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Commutativity and Ideals in Strongly Graded Rings

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Abstract In some recent papers by the first two authors it was shown that for any algebraic crossed product $\mathcal{A}$, where $\mathcal{A}_0$, the subring in the degree zero component of the grading, is a commutative ring, each non-zero two-sided ideal in $\mathcal{A}$ has a non-zero intersection with the commutant $C_{\mathcal{A}}(\mathcal{A}_0)$ of $\mathcal{A}_0$ in $\mathcal{A}$. This result has also been generalized to crystalline graded rings; a more general class of graded rings to which algebraic crossed products belong. In this paper we generalize this result in another direction, namely to strongly graded rings (in some literature referred to as *generalized crossed products*) where the subring $\mathcal{A}_0$, the degree zero component of the grading, is a commutative ring. We also give a description of the intersection between arbitrary ideals and commutants to bigger subrings than $\mathcal{A}_0$, and this is done both for strongly graded rings and crystalline graded rings.

Keywords Strongly graded rings · Commutativity · Ideals

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1 Introduction

Dynamical systems, generated by the iteration of homeomorphisms of compact Hausdorff spaces, lead to crossed product algebras of continuous functions on the space, by the action

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of the additive group of integers via composition of continuous functions with the iterations of the homeomorphisms. In the context of C^*-algebras, the interplay between topological properties of the dynamical system such as minimality, transitivity, freeness on one hand, and properties of ideals, subalgebras and representations of the corresponding crossed product on the other hand, has been a subject of intensive investigation at least since the 1960's, both for single map dynamics and for more general actions of groups and semigroups, that is in particular iterations of several transformations called iterated function systems in the literature on fractals and dynamical systems. Such constant and growing interest to this interplay between dynamics, actions and non-commutative algebras can be explained by the fundamental importance of this interplay and its implications for operator representations of the corresponding crossed product algebras, spectral and harmonic analysis and non-commutative analysis and non-commutative geometry fundamental for the mathematical foundations of quantum mechanics, quantum field theory, string theory, integrable systems, lattice models, quantization, symmetry analysis, renormalization, and recently in analysis and geometry of fractals and in wavelet analysis and its applications in signal and image processing (see [1–5, 7, 9–14, 19–21, 28–30, 33, 39, 42, 44, 55, 56, 58] and references therein).

There has been a substantial progress on the interplay between C^*-algebras and dynamics of iterations of continuous transformations and more general actions of groups on compact Hausdorff spaces [2, 10, 44, 55–57]. However, the investigation of actions of not necessarily continuous transformations on more general and more irregular spaces than Hausdorff spaces requires an extension of this interplay beyond C^*-algebras to a purely algebraic framework of general algebras and rings. Only partial progress in this important direction has been made. In [45–48], extensions and modifications of this result and the interplay between dynamics and maximal commutativity properties of the canonical coefficient subalgebra, the degree zero component of the grading, and its intersection with ideals was investigated for dynamical systems that are not topologically free on more general spaces than Hausdorff spaces both in the context of algebraic crossed products by $\mathbb{Z}$ and for the corresponding Banach algebra and C^*-algebra crossed products in the case of single homeomorphism dynamical systems or more general dynamical systems generated by an invertible map. Also in these works, this interplay has been considered from the point of view of representations as well as with respect to duality in the crossed product algebras. Some results, that could be considered as related to this direction of interplay, have been scattered within the purely algebraic literature on graded rings and algebras [6, 8, 15–18, 22–27, 31–35, 40, 41, 43, 49–54]. In many of these related results, very special properties are assumed for the coefficient subring or for the whole crossed product or graded ring or algebra, such as being a ring without zero-divisors, semi-simple or simple ring, etc. This has been motivated in most cases by the desire to use the algebraic constructions, tools and techniques that were available at the time. However, it turns out that these restrictions often exclude for example many important classes of algebras arising in physics and associated to actions on algebras and rings of functions on infinite spaces or other algebras and rings with zero-divisors, and many other situations. Thus, it is desirable to investigate the above mentioned interplay between actions and properties of ideals and subalgebras for general graded rings and crossed product rings and algebras and their generalizations, without any restrictive artificially imposed conditions. It turns out that many interesting properties and results hold in such much greater generality and also as a consequence, new previously not noticed results and constructions come to light.

In this paper we focus on the connections between the structure of ideals and the commutant of subrings in generalizations of crossed product rings and in general classes of graded rings. This, in particular, provides a general understanding of the conditions for maximal commutativity of the degree zero component subalgebra of the grading and properties of more general subalgebras important for representation theory. In this paper we substantially extend the approach and some of the key results in [36–38] to more general subrings than the degree zero component of the grading in crossed products, or more general graded rings.

In Sect. 2 we briefly recall the basics of graded rings and crossed products. Given a ring $\mathcal{R}$ and a subset $\mathcal{S} \subseteq \mathcal{R}$, we denote by

$$C_{\mathcal{R}}(\mathcal{S}) = \{r \in \mathcal{R} \mid rs = sr, \ \forall s \in \mathcal{S}\}$$

the commutant of $\mathcal{S}$ in $\mathcal{R}$. The following theorem was shown in [36].

Theorem 1 *If $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$ is a G-crossed product where $\mathcal{R}_e$ is ccommutative, then*

$$I \cap C_{\mathcal{R}}(\mathcal{R}_e) \neq \{0\}$$

for every non-zero two-sided ideal I in $\mathcal{R}$.

Given a normal subgroup N of G one can consider the subring $\mathcal{R}_N = \bigoplus_{n \in N} \mathcal{R}_n$ in $\mathcal{R}$ and obtain a generalization of Theorem 1 by considering the intersection between arbitrary non-zero ideals and $C_{\mathcal{R}}(\mathcal{R}_N)$. This is done in Sect. 3 (Theorem 2).

In Sect. 4 we consider general strongly graded rings $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$, which are not necessarily crossed products. Given any subgroup H of G we give a description of the commutant of $\mathcal{R}_H$ in $\mathcal{R}$ (Theorem 3) and prove the main theorem (Theorem 4). We obtain some interesting corollaries (Corollaries 2, 4 and 5) which generalize the results obtained in Sect. 2 and generalize Theorem 1 to general strongly graded rings.

In Sect. 5 we recall the definition and basic properties of *crystalline graded rings*, a class of graded rings which are not necessarily strongly graded (for more details see [34, 35, 38]). Given a subgroup H of G we give a description of the commutant of $\mathcal{A}_H$ in the crystalline graded ring $\mathcal{A}$ and give sufficient conditions for each non-zero two-sided ideal I in $\mathcal{A}$ to have a non-zero intersection with $C_{\mathcal{A}}(\mathcal{A}_H)$ (Theorem 5).

2 Preliminaries

Throughout this paper all rings are assumed to be unital and associative and we let G be an arbitrary group with neutral element e.

A ring $\mathcal{R}$ is said to be *G-graded* if

$$\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g \quad \text{and} \quad \mathcal{R}_g \mathcal{R}_h \subseteq \mathcal{R}_{gh}$$

for all $g, h \in G$, where $\{\mathcal{R}_g\}_{g \in G}$ is a family of additive subgroups in $\mathcal{R}$. The additive subgroup $\mathcal{R}_g$ is called the homogeneous component of $\mathcal{R}$ of degree $g \in G$. Moreover, if $\mathcal{R}_g \mathcal{R}_h = \mathcal{R}_{gh}$ holds for all $g, h \in G$, then $\mathcal{R}$ is said to be *strongly graded by G* and if we in addition have

$$U(\mathcal{R}) \cap \mathcal{R}_g \neq \emptyset$$

for each $g \in G$, where $U(\mathcal{R})$ denotes the group of multiplication invertible elements in $\mathcal{R}$, then $\mathcal{R}$ is said to be a *G-crossed product*.

Suppose that we are given a group G, a ring $\mathcal{R}_e$ and two maps $\sigma : G \to \mathrm{Aut}(\mathcal{R}_e)$ and $\alpha : G \times G \to U(\mathcal{R}_e)$ satisfying the following conditions

$$\sigma_g(\sigma_h(a))\,\alpha(g,h) = \alpha(g,h)\,\sigma_{gh}(a) \tag{1}$$

$$\alpha(g,h)\,\alpha(gh,s) = \sigma_g(\alpha(h,s))\,\alpha(g,hs) \tag{2}$$

$$\alpha(g,e) = \alpha(e,g) = 1_{\mathcal{R}_e} \tag{3}$$

for all $g, h, s \in G$ and $a \in \mathcal{R}_e$. We may then choose a family of symbols $\{v_g\}_{g\in G}$ and define $\mathcal{R}'$ to be the free left $\mathcal{R}_e$-module with basis $\{v_g\}_{g\in G}$ and define a multiplication on the set $\mathcal{R}'$ by

$$(a_1\, v_g)(a_2\, v_h) = a_1\, \sigma_g(a_2)\, \alpha(g,h)\, v_{gh}$$

for $a_1, a_2 \in \mathcal{R}_e$ and $g, h \in G$. It turns out that $\mathcal{R}'$ is an associative and unital ring with this multiplication and that it is in fact a G-crossed product, where the homogeneous component of degree $g \in G$ is given by $\mathcal{R}_e\, v_g$.

Conversely, given a G-crossed product $\mathcal{R} = \bigoplus_{g\in G} \mathcal{R}_g$, one can choose a family of elements $\{u_g\}_{g\in G}$ in $\mathcal{R}$ such that $u_g \in U(\mathcal{R}) \cap R_g$ for each $g \in G$ and put $u_e = 1_{\mathcal{R}}$. It is clear that $\mathcal{R}_g = \mathcal{R}_e\, u_g = u_g\, \mathcal{R}_e$ and that the set $\{u_g\}_{g\in G}$ is a basis for $\mathcal{R}$ as a left (and right) $\mathcal{R}_e$-module. We may now define a map

$$\sigma : G \to \mathrm{Aut}(\mathcal{R}_e)$$

by $u_g\, a = \sigma_g(a)\, u_g$ for all $a \in \mathcal{R}$ and $g \in G$. Furthermore, we define a map

$$\alpha : G \times G \to U(\mathcal{R}_e)$$

by $\alpha(g,h) = u_g\, u_h\, u_{gh}^{-1}$ and it is straight forward to check that these maps satisfy conditions (1)–(3) above. Furthermore, one can use these maps together with G and $\mathcal{R}_e$ and make the previous construction and obtain a G-crossed product $\mathcal{R}'$ which actually turns out to be isomorphic to the G-crossed product $\mathcal{R}$ that we started with. For more details on this we refer to [33, Propositions 1.4.1 and 1.4.2].

Remark 1 The above crossed product will be denoted by $\mathcal{R}_e \rtimes_\alpha^\sigma G$, to indicate the maps σ and α.

3 Subrings Graded by Subgroups

Given a G-graded ring $\mathcal{R} = \bigoplus_{g\in G} \mathcal{R}_g$ and a non-empty subset X of G, we denote

$$\mathcal{R}_X = \bigoplus_{x\in X} \mathcal{R}_x.$$

In particular, if H is a subgroup of G, then $\mathcal{R}_H = \bigoplus_{h\in H} \mathcal{R}_h$ is a subring in $\mathcal{R}$, and it is in fact an H-graded ring. The following lemma can be found in [22, Proposition 1.7].

Lemma 1 *If $\mathcal{R} = \bigoplus_{g\in G} \mathcal{R}_g$ is a G-graded ring and N is a normal subgroup of G, then $\mathcal{R}$ can be regarded as a G/N-graded ring, where the homogeneous components are given by*

$$\mathcal{R}_{gN} = \bigoplus_{x\in gN} \mathcal{R}_x$$

for $gN \in G/N$. Moreover, if $\mathcal{R}$ is a crossed product of G over $\mathcal{R}_e$, then $\mathcal{R}$ can also be regarded as a crossed product of G/N over

$$\mathcal{R}_N = \bigoplus_{x \in N} \mathcal{R}_x.$$

Proposition 1 *Let $\mathcal{R} = \bigoplus_{g\in G} \mathcal{R}_g = \mathcal{R}_e \rtimes_\alpha^\sigma G$ be a G-crossed product and suppose that N is a subgroup of G. If the following conditions are satisfied*:

(i) *$\mathcal{R}_e$ is commutative*
(ii) $N \subseteq Z(G) \bigcap \ker(\sigma)$
(iii) $\alpha(x, y) = \alpha(y, x)$ *for all* $(x, y) \in N \times N$

then $\mathcal{R}_N$ is commutative.

Proof Let the family of elements $\{u_g\}_{g\in G}$ be chosen as in Sect. 2. To prove that $\mathcal{R}_N$ is commutative, it suffices to show that for any $g, h \in N$ and $a_g, b_h \in \mathcal{R}_e$ the two elements $a_g\, u_g$ and $b_h\, u_h$ commute. By our assumptions, we have

$$\begin{aligned}(a_g\, u_g)\,(b_h\, u_h) &= a_g\, \sigma_g(b_h)\, \alpha(g, h) u_{gh} = a_g\, b_h\, \alpha(g, h) u_{gh}\\ &= b_h\, a_g\, \alpha(h, g) u_{hg} = b_h\, \sigma_h(a_g)\, \alpha(h, g)\, u_{hg} = (b_h\, u_h)\,(a_g\, u_g)\end{aligned}$$

and hence $\mathcal{R}_N$ is commutative. □

Theorem 2 *If $\mathcal{R} = \bigoplus_{g\in G} \mathcal{R}_g = \mathcal{R}_e \rtimes_\alpha^\sigma G$ is a G-crossed product and the following conditions are satisfied*:

(i) *$\mathcal{R}_e$ is commutative*
(ii) *N is a subgroup of G, such that $N \subseteq Z(G) \bigcap \ker(\sigma)$*
(iii) $\alpha(x, y) = \alpha(y, x)$ *for all* $(x, y) \in N \times N$

then

$$I \cap C_{\mathcal{R}}(\mathcal{R}_N) \neq \{0\}$$

for every non-zero two-sided ideal I in $\mathcal{R}$.

Proof It is clear that N is normal in G and it follows from Lemma 1 that $\mathcal{R}_e \rtimes_\alpha^\sigma G = \mathcal{R}_N \rtimes_{\alpha'}^{\sigma'} G/N$ for some maps σ' and α'. By our assumptions and Proposition 1 we see that $\mathcal{R}_N$ is commutative, and hence by Theorem 1 it follows that each non-zero two-sided ideal in $\mathcal{R}$ has a non-zero intersection with $C_{\mathcal{R}}(\mathcal{R}_N)$. □

Corollary 1 *If $\mathcal{R} = \bigoplus_{g\in G} \mathcal{R}_g = \mathcal{R}_e \rtimes^\sigma G$ is a G-graded skew group ring where $\mathcal{R}_e$ is commutative and $N \subseteq Z(G) \cap \ker(\sigma)$ is a subgroup of G, then*

$$I \cap C_{\mathcal{R}}(\mathcal{R}_N) \neq \{0\}$$

for every non-zero two-sided ideal I in $\mathcal{R}$.

Remark 2 Let $\mathcal{R} = \bigoplus_{g\in G} \mathcal{R}_g$ be a G-graded ring. If

$$\{e\} \subseteq \cdots \subseteq G_k \subseteq Z(G) \subseteq G_n \subseteq \cdots \subseteq G$$

is an increasing chain of subgroups of G, then we get

$$\mathcal{R}_e \subseteq \cdots \subseteq \mathcal{R}_{G_k} \subseteq \mathcal{R}_{Z(G)} \subseteq \mathcal{R}_{G_n} \subseteq \cdots \subseteq \mathcal{R}$$

as an increasing chain of subrings in $\mathcal{R}$ and the corresponding

$$C_{\mathcal{R}}(\mathcal{R}_e) \supseteq \cdots \supseteq C_{\mathcal{R}}(\mathcal{R}_{G_k}) \supseteq C_{\mathcal{R}}(\mathcal{R}_{Z(G)}) \supseteq C_{\mathcal{R}}(\mathcal{R}_{G_n}) \supseteq \cdots \supseteq C_{\mathcal{R}}(\mathcal{R}) = Z(\mathcal{R})$$

as a decreasing chain of subrings in $\mathcal{R}$. The existence of non-trivial subgroups N of G satisfying the conditions of Theorem 2 therefore provides more precise information about the ideals in the crossed product than the previous Theorem 1. By the arguments made above it is clear that $N = Z(G) \cap \ker(\sigma)$, the biggest normal subgroup to fit into our theorems, is the most interesting case to consider since it makes $C_{\mathcal{R}}(\mathcal{R}_N)$ as small as possible.

4 Strongly Graded Rings

In this section we let $\mathcal{R} = \bigoplus_{g\in G} \mathcal{R}_g$ be a strongly G-graded ring, not necessarily a crossed product. It follows that $1_{\mathcal{R}} \in \mathcal{R}_e$ since $\mathcal{R}$ is G-graded (see [33, Proposition 1.1.1]), and that $\mathcal{R}_g \mathcal{R}_{g^{-1}} = \mathcal{R}_e$ for each $g \in G$ since $\mathcal{R}$ is strongly G-graded. Thus, for each $g \in G$ there exists a positive integer n_g and elements $a_g^{(i)} \in \mathcal{R}_g$, $b_{g^{-1}}^{(i)} \in \mathcal{R}_{g^{-1}}$ for $i \in \{1, \dots, n_g\}$, such that

$$\sum_{i=1}^{n_g} a_g^{(i)} b_{g^{-1}}^{(i)} = 1_{\mathcal{R}}. \tag{4}$$

For every $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$, and in particular for every $\lambda \in Z(\mathcal{R}_e) \subseteq C_{\mathcal{R}}(\mathcal{R}_e)$, and $g \in G$ we define

$$\sigma_g(\lambda) = \sum_{i=1}^{n_g} a_g^{(i)} \lambda b_{g^{-1}}^{(i)}. \tag{5}$$

Remark 3 The definition of σ_g is independent of the choice of the $a_g^{(i)}$'s and $b_{b^{-1}}^{(i)}$'s. Indeed, given positive integers n_g, n'_g and elements $a_g^{(i)}, c_g^{(j)} \in \mathcal{R}_g$, $b_{g^{-1}}^{(i)}, d_{g^{-1}}^{(j)} \in \mathcal{R}_{g^{-1}}$ for $i \in \{1, \dots, n_g\}$ and $j \in \{1, \dots, n'_g\}$ such that

$$\sum_{i=1}^{n_g} a_g^{(i)} b_{g^{-1}}^{(i)} = 1_{\mathcal{R}} \quad \text{and} \quad \sum_{j=1}^{n'_g} c_g^{(j)} d_{g^{-1}}^{(j)} = 1_{\mathcal{R}},$$

for $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$ we get

$$\begin{aligned}
&\left(\sum_{i=1}^{n_g} a_g^{(i)} \lambda b_{g^{-1}}^{(i)}\right) - \left(\sum_{j=1}^{n'_g} c_g^{(j)} \lambda d_{g^{-1}}^{(j)}\right) \\
&\quad = 1_{\mathcal{R}} \left(\sum_{i=1}^{n_g} a_g^{(i)} \lambda b_{g^{-1}}^{(i)}\right) - \left(\sum_{j=1}^{n'_g} c_g^{(j)} \lambda d_{g^{-1}}^{(j)}\right) 1_{\mathcal{R}}
\end{aligned}$$

$$= \sum_{j=1}^{n'_g} \sum_{i=1}^{n_g} c_g^{(j)} \underbrace{d_{g^{-1}}^{(j)} a_g^{(i)}}_{\in \mathcal{R}_e} \lambda\, b_{g^{-1}}^{(i)} - \sum_{j=1}^{n'_g} \sum_{i=1}^{n_g} c_g^{(j)} \lambda\, d_{g^{-1}}^{(j)} a_g^{(i)} b_{g^{-1}}^{(i)}$$

$$= \sum_{j=1}^{n'_g} \sum_{i=1}^{n_g} c_g^{(j)} \lambda\, d_{g^{-1}}^{(j)} a_g^{(i)} b_{g^{-1}}^{(i)} - \sum_{j=1}^{n'_g} \sum_{i=1}^{n_g} c_g^{(j)} \lambda\, d_{g^{-1}}^{(j)} a_g^{(i)} b_{g^{-1}}^{(i)} = 0.$$

Lemma 2 *Let $\mathcal{R} = \bigoplus_{g\in G} \mathcal{R}_g$ be a strongly G-graded ring. If $a \in \mathcal{R}$ is such that*

$$a\,\mathcal{R}_g = \{0\}$$

for some $g \in G$, then $a = 0$.

Proof Suppose that $a\,\mathcal{R}_g = \{0\}$ for some $g \in G$, $a \in \mathcal{R}$. We then have $a\,\mathcal{R}_g\,\mathcal{R}_{g^{-1}} = \{0\}$ or equivalently $a\,\mathcal{R}_e = \{0\}$. From the fact that $1_{\mathcal{R}} \in \mathcal{R}_e$, we conclude that $a = 0$. □

For the convenience of the reader we include the following lemma from [22, Proposition 1.8].

Lemma 3 *Let $\mathcal{R} = \bigoplus_{g\in G} \mathcal{R}_g$ be a strongly G-graded ring, $g \in G$ and write $\sum_{i=1}^{n_g} a_g^{(i)} b_{g^{-1}}^{(i)} = 1_{\mathcal{R}}$ for some $n_g > 0$ and $a_g^{(i)} \in \mathcal{R}_g$, $b_{g^{-1}}^{(i)} \in \mathcal{R}_{g^{-1}}$ for $i \in \{1, \ldots, n_g\}$. For each $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$ define $\sigma_g(\lambda)$ by $\sigma_g(\lambda) = \sum_{i=1}^{n_g} a_g^{(i)} \lambda\, b_{g^{-1}}^{(i)}$. The following properties hold*:

(i) *$\sigma_g(\lambda)$ is a unique element of $\mathcal{R}$ satisfying*

$$r_g\,\lambda = \sigma_g(\lambda)\, r_g, \quad \forall r_g \in \mathcal{R}_g. \tag{6}$$

Furthermore, $\sigma_g(\lambda) \in C_{\mathcal{R}}(\mathcal{R}_e)$ and if $\lambda \in Z(\mathcal{R}_e)$, then $\sigma_g(\lambda) \in Z(\mathcal{R}_e)$.

(ii) *The group G acts as automorphisms of the rings $C_{\mathcal{R}}(\mathcal{R}_e)$ and $Z(\mathcal{R}_e)$, with each $g \in G$ sending any $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$ and $\lambda \in Z(\mathcal{R}_e)$, respectively, into $\sigma_g(\lambda)$.*

(iii) *$Z(\mathcal{R}) = \{\lambda \in C_{\mathcal{R}}(\mathcal{R}_e) \mid \sigma_g(\lambda) = \lambda,\ \forall g \in G\}$, i.e. $Z(\mathcal{R})$ is the fixed subring $C_{\mathcal{R}}(\mathcal{R}_e)^G$ of $C_{\mathcal{R}}(\mathcal{R}_e)$ with respect to the action of G.*

Proof (i) Let $g \in G$. If $r_g \in \mathcal{R}_g$, then $b_{g^{-1}}^{(i)} r_g \in \mathcal{R}_{g^{-1}}\mathcal{R}_g = \mathcal{R}_e$ and so $b_{g^{-1}}^{(i)} r_g$ commutes with $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$ for each $i \in \{1, \ldots, n_g\}$. It follows that

$$\sigma_g(\lambda)\, r_g = \sum_{i=1}^{n_g} a_g^{(i)} \lambda\, b_{g^{-1}}^{(i)} r_g = \sum_{i=1}^{n_g} a_g^{(i)} b_{g^{-1}}^{(i)} r_g\,\lambda = r_g\,\lambda.$$

Take an arbitrary $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$ and let $x \in \mathcal{R}$ be an element satisfying $a_g^{(i)} \lambda = x\, a_g^{(i)}$ for all $i \in \{1, \ldots, n_g\}$. This yields

$$\sigma_g(\lambda) = \sum_{i=1}^{n_g} a_g^{(i)} \lambda\, b_{g^{-1}}^{(i)} = \sum_{i=1}^{n_g} x\, a_g^{(i)} b_{g^{-1}}^{(i)} = x$$

which shows that $\sigma_g(\lambda)$ is a unique element satisfying (6). By the strong gradation it follows that if $\lambda \in \mathcal{R}_e$, then $\sigma_g(\lambda) \in \mathcal{R}_e$. In particular if $\lambda \in Z(\mathcal{R}_e) \subseteq C_{\mathcal{R}}(\mathcal{R}_e)$, then $\sigma_g(\lambda) \in Z(\mathcal{R}_e)$.

Indeed, for $\lambda \in Z(\mathcal{R}_e)$ and $c \in \mathcal{R}_e$ we have

$$c\,\sigma_g(\lambda) = 1_{\mathcal{R}}\,c\,\sigma_g(\lambda) = \sum_{i=1}^{n_g}\sum_{j=1}^{n'_g} a_g'^{(j)}\,\underbrace{b_{g^{-1}}'^{(j)}\,c\,a_g^{(i)}}_{\in\mathcal{R}_e}\,\lambda\,b_{g^{-1}}^{(i)}$$

$$= \sum_{i=1}^{n_g}\sum_{j=1}^{n'_g} a_g'^{(j)}\,\lambda\,b_{g^{-1}}'^{(j)}\,c\,a_g^{(i)}\,b_{g^{-1}}^{(i)} = \sigma_g(\lambda)\,c\,1_{\mathcal{R}} = \sigma_g(\lambda)\,c$$

where $\sum_{j=1}^{n'_g} a_g'^{(j)}\,b_{g^{-1}}'^{(j)} = 1_{\mathcal{R}}$, and hence it only remains to verify that $\sigma_g(\lambda) \in C_{\mathcal{R}}(\mathcal{R}_e)$ for an arbitrary $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$. If $r_g \in \mathcal{R}_g$ and $z \in \mathcal{R}_e$, then $z\,r_g \in \mathcal{R}_e\mathcal{R}_g = \mathcal{R}_g$, so we have

$$(\sigma_g(\lambda)\,z)\,r_g = \sigma_g(\lambda)\,(z\,r_g) = (z\,r_g)\,\lambda$$
$$= z\,(r_g\,\lambda) = z\,(\sigma_g(\lambda)\,r_g) = (z\,\sigma_g(\lambda))\,r_g$$

which means that $(\sigma_g(\lambda)\,z - z\,\sigma_g(\lambda))\,\mathcal{R}_g = \{0\}$. By Lemma 2 we conclude that $\sigma_g(\lambda)\,z = z\,\sigma_g(\lambda)$ and hence $\sigma_g(\lambda) \in C_{\mathcal{R}}(\mathcal{R}_e)$.

(ii) Since $1_{\mathcal{R}} \in \mathcal{R}_e$, we have for each $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$ that

$$\lambda = 1_{\mathcal{R}}\,\lambda = \sigma_e(\lambda)\,1_{\mathcal{R}} = \sigma_e(\lambda).$$

If $g, h \in G$, $r_g \in \mathcal{R}_g$ and $r_h \in \mathcal{R}_h$, then $r_g\,r_h \in \mathcal{R}_{gh}$ and for $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$ we have

$$\sigma_{gh}(\lambda)\,(r_g\,r_h) = (r_g\,r_h)\,\lambda = r_g\,(r_h\,\lambda) = r_g\,(\sigma_h(\lambda)\,r_h)$$
$$= (r_g\,\sigma_h(\lambda))\,r_h = (\sigma_g(\sigma_h(\lambda))\,r_g)\,r_h = \sigma_g(\sigma_h(\lambda))\,(r_g\,r_h)$$

The products of the form $r_g\,r_h$ generate the submodule $\mathcal{R}_{gh}$ and by Lemma 2 we conclude that

$$\sigma_g(\sigma_h(\lambda)) = \sigma_{gh}(\lambda)$$

proving that $(g,\lambda) \mapsto \sigma_g(\lambda)$ is an action of G on the set $C_{\mathcal{R}}(\mathcal{R}_e)$. Now take an arbitrary $g \in G$ and fix it. By the definition of $\sigma_g(\lambda)$, the map $\lambda \mapsto \sigma_g(\lambda)$ is clearly additive. For some positive integer $n_{g^{-1}}$ we may choose $c_{g^{-1}}^{(j)} \in \mathcal{R}_{g^{-1}}$ and $d_g^{(j)} \in \mathcal{R}$ for $j \in \{1,\dots,n_{g^{-1}}\}$, such that $1_{\mathcal{R}} = \sum_{j=1}^{n_{g^{-1}}} c_{g^{-1}}^{(j)}\,d_g^{(j)}$ and define $\sigma_{g^{-1}}$ following (5). Then, for each $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$, we get

$$\sigma_{g^{-1}}(\sigma_g(\lambda)) = \sum_{j=1}^{n_{g^{-1}}} c_{g^{-1}}^{(j)}\left(\sum_{i=1}^{n_g} a_g^{(i)}\,\lambda\,b_{g^{-1}}^{(i)}\right) d_g^{(j)} = \sum_{j=1}^{n_{g^{-1}}} c_{g^{-1}}^{(j)}\left(\sum_{i=1}^{n_g} a_g^{(i)}\,\lambda\,\underbrace{b_{g^{-1}}^{(i)}\,d_g^{(j)}}_{\in\mathcal{R}_e}\right)$$

$$= \sum_{j=1}^{n_{g^{-1}}} c_{g^{-1}}^{(j)}\left(\sum_{i=1}^{n_g} a_g^{(i)}\,b_{g^{-1}}^{(i)}\,d_g^{(j)}\,\lambda\right) = \sum_{j=1}^{n_{g^{-1}}} c_{g^{-1}}^{(j)}\,\underbrace{\left(\sum_{i=1}^{n_g} a_g^{(i)}\,b_{g^{-1}}^{(i)}\right)}_{=1_{\mathcal{R}}} d_g^{(j)}\,\lambda$$

$$= \underbrace{\sum_{j=1}^{n_{g^{-1}}} c_{g^{-1}}^{(j)}\,d_g^{(j)}}_{=1_{\mathcal{R}}}\,\lambda = \lambda$$

and hence $\sigma_{g^{-1}}$ is the inverse of σ_g. For any $\lambda, t \in C_{\mathcal{R}}(\mathcal{R}_e)$ and $r_g \in \mathcal{R}_g$, we get

$$\begin{aligned}\sigma_g(\lambda t)\, r_g &= r_g\,(\lambda t) = (r_g\,\lambda)\, t = (\sigma_g(\lambda)\, r_g)\, t \\ &= \sigma_g(\lambda)\,(r_g\, t) = \sigma_g(\lambda)\,(\sigma_g(t)\, r_g) = (\sigma_g(\lambda)\,\sigma_g(t))\, r_g.\end{aligned}$$

By Lemma 2 this implies $\sigma_g(\lambda t) = \sigma_g(\lambda)\,\sigma_g(t)$. Therefore, for each $g \in G$, the map $\lambda \mapsto \sigma_g(\lambda)$ is an automorphism of the ring $C_{\mathcal{R}}(\mathcal{R}_e)$.

(iii) Since $\mathcal{R} = \bigoplus_{g\in G} \mathcal{R}_g$ is strongly G-graded, we have

$$Z(\mathcal{R}) = \bigcap_{g\in G} C_{\mathcal{R}}(\mathcal{R}_g) = \{\lambda \in C_{\mathcal{R}}(\mathcal{R}_e) \mid \lambda \in C_{\mathcal{R}}(\mathcal{R}_g), \quad \forall g \in G\}$$

and the result now follows from the fact that an element $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$ centralizes $\mathcal{R}_g$, $g \in G$, if and only if $\sigma_g(\lambda) = \lambda$. Indeed, if $\sigma_g(\lambda) = \lambda$ for some $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$, then clearly λ centralizes $\mathcal{R}_g$. Conversely, if we suppose that $\mathcal{R}_g$ is centralized by some $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$, then we have $(\sigma_g(\lambda) - \lambda)\,\mathcal{R}_g = \{0\}$ and hence by Lemma 2 we have $\sigma_g(\lambda) = \lambda$. □

Remark 4 We have shown that G acts as automorphisms of $C_{\mathcal{R}}(\mathcal{R}_e)$. However, note that since $\mathcal{R}_e$ is not assumed to be commutative, we may have $\mathcal{R}_e \not\subseteq C_{\mathcal{R}}(\mathcal{R}_e)$ and hence G does not necessarily act as automorphisms of $\mathcal{R}_e$. This should be compared to the case of an algebraic crossed product as described in the previous section. For crossed products, if $\mathcal{R}_e$ is commutative, then we see that G acts as automorphisms of $\mathcal{R}_e$, but in general this is not true.

Lemma 4 *Let $\mathcal{R} = \bigoplus_{g\in G} \mathcal{R}_g$ be a strongly G-graded ring and $\sigma : G \to \mathrm{Aut}(C_{\mathcal{R}}(\mathcal{R}_e))$ defined as in (5). If $\mathcal{R}_e$ is maximal commutative in $\mathcal{R}$, then* $\ker(\sigma) = \{e\}$.

Proof By our assumption $\mathcal{R}_e = C_{\mathcal{R}}(\mathcal{R}_e)$ is maximal commutative in $\mathcal{R}$ and hence for each $g \neq e$ and all $r_g \in \mathcal{R}_g$, there must exist some $\lambda \in \mathcal{R}_e$ such that $\lambda\, r_g \neq r_g\, \lambda = \sigma_g(\lambda)\, r_g$, using the definition of $\sigma : G \to \mathrm{Aut}(\mathcal{R}_e)$. Hence $\sigma_g \neq \mathrm{id}_{\mathcal{R}_e}$ for each $g \neq e$. □

We shall now state an obvious, but useful lemma.

Lemma 5 *If $\mathcal{R} = \bigoplus_{g\in G} \mathcal{R}_g$ is a strongly G-graded ring, then*

$$\begin{aligned}C_{\mathcal{R}}(\mathcal{R}_e) &= \left\{\lambda = \sum_{g\in G} \lambda_g \in \mathcal{R} \,\middle|\, \lambda_g \in \mathcal{R}_g,\ r_e\,\lambda_g = \lambda_g\, r_e, \quad \forall g \in G, \forall r_e \in \mathcal{R}_e\right\} \\ &= \left\{\lambda = \sum_{g\in G} \lambda_g \in \mathcal{R} \,\middle|\, \lambda_g \in \mathcal{R}_g \cap C_{\mathcal{R}}(\mathcal{R}_e), \quad \forall g \in G\right\} \\ &= \bigoplus_{g\in G} \left(\mathcal{R}_g \cap C_{\mathcal{R}}(\mathcal{R}_e)\right).\end{aligned}$$

The following theorem is a generalization of (iii) of Lemma 3.

Theorem 3 *Let $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$ be a strongly G-graded ring, H a subgroup of G and denote $\mathcal{R}_H = \bigoplus_{h \in H} \mathcal{R}_h$. If $\sigma : G \to \mathrm{Aut}(C_{\mathcal{R}}(\mathcal{R}_e))$ is the action defined in (5), then it follows that*

$$\begin{aligned} C_{\mathcal{R}}(\mathcal{R}_H) &= \left\{ \lambda = \sum_{g \in G} \lambda_g \in \mathcal{R} \;\middle|\; \lambda_g \in C_{\mathcal{R}}(\mathcal{R}_e) \cap \mathcal{R}_g,\ \sigma_h(\lambda_g) = \lambda_{hgh^{-1}},\ \forall g \in G, \forall h \in H \right\} \\ &= \{\lambda \in C_{\mathcal{R}}(\mathcal{R}_e) \mid \sigma_h(\lambda) = \lambda,\ \forall h \in H\}. \end{aligned}$$

Proof Let $\lambda = \sum_{g \in G} \lambda_g \in C_{\mathcal{R}}(\mathcal{R}_H)$, with $\lambda_g \in \mathcal{R}_g$, be arbitrary. Since $\mathcal{R}_e \subseteq \mathcal{R}_H$, we have $\lambda \in C_{\mathcal{R}}(\mathcal{R}_e)$ and from Lemma 5 we see that $\lambda_g \in C_{\mathcal{R}}(\mathcal{R}_e)$ for each $g \in G$. For every $r_h \in \mathcal{R}_h$, $h \in H$, we have

$$r_h \sum_{g \in G} \lambda_g = \sum_{g \in G} \lambda_g r_h$$

since $\lambda \in C_{\mathcal{R}}(\mathcal{R}_H)$, but $\lambda_g \in C_{\mathcal{R}}(\mathcal{R}_e)$ so from (6) we get

$$\sum_{g \in G} \sigma_h(\lambda_g) r_h = \sum_{g \in G} \lambda_g r_h$$

which is an equality in $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$. If we look in the $\mathcal{R}_h \mathcal{R}_g \mathcal{R}_{h^{-1}} \mathcal{R}_h = \mathcal{R}_{hg}$ component, for all $g \in G, h \in H$, we deduce that

$$\sigma_h(\lambda_g)\, r_h = \lambda_{hgh^{-1}}\, r_h, \quad \forall\, r_h \in \mathcal{R}_h$$

since the sum $\mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g$ is direct. Applying the above equality to the elements $a_h^{(i)}$ of $\mathcal{R}_h$ in (4), we get

$$\sigma_h(\lambda_g)\, a_h^{(i)} = \lambda_{hgh^{-1}}\, a_h^{(i)}$$

for each $i \in \{1, \dots, n_h\}$, which implies

$$\begin{aligned} \sigma_h(\lambda_g) = \sigma_h(\lambda_g)\, 1_{\mathcal{R}} = \sigma_h(\lambda_g) \sum_{i=1}^{n_h} a_h^{(i)}\, b_{h^{-1}}^{(i)} &= \sum_{i=1}^{n_h} \sigma_h(\lambda_g)\, a_h^{(i)}\, b_{h^{-1}}^{(i)} \\ = \sum_{i=1}^{n_h} \lambda_{hgh^{-1}}\, a_h^{(i)}\, b_{h^{-1}}^{(i)} = \lambda_{hgh^{-1}} \sum_{i=1}^{n_h} a_h^{(i)}\, b_{h^{-1}}^{(i)} &= \lambda_{hgh^{-1}}. \end{aligned}$$

Conversely, let $\lambda = \sum_{g \in G} \lambda_g \in \mathcal{R}$, where $\lambda_g \in C_{\mathcal{R}}(\mathcal{R}_e) \cap \mathcal{R}_g$ and $\sigma_h(\lambda_g) = \lambda_{hgh^{-1}}$, for all $g \in G, h \in H$. Then, for every $r_h \in \mathcal{R}_h$,

$$r_h\, \lambda = \sum_{g \in G} r_h\, \lambda_g = \sum_{g \in G} \sigma_h(\lambda_g)\, r_h = \sum_{g \in G} \lambda_{hgh^{-1}}\, r_h = \sum_{k \in G} \lambda_k\, r_h = \lambda\, r_h$$

and hence $\lambda \in C_{\mathcal{R}}(\mathcal{R}_H)$. This concludes the proof. □

Remark 5 If $\mathcal{R}_e$ is commutative, then $\mathcal{R}_e = Z(\mathcal{R}_e)$. Thus, if $\mathcal{R}_e$ is commutative, then G acts as automorphisms of the ring $\mathcal{R}_e$.

Theorem 4 *Let* $\mathcal{R} = \bigoplus_{g\in G} \mathcal{R}_g$ *be a strongly* G*-graded ring where* $\mathcal{R}_e$ *is commutative and* $\ker(\sigma)$ *is the kernel of the previously defined action* $\sigma : G \to \mathrm{Aut}(\mathcal{R}_e)$, *i.e.* $\ker(\sigma) = \{g \in G \mid \sigma_g(\lambda_e) = \lambda_e,\ \forall \lambda_e \in \mathcal{R}_e\}$. *If* H *is a subgroup of* G *which is contained in* $\ker(\sigma) \cap Z(G)$, *then*

$$I \cap C_{\mathcal{R}}(\mathcal{R}_H) \neq \{0\}$$

for every non-zero two-sided ideal I *in* $\mathcal{R}$.

Proof Let I be an arbitrary non-zero two-sided ideal in $\mathcal{R}$. For every $h \in H$ and $r_h \in \mathcal{R}_h$ we define a kill operator

$$D_{r_h} : \mathcal{R} \to \mathcal{R},\ \ D_{r_h}\left(\sum_{g\in G} \lambda_g\right) = r_h \sum_{g\in G} \lambda_g - \sum_{g\in G} \lambda_g r_h = \sum_{k\in G} d_k.$$

Note that for every non-zero summand $\lambda_g \in \mathcal{R}_g$ of $\lambda = \sum_{g\in G} \lambda_g$, we take a summand $d_{hg} = r_h\lambda_g - \lambda_g r_h \in \mathcal{R}_{gh} = \mathcal{R}_{hg}$ of $D_{r_h}(\lambda)$ which may be zero or nonzero, but

$$d_h = r_h \lambda_e - \lambda_e r_h = \sigma_h(\lambda_e) r_h - \lambda_e r_h = \lambda_e r_h - \lambda_e r_h = 0.$$

Thus, for $\lambda = \sum_{g\in G} \lambda_g \in \mathcal{R}$ with $\lambda_e \neq 0$ and $D_{r_h}(\lambda) = \sum_{k\in G} d_k$, we get

$$\#\operatorname{supp}(\lambda) = \#\{s \in G \mid \lambda_s \neq 0\} > \#\{s \in G \mid d_s \neq 0\} = \#\operatorname{supp}(D_{r_h}(\lambda)).$$

Furthermore, note that for all $r_h \in \mathcal{R}_h$, I is invariant under D_{r_h} and

$$C_{\mathcal{R}}(\mathcal{R}_H) = \bigcap_{h\in H, r_h\in\mathcal{R}_h} \ker(D_{r_h}).$$

Now, let $\lambda = \sum_{g\in G} \lambda_g \in I$ be non-zero. We can assume that $\lambda_e \neq 0$. Otherwise there exists some non-zero $\lambda' = \sum_{g\in G} \lambda'_g \in I$ with $\lambda'_e \neq 0$. Indeed, $\lambda \neq 0$ so there exists $t \in G$ such that $\lambda_t \neq 0$. There exists, as well, some $j \in \{1, \ldots, n_t\}$ such that $b^{(j)}_{t^{-1}}\lambda_t \neq 0$, where $b^{(j)}_{t^{-1}} \in \mathcal{R}_{t^{-1}}$ is as in (4), because if $b^{(i)}_{t^{-1}}\lambda_t = 0$, $\forall i \in \{1, \ldots, n_t\}$, then

$$\lambda_t = 1_{\mathcal{R}} \cdot \lambda_t = \sum_{i=1}^{n_t} a^{(i)}_t b^{(i)}_{t^{-1}} \lambda_t = 0.$$

Thus, for every non-zero element λ of I we can have an element $b^{(j)}_{t^{-1}}\lambda = \lambda' = \sum_{g\in G} \lambda'_g$ of I with $\lambda'_e = b^{(j)}_{t^{-1}}\lambda_t \neq 0$, and $\#\operatorname{supp}(\lambda) \geq \#\operatorname{supp}(\lambda') \geq 1$.

We return to the element $\lambda = \sum_{g\in G} \lambda_g \in I$ with $\lambda_e \neq 0$. If $\lambda \in C_{\mathcal{R}}(\mathcal{R}_H)$ we have nothing to prove. If $\lambda \notin C_{\mathcal{R}}(\mathcal{R}_H)$, then there exists $h \in H$ and $r_h \in \mathcal{R}_h$ such that $D_{r_h}(\lambda) \neq 0$. But $D_{r_h}(\lambda) \in I$ so we have a new element of I with smaller support. If we continue in the same way, the procedure must eventually end, because $\operatorname{supp}(\lambda) < \infty$. So, there will be a stop of this procedure which gives an element $\mu = \sum_{g\in G} \mu_g \in I \cap C_{\mathcal{R}}(\mathcal{R}_H)$, with $\mu_e \neq 0$. □

The following corollary generalizes Theorem 2 to the situation when $\mathcal{R}_N$ need not necessarily be commutative.

Corollary 2 *If* $\mathcal{R} = \bigoplus_{g\in G} \mathcal{R}_g = \mathcal{R}_e \rtimes^{\sigma'}_{\alpha'} G$ *is a* G*-crossed product and both of the following conditions are satisfied*:

(i) *$\mathcal{R}_e$ is commutative*
(ii) *N is a subgroup of G, such that $N \subseteq Z(G) \cap \ker(\sigma')$*

then

$$I \cap C_{\mathcal{R}}(\mathcal{R}_N) \neq \{0\}$$

for every non-zero two-sided ideal I in $\mathcal{R}$.

Proof For each $g \in G$ we may choose $u_g \in U(R) \cap \mathcal{R}_g$. It follows from [33, Proposition 1.1.1] that $(u_g)^{-1} \in \mathcal{R}_{g^{-1}}$. Clearly $u_g u_g^{-1} = 1_{\mathcal{R}}$ and following (5) we define $\sigma_g(a) = u_g\, a\, u_g^{-1}$ for all $a \in C_{\mathcal{R}}(\mathcal{R}_e)$. In particular $\mathcal{R}_e \subseteq C_{\mathcal{R}}(\mathcal{R}_e)$ since $\mathcal{R}_e$ is commutative, and it is now clear that the restriction of σ_g to $\mathcal{R}_e$ is equal to σ'_g for each $g \in G$. From Theorem 4 it now follows that each non-zero two-sided ideal in $\mathcal{R}$ has a non-zero intersection with $C_{\mathcal{R}}(\mathcal{R}_N)$. □

The following corollary generalizes [36, Theorem 2] from G-crossed products to strongly G-graded rings.

Corollary 3 *If $\mathcal{R} = \bigoplus_{g\in G} \mathcal{R}_g$ is a strongly G-graded ring where $\mathcal{R}_e$ is commutative, then*

$$I \cap C_{\mathcal{R}}(\mathcal{R}_e) \neq \{0\}$$

for every non-zero two-sided ideal I in $\mathcal{R}$.

Proof Consider the subgroup $\{e\}$ of G. Clearly $\{e\} \subseteq Z(G) \cap \ker(\sigma)$ and since $\mathcal{R}_e$ is commutative it follows from Theorem 4 that $I \cap C_{\mathcal{R}}(\mathcal{R}_e) \neq \{0\}$ for each non-zero two-sided ideal I in $\mathcal{R}$. □

Corollary 4 *If $\mathcal{R} = \bigoplus_{g\in G} \mathcal{R}_g$ is a strongly G-graded ring where $\mathcal{R}_e$ is maximal commutative in $\mathcal{R}$, then*

$$I \cap \mathcal{R}_e \neq \{0\}$$

for every non-zero two-sided ideal I in $\mathcal{R}$.

Proof By the assumption $\mathcal{R}_e$ is maximal commutative in $\mathcal{R}$, i.e. $C_{\mathcal{R}}(\mathcal{R}_e) = \mathcal{R}_e$, and hence the desired conclusion follows immediately from Corollary 3. □

Corollary 5 *If $\mathcal{R} = \bigoplus_{g\in G} \mathcal{R}_g$ is a twisted group ring, where $\mathcal{R}_e$ is commutative and G is abelian, then*

$$I \cap Z(\mathcal{R}) \neq \{0\}$$

for every non-zero two-sided ideal I in $\mathcal{R}$.

Proof Since $\mathcal{R}$ is a twisted group ring, all homogeneous elements commute with $\mathcal{R}_e$. Hence, $\ker(\sigma) = G$ and moreover $G = Z(G)$ since G is abelian. Consider the subgroup $\ker(\sigma) \cap Z(G) = G \cap G = G$ of G and note that $C_{\mathcal{R}}(\mathcal{R}_G) = Z(\mathcal{R})$. By our assumptions $\mathcal{R}_e$ is commutative and it now follows from Theorem 4 that

$$I \cap Z(\mathcal{R}) \neq \{0\}$$

for each non-zero two-sided ideal I in $\mathcal{R}$. □

Remark 6 It was shown in [43, Theorem 2] that if $\mathcal{R}$ is a semiprime P.I. ring, then $I \cap Z(R) \neq \{0\}$ for each non-zero ideal I in $\mathcal{R}$.

5 Crystalline Graded Rings

We shall begin this section by recalling the definition of a crystalline graded ring. We would also like to emphasize that rings of this class are in general not strongly graded.

Definition 1 (Pre-crystalline graded ring) An associative and unital ring $\mathcal{A}$ is said to be *pre-crystalline graded* if:

(i) there is a group G (with neutral element e),
(ii) there is a map $u : G \to \mathcal{A}, \quad g \mapsto u_g$ such that $u_e = 1_{\mathcal{A}}$ and $u_g \neq 0$ for every $g \in G$,
(iii) there is a subring $\mathcal{A}_0 \subseteq \mathcal{A}$ containing $1_{\mathcal{A}}$,

such that the following conditions are satisfied:

(P1) $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_0\, u_g$;
(P2) For every $g \in G$, $u_g\, \mathcal{A}_0 = \mathcal{A}_0\, u_g$ and this is a free left $\mathcal{A}_0$-module of rank one;
(P3) The decomposition in P1 makes $\mathcal{A}$ into a G-graded ring with $\mathcal{A}_0 = \mathcal{A}_e$.

Lemma 6 (see [34]) *With notation and definitions as above*:

(i) *For every $g \in G$, there is a set map $\sigma_g : \mathcal{A}_0 \to \mathcal{A}_0$ defined by $u_g\, a = \sigma_g(a)\, u_g$ for $a \in \mathcal{A}_0$. The map σ_g is a surjective ring morphism. Moreover, $\sigma_e = \mathrm{id}_{\mathcal{A}_0}$.*
(ii) *There is a set map $\alpha : G \times G \to \mathcal{A}_0$ defined by $u_s\, u_t = \alpha(s,t)\, u_{st}$ for $s, t \in G$. For any triple $s, t, w \in G$ and $a \in \mathcal{A}_0$ the following equalities hold*:

$$\alpha(s,t)\alpha(st,w) = \sigma_s(\alpha(t,w))\alpha(s,tw) \tag{7}$$

$$\sigma_s(\sigma_t(a))\alpha(s,t) = \alpha(s,t)\sigma_{st}(a) \tag{8}$$

(iii) *For every $g \in G$ we have $\alpha(g,e) = \alpha(e,g) = 1_{\mathcal{A}_0}$ and $\alpha(g,g^{-1}) = \sigma_g(\alpha(g^{-1},g))$.*

A pre-crystalline graded ring $\mathcal{A}$ with the above properties will be denoted by $\mathcal{A}_0 \diamondsuit_\sigma^\alpha G$ and each element of this ring is written as a sum $\sum_{g \in G} r_g\, u_g$ with coefficients $r_g \in \mathcal{A}_0$, of which only finitely many are non-zero. In [34] it was shown that for pre-crystalline graded rings, the elements $\alpha(s,t)$ are normalizing elements of $\mathcal{A}_0$, i.e. $\mathcal{A}_0\, \alpha(s,t) = \alpha(s,t)\, \mathcal{A}_0$ for each $s, t \in G$. For a pre-crystalline graded ring $\mathcal{A}_0 \diamondsuit_\sigma^\alpha G$, we let $S(G)$ denote the multiplicative set in $\mathcal{A}_0$ generated by $\{\alpha(g, g^{-1}) \mid g \in G\}$ and let $S(G \times G)$ denote the multiplicative set generated by $\{\alpha(g,h) \mid g, h \in G\}$.

Lemma 7 (see [34]) *If $\mathcal{A} = \mathcal{A}_0 \diamondsuit_\sigma^\alpha G$ is a pre-crystalline graded ring, then the following assertions are equivalent*:

(i) *$\mathcal{A}_0$ is $S(G)$-torsion free.*
(ii) *$\mathcal{A}$ is $S(G)$-torsion free.*
(iii) *$\alpha(g, g^{-1})\, a_0 = 0$ for some $g \in G$ implies $a_0 = 0$.*
(iv) *$\alpha(g,h)\, a_0 = 0$ for some $g, h \in G$ implies $a_0 = 0$.*
(v) *$\mathcal{A}_0\, u_g = u_g\, \mathcal{A}_0$ is also free as a right $\mathcal{A}_0$-module, with basis u_g, for every $g \in G$.*
(vi) *For every $g \in G$, σ_g is bijective and hence a ring automorphism of $\mathcal{A}_0$.*

Definition 2 (Crystalline graded ring) A pre-crystalline graded ring $\mathcal{A}_0\Diamond_\sigma^\alpha G$, which is $S(G)$-torsion free, is said to be a *crystalline graded ring*.

Examples of crystalline graded rings are given by the algebraic crossed products, the generalized twisted group rings, the Weyl algebras, the quantum Weyl algebra, the generalized Weyl algebras, quantum sl_2, etc. For more examples we refer to [34].

Proposition 2 *Let $\mathcal{A}=\mathcal{A}_0\Diamond_\sigma^\alpha G$ be a pre-crystalline graded ring, H a subgroup of G and consider the subring $\mathcal{A}_H=\mathcal{A}_0\Diamond_\sigma^\alpha H$ in $\mathcal{A}$. The commutant of $\mathcal{A}_H$ in $\mathcal{A}$ is*

$$C_{\mathcal{A}}(\mathcal{A}_H)=\left\{\sum_{g\in G} r_g\, u_g \in \mathcal{A} \;\middle|\; r_{th^{-1}}\,\alpha(th^{-1},h)=\sigma_h(r_{h^{-1}t})\,\alpha(h,h^{-1}t),\right.$$

$$\left. r_t\,\sigma_t(a)=a\,r_t,\quad \forall a\in\mathcal{A}_0,\ \forall h\in H,\ \forall t\in G\right\}.$$

Proof Suppose that $\sum_{g\in G} r_g\,u_g\in C_{\mathcal{A}}(\mathcal{A}_H)$. Clearly $\mathcal{A}_0\subseteq\mathcal{A}_H$ and hence for any $a\in\mathcal{A}_0$, we have

$$a\Big(\sum_{g\in G} r_g\,u_g\Big)=\Big(\sum_{g\in G} r_g\,u_g\Big)a \quad\Longleftrightarrow\quad \sum_{g\in G} a\,r_g\,u_g=\sum_{g\in G} r_g\,\sigma_g(a)\,u_g$$

$$\Longleftrightarrow\quad a\,r_g=r_g\,\sigma_g(a),\quad \forall g\in G.$$

Furthermore, let $h\in H$ be arbitrary. Since $u_h\in\mathcal{A}_H$ we have

$$u_h\Big(\sum_{g\in G} r_g\,u_g\Big)=\Big(\sum_{g\in G} r_g\,u_g\Big)u_h$$

$$\Longleftrightarrow\quad \sum_{g\in G}\sigma_h(r_g)\,\alpha(h,g)\,u_{hg}=\sum_{g\in G} r_g\,\alpha(g,h)\,u_{gh}$$

$$\Longleftrightarrow\quad \sum_{t\in G}\sigma_h(r_{h^{-1}t})\,\alpha(h,h^{-1}t)\,u_t=\sum_{t\in G} r_{th^{-1}}\,\alpha(th^{-1},h)\,u_t$$

$$\Longleftrightarrow\quad \sigma_h(r_{h^{-1}t})\,\alpha(h,h^{-1}t)=r_{th^{-1}}\,\alpha(th^{-1},h),\quad \forall t\in G.$$

Conversely, suppose that the coefficients of an element $\sum_{g\in G} r_g\,u_g$ satisfy the following two conditions:

1. $r_t\,\sigma_t(a)=a\,r_t$ for all $a\in\mathcal{A}_0$ and $t\in G$.
2. $r_{th^{-1}}\,\alpha(th^{-1},h)=\sigma_h(r_{h^{-1}t})\,\alpha(h,h^{-1}t)$ for all $h\in H$, $t\in G$.

By carrying out calculations similar to the ones presented above, for any $\sum_{h\in H} b_h\,u_h\in\mathcal{A}_H$ we get

$$\Big(\sum_{h\in H} b_h\,u_h\Big)\Big(\sum_{g\in G} r_g\,u_g\Big)=\sum_{h\in H} b_h\Big(\sum_{g\in G} r_g\,u_g\Big)u_h=\sum_{h\in H}\Big(\sum_{g\in G} r_g\,u_g\Big)b_h\,u_h$$

$$=\Big(\sum_{g\in G} r_g\,u_g\Big)\Big(\sum_{h\in H} b_h\,u_h\Big)$$

which shows that $\sum_{g\in G} r_g\,u_g\in C_{\mathcal{A}}(\mathcal{A}_H)$. This concludes the proof. □

Remark 7 By putting $H = \{e\}$ respectively $H = G$ we get expressions for $C_{\mathcal{A}}(\mathcal{A}_0)$ respectively $Z(\mathcal{A})$.

Theorem 5 *If $\mathcal{A} = \mathcal{A}_0 \Diamond_\sigma^\alpha G$ is a crystalline graded ring, where $\mathcal{A}_0$ is commutative and H is a subgroup of G contained in $Z(G) \cap \ker(\sigma)$, then*

$$I \cap C_{\mathcal{A}}(\mathcal{A}_H) \neq \{0\}$$

for every non-zero two-sided ideal I in $\mathcal{A}$.

Proof Let I be an arbitrary non-zero two-sided ideal in $\mathcal{A}$ and assume that $\mathcal{A}_0$ is commutative. For each $g \in G$ we define a map

$$T_g : \mathcal{A} \to \mathcal{A}, \quad \sum_{s \in G} a_s\, u_s \mapsto \left(\sum_{s \in G} a_s\, u_s \right) u_g .$$

Note that, for each $g \in G$, I is invariant under T_g. We have

$$T_g \left(\sum_{s \in G} a_s\, u_s \right) = \left(\sum_{s \in G} a_s\, u_s \right) u_g = \sum_{s \in G} a_s\, \alpha(s, g)\, u_{sg}$$

for every $g \in G$. Suppose that $\sum_{s \in G} a_s\, u_s$ is such that $a_e = 0$ but $a_p \neq 0$ for some $p \neq e$. Then, we get $T_{p^{-1}}(\sum_{s \in G} a_s\, u_s) = \sum_{s \in G} a_s\, \alpha(s, p^{-1})\, u_{sp^{-1}}$. In particular we see that the coefficient in front of u_e is given by $a_p\, \alpha(p, p^{-1})$ and since $\mathcal{A}$ is assumed to have no $S(G)$-torsion and $\mathcal{A}_0$ is assumed to be commutative, we see by (iii) in Lemma 7 that $a_p\, \alpha(p, p^{-1}) \neq 0$. It is now clear that for each non-zero element $c \in \mathcal{A}$ it is always possible to choose some $g \in G$ and let T_g operate on c to end up with an element where the coefficient in front of u_e is non-zero. For each $b \in \mathcal{A}_0$ and $h \in H$, we define a map

$$D_{b u_h} : \mathcal{A} \to \mathcal{A}, \quad \sum_{s \in G} a_s\, u_s \mapsto b\, u_h \left(\sum_{s \in G} a_s\, u_s \right) - \left(\sum_{s \in G} a_s\, u_s \right) b\, u_h .$$

Note that, for each $b \in \mathcal{A}_0$ and $h \in H$, I is invariant under $D_{b u_h}$. Furthermore, due to the fact that $H \subseteq Z(G) \cap \ker(\sigma)$, we have

$$\begin{aligned} D_{b u_h} \left(\sum_{s \in G} a_s\, u_s \right) &= \left(\sum_{s \in G} b\, \sigma_h(a_s)\, \alpha(h, s)\, u_{hs} \right) - \left(\sum_{s \in G} a_s\, \sigma_s(b)\, \alpha(s, h)\, u_{sh} \right) \\ &= \sum_{s \in G} (b\, a_s\, \alpha(h, s) - a_s\, \sigma_s(b)\, \alpha(s, h))\, u_{hs} = \sum_{t \in G \setminus \{h\}} d_t\, u_t \end{aligned}$$

since $d_h = (b\, a_e\, \alpha(h, e) - a_e\, \sigma_e(b)\, \alpha(e, h)) = 0$. It is important to note that

$$C_{\mathcal{A}}(\mathcal{A}_H) = \bigcap_{b \in \mathcal{A}_0, h \in H} \ker(D_{b u_h})$$

and hence for any $\sum_{s \in G} a_s\, u_s \in \mathcal{A} \setminus C_{\mathcal{A}}(\mathcal{A}_H)$ we are always able to choose $b \in \mathcal{A}_0$ and $h \in H$ and the corresponding $D_{b u_h}$ and have $\sum_{s \in G} a_s\, u_s \notin \ker(D_{b u_h})$. Therefore we can always pick an operator $D_{b u_h}$ which kills the coefficient d_h (coming from a_e) without killing everything. Hence, if $a_e \neq 0$, the number of non-zero coefficients of the resulting element will always be reduced by at least one.

The ideal I is assumed to be non-zero, which means that we can pick some non-zero element $\sum_{s\in G} r_s\, u_s \in I$. If $\sum_{s\in G} r_s\, u_s \in C_{\mathcal{A}}(\mathcal{A}_H)$, then we are finished, so assume that this is not the case. Note that $r_s \neq 0$ for finitely many $s \in G$. Recall that the ideal I is invariant under T_g and $D_{b\,u_h}$ for all $g \in G$, $b \in \mathcal{A}_0$ and $h \in H$. We may now use the operators $\{T_g\}_{g\in G}$ and $\{D_{b\,u_h}\}_{b\in\mathcal{A}_0, h\in H}$ to generate new elements of I. More specifically, we may use the T_g:s to translate our element $\sum_{s\in G} r_s\, u_s$ into a new element which has a non-zero coefficient in front of u_e (if needed) after which we use the $D_{b\,u_h}$ operator to kill this coefficient and end up with yet another new element of I which is non-zero but has a smaller number of non-zero coefficients. We may repeat this procedure and in a finite number of iterations arrive at an element of I which lies in $C_{\mathcal{A}}(\mathcal{A}_H) \setminus \mathcal{A}_0$, and if not we continue the above procedure until we reach an element in $\mathcal{A}_0 \setminus \{0\}$. In particular $\mathcal{A}_0 \subseteq C_{\mathcal{A}}(\mathcal{A}_H)$ since $\mathcal{A}_0$ is commutative and hence $I \cap C_{\mathcal{A}}(\mathcal{A}_H) \neq \{0\}$. □

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