag{16}$$

Proof The functions g and h are defined up to gauge transformation $g \mapsto \eta(g), h \mapsto \zeta(h)$ and up to permutation $g \mapsto h, h \mapsto g$. Here η and ζ are arbitrary functions such that $\eta'\zeta' \neq 0$. Equation (16) is invariant with respect to such transformations.

Necessity follows from the fact that for the telegraph equation we have $g = q_1$, $h = q_2$, and $\Phi(q_1, q_2) = 1$.

Now we prove the sufficiency. Due to Theorem 1, the equation is equivalent to (8) with $k(x, y) = \Phi(x, y)$. It follows from the fact that Laplace forms are contact invariant of differential equations. Due to Theorem 2, the last equation is equivalent to the telegraph equation. □

4 Elliptic Equations

4.1 Equations $v_{xx} + v_{xx} = f(x, y, v, v_x, v_y)$

Since the Laplace forms for elliptic equations are complex conjugate [15], all elliptic Monge-Ampère equations split into two classes: $H_{1,1}$ and $H_{2,2}$.

Equations of the class $H_{1,1}$ were considered in [26]. Any equation of the class $H_{2,2}$ is contact equivalent to an equation of the type [26]

$$v_{xx} + v_{yy} = f(x, y, v, v_x, v_y) \tag{17}$$

for a smooth function f [26].

The following lemma describes the class of contact transformations which preserve the class of equations of type (17).

Lemma 1 *A contact transformation takes an equation of type* (17) *to equation of the same type if and only if this transformation is a prolongation to J^1M of the following point transformation*:

$$(x, y, v) \mapsto (X(x, y), Y(x, y), V(x, y, v)), \tag{18}$$

where X and Y are harmonic conjugate functions, and V is a smooth function.

Proof The first integrals of the complex distributions $C_-^{(2)}$ and $C_+^{(2)}$ for (17) are $z = x + \iota y$ and $\overline{z} = x - \iota y$, where $\iota = \sqrt{-1}$. Therefore, the transformations should be one of two types: $z \mapsto P(z)$ or $z \mapsto Q(\overline{z})$, where P and Q are some complex functions.

It is easy to see that the functions $P(z)$ and $Q(\overline{z})$ are holomorphic and anti-holomorphic, respectively. Therefore, $P(z) = X(x, y) + \iota Y(x, y)$ and $Q(\overline{z}) = X(x, y) - \iota Y(x, y)$, where X and Y are harmonic conjugate functions. The transformation of the first type is $(x, y) \mapsto (X, Y)$ and the transformation of the second type is $(x, y) \mapsto (X, -Y)$. The functions X and $-Y$ are harmonic conjugate also.

The function V does not depend on p_1 and p_2. Otherwise, the corresponding transformation cannot be contact. □

Corollary 1 *A contact transformation takes a linear equation of type*

$$v_{xx} + v_{yy} = a(x, y)v_x + b(x, y)v_y + c(x, y)v + g(x, y) \tag{19}$$

to an equation of the same type if and only if it is a prolongation to J^1M of the following point transformation:

$$(x, y, v) \mapsto (X(x, y), Y(x, y), A(x, y)v + B(x, y)), \tag{20}$$

where X and Y are harmonic conjugate functions and A, B are smooth functions.

4.2 Normal Form $v_{xx} + v_{xx} = k(x, y)v + f(x, y)$

Let (q^0, u^0, p^0) be local coordinates of a point $a_0 \in J^1M$. Suppose that the function k does not vanish at the point $q^0 \in M$, i.e., the equation

$$v_{xx} + v_{yy} = k(x, y)v + f(x, y) \tag{21}$$

belongs to the class $H_{2,2}$.

Theorem 4 *An elliptic Monge-Ampère equation of the class $H_{2,2}$ is locally contact equivalent to (21) at a point $a_0 \in J^1M$ if and only if the following conditions for the Laplace forms of E:*

1. $\lambda_+ \wedge \lambda_- = 0$,
2. $d\lambda_+ = d\lambda_- = 0$,
3. $\lambda_+ + \lambda_- = 0$

hold.

Proof Necessity one can check by straightforward computations. Let's prove the sufficiency.

Due to the theorem of contact linearization [15], the equation E is locally contact equivalent to the linear (19).

Since the Laplace forms of (19) are

$$\lambda_\pm = \frac{1}{4}\left(b_{q_1} - a_{q_2} \pm \left(\frac{1}{2}(a^2 + b^2) + 2c - a_{q_1} - b_{q_2}\right)\iota\right)dq_1 \wedge dq_2, \tag{22}$$

the third condition of the theorem implies equality $b_{q_1} = a_{q_2}$. Then $a = \varphi_{q_1}$ and $b = \varphi_{q_2}$ for some smooth function $\varphi = \varphi(q_1, q_2)$. The function φ is defined up to an additive constant. We choose this constant in such a way that $\varphi(q^0) = 0$.

The effective form corresponding to (19) is

$$\omega = dq_1 \wedge dp_2 - dq_2 \wedge dp_1 - (\varphi_{q_1} p_1 + \varphi_{q_1} p_2 + cu + g)dq_1 \wedge dq_2.$$

Applying the contact transformation

$$\phi : \begin{cases} q_1 \mapsto q_1, \\ q_2 \mapsto q_2, \\ u \mapsto (u + \alpha(q_1^0 - q_1) + \beta(q_2^0 - q_2))e^{\frac{\varphi}{2}}, \\ p_1 \mapsto \left(p_1 - \alpha + \dfrac{\varphi_{q_1}}{2}(u + \alpha(q_1^0 - q_1) + \beta(q_2^0 - q_2))\right)e^{\frac{\varphi}{2}}, \\ p_2 \mapsto \left(p_2 - \beta + \dfrac{\varphi_{q_2}}{2}(u + \alpha(q_1^0 - q_1) + \beta(q_2^0 - q_2))\right)e^{\frac{\varphi}{2}}, \end{cases}$$

where $\alpha = \frac{1}{2}u^0\varphi_{q_1}(q^0)$ and $\beta = \frac{1}{2}u^0\varphi_{q_2}(q^0)$, to the 2-form ω and taking the effective part of $\phi^*(\omega)$, we get:

$$\phi^*(\omega)_\varepsilon = (k(q)u + f(q))dq_1 \wedge dq_2 + dq_1 \wedge dp_2 - dq_2 \wedge dp_1,$$

where

$$k = \frac{1}{2}(\varphi_{q_2q_2} + \varphi_{q_1q_1}) - \frac{1}{4}(\varphi^2_{q_1} + \varphi^2_{q_2}) - c = -\frac{1}{4}(a^2 + b^2) + \frac{1}{2}(a_{q_1} + b_{q_2}) - c,$$
$$f = (\alpha(q^0_1 - q_1) + \beta(q^0_2 - q_2)) - ge^{-\frac{\varphi}{2}}k,$$

and the corresponding equation has form (21). □

Remark 1 If the coefficients A, B, C, D and E of (1) are analytic functions, then, under conditions of the previous theorem, the term f in (21) can be eliminated, similar to the case of hyperbolic equations (see Theorem 1).

4.3 The Helmholtz Normal Form

The Helmholtz equation has the form

$$v_{xx} + v_{yy} = \kappa v + f(x, y), \tag{23}$$

where κ is some constant ($\kappa \neq 0$).

Let us find conditions under which Monge-Ampère equations of the class $H_{2,2}$ are equivalent to the Helmholtz equation.

We need the following

Lemma 2 *For any harmonic function $w(x, y)$, there exists a harmonic function $h(x, y)$ such that*

$$h^2_x + h^2_y = e^w. \tag{24}$$

Proof It is sufficient to prove that the overdetermined system

$$\begin{cases} F = h_{xx} + h_{yy} = 0, \\ G = h^2_x + h^2_y - e^w = 0 \end{cases} \tag{25}$$

with respect to the function h is compatible for any harmonic function w. To prove this we calculate the Kruglikov–Lychagin–Mayer bracket [9] for this system:

$$[F, G] = 2(h^2_{xx} + 2h^2_{xy} + h^2_{yy}) - e^w(w^2_x + w^2_y). \tag{26}$$

This bracket vanishes due to the system. Therefore, system (25) is formally integrable. Since this system is finite type, it has a smooth solution (see [9]). □

Theorem 5 *The linear elliptic equation*

$$v_{xx} + v_{yy} = a(x, y)v_x + b(x, y)v_y + c(x, y)v + g(x, y) \tag{27}$$

of the class $H_{2,2}$ is locally equivalent to the Helmholtz equation at a point $q^0 \in M$ if and only if $a_y = b_x$ and the function

$$k = \frac{1}{4}(a^2 + b^2) + c - \frac{1}{2}(a_x + b_y)$$

satisfies the following differential equation:

$$k(k_{xx}+k_{yy})=k_x^2+k_y^2. \tag{28}$$

Proof Necessity one can check by straightforward computations. Let's prove the sufficiency.

Due to Theorem 4, (27) is equivalent to (21). Equation (28) can be written as

$$\Delta(\ln|k|)=0,$$

where Δ is the Laplace operator. Therefore, the function k has the form $k(x,y)=\varepsilon e^{w(x,y)}$, where w is a harmonic function and $\varepsilon=\pm 1$.

Due to Lemma 2, there exists a harmonic function $h=h(x,y)$ such that $k=\varepsilon(h_x^2+h_y^2)$. Then the corresponding effective form for (21) is

$$\omega=dq_1\wedge dp_2-dq_2\wedge dp_1-(\varepsilon(h_{q_1}^2+h_{q_2}^2)u+f)dq_1\wedge dq_2.$$

Let $g=g(x,y)$ be a function, which is harmonic conjugate to h, i.e., $h_x=g_y$ and $h_y=-g_x$. The functions h and g are determined up to additive constants. Choose them in such a way that $h(q^0)=q_1^0$, $g(q_2^0)=q_0^2$. Due to Lemma 1, we can construct the following transformation of M:

$$(q_1,q_2)\mapsto(Q_1=h(q),\ Q_2=g(q)).$$

It preserves the point $q^0\in M$. Since $k(q^0)\neq 0$ and the Jacobian

$$\frac{\partial(Q_1,Q_2)}{\partial(q_1,q_2)}=h_{q_1}g_{q_2}-h_{q_2}g_{q_1}=h_{q_1}^2+h_{q_2}^2\neq 0$$

in some neighborhood of q^0. For the inverse transformation

$$(Q_1,Q_2)\mapsto(q_1=H(Q),q_2=G(Q)),$$

the functions H and G are harmonic conjugate and

$$h_{q_1}^2+h_{q_2}^2=\frac{1}{H_{Q_1}^2+H_{Q_2}^2}.$$

Let us take the following local point transformation

$$\phi:(q_1,q_2,u)\mapsto(Q_1=h(q),\ Q_2=g(q),U=u)$$

and prolong it to a contact transformation. Then

$$\phi^*(\omega)_\varepsilon=dQ_1\wedge dP_1-dQ_2\wedge dP_2-(\varepsilon U+\widetilde{f}(Q))dQ_1\wedge dQ_2,$$

where

$$\widetilde{f}(Q)=\frac{f(H(Q),G(Q))}{H_{Q_1}^2+H_{Q_2}^2}.$$

The corresponding equation is (23) with $\kappa=-1$ or $\kappa=1$. $\square$

The following theorem gives a criterion of equivalence of Monge-Ampère equations to the Helmholtz equation.

Theorem 6 *An elliptic Monge-Ampère equation is locally contact equivalent to the Helmholtz equation at a point* $a_0 \in J^1 M$ *if and only if its Laplace forms are*

$$\lambda_+ = \Phi(g, \overline{g}) dg \wedge d\overline{g}, \qquad \lambda_- = -\Phi(g, \overline{g}) dg \wedge d\overline{g} \tag{29}$$

were g *and* $\overline{g}$ *are first integrals of the complex distributions* $C_+^{(2)}$ *and* $C_-^{(2)}$, *and the function* $\Phi(g, h)$ *does not vanish at the point* a_0 *and satisfies the following differential equation*:

$$\Phi \Phi_{g\overline{g}} - \Phi_g \Phi_{\overline{g}} = 0. \tag{30}$$

Proof Necessity one can check by straightforward calculations. To prove the sufficiency remark that in new variables $\widetilde{x} = \mathrm{Re}(g)$ and $\widetilde{y} = \mathrm{Im}(g)$ the Monge-Ampère equation has the form

$$v_{\widetilde{x}\widetilde{x}} + v_{\widetilde{y}\widetilde{y}} = f(\widetilde{x}, \widetilde{y}, v, v_{\widetilde{x}}, v_{\widetilde{y}}),$$

or, if we omit the tilde, form (17). Equation (30) takes the form

$$\Phi \Phi_{z\overline{z}} - \Phi_z \Phi_{\overline{z}} = 0, \tag{31}$$

where $z = x + \iota y$. Since the Laplace forms are complex conjugate, $\Phi(z, \overline{z})$ is a real-value function. Define the function $K(x, y) = 4\Phi(z, \overline{z})$. Then

$$\lambda_\pm = \pm \frac{\iota}{2} K(q) dq_1 \wedge dq_2,$$

and the function K satisfies the equation

$$K(K_{xx} + K_{yy}) = K_x^2 + K_y^2. \tag{32}$$

Due to Theorems 4 and 5, the equation is locally contact equivalent to one of the following equations:

$$v_{xx} + v_{yy} = v + f(x, y), \quad \text{or} \quad v_{xx} + v_{yy} = -v + f(x, y),$$

where f is a smooth function. □

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