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On the Classification of N-Points Concentrating Solutions for Mean Field Equations and the Symmetry Properties of the N-Vortex Singular Hamiltonian on the Unit Disk

Daniele Bartolucci

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Abstract Motivated by the analysis of the multiple bubbling phenomenon (Bartolucci et al. in Commun. Partial Differ. Equ. 29(7–8):1241–1265, 2004) for a singular mean field equation on the unit disk (Bartolucci and Montefusco in Nonlinearity 19:611–631, 2006), for any $N \geq 3$ we characterize a subset of the $2\pi/N$-symmetric part of the critical set of the N-vortex singular Hamiltonian. In particular we prove that this critical subset is of saddle type. As a consequence of our result, and motivated by a recently posed open problem (Bartolucci et al. in Commun. Partial Differ. Equ. 29(7–8):1241–1265, 2004), we can prove the existence of a multiple bubbling sequence of solutions for the singular mean field equation.

Keywords Mean field equations · Concentrating and blow-up solutions · N-vortex Hamiltonian

1 Introduction

We are motivated by the analysis of the multiple bubbling phenomenon [2] for the singular mean field equation

$$\begin{cases} -\Delta u = \lambda \frac{e^u}{\int_\Omega e^u} - 4\pi\alpha\delta_{p=0} & \text{in } \Omega, \\ \mathrm{osc}_{\partial\Omega} u \leq C & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is any open smooth and bounded domain, $p = 0 \in \Omega$, $\lambda > 0$ and $\alpha > 0$. The analysis of the mean field equation (1.1) on two dimensional domains and on compact two manifolds has recently attracted a lot of attention due its many applications in mathematical physics. We refer the reader to [2–4, 7–13, 15–17, 21, 23, 29, 30, 32, 33], and the references quoted therein for further details. In particular we refer to [8, 9, 19] and the introduction of [3] for the application of (1.1) to the analysis of vortex-type configurations in turbulent Euler flows.

D. Bartolucci (✉)
Department of Mathematics, University of Rome “Tor Vergata”, Via della Ricerca Scientifica 1, 00133 Roma, Italy
e-mail: bartoluc@mat.uniroma2.it

Let us recall that, by using the concentration compactness theory for solutions of Liouville type equations [4, 7], one can see that any unbounded sequence of solutions for (1.1), satisfying $\mathrm{osc}_{\partial\Omega} u_n \leq C$ and $\lambda_n = O(1)$, admits a subsequence which satisfies,

$$\lambda_n \frac{e^{u_n}}{\int_\Omega e^{u_n}} \rightharpoonup \gamma\delta_{z=0} + 8\pi \sum_{j=1}^{N} \delta_{z=\varsigma_j}, \quad \text{as } n \to +\infty, \tag{1.2}$$

weakly in the sense of measures in Ω, for some $N \in \mathbb{N} \cup \{0\}$, $\gamma \in \{0, 8\pi(1+\alpha)\}$, $\{\varsigma_1, \ldots, \varsigma_N\} \subset \Omega \setminus \{p\}$ and $\gamma + 8\pi N \neq 0$. Clearly $\lambda_n \to \lambda = \gamma + 8\pi N$.

In particular, let us assume that concentration occurs at p, that is $\gamma = 8\pi(1+\alpha)$. By applying the results in [2], it follows that if $\alpha \notin \mathbb{N}$, then, for any n large enough, there exist $C > 0$ such that, for any given and small enough $r > 0$, we have

$$\left| u_n(z) - \log \frac{\mu_n^2 |z|^{2\alpha}}{(1 + \zeta_n \mu_n |z|^{2(1+\alpha)})^2} \right| \leq C, \quad \forall z \in B_r(p), \tag{1.3}$$

where $\mu_n^2 \to +\infty$, as $n \to +\infty$, and $\zeta_n = \frac{\lambda_n}{8(1+\alpha)^2}$. On the other side, it has been observed in [2] that the assumption $\alpha \notin \mathbb{N}$ is not a mere technical condition. Indeed, it is still possible for example that a sequence of solutions exist such that $\alpha = m \in \mathbb{N}$, and

$$\left| u_n(z) - \log \frac{\mu_n^2 |z|^{2m}}{(1 + \zeta_n \mu_n |z^{m+1} - b_n^{m+1}|^2)^2} \right| \leq C, \quad \forall z \in B_r(p), \tag{1.4}$$

with ζ_n as above, and

$$\mu_n \to +\infty, \quad b_n \to 0^+, \quad \mu_n |b_n|^{m+1} \to +\infty, \quad \text{as } n \to +\infty. \tag{1.5}$$

We refer to [2] for more details concerning this problem. This kind of multiple bubbling behavior corresponds to the case where we have $m+1$ non overlapping bubbles, converging simultaneously to the origin, but blowing up much faster and is in striking contrast with the case where solutions concentrates at points $\varsigma \neq p$. Indeed, it has been proved in [21] that solutions take up the form (1.3) with $\alpha = 0$ in this case. It is a challenging open problem, already posed in [2], to establish whether or not (1.4) describes the *full* set of non radial bubbling sequences for $\alpha \in \mathbb{N}$. As a first step toward the understanding of this problem, it seems interesting to analyze the "simpler" case, that is $\Omega = D = \{(x, y) \in \mathbb{R}^2 : |(x, y)| < 1\}$, the two dimensional open unit disk, with the singularity located at the origin $p = 0$,

$$\begin{cases} -\Delta u_\alpha = \lambda \frac{e^{u_\alpha}}{\int_D e^{u_\alpha}} - 4\pi\alpha\delta_{p=0} & \text{in } D, \\ u_\alpha = 0 & \text{on } \partial D. \end{cases} \tag{1.6}$$

Indeed, in this particular situation, one can try to classify all the blow up sequences. This program has been recently initiated in [3], in connection with a problem of independent interest, that is the mean field limit [8, 9] for a turbulent Euler flow with one negative vortex sink. In [3], the analysis of non radial [31], concentrating solutions [2, 11, 13, 17, 21], for the singular mean field equation (1.6) has been worked out in case $N = 1$ and $N = 2$ in (1.2). Here we will make a further step in this direction.

Let us recall that, by using the Pohozaev identity, one can see that there is no solution for (1.6) with $\lambda \geq 8\pi(1+\alpha)$. As a consequence, it is easy to see that, for any concentrating sequence of solutions for (1.6), which then satisfies (1.2), either $N = 0$ and then $\gamma = 8\pi(1+\alpha)$, or $\gamma = 0$ and then $1 \leq N \leq 1 + \alpha$.

We observe at this point that by a result in [13], we know that for any $1 \le N < 1+\alpha$ there exist a sequence of solutions for (1.6) which satisfies (1.2) with $\gamma = 0$. Among other things, it has been proved in [3] that these concentrating sequences of solutions for (1.6) satisfying (1.2) with $1 \le N < 1+\alpha$ exist in case $N = 1, 2$ if and only if either,

$$N = 1, \quad \text{and} \quad |\varsigma_1| = \rho_1(\alpha) = \left(\frac{\alpha}{\alpha+2}\right)^{1/2},$$

or,

$$N = 2, \quad \text{and} \quad \varsigma_1 = -\varsigma_2, \qquad |\varsigma_1| = |\varsigma_2| = \rho_2(\alpha) = \left(\frac{\alpha-1}{\alpha+3}\right)^{1/4}.$$

In particular, it has been conjectured in [3] that for any $3 \le N < 1+\alpha$, there exist a concentrating sequence of solutions for (1.6), satisfying (1.2) with $\gamma = 0$, such that $\{\varsigma_1, \dots, \varsigma_N\}$ are the vertices of a regular N-polygon which satisfies,

$$|\varsigma_1| = \cdots = |\varsigma_N| = \rho_N(\alpha) = \left(\frac{\alpha-N+1}{\alpha+N+1}\right)^{1/2N}. \tag{1.7}$$

Our first result is the following:

Theorem 1.1 *For any $3 \le N < 1+\alpha$, there exist a sequence of solutions $\{u_{\alpha,n}\}$ for (1.6), which satisfies (1.2) with $\{\varsigma_1, \dots, \varsigma_N\}$ being the vertices of regular N-polygon, if and only if (1.7) holds true.*

We observe at this point that, going trough the arguments in [13, 17], and using the symmetries of the problem, one can prove that indeed for any such N there exist a $2\pi/N$-symmetric sequence of solutions whose concentration points $\{\varsigma_1, \dots, \varsigma_N\}$ lye on the vertices of a regular N-polygon. It is an interesting open problem, which we will not discuss here, to establish the uniqueness (modulo rotations) of these solutions.

Thus, to establish Theorem 1.1, what we really need to show is that $\rho = \rho_N(\alpha) = (\frac{\alpha-N+1}{\alpha+N+1})^{1/2N}$ is a necessary condition for the existence of such a concentration sequence. This task may be accomplished by the analysis of the singular N-vortex Hamiltonian. Indeed, it is well known, see for example [22], that those concentration points $\{\varsigma_1, \dots, \varsigma_N\}$ of blowing up solutions, must be critical points of the N-vortex Hamiltonian (1.8) below.

For any $(x, y) \in D$, we set $z = x + \imath y$, $\overline{z} = x - \imath y$ and denote by $D^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ the punctured disc. Fix $N \ge 2$ and set $\Theta_N = \{\{z_1, \dots, z_N\} \subset (D)^N : z_m = z_j, \forall m \neq j\}$, and $\Omega_N = (D^*)^N \setminus \Theta_N$. For any given $\alpha > 0$, we define the N-vortex singular Hamiltonian, $\mathcal{F}_\alpha : \Omega_N \mapsto \mathbb{R}$,

$$\mathcal{F}_\alpha(z_1, \dots, z_N) = 2\alpha \sum_{m=1}^{N} \log|z_m| + 2\sum_{m=1}^{N} \log(1 - |z_m|^2) + 2\sum_{m\neq j}^{N} \log\frac{|z_m \overline{z}_j - 1|}{|z_m - z_j|}. \tag{1.8}$$

Of course, $\mathcal{F}_0 : D^N \setminus \Theta_N \mapsto \mathbb{R}$, in case $\alpha = 0$.

The analysis of $\mathcal{F}_\alpha$ is a problem of independent interest. It is motivated by the study of the point vortex model for two dimensional Euler flows [25]. Let us consider an Euler flow in D, with null velocity flux through the boundary and vorticity concentrated in small "blobs" around N given points $(z_1, \dots, z_N) \in D^N \setminus \Theta_N$. Assume that the vortex intensities are all equal to 1. It can be rigorously proved, see [24], that the effect of the Euler dynamics is to let the vorticity centers positions $(z_1(t), \dots, z_N(t))$ obey to an approximate Hamiltonian dynamics, whose Hamilton function (with respect to the symplectic form $\sum_{j=1}^{N} dz_j \wedge d\overline{z}_j$) is indeed $\frac{1}{8\pi}\mathcal{F}_0$. In case $\alpha > 0$, we are including in the system the effect of a negative sink, located at the origin $z = 0$, with vorticity intensity $-\frac{\alpha}{2}$. We refer the reader to [5, 6, 20, 24, 25, 27] for more details and references concerning the N-vortex problem.

We will prove Theorem 1.1 by means of the following:

Theorem 1.2 *Let $(\eta_1, \dots, \eta_n) \in \Omega_N$ be the vertices of a regular N-polygon satisfying,*

$$|\eta_1| = \cdots = |\eta_N| = \rho.$$

Then $(\eta_1, \dots, \eta_n)$ is a critical point for $\mathcal{F}_\alpha$, if and only if $N < 1 + \alpha$ and

$$\rho = \rho_N(\alpha) = \left(\frac{\alpha - N + 1}{\alpha + N + 1}\right)^{1/2N}.$$

It seems natural at this point to seek an answer for the following:

Question 1 Does problem (1.6) admits $2\pi/m$-symmetric concentrating solutions, satisfying (1.2) with $3 \le N < 1 + \alpha$ and $m < N$?

In other words we cannot exclude a priori the existence of non radial N-point concentrating solutions, whose concentration points does not lye on the vertices of a regular N-polygon. Of course, the answer is negative in case $N = 1, 2$ by the above mentioned results in [3]. In particular, this problem is closely related to the following:

Question 2 Does $\mathcal{F}_\alpha$ with $3 \le N < 1+\alpha$ admits $2\pi/m$-symmetric critical points, satisfying $m < N$?

We observe that indeed many of these $2\pi/m$-symmetric critical points are known to exist for the N-vortex Hamiltonian on $\mathbb{R}^2$, see for example [20].

Questions 1 and 2, besides their interest for the vortex problem, are quite relevant for the analysis of the multiple bubbling phenomenon associated with (1.4). We will be more precise concerning this point in the final part of this introduction.

Our next aim, is then to make a first step toward the understanding of the global properties of $\mathcal{F}_\alpha$. We think at this result as to the first step toward the full description of the critical set of $\mathcal{F}_\alpha$ and then, via the results in [13, 17] of the concentrating solutions for (1.6). Indeed, a crucial point in the construction of these concentrating solutions is the existence of "stable" critical points for $\mathcal{F}_\alpha$. In this context, "stability" of critical points means that, roughly speaking, either sub-super levels corresponding to the given critical level are not topologically equivalent, or that the critical point is stable under C^1-perturbations of the N-vortex Hamiltonian. We refer to [13, 17] for further details.

To state our result concerning the global structure of $\mathcal{F}_\alpha$, let us set,

$$\xi_j = e^{\imath 2\pi \frac{j-1}{N}}, \quad j \in \{1, \dots, N\},$$

to be the N-roots of unity and define,

$$\begin{cases} \Omega_N^{\mathrm{reg}} = \{(z_1, \dots, z_N) \in \Omega_N \,|\, z_j = r_j \xi_j e^{\imath \omega_j},\ \forall j = 1, \dots, N\}, \\ \quad r_j \in (0, 1),\ \omega_j \in (-\frac{2\pi}{N}, \frac{2\pi}{N}),\ \forall j = 1, \dots, N, \end{cases}$$

and

$$A_N = \left\{ (r_1, \ldots, r_N; \omega_1, \ldots, \omega_N) \in (0,1)^N \times \left(-\frac{2\pi}{N}, \frac{2\pi}{N} \right)^N \right\}.$$

Clearly

$$(r_1, \ldots, r_N; \omega_1, \ldots, \omega_N) \in A_N \quad \Longleftrightarrow \quad (r_1 \xi_1 e^{\imath \omega_1}, \ldots, r_N \xi_N e^{\imath \omega_N}) \in \Omega_N^{\mathrm{reg}}.$$

Set $H_{N,\alpha} : A_N \mapsto \mathbb{R}$,

$$H_{N,\alpha}(r_1, \ldots, r_N; \omega_1, \ldots, \omega_N) = \mathcal{F}_\alpha(r_1 \xi_1 e^{\imath \omega_1}, \ldots, r_N \xi_N e^{\imath \omega_N}). \tag{1.9}$$

We remark that the analysis of the critical set in case $N = 1, 2$ is much easier and has been already worked out in [3]. We will then provide full statements corresponding to the case $N = 2$ for reader's convenience.

Theorem 1.3 *For any $N \geq 2$ and for any $\alpha > N - 1$, define*

$$\rho_N(\alpha) = \left(\frac{\alpha - N + 1}{\alpha + N + 1} \right)^{1/2N}.$$

(a) *For any $N \geq 2$, for any $\alpha \in (0, +\infty)$ and for any $\rho \in (0, 1)$, there exist a unique function $T_{N,\rho,\alpha} : (-\frac{2\pi}{N}, \frac{2\pi}{N}) \mapsto \mathbb{R}$, which does not depend on the choice of $j \in \{1, \ldots, N\}$, such that,*

$$T_{N,\rho,\alpha}(\omega_j) = H_{N,\alpha}(\rho, \ldots, \rho; 0, \ldots, \omega_j, \ldots, 0), \quad \forall j = 1, \ldots, N.$$

Moreover $T_{N,\rho,\alpha}$ is even and strictly increasing in $(0, \frac{2\pi}{N})$. In particular $T_{N,\rho,\alpha}(\omega) \to +\infty$ as $\omega \to -\frac{2\pi}{N}^+$ and $\omega \to +\frac{2\pi}{N}^-$ and has a unique critical point at $\omega = 0$, where it attains its unique absolute (strict) minimum.

(b) *For any $N \geq 2$, for any $\alpha \in (0, +\infty)$ and for any $\rho \in (0, 1)$, there exist a unique function $R_{N,\rho,\alpha} : (0, 1) \mapsto \mathbb{R}$, which does not depend on the choice of $j \in \{1, \ldots, N\}$, such that,*

$$R_{N,\rho,\alpha}(r_j) = H_{N,\alpha}(\rho, \ldots, r_j, \ldots, \rho; 0, \ldots, 0), \quad \forall j = 1, \ldots, N.$$

For any $N \geq 2$ and for any $\alpha \in (0, +\infty)$, $R_{N,\rho,\alpha}(r) \to -\infty$ as $r \to 1^-$, and $r \to 0^+$. In particular, $R_{N,\rho,\alpha}$ admits $r = \rho$ as a critical point if and only if $\alpha > N - 1$ and $\rho = \rho_N(\alpha)$.

Note that Theorem 1.3 immediately implies Theorem 1.2. We may observe at this point that some physical argument [9] suggests that critical points of $\mathcal{F}_\alpha$ for $N \geq 2$ should not be relative maximums. We will prove this fact by a refined analysis of the function $R_{N,\rho,\alpha}$.

Theorem 1.4 *Let $R_{N,\rho,\alpha}$ be the function defined in Theorem 1.3. Then, $r = \rho_N(\alpha)$ is a strict relative maximum for $R_{N,\rho_N(\alpha),\alpha}$. Moreover, either,*

- *$N = 2$, and then for any $\alpha \in (0, +\infty)$ and for any $\rho \in (0, 1)$, $R_{2,\rho,\alpha}$ has a unique critical point $r = \overline{r}(2, \rho, \alpha) \in (0, 1)$ where it attains its unique relative and strict absolute maximum. In particular, $\overline{r}(2, \rho, \alpha) = \rho$ if and only if $\alpha > 1$ and $\rho = \rho_2(\alpha)$, or,*

- *$N \geq 3$, and then for any $\alpha \in [2N-4, +\infty)$ and for any $\rho \in (0,1)$, $R_{N,\rho,\alpha}$ has a unique critical point $r = \overline{r}(N,\rho,\alpha) \in (0,1)$ where it attains its unique relative and strict absolute maximum. In particular, either $N = 3$ and then $\overline{r}(3,\rho,\alpha) = \rho$ if and only if $\alpha > 2$ and $\rho = \rho_3(\alpha)$, or $N \geq 4$ and $\alpha \geq 2N-4$, and then $\overline{r}(N,\rho,\alpha) = \rho$ if and only if $\rho = \rho_N(\alpha)$.*

Remark In case $\alpha = 0$, we can prove a much stronger result, i.e. $\mathcal{F}_\alpha$ has no critical points at all for $N \geq 2$. An elementary proof of this fact has been suggested to us by E. Caglioti. Motivated by the N-vortex problem, we will discuss the structure of the critical set for the N-vortex Hamiltonian with $\alpha = 0$ on convex domains in a forthcoming paper [1].

We remark that, by using the principle of symmetrical criticality [26], the existence of $2\pi/N$-symmetric critical points for $\mathcal{F}_\alpha$ can be inferred *a priori* by the analysis of $T_{N,\rho,\alpha}$ and $f_{N,\alpha}(r)$ defined in (3.1) below. We refer the reader to [20] for some remarkable application of this method to the analysis of the N-vortex Hamiltonian on $\mathbb{R}^2$. Of course this approach no longer suffice to derive neither the full statement of Theorem 1.2, nor the qualitative behavior for $\mathcal{F}_\alpha$ as stated in Theorems 1.3, 1.4.

Concerning the proof of Theorem 1.4, we remark that the analysis of the derivatives of $R_{N,\rho,\alpha}$ involves the study of the sign and roots of a polynomial of degree $2N$, whose coefficients are polynomials in ρ of degree ranging from $N-1$ to $2N-2$, see the definition (4.2) of the polynomial $P_N(r;\rho,\alpha)$ in Lemma 4.1. We will use a recursion formula to find out the explicit expressions of those coefficients, see Lemma 4.1. Then, in order to prove that $R_{N,\rho,\alpha}$ has at most one critical point, we analyze the number of sign changes of the coefficients sequence, see Lemmas 4.3, 4.4, and apply the celebrated Descartes' Rule of Signs [28]. We stated the case $N = 2$ separately in Theorem 1.4 since we obtain in this particular situation the kind of result that is likely to hold in case $N \geq 3$ as well.

Indeed, we believe that $R_{N,\rho,\alpha}$ admits a unique critical point also in case $N \geq 3$ and $\alpha \in (0, 2N-4)$. A strong support to this conjecture comes from numerical arguments. Otherwise, by using the above mentioned explicit expressions for the coefficients, one may argue directly and discuss one by one the cases where N is low. We will not discuss this issues here. Instead, we support our conjecture by showing that for any $N \geq 3$ and $\alpha \in (0, +\infty)$, then $R_{N,\rho,\alpha}$ admits a unique critical point provided ρ is large enough, see Remark 5.1 in Sect. 5. Actually, this result is stronger than Theorem 1.3, but we state it separately to simplify the exposition.

Finally, we will use the knowledge of the explicit expression of the concentration radius $\rho_N(\alpha)$, to prove the existence of *multiple bubbling* sequence of solutions for (1.6) which concentrates at the origin. The incoming discussion will also clarify the relevance of Questions 1 and 2 above for the analysis of the multiple bubbling phenomenon.

Assume that $\{\alpha_n\} \subset [0, +\infty)$, $\alpha_n \to \alpha \in [0, +\infty)$, $\lambda_n \to \lambda \in [0, 8\pi(1+\alpha)]$ and let $\{u_{\alpha_n,n}\}$ be a sequence of solutions for (1.6) satisfying (1.2). As already remarked above, by using the quantization results in [4], we see that if $\gamma \neq 0$, then necessarily $\lambda = \gamma = 8\pi(1+\alpha)$ and $N = 0$. On the other side, if $m \in \mathbb{N}$ and $\alpha_n \to m^+$, then the concentration radius $\rho_{m+1}(\alpha_n)$ of the sequences blowing up at the vertices of an $N = m+1$ regular polygon satisfies,

$$\rho_{m+1}(\alpha_n) = \left(\frac{\alpha_n - m}{\alpha_n + m + 2}\right)^{\frac{1}{2(m+1)}} \to \rho_{m+1}(m) = 0, \quad \text{as } n \to +\infty.$$

By using this observation, it has been proved in [3] that multiple bubbling solutions blowing up at the origin exist if $\alpha_n \to 1^+$, i.e. if $m = 1$. By using Theorem 1.1 and the results in [13, 17, 21], we obtain the following,

Theorem 1.5 *For any $m \geq 2$, there exist a sequence $\alpha_n \to m^+$, as $n \to +\infty$, and a sequence of solutions $\{u_n\}$ for (1.6) such that,*

$$\lambda_n \to 8\pi(1+m) \quad \textit{and} \quad \lambda_n \frac{e^{u_n}}{\int_D e^{u_n}} \to 8\pi(1+m)\delta_{p=0}, \quad \textit{as } n \to +\infty,$$

weakly in the sense of measures in D and in $C^2_{\mathrm{loc}}(\overline{D}^*)$. Moreover, there exist $\overline{n} \in \mathbb{N}$, a sequence of positive numbers $\{\sigma_n\}$ and sequences of complex numbers $\{z_n^{(j)}\}_{j=1,\ldots,N}$ such that:

$(S)_1$ *For any $j = 1, \ldots, m+1$ and for any $n > \overline{n}$, $z_n^{(j)}$ is the unique absolute maximum point for $u_n(z) - 2\alpha_n \log|z|$ in $B_{\sigma_n}(z_n^{(j)}) := \{z \in D \,|\, |z - z_n^{(j)}| < \sigma_n\}$ and*

$$u_n(z_n^{(j)}) - 2\alpha_n \log|z_n^{(j)}| \to +\infty, \quad \textit{as } n \to +\infty.$$

$(S)_2$ *For any $j = 1, \ldots, m+1$,*

$$z_n^{(j)} \to 0, \quad \textit{and} \quad \sigma_n \to 0^+, \quad \frac{\sigma_n}{|z_n^{(j)}|} \to 0^+, \quad \frac{|z_n^{(j)} - \rho_{m+1}(\alpha_n)e^{2\pi\iota\frac{j-1}{m+1}}|}{\sigma_n} \to 0^+$$

as $n \to +\infty$.

$(S)_3$ *For any $j \neq i$, $B_{\sigma_n}(z_n^{(j)}) \cap B_{\sigma_n}(z_n^{(i)}) = \emptyset$, for any $n > \overline{n}$.*
$(S)_4$ *For any $j = 1, \ldots, m+1$,*

$$\lambda_n \int_{B_{\sigma_n}(z_n^{(j)})} \frac{e^{u_n}}{\int_D e^{u_n}} \to 8\pi, \quad \textit{as } n \to +\infty.$$

While we were preparing the final version of this manuscript we have been aware by G. Tarantello of the following preprint [14], where the existence of multiple bubbling-type solutions for mean field equations with a Dirac source is derived on simply connected domains.

We conclude this introduction with a short discussion concerning the main ideas behind the results we obtained in this paper and more general interdisciplinary motivations.

The main idea behind this work is very simple. We use some information concerning the solution's set of certain elliptic equations to make some guesses concerning the structure of the critical set of the singular N-vortex Hamiltonian. Then, by using the Descartes' Rule of Signs, the analysis of those guesses is worked out rigorously by the study of the zeroes of various special polynomials, providing also some feed-back about the equation itself. For example, we obtain the explicit expressions of the coordinates of the vortex points for a "simple" model of the mean field equation for Euler flows with one negative sink.

We believe that this analysis, behind its interest for the study of semilinear elliptic equations, has many interesting connections with other disciplines. Of course the explicit expressions of the vortex points mentioned above, as well as the study of the structure of the singular N-vortex Hamiltonian, can be of some interest from the point of view of Fluids Physics. In particular, we believe that a better knowledge of the N-vortex configurations in this simple case may help the understanding of the local structures often observed in nature in connection with almost two dimensional turbulence, as for example the transition layer of two dimensional shear flows.

Moreover, it turns out that a more detailed analysis of the problem calls up for the study of the properties of general polynomials of high degree with a peculiar structure induced by the interaction of the N-vortices with the singularity. In particular, one may use the information gained about the critical set of N-vortex Hamiltonian to make some guesses concerning the structure of the zeroes of those non-trivial special polynomials. Of course, the rigorous analysis of those guesses will provide a feed-back concerning the N-vortex Hamiltonian and the corresponding mean field equation.

This paper is organized as follows. In Sect. 2 we prove Theorems 1.1 and 1.5. In Sect. 3 we prove Theorems 1.3 and 1.2. In Sect. 4 we provide the proofs of various results, needed in Sect. 5 for the proof of Theorem 1.4.

2 The Proofs of Theorems 1.1 and 1.5

In this section, by using Theorem 1.2, we prove both Theorems 1.1 and 1.5.

The Proof of Theorem 1.1 By arguing as in [13, 17], and by using the symmetries of the problem, for any $3 \leq N < 1+\alpha$, one can reduce the problem of finding a $2\pi/N$-symmetric sequence of concentrating solutions for (1.6), satisfying (1.2) with $\{\varsigma_1, \dots, \varsigma_N\}$ being the vertices of a regular N-polygon, to that of the existence of an absolute maximum for the function $f_{N,\alpha}$ defined in (3.1) below. The existence of that maximum can be easily derived, see either [18] or Proposition 3.2 below. Otherwise, by using Theorems 1.3 and 1.4, one can easily check that for any $3 \leq N < 1+\alpha$, the set of vertices $\{\eta_1, \dots, \eta_N\}$ of a regular N-polygon of radius $\rho = \rho_N(\alpha) = (\frac{\alpha-N+1}{\alpha+N+1})^{1/2N}$, is a non degenerate critical set for $\mathcal{F}_\alpha$ according to the definition of [13]. In any case, we come up with the existence of at least one concentrating sequence of solutions satisfying the desired properties for some $\rho \in (0,1)$. Thus, we just need to prove that if such a kind of concentrating solutions exist, then $\rho = \rho_N(\alpha)$. Since the blow up points $\{\varsigma_1, \dots, \varsigma_N\}$ must be critical points of $\mathcal{F}_\alpha$, see for example [22], then the conclusion follows immediately by Theorem 1.2. □

The Proof of Theorem 1.5 Let $m \geq 2$ be any given integer. By using Theorem 1.1, for any $\alpha > m$, we obtain a sequence of solutions $\{u_{\alpha,n}\}$ for (1.6), satisfying (1.2), whose concentration points lie on the vertices of a regular N-polygon $\{\eta_1, \dots, \eta_N\}$, with $N = m+1$ and $\rho = \rho_{m+1}(\alpha)$. Clearly, $\lambda_n \to 8\pi(1+m)$, as $n \to +\infty$.

Let $\{\alpha_k\}$ be any definitively monotone sequence satisfying $\alpha_k \to m^+$, as $k \to +\infty$. For any given $k \in \mathbb{N}$, by the results in [21], there exist subsequences, still denoted by $\{u_{\alpha_k,n}\}$ and $\{\lambda_{n,k}\}$, and there exist $\overline{n}_k \in \mathbb{N}$, $\overline{\sigma}_k > 0$, and $m+1$ sequences $\{\{z_{n,k}^{(1)}\}, \dots, \{z_{n,k}^{(m+1)}\}\}$, such that:

$(i-1)$ For any $j = 1, \dots, m+1$ and for any $n > \overline{n}_k$, $z_{n,k}^{(j)}$ is the unique absolute maximum point for $u_{n,\alpha_k}(z) - 2\alpha_k \log|z|$ in $B_{\overline{\sigma}_k}(z_{n,k}^{(j)}) := \{z \in D \,|\, |z - z_{n,k}^{(j)}| < \overline{\sigma}_k\}$ and

$$u_n(z_{n,k}^{(j)}) - 2\alpha_k \log|z_{n,k}^{(j)}| \to +\infty, \quad \text{as } n \to +\infty.$$

$(i-2)$ For any $j = 1, \dots, m+1$, $z_{n,k}^{(j)} \to \eta_j$, as $n \to +\infty$.

$(i-3)$ For any $j = 1, \dots, m+1$, $j \neq l$, $B_{\overline{\sigma}_k}(z_{n,k}^{(j)}) \cap B_{\overline{\sigma}_k}(z_{n,k}^{(l)})$.

$(i-4)$ For any $j = 1, \dots, m+1$,

$$\lambda_{n,k} \int_{B_{\overline{\sigma}_k}(z_{n,k}^{(j)})} \frac{e^{u_{\alpha_k,n}}}{\int_D e^{u_{\alpha_k,n}}} \to 8\pi, \quad \text{as } n \to +\infty.$$

At this point, observe that, for any $k \in \mathbb{N}$, we can find a sequence $\{\sigma_n^{(k)}\}$, such that,

$$0 < \sigma_n^{(k)} < \sigma_k, \quad \sigma_n^{(k)} \to 0^+, \quad \frac{\overline{\sigma}_n^{(k)}}{|z_{n,k}^{(j)}|} \to 0^+, \quad \text{as } n \to +\infty,$$

$$\lambda_{n,k} \int_{B_{\overline{\sigma}_n^{(k)}}(z_{n,k}^{(j)})} \frac{e^{u_{\alpha_k,n}}}{\int_D e^{u_{\alpha_k,n}}} \to 8\pi, \quad \text{as } n \to +\infty,$$

and

$$\frac{|z_{n,k}^{(j)} - \rho_{m+1}(\alpha_k) e^{2\pi\imath \frac{j-1}{m+1}}|}{\sigma_n^{(k)}} \to 0^+ \text{ as } n \to +\infty.$$

Hence, by means of a diagonal argument, we can find sub-subsequences,

$$\{\sigma_n\} := \{\sigma_n^{(k_n)}\}, \qquad \{z_n^{(j)}\} := \{z_{n,k_n}^{(j)}\}, \qquad \{\alpha_n\} := \{\alpha_{k_n}\},$$
$$\{u_n\} := \{u_{\alpha_n,n}\}, \qquad \{\lambda_n\} := \{\lambda_{k_n}\},$$

such that $(S)_1$, $(S)_2$, $(S)_3$, $(S)_4$ are satisfied. □

3 Analysis of the Critical Set of $\mathcal{F}_\alpha$

In this section we prove Theorems 1.2 and 1.3. We divide the proof in three main steps, corresponding to Propositions 3.1, 3.5, 3.6. It will be clear that Propositions 3.1, 3.5 and 3.6, together with the very definition of $H_{N,\alpha}$ imply Theorem 1.3, which in turn implies Theorem 1.2. As already mentioned in the introduction, one can also prove Theorem 1.2 by using Propositions 3.2 and 3.5 together with the principle of symmetrical criticality [26].

Proposition 3.1 *Let $N \geq 2$, $\alpha > 0$ and assume that $(\eta_1, \ldots, \eta_N)$ is a critical point for $\mathcal{F}_\alpha$ lying on the vertices of a regular N-polygon inscribed in a disk of radius ρ. Then $\alpha > N - 1$ and*

$$\rho = \rho_N(\alpha) = \left(\frac{\alpha - N + 1}{\alpha + N + 1}\right)^{1/2N}.$$

Proof Let $H_{N,\alpha}$ be defined in (1.9) and set

$$f_{N,\alpha}(r) = H_{N,\alpha}(r, \ldots, r; 0, \ldots, 0) \equiv \mathcal{F}_\alpha(r\,\xi_1, \ldots, r\,\xi_N), \quad r \in (0, 1). \tag{3.1}$$

Since, by assumption, $(\rho\xi_1, \ldots, \rho\xi_N)$ is a critical point for $\mathcal{F}_\alpha(z_1, \ldots, z_N)$, and since $\mathcal{F}_\alpha$ is smooth in Ω_N, then ρ is a critical point for $f_{N,\alpha}$. The following proposition clearly implies the assertion.

Proposition 3.2 *f_N takes the following form,*

$$f_{N,\alpha}(r) = 2\log[\Phi_{N,\alpha}(r)] - 2\log[C(N)], \tag{3.2}$$

where $C(N)$ is a positive constant and

$$\Phi_{N,\alpha}(r) = r^{N(\alpha-N+1)}(1 - r^{2N})^N, \quad r \in (0, 1).$$

In particular,

$$\frac{d}{dr} f_{N,\alpha}(\rho) = 0, \quad \textit{if and only if} \quad \rho = \left(\frac{\alpha - N + 1}{\alpha + N + 1}\right)^{1/2N}.$$

Proof We first derive an explicit representation formula for $\Phi_{N,\alpha}$ and $C(N)$ in (3.2).

Lemma 3.3 *Let* $\Phi_{N,\alpha}$ *and* $C(N)$ *be defined by* (3.2)*. Then,*

$$\Phi_{N,\alpha}(r) = r^{N(\alpha-N+1)}(1-r^2)^N \prod_{k=1}^{N-1} |r^2 e^{\imath 2\pi \frac{k}{N}} - 1|^{2(N-k)}, \quad r \in (0,1),$$

and

$$C(N) = \prod_{k=1}^{N-1} |e^{\imath 2\pi \frac{k}{N}} - 1|^{2(N-k)}.$$

Proof By the very definition of $f_{N,\alpha}$ we have

$$\begin{aligned}
f_{N,\alpha}(r) &= 2\alpha N \log r + 2N \log(1-r^2) + 2 \sum_{m\neq j}^{1,\dots,N} \log \frac{|r^2 e^{\imath 2\pi \frac{m-j}{N}} - 1|}{|r e^{\imath 2\pi \frac{m-1}{N}} - r e^{\imath 2\pi \frac{j-1}{N}}|} \\
&= 2\log[r^{\alpha N}(1-r^2)^N] + 4 \sum_{m>j}^{1,\dots,N} \log\left| r e^{\imath 2\pi \frac{m-j}{N}} - \frac{1}{r}\right| - 4 \sum_{m>j}^{1,\dots,N} \log |e^{\imath 2\pi \frac{m-j}{N}} - 1| \\
&= 2\log[r^{\alpha N}(1-r^2)^N] + 4 \sum_{k=1}^{N-1} (N-k) \log\left| r e^{\imath 2\pi \frac{k}{N}} - \frac{1}{r}\right| \\
&\quad - 4 \sum_{k=1}^{N-1} (N-k) \log |e^{\imath 2\pi \frac{k}{N}} - 1|.
\end{aligned}$$

The last relation already yields the explicit expression for $C(N)$, while the desired conclusion follows if we observe that,

$$\begin{aligned}
4 \sum_{k=1}^{N-1} (N-k) \log\left| r e^{\imath 2\pi \frac{k}{N}} - \frac{1}{r}\right| &= 2\log\left(\prod_{k=1}^{N-1} r^{2(k-N)} |r^2 e^{\imath 2\pi \frac{k}{N}} - 1|^{2(N-k)}\right) \\
&= 2\log\left(r^{-N(N-1)} \prod_{k=1}^{N-1} |r^2 e^{\imath 2\pi \frac{k}{N}} - 1|^{2(N-k)}\right).
\end{aligned}$$

□

We will conclude the proof of Proposition 3.2 by virtue of the following,

Lemma 3.4 *Define*

$$\Psi_N(r) = r^{-N(\alpha-N+1)} \Phi_{N,\alpha}(r) = (1-r^2)^N \prod_{k=1}^{N-1} |r^2 e^{\imath 2\pi \frac{k}{N}} - 1|^{2(N-k)}, \quad r \in \mathbb{R}^+.$$

Then

$$\Psi_N(r) = (1 - r^{2N})^N.$$

Proof Set

$$p_N(w) := (1 - w)^N \prod_{k=1}^{N-1} \left(w^2 - 2w \cos\left(2\pi \frac{k}{N}\right) + 1\right)^{N-k}, \quad w \in \mathbb{C}.$$

It is readily seen that p_N is a complex polynomial of degree N^2 whose roots are the N-roots of unity each one with multiplicity N. Since the coefficient of the term w^{N^2} is $(-1)^N$, it follows by the unique factorization theorem that

$$p_N(w) = \left[\prod_{k=1}^{N} (\xi_k - w)\right]^N \equiv \left(1 - w^N\right)^N, \quad \forall w \in \mathbb{C}.$$

The conclusion then follows since $\Psi_N(r)$ is a real polynomial of degree $2N^2$ and

$$\Psi_N(r) = p_N(r^2), \quad \forall r \in \mathbb{R}. \qquad \square$$

By virtue of these lemmas, we conclude that $f_{N,\alpha}(r)$ takes the form

$$f_{N,\alpha}(r) = 2\log[\Phi_{N,\alpha}(r)] - 2\log[C(N)],$$

where

$$\Phi_{N,\alpha}(r) = r^{N(\alpha-N+1)}(1 - r^{2N})^N, \quad r \in (0,1).$$

By a straightforward evaluation we obtain,

$$\frac{d}{dr} f_N(\rho) = \frac{2}{\Phi_{N,\alpha}(\rho)} \frac{d}{dr}\Phi_{N,\alpha}(\rho) = \frac{2N}{\rho(1-\rho^{2N})}[(\alpha - N + 1) - (\alpha + N + 1)\rho^{2N}],$$

and the conclusion of Proposition 3.2 follows. $\square$

$\square$

We introduce the auxiliary function

$$F_\alpha(r_1, \ldots, r_N; e^{\iota\omega_1}, \ldots, e^{\iota\omega_N}) := \mathcal{F}_\alpha(r_1\xi_1 e^{\iota\omega_1}, \ldots, r_N\xi_N e^{\iota\omega_N}). \tag{3.3}$$

Although we do not really need to introduce this new function, we believe that it makes the proofs less involved. Clearly, by (1.9), we have

$$H_{N,\alpha}(r_1, \ldots, r_N; \omega_1, \ldots, \omega_N) = F_\alpha(r_1, \ldots, r_N; e^{\iota\omega_1}, \ldots, e^{\iota\omega_N}). \tag{3.4}$$

Proposition 3.5 *For any* $N \geq 2$, *for any* $\alpha \in [0, +\infty)$ *and for any* $\rho \in (0,1)$, *there exist a unique function* $T_{N,\rho,\alpha} : (-\frac{2\pi}{N}, \frac{2\pi}{N}) \mapsto \mathbb{R}$, *which does not depend on the choice of* $j \in \{1, \ldots, N\}$, *such that*

$$T_{N,\rho,\alpha}(\omega_j) = H_{N,\alpha}(\rho, \ldots, \rho; 0, \ldots, \omega_j, \ldots, 0), \quad \forall j = 1, \ldots, N.$$

Moreover $T_{N,\rho,\alpha}$ is even and strictly increasing in $(0,\frac{2\pi}{N})$. In particular $T_{N,\rho,\alpha}(\omega)\to+\infty$ as $\omega\to-\frac{2\pi}{N}^{+}$ and $\omega\to+\frac{2\pi}{N}^{-}$ and has a unique critical point at $\omega=0$, where it attains its unique relative and strict absolute minimum.

Proof Let F_α be defined by (3.3). By using (1.8), we see that $F_\alpha(\rho,\dots,\rho;\xi_1 e^{\imath\omega},\dots,\xi_N)$ is a well defined function of $\omega\in(-\frac{2\pi}{N},\frac{2\pi}{N})$. Moreover, $\forall\phi\in[0,2\pi)$, $F_\alpha(\rho,\dots,\rho; e^{\imath\omega_1},\dots,e^{\imath\omega_N})$ is invariant under the phase shift $\omega_j\mapsto\omega_j+\phi$, $\forall j\in\{1,\dots,N\}$, and with respect to arbitrary permutations of the phase variables $\{e^{\imath\omega_j}\}$. Hence, by elementary symmetry considerations we conclude that there exist a *unique* function $T_{N,\rho,\alpha}:(-\frac{2\pi}{N},\frac{2\pi}{N})\mapsto\mathbb{R}$, which *does not* depend by j, such that

$$T_{N,\rho,\alpha}(\omega)=F_\alpha(\rho,\dots,\rho;\xi_1,\dots,\xi_j e^{\imath\omega},\dots,\xi_N),\quad \forall j\in\{1,\dots,N\}.$$

In particular, without loss of generality, we may assume $j=1$ and observe that, by using (1.8),

$$T_{N,\rho,\alpha}(\omega)=F_\alpha(\rho,\dots,\rho;\xi_1 e^{\imath\omega},\dots,\dots,\xi_N)=\tau_\rho(\omega)+C(N,\rho,\alpha),$$

where $C(N,\rho,\alpha)$ is a constant which depends only by ρ,α and $\{\xi_1,\dots,\xi_N\}$, while

$$\tau_\rho(\omega)=4\sum_{j=2}^{N}\log\frac{|\rho^2 e^{\imath\omega}e^{\imath 2\pi\frac{j-1}{N}}-1|}{|e^{\imath\omega}e^{\imath 2\pi\frac{j-1}{N}}-1|}=4\log\left(\prod_{j=2}^{N}\frac{|\rho^2 e^{\imath\omega}e^{\imath 2\pi\frac{j-1}{N}}-1|}{|e^{\imath\omega}e^{\imath 2\pi\frac{j-1}{N}}-1|}\right).$$

It is also easy to verify that $T_{N,\rho,\alpha}$ is even, since indeed, for any $\omega\in[0,\frac{2\pi}{N})$, we clearly have,

$$\tau_\rho(\omega)=\tau_\rho(-\omega).$$

It follows immediately that $\tau_\rho(\omega)$, and then $T_{N,\rho,\alpha}(\omega)$, admits $\omega=0$ as either a relative minimum or relative maximum. Then, for any $j\in 2,\dots,N$, let us define

$$\phi_N(j)=2\pi\frac{j-1}{N},\qquad \mathrm{w}_j(\omega)=\frac{\rho^2 e^{\imath(\omega-\phi_N(j))}-1}{e^{\imath(\omega-\phi_N(j))}-1},$$

and observe that

$$\frac{d}{d\omega}|\mathrm{w}_j(\omega)|^2=-2\frac{(\rho^2-1)^2}{(e^{\imath(\omega-\phi_N(k))}-1)^2}\sin(\omega-\phi_N(j))>0,$$

for any $\omega\in(0,\frac{2\pi}{N})$, and for any $j=2,\dots,N-1$. Thus $\tau_\rho(\omega)$ is strictly monotone increasing in $(0,\frac{2\pi}{N})$. Since $T_{N,\rho,\alpha}(\omega)\to+\infty$ as $\omega\to\frac{2\pi}{N}^{-}$, then the conclusion of Proposition 3.5 follows. □

Proposition 3.6 *Set*

$$\rho_N(\alpha)=\left(\frac{\alpha-N+1}{\alpha+N+1}\right)^{1/2N}.$$

For any $N\geq 2$, for any $\alpha\in(0,+\infty)$ and for any $\rho\in(0,1)$, there exist a unique function $R_{N,\rho,\alpha}:(0,1)\mapsto\mathbb{R}$, which does not depend on the choice of $j\in\{1,\dots,N\}$ such that,

$$R_{N,\rho,\alpha}(r_j)=H_{N,\alpha}(\rho,\dots,r_j,\dots,\rho;0,\dots,0),\quad \forall j=1,\dots,N.$$

For any $N \geq 2$ and for any $\alpha \in (0, +\infty)$, $R_{N,\rho,\alpha}(r) \to -\infty$ as $r \to 1^-$, and $r \to 0^+$. In particular, $R_{N,\rho,\alpha}$ admits $r = \rho$ as a critical point if and only if $\alpha > N - 1$ and $\rho = \rho_N(\alpha)$.

Proof By using (1.8), for any given $N \geq 2$, $j \in \{1, \dots, N\}$ and $\rho \in (0, 1)$, we conclude that $F_\alpha(\rho, \dots, r_j, \dots, \rho; \xi_1, \dots, \dots, \xi_N)$ is well defined in $(0, 1)$ as a function of $r_j \in (0, 1)$. As in the proof of Proposition 3.5, by elementary symmetry considerations, we conclude that for any $\rho \in (0, 1)$, there exist a *unique* function $R_{N,\rho,\alpha} : (0, 1) \mapsto \mathbb{R}$, which *does not* depend by j, such that

$$R_{N,\rho,\alpha}(r_j) = F_\alpha(\rho, \dots, r_j, \dots, \rho; \xi_1, \dots, \xi_N), \quad \forall j \in \{1, \dots, N\}.$$

In particular, without loss of generality, we may assume that $j = 1$, and observe that, by setting $r \equiv r_1$, and by using (1.8), we have

$$R_{N,\rho,\alpha}(r) = F_\alpha(r, \rho, \dots, \rho; \xi_1, \dots, \xi_N) = K_{N,\rho,\alpha}(r) + \gamma(\rho, N), \tag{3.5}$$

where $\gamma(\rho, N)$ depends only by ρ and $\{\xi_1, \dots, \xi_N\}$, while

$$\begin{aligned} K_{N,\rho,\alpha}(r) &= 2 \log r^\alpha (1 - r^2) + 4 \sum_{j=2}^{N} \log \frac{|r\rho e^{\imath 2\pi \frac{j-1}{N}} - 1|}{|r e^{\imath 2\pi \frac{j}{N}} - \rho e^{\imath 2\pi \frac{1}{N}}|} \\ &= 2 \log r^\alpha (1 - r^2) + 2 \log \left(\prod_{j=2}^{N} \frac{|r\rho e^{\imath 2\pi \frac{j-1}{N}} - 1|^2}{|r e^{\imath 2\pi \frac{j-1}{N}} - \rho|^2} \right). \end{aligned}$$

By using the unique factorization theorem as in Lemma 3.3, it is not difficult to verify that

$$\begin{aligned} &K_{N,\rho,\alpha}(r) = 2 \log r^\alpha + 2 \log (\psi_{N,\rho}(r)), \\ &\psi_{N,\rho}(r) = (1 - r^2) \left(\frac{\rho^N r^N - 1}{\rho r - 1} \right)^2 \left(\frac{r - \rho}{r^N - \rho^N} \right)^2 = (1 - r^2) \left(\frac{\sum_{k=0}^{N-1} r^k \rho^k}{\sum_{k=0}^{N-1} r^k \rho^{N-k-1}} \right)^2. \end{aligned}$$

By using the last equality, it follows by a lengthy but straightforward calculation that

$$\begin{aligned} \frac{d}{dr} K_{N,\rho,\alpha}(\rho) = &\frac{2}{\rho^\alpha \psi_{N,\rho}(\rho)} \frac{\rho^{\alpha - 2N + 1}(1 - \rho^{2N})}{N^2(1 - \rho^2)^2} \\ &\times [\alpha - N + 1 - (\alpha - N + 1)\rho^2 \\ &- (\alpha + N + 1)\rho^{2N} + (\alpha + N + 1)\rho^{2N+2}]. \end{aligned}$$

Note that the last relation can be also obtained by using Lemma 4.1 below. Since,

$$\begin{aligned} &\alpha - N + 1 - (\alpha - N + 1)\rho^2 - (\alpha + N + 1)\rho^{2N} + (\alpha + N + 1)\rho^{2N+2} \\ &= (\alpha - N + 1 - (\alpha + N + 1)\rho^{2N})(1 - \rho^2), \end{aligned}$$

we easily conclude that $\frac{d}{dr} K_{N,\rho,\alpha}(\rho) = 0$ if and only if

$$\rho = \rho_N(\alpha) = \left(\frac{\alpha - N + 1}{\alpha + N + 1} \right)^{\frac{1}{2N}}.$$

It follows immediately that either $\alpha \in [0, N-1]$ and then for any given $\rho \in (0, 1)$, we conclude that $r = \rho$ is not a critical point for $R_{N,\rho,\alpha}$, or $\alpha > N-1$ and then $r = \rho$ is a critical point for $R_{N,\rho,\alpha}$ if and only if $\rho = \rho_N(\alpha)$. Note in particular that, for any $\rho \in (0, 1)$, we have that either $\alpha \in [0, +\infty)$, and then $R_{N,\rho,\alpha}(r) \to -\infty$ as $r \to 1^-$, or $r \to 0^+$ and then $R_{N,\rho,\alpha}(r) \to -\infty$ only if $\alpha \in (0, +\infty)$. Then, the desired conclusion follows by using (3.4). □

4 The Analysis of the Radial Function $R_{N,\rho,\alpha}$

In this section we will make a further step in the analysis of the graph of the radial function $R_{N,\rho,\alpha}$ defined in (3.5). This refined analysis is needed to complete the proof of Theorem 1.4.

The study of the set of critical points of $R_{N,\rho,\alpha}$ calls up for the study of the roots of a general polynomial $P_N(r; \rho, \alpha)$ of degree $2N$, see (4.2) below, whose coefficients are polynomials in ρ of degree ranging from $N-1$ to $2N-2$. Our main goal is to prove that if $\alpha > 0$ (actually $\alpha \geq 2N-4$ if $N \geq 3$), then $P_N(r; \rho, \alpha)$ has one and only one root for $r \in (0, 1)$. The idea of the proof is to apply the celebrated Descartes' Rule of Signs to P_N [28], see the proof of Theorem 1.4 in Sect. 5 for further details.

For reader's convenience we recall that, see either (3.5) above,

$$R_{N,\rho,\alpha}(r) = F_\alpha(r, \rho, \ldots, \rho; \xi_1, \ldots, \xi_N) = K_{N,\rho,\alpha}(r) + \gamma(\rho, N),$$

where $\gamma(\rho, N)$ depends only by ρ and $\{\xi_1, \ldots, \xi_N\}$, while

$$K_{N,\rho,\alpha}(r) = 2\log r^\alpha + 2\log(\psi_{N,\rho}(r)),$$

$$\psi_{N,\rho}(r) = (1-r^2)\left(\frac{\rho^N r^N - 1}{\rho r - 1}\right)^2 \left(\frac{r-\rho}{r^N - \rho^N}\right)^2 = (1-r^2)\left(\frac{\sum_{k=0}^{N-1} r^k \rho^k}{\sum_{k=0}^{N-1} r^k \rho^{N-k-1}}\right)^2.$$

Let us define

$$h_{N,\rho}(r) = \frac{\rho^N r^N - 1}{\rho r - 1} = \sum_{k=0}^{N-1} r^k \rho^k,$$

$$g_{N,\rho}(r) = \frac{r^N - \rho^N}{r - \rho} = \sum_{k=0}^{N-1} r^k \rho^{N-k-1}.$$

Concerning the derivative of $K_{N,\rho,\alpha}$, we have the following,

Lemma 4.1 *For any $N \geq 3$, for any $\alpha > 0$ and for any $\rho \in (0, 1)$, the first derivative of $R_{N,\rho,\alpha}$ takes the form:*

$$\frac{d}{dr} R_{N,\rho,\alpha}(r) = \frac{d}{dr} K_{N,\rho,\alpha}(r) = \frac{2}{r\psi_{N,\rho}(r)} \frac{h_{N,\rho}(r)}{g^3_{N,\rho}(r)} P_N(r; \rho, \alpha), \tag{4.1}$$

where

$$P_N(r; \rho, \alpha) = \sum_{m=0}^{2N} a_m^{(N)}(\rho, \alpha) r^m, \tag{4.2}$$

and

$$a_0^{(N)}(\rho,\alpha) = \alpha\rho^{N-1}; \tag{4.3}$$

$$a_1^{(N)}(\rho,\alpha) = \rho^{N-2}\left[(-2+\alpha)+(2+\alpha)\rho^2\right]; \tag{4.4}$$

$$\begin{cases} a_{N-k}^{(N)}(\rho,\alpha) = \rho^{k-1}[(-2(N-k)+\alpha) - 2\sum_{j=1}^{N-k-1}\rho^{2j} + (2(N-k)+\alpha)\rho^{2(N-k)}], \\ \quad k = 1,\dots,N-2; \end{cases} \tag{4.5}$$

$$a_N^{(N)}(\rho,\alpha) = -2\sum_{j=0}^{N-2}\rho^{2j+1}; \tag{4.6}$$

$$\begin{cases} a_{N+k}^{(N)}(\rho,\alpha) = \rho^{k-1}[(2(N-k-1)-\alpha) - 2\sum_{j=1}^{N-k-1}\rho^{2j} - (2(N-k+1)+\alpha)\rho^{2(N-k)}], \\ \quad k = 1,\dots,N-2; \end{cases} \tag{4.7}$$

$$a_{2N-1}^{(N)}(\rho,\alpha) = \rho^{N-2}[-\alpha-(4+\alpha)\rho^2]; \tag{4.8}$$

$$a_{2N}^{(N)}(\rho,\alpha) = -(2+\alpha)\rho^{N-1}. \tag{4.9}$$

In case $N = 2$, *the first derivative of* $K_{2,\rho,\alpha}$ *takes the form* (4.1)–(4.2) *with the coefficients* $\{a_m^{(2)}(\rho,\alpha)\}_{m=0,\dots,4}$ *taking the forms* (4.3), (4.4), (4.6), (4.8), (4.9) *evaluated at* $N = 2$ *respectively.*

Proof It is straightforward to verify that

$$\frac{d}{dr}R_{N,\rho,\alpha}(r) = \frac{d}{dr}K_{N,\rho,\alpha}(r) = \frac{2r^{\alpha-1}}{r^\alpha\psi_{N,\rho}(r)}\frac{h_{N,\rho}(r)}{g_{N,\rho}^3(r)}P_N(r;\rho,\alpha),$$

where

$$\begin{aligned} P_N(r;\rho,\alpha) &= (\alpha-(2+\alpha)r^2)g_{N,\rho}(r)h_{N,\rho}(r) \\ &\quad + 2r(1-r^2)\left(g_{N,\rho}(r)\frac{d}{dr}h_{N,\rho}(r) - h_{N,\rho}(r)\frac{d}{dr}g_{N,\rho}(r)\right). \end{aligned} \tag{4.10}$$

In case $N = 2, 3, 4$ the conclusion follows by a straightforward calculation. Next, observe that, for any $N \geq 2$,

$$P_{N+1}(r;\rho,\alpha) = \rho P_N(r;\rho,\alpha) + (\alpha-(2+\alpha)r^2)B_{N,\rho}(r) + 2r(1-r^2)C_{N,\rho}(r),$$

where

$$B_{N,\rho}(r) = \left(\rho^{N+1}g_{N,\rho}(r) + h_{N,\rho}(r)\right)r^N + \rho^N r^{2N},$$

and

$$C_{N,\rho}(r) = \left(\frac{d}{dr}h_{N,\rho}(r) - \rho^{N+1}\frac{d}{dr}g_{N,\rho}(r)\right)r^N + \left(N\rho^{N+1}g_{N,\rho}(r) - Nh_{N,\rho}(r)\right)r^{N-1}.$$

We use the last identities together with the explicit sum expansions for $g_{N,\rho}$ and $h_{N,\rho}$ to obtain an induction rule for the coefficients $\{a_m^{(N+1)}(\rho,\alpha)\}_{m=0,\dots,2(N+1)}$. In case $N \geq 5$, we obtain

$$a_m^{(N+1)}(\rho,\alpha) = \rho a_m^{(N)}(\rho,\alpha), \quad m = 1,\dots,N-1; \tag{4.11}$$

$$a_N^{(N+1)}(\rho,\alpha) = \rho a_N^{(N)}(\rho,\alpha) + (-2N+\alpha) + (2N+\alpha)\rho^{2N}; \tag{4.12}$$

$$a_{N+1}^{(N+1)}(\rho,\alpha) = \rho a_{N+1}^{(N)}(\rho,\alpha) + (-2(N-1)+\alpha)\rho + (2(N-1)+\alpha)\rho^{2N-1}; \tag{4.13}$$

$$\begin{aligned} a_{N+2}^{(N+1)}(\rho,\alpha) = \rho a_{N+2}^{(N)}(\rho,\alpha) &+ (2N-(2+\alpha)) + (-2(N-2)+\alpha)\rho^2 \\ &+ (2(N-2)+\alpha)\rho^{2N-2} - (2N+(2+\alpha))\rho^{2N}; \end{aligned} \tag{4.14}$$

$$\begin{aligned} a_{N+3}^{(N+1)}(\rho,\alpha) = \rho a_{N+3}^{(N)}(\rho,\alpha) &+ (2(N-1)-(2+\alpha))\rho + (-2(N-3)+\alpha)\rho^3 \\ &+ (2(N-3)+\alpha)\rho^{2N-3} - (2(N-1)+(2+\alpha))\rho^{2N-1}; \end{aligned} \tag{4.15}$$

$$\begin{cases} a_{N+k}^{(N+1)}(\rho,\alpha) = \rho a_{N+k}^{(N)}(\rho,\alpha) + (-(3N-k)+\alpha)\rho^k + (2(N-m+1)-\alpha)\rho^{k+2} \\ \qquad\qquad + ((3N-k)+\alpha)\rho^{2N-k} - (2(N-k+3)+\alpha)\rho^{2(N+1)-k}, \\ k = 4,\dots,N-1; \end{cases} \tag{4.16}$$

$$a_{2N}^{(N+1)}(\rho,\alpha) = \rho a_{2N}^{(N)}(\rho,\alpha) + (2-\alpha)\rho^{N-2} - \alpha\rho^N - (6+\alpha)\rho^{N+2} \tag{4.17}$$

$$a_{2N+1}^{(N+1)}(\rho,\alpha) = -\alpha\rho^{N-1} - (4+\alpha)\rho^{N+1}; \tag{4.18}$$

$$a_{2N}^{(N+1)}(\rho,\alpha) = -(2+\alpha)\rho^N. \tag{4.19}$$

The conclusion follows by substituting the expressions (4.3), ..., (4.9) in (4.11), ..., (4.19). □

As a first application of Lemma 4.1, we have the following,

Lemma 4.2 *Let $R_{N,\rho,\alpha}$ be the one variable function defined in Proposition 3.6. For any $N \geq 2$ and for any $\alpha > N-1$, $r = \rho_N(\alpha)$ is a relative strict maximum point for $R_{N,\rho_N(\alpha),\alpha}$.*

Proof It follows by Lemma 4.1 that,

$$\frac{d}{dr} R_{N,\rho,\alpha}(r) = \frac{2}{r\psi_{N,\rho}(r)} \frac{h_{N,\rho}(r)}{g_{N,\rho}^3(r)} P_N(r;\rho,\alpha).$$

It is straightforward to verify that,

$$0 < \frac{2}{r\psi_{N,\rho}(r)} \frac{h_{N,\rho}(r)}{g_{N,\rho}^3(r)} < +\infty, \quad \forall r \in (0,1).$$

Hence the sign and the roots of $\frac{d}{dr} R_{N,\rho_N(\alpha),\alpha}(r)$, $r \in (0,1)$ coincide with the sign and the roots of $P_N(r;\rho,\alpha)$, $r \in (0,1)$. By using Proposition 3.6, we know that $P_N(\rho_N(\alpha);$

$\rho_N(\alpha),\alpha)=0$. Hence, to conclude the proof, it is enough to prove that, for any $N\geq 2$ and for any $\alpha>N-1$,

$$\frac{d}{dr}P_N(\rho_N(\alpha);\rho_N(\alpha),\alpha)<0. \tag{4.20}$$

We will prove (4.20) for any $N\geq 4$. The cases $N=2,3$ can be easily worked out by the same argument. We first use Lemma 4.1 to obtain,

$$\frac{d}{dr}P_N(r;\rho,\alpha)=\sum_{m=1}^{2N} m a_m^{(N)}(\rho,\alpha)r^{m-1}.$$

By using (4.3)–(4.9), we also have

$$\begin{aligned}
&\rho^{-(N-2)}\frac{d}{dr}P_N(\rho;\rho,\alpha)\\
&\quad=(-2+\alpha)+(2+\alpha)\rho^2\\
&\qquad+\sum_{k=1}^{N-2}(N-k)\left[-2(N-k)+\alpha-2\sum_{j=1}^{N-k-1}\rho^{2j}+(2(N-k)+\alpha)\rho^{2(N-k)}\right]\\
&\qquad-2N\sum_{j=1}^{N-2}\rho^{2j+2}\\
&\qquad+\sum_{k=1}^{N-2}(N+k)\left[(2(N-k-1)-\alpha)\rho^{2k}-2\sum_{j=1}^{N-k-1}\rho^{2j+2k}-(2(N-k+1)+\alpha)\rho^{2N}\right]\\
&\qquad-\alpha(2N-1)\rho^{2N-2}-((4+\alpha)(2N-1)-(2+\alpha)2N)\rho^{2N}.
\end{aligned}$$

Then, for any $N\geq 4$, we obtain,

$$\begin{aligned}
&\rho^{-(N-2)}\frac{d}{dr}P_N(\rho;\rho,\alpha)\\
&\quad=\alpha\frac{N(N-1)}{2}-\frac{N(N-1)(2N-1)}{3}\\
&\qquad-\sum_{j=1}^{N-1}N[\alpha-(N-2j-1)]\rho^{2j}-\left[\alpha\frac{N(3N+1)}{2}+\frac{2N(N+1)(2N+1)}{3}\right]\rho^{2N}.
\end{aligned}$$

Clearly,

$$\alpha-(N-2j-1)\geq\alpha-(N-3),\quad \forall j=1,\dots,N-1.$$

Hence, for any $\alpha>N-1$, the coefficients corresponding to the powers $\rho^{2j},\ j=1,\dots,N$, are *all strictly negative*. Unfortunately this is not the case for the first term. Indeed,

$$\alpha\frac{N(N-1)}{2}-\frac{N(N-1)(2N-1)}{3}\leq 0\quad\Longleftrightarrow\quad \alpha\leq\frac{2(2N-1)}{3}.$$

Hence, (4.20) holds true for any $\alpha \leq \frac{2(2N-1)}{3}$ and we are left with the case $\alpha > \frac{2(2N-1)}{3}$. We use the fact that,

$$\rho^2 > \rho^2 > \cdots > \rho^{2(N-2)} > \rho^{2N}, \quad \forall \rho \in (0,1).$$

Then, we obtain,

$$\begin{aligned}
&\rho^{-(N-2)} \frac{d}{dr} P_N(\rho; \rho, \alpha) \\
&\quad < \alpha \frac{N(N-1)}{2} - \frac{N(N-1)(2N-1)}{3} \\
&\qquad - \left\{ \sum_{j=1}^{N-1} N\left[\alpha - (N-2j-1)\right] \right\} \rho^{2N} - \left[\alpha \frac{N(3N+1)}{2} - \frac{2N(N+1)(2N+1)}{3} \right] \rho^{2N}.
\end{aligned}$$

The sum in parentheses is easily evaluated to obtain,

$$-\sum_{j=1}^{N-1} N[\alpha - (N-2j-1)] = \alpha N(N-1) + N(N-1).$$

We conclude that,

$$\begin{aligned}
&\rho \frac{d}{dr} P_N(\rho; \rho, \alpha) \\
&\quad < \alpha \frac{N(N-1)}{2} - \frac{N(N-1)(2N-1)}{3} \\
&\qquad - \left[\alpha \left(\frac{N(3N+1)}{2} + N(N-1) \right) + \frac{2N(N+1)(2N+1)}{3} + N(N-1) \right] \rho^{2N}.
\end{aligned}$$

By using the explicit expression of $\rho_N(\alpha)$, we come up with the following sufficient condition for (4.20) to hold true:

$$\begin{aligned}
&\alpha \frac{N(N-1)}{2} - \frac{N(N-1)(2N-1)}{3} \\
&\quad - \left[\alpha \left(\frac{N(3N+1)}{2} + N(N-1) \right) + \frac{2N(N+1)(2N+1)}{3} + N(N-1) \right] \frac{\alpha - N + 1}{\alpha + N + 1} \\
&\quad < 0.
\end{aligned}$$

After some straightforward simplifications, this condition is seen to be equivalent to

$$-2\alpha^2 + (N-5)\alpha + \frac{2}{3}N^2 + 2 - 2\frac{8}{3} < 0.$$

Solving the inequality yields,

$$\alpha > \frac{1}{12}(3N - 15 + \sqrt{3}\sqrt{19N^2 + 18N + 11}).$$

It is then enough to check that for any $N \geq 4$,

$$\frac{2(2N-1)}{3} > \frac{1}{12}(3N - 15 + \sqrt{3}\sqrt{19N^2 + 18N + 11}),$$

which is equivalent to,

$$28N^2 - 59N + 4 > 0.$$

The conclusion follows since the roots of the equation are,

$$N_+ = \frac{1}{56}(59 + \sqrt{337}) \simeq 2.03701, \qquad N_- = \frac{1}{56}(59 - \sqrt{337}) \simeq 0.0701307. \qquad \square$$

As a first step toward the application of the Descartes' Rule of Signs to P_N, we will prove that for any $N \geq 2$, and for any $\rho \in (0,1)$ and $\alpha > 0$, then there is one and only one sign change in the sequence $\{a_m^{(N)}(\rho,\alpha)\}_{m=0,\dots,N}$. To achieve this goal, we will first prove that $\rho^{-(N-m-1)} a_m^{(N)}(\rho,\alpha) > \rho^{-(N-(m+1)-1)} a_{m+1}^{(N)}(\rho,\alpha)$, for any $m = 1,\dots,N-2$.

Lemma 4.3 *For any $N \geq 2$, for any $\alpha > 0$ and for any $\rho \in (0,1)$, it holds*

$$\rho^{-(N-m-1)} a_m^{(N)}(\rho,\alpha) > \rho^{-(N-(m+1)-1)} a_{m+1}^{(N)}(\rho,\alpha), \qquad \forall m = 1,\dots,N-2.$$

Moreover, for any $N \geq 2$, $\rho \in (0,1)$ and $\alpha > 0$, there is one and only one sign change in the sequence $\{a_m^{(N)}(\rho,\alpha)\}_{m=0,\dots,N}$.

Proof We discuss the case where $N \geq 3$. The case $N = 2$ is much easier.
For $m = 1$, we use (4.4) and (4.5) with $k = N-2$, and obtain

$$\begin{aligned}
\rho^{-(N-2)} a_1^{(N)}(\rho,\alpha) &= (-2+\alpha) + (2+\alpha)\rho^2 > (-4+\alpha) + (2+\alpha)\rho^2 \\
&= (-4+\alpha) - 2\rho^2 + (4+\alpha)\rho^2 > (-4+\alpha) - 2\rho^2 + (4+\alpha)\rho^4 \\
&= \rho^{-(N-3)} a_2^{(N)}(\rho,\alpha).
\end{aligned}$$

Note that the last inequality follows since $\rho \in (0,1)$.

We use the same argument in case $m \geq 2$. Fix m to be any integer satisfying $m \in \{2,\dots,N-2\}$. We use (4.5) with $k = N-m$ and $k = N-m+1$, to conclude that,

$$\begin{aligned}
\rho^{-N-m-1} a_m^{(N)}(\rho,\alpha) &= (-2m+\alpha) - 2\sum_{j=1}^{m-1} \rho^{2j} + (2m+\alpha)\rho^{2m} \\
&> (-2(m+1)+\alpha) - 2\sum_{j=1}^{m-1} \rho^{2j} + (2m+\alpha)\rho^{2m} \\
&= (-2(m+1)+\alpha) - 2\sum_{j=1}^{m-1} \rho^{2j} - 2\rho^{2m} + (2(m+1)+\alpha)\rho^{2m} \\
&> (-2(m+1)+\alpha) - 2\sum_{j=1}^{m} \rho^{2j} + (2(m+1)+\alpha)\rho^{2(m+1)} \\
&= \rho^{-N-m} a_{m+1}^{(N)}(\rho,\alpha).
\end{aligned}$$

The last inequality follows since $\rho \in (0,1)$. This fact concludes the proof of the monotonicity of the sequence $\{\rho^{-N-m-1} a_m^{(N)}(\rho,\alpha)\}_{m=1,\dots,N-1}$.

Next observe that, by using (4.3) and (4.6), then we have $a_0^{(N)}(\rho,\alpha) > 0$ and $a_N^{(N)}(\rho,\alpha) < 0$ for any $N \geq 2$ and $\rho \in (0,1)$. By monotonicity, we immediately conclude that if $\alpha > 0$ there is one and only one sign change in the sequence $\{a_m^{(N)}(\rho,\alpha)\}_{m=0,\dots,N}$. $\square$

Next, we provide sufficient conditions to guarantee that $\{a_m^{(N)}(\rho,\alpha)\}_{m=N,\dots,2N}$ does not change sign.

Lemma 4.4 *Assume that $N \geq 2$ and $\alpha \in (0,+\infty)$. Then, for any $\rho \in (0,1)$ which satisfies,*

$$\rho^{2(N-1)} > \frac{2N-4-\alpha}{2N+4+\alpha},$$

we have,

$$\{a_m^{(N)}(\rho,\alpha)\}_{m=N,\dots,2N} \subset (-\infty,0).$$

Proof By using (4.6), (4.8), (4.9), it follows immediately that $a_N^{(N)}(\rho,\alpha) < 0$, $a_{2N-1}^{(N)}(\rho,\alpha) < 0$ and $a_{2N}^{(N)}(\rho,\alpha) < 0$ for any $N \geq 2$, $\alpha > 0$ and for any $\rho \in (0,1)$. This observation clearly proves the lemma in case $N = 2$. Next, by using (4.7), we have,

$$a_{N+k}^{(N)}(\rho,\alpha) = \rho^{k-1}\left[(2(N-k-1)-\alpha) - 2\sum_{j=1}^{N-k-1}\rho^{2j} - (2(N-k+1)+\alpha)\rho^{2(N-k)}\right],$$

for any $k = 1,\dots,N-2$. Clearly, for any $\rho \in (0,1)$, we have $\rho^2 > \cdots > \rho^{2N-2} > \rho^{2N}$. Hence, for any $k = 1,\dots,N-2$, we obtain,

$$\rho^{-(k-1)}a_{N+k}^{(N)}(\rho,\alpha) < (2(N-k-1)-\alpha) + (4(N-k)+\alpha)\rho^{2(N-k)},$$

and in particular, $\rho^{-(k-1)}a_{N+k}^{(N)}(\rho,\alpha) < 0$, whenever,

$$\rho^{2(N-k)} > \frac{2(N-k-1)-\alpha}{4(N-k)+\alpha}.$$

The conclusion follows since,

$$\begin{aligned}
\max_{k=1,\dots,N-2}\left(\frac{2(N-k-1)-\alpha}{4(N-k)+\alpha}\right)^{\frac{1}{2(N-k)}} &= \max_{m=1,\dots,N-1}\left(\frac{2(m-1)-\alpha}{4(2N-m)+\alpha}\right)^{\frac{1}{2m}} \\
&\leq \max_{m=1,\dots,N-1}\left(\frac{2(m-1)-\alpha}{4(2N-m)+\alpha}\right)^{\frac{1}{2(N-1)}} \\
&= \left(\frac{2((N-1)-1)-\alpha}{4(2N-(N-1))+\alpha}\right)^{\frac{1}{2(N-1)}} \\
&= \left(\frac{2N-4-\alpha}{4N+4+\alpha}\right)^{\frac{1}{2(N-1)}}.
\end{aligned}$$

Then, we conclude in particular that $\{a_m^{(N)}(\rho,\alpha)\}_{m=N,\dots,2N} \subset (-\infty,0)$, for any $N \geq 3$ and $\alpha \in [2N-4,+\infty)$. $\square$

5 The Proof of Theorem 1.4

The Proof of Theorem 1.4 The first assertion follows immediately by Lemma 4.2.

Next observe that, by Lemma 4.1,

$$\frac{d}{dr}R_{N,\rho,\alpha}(r) = \frac{d}{dr}K_{N,\rho,\alpha}(r) = \frac{2}{r\psi_{N,\rho}(r)}\frac{h_{N,\rho}(r)}{g^3_{N,\rho}(r)}P_N(r;\rho,\alpha).$$

It is straightforward to verify that,

$$0 < \frac{2}{r\psi_{N,\rho}(r)}\frac{h_{N,\rho}(r)}{g^3_{N,\rho}(r)} < +\infty, \quad \forall r \in (0,1).$$

Hence, $r \in (0,1)$ is a critical point for $R_{N,\rho,\alpha}(r)$, if and only if $P_N(r;\rho,\alpha) = 0$.

We discuss the assertions of Theorem 1.4 separately.

CASE $N = 2$ and $\alpha \in (0 + \infty)$.

By Proposition 3.6, for any $\rho \in (0,1)$ and $\alpha \in (0,+\infty)$, $R_{2,\rho,\alpha}(r) \to -\infty$ as $r \to 0^+$ and $r \to 1^-$. Since $R_{2,\rho,\alpha}$ is smooth, it admits at least one interior absolute maximum point, i.e. $P_2(r;\rho,\alpha)$ admits at least one root in $(0,1)$. We are going to prove that $R_{2,\rho,\alpha}$ admits at most one critical point, i.e. that $P_2(r;\rho,\alpha)$ admits at most one root in $(0,1)$. We recall the celebrated Descartes' Rule of Signs [28]:

An equation can have as many true [positive] roots as it contains changes of sign from $+$ *to* $-$ *or from* $-$ *to* $+$.

By using either Lemma 4.1 or Lemmas 4.3 and 4.4 we conclude that for any $\alpha \in (0,+\infty)$ and for any $\rho \in (0,1)$, we have $a_0^{(2)}(\rho,\alpha) > 0$ and $\{a_m^{(N)}(\rho,\alpha)\}_{m=1,2,3,4} \subset (-\infty,0)$. It follows by the Descartes' Rule of Signs that $P_2(r;\rho,\alpha)$ admits at most one positive root. Hence, for any $\alpha \in (0,+\infty)$ and for any $\rho \in (0,1)$ there exist a unique critical point $\overline{r} = \overline{r}(2,\rho,\alpha) \in (0,1)$ for $R_{2,\rho,\alpha}$. Of course, $\overline{r}(2,\rho,\alpha)$ is the unique relative and strict absolute maximum for $R_{2,\rho,\alpha}$ in $(0,1)$. Finally, observe that by Theorem 1.3-(b), $\overline{r}(2,\rho,\alpha) = \rho$ if and only if $\alpha > 1$ and $\rho = \rho_2(\alpha)$.

CASE $N \geq 3$ and $\alpha \in [2N-4,+\infty)$.

By Proposition 3.6, $R_{N,\rho,\alpha}(r) \to -\infty$ as $r \to 0^+$ and $r \to 1^-$. Since $R_{N,\rho,\alpha}$ is smooth, it admits at least one interior absolute maximum point, i.e. $P_N(r;\rho,\alpha)$ admits at least one root in $(0,1)$. We are going to prove that $R_{N,\rho,\alpha}$ admits at most one critical point, i.e. that $P_N(r;\rho,\alpha)$ admits at most one root in $(0,1)$.

It follows by Lemma 4.3 that for any $N \geq 3$, $\alpha > 0$ and for any $\rho \in (0,1)$, there is one and only one change of sign in the sequence $\{a_m^{(N)}(\rho,\alpha)\}_{m=0,\ldots,N}$. By using Lemma 4.4, we see that for any $N \geq 3$, $\alpha \in [2N-4,\infty)$ and for any $\rho \in (0,1)$, $\{a_m^{(N)}(\rho,\alpha)\}_{m=N,\ldots,2N} \subset (-\infty,0)$. By using the Descartes' Rule of Signs once more we conclude that $P_N(r;\rho,\alpha)$ admits at most one positive root. Hence, for any $N \geq 3$, $\alpha \in [2N-4,\infty)$ and for any $\rho \in (0,1)$ there exist a unique critical point $\overline{r} = \overline{r}(N,\rho,\alpha) \in (0,1)$ for $R_{N,\rho,\alpha}$. Of course, $\overline{r}(N,\rho,\alpha)$ is the unique relative and strict absolute maximum for $R_{N,\rho,\alpha}$ in $(0,1)$. Finally, observe that by Theorem 1.3-(b), either $N = 3$, and then $\overline{r}(3,\rho,\alpha) = \rho$ if and only if $\alpha > 2$ and $\rho = \rho_3(\alpha)$, or $N \geq 4$, and then, for any $\alpha \geq 2N-4$, $\overline{r}(N,\rho,\alpha) = \rho$ if and only if $\rho = \rho_N(\alpha)$. □

Remark 5.1 It is clear by the proof that indeed Lemma 4.4 implies that for any $N \geq 3$ and $\alpha \in (0,+\infty)$, $R_{N,\rho,\alpha}$ has a unique root in $(0,1)$ whenever $\rho^{2(N-1)} > \frac{2N-4-\alpha}{2N+4+\alpha}$.

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