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Lie Symmetry Analysis and Exact Solutions for the Extended mKdV Equation

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Abstract By using Lie symmetry analysis and the method of dynamical systems for the extended mKdV equation, the all exact solutions based on the Lie group method are given. Especially, the bifurcations and traveling wave solutions are obtained. To guarantee the existence of the above solutions, all parameter conditions are determined. Furthermore, the exact analytic solutions are considered by using the power series method. Such solutions for the equation are important in both applications and the theory of nonlinear science.

Keywords Lie symmetry analysis · Bifurcation · Traveling wave solution · Exact analytic solution · Extended mKdV equation

Mathematics Subject Classification (2000) 17B80 · 35B10 · 35Q53

1 Introduction

The KdV type equations with the quadratic nonlinearity are important nonlinear models, which have been derived in many unrelated branches of sciences and engineering including the pulse-width modulation, mass transports in a chemical response theory, dust acoustic

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solitary structures in magnetized dusty plasmas and nonlinear long dynamo waves observed in the Sun. However, the higher-order nonlinear terms must be taken into account in some complicated situations like at the critical density or in the vicinity of the critical velocity. The modified KdV (mKdV) equation, on the other hand, has recently been discovered, e.g., to model the dust-ion-acoustic waves in such cosmic environments as those in the supernova shells and Saturn's F-ring, etc. The solutions of such equations have been studied extensively. Especially, the soliton solutions and the periodic wave solutions for the classical KdV equations $u_t + u_{xxx} + uu_x = 0$, $u_t = u_{xxx} + 6uu_x$ and the mKdV equation $u_t + u_{xxx} + 6u^2u_x = 0$, etc., are investigated [1–8]. In this paper, by using Lie symmetry analysis and the method of dynamical systems, we will consider the following extended form of the mKdV equation:

$$u_t + a_1u_{xxx} + a_2u_x + a_3uu_x + a_4u^2u_x = 0, \tag{1}$$

where $u = u(x,t)$ denotes the unknown function, the parameters $a_i \in \mathbb{R}$ $(i = 1, \dots, 4)$ and $a_1a_4 \neq 0$ (see Remark 1.1).

Clearly, when $a_1 = 1$, $a_2 = a_4 = 0$ and $a_3 = 1$, (1) is the KdV equation $u_t + u_{xxx} + uu_x = 0$. When $a_1 = -1$, $a_2 = a_4 = 0$ and $a_3 = -6$, (1) is the classical KdV equation $u_t = u_{xxx} + 6uu_x$. When $a_1 = 1$, $a_2 = a_3 = 0$ and $a_4 = 6$, (1) is the famous modified KdV (mKdV) equation $u_t + u_{xxx} + 6u^2u_x = 0$. Recently, by using the method of dynamical systems, J. Li considered a $(n+1)$-dimensional multiple sine-Gordon equation, [9, 10]. By using the tanh method and a variable separated ODE method, Wazwaz, [11] derived several exact traveling wave solutions for $\beta = 0$ and $m = 1$ in [9]. By using Lie symmetry analysis and power series method, H. Liu et al investigated the exact solutions for general Burgers' equation [12].

The outline of this paper is as follows. Firstly, the vector field and reduced equations for the extended mKdV equation will be given by Lie symmetry analysis. Secondly, we will consider the existence and dynamical behavior of the bounded traveling wave solutions of (1) in different regions of the parametric space by using the method of dynamical systems. Thirdly, the exact analytic solutions for the reduced equation are obtained. Finally, we conclude and make some remarks.

Remark 1.1 Note that $a_1 \neq 0$ is necessary for KdV type equations. If $a_4 = 0$, then (1) is rather a KdV type equation than a mKdV equation.

2 Lie Symmetry Analysis and Reduced Equations

First of all, by using Lie symmetry analysis method [1, 2, 12], we can get the vector field of (1) is as follows:

$$V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial t},$$
$$V_3 = [-2a_4x + (a_3^2 - 4a_2a_4)t]\frac{\partial}{\partial x} - 6a_4t\frac{\partial}{\partial t} + (2a_4u + a_3)\frac{\partial}{\partial u}.$$

It is easy to check that $\{V_1, V_2, V_3\}$ is closed under the Lie bracket. Thus, a basis for the Lie algebra is $\{V_1, V_2, V_3\}$, which is a three-dimensional Lie algebra. We can see that V_1 is a space translation, V_2 is a time translation, and V_3 is a scaling transformation.

In particular, when $a_1 = 1$, $a_2 = a_3 = 0$ and $a_4 = 6$, we obtain the vector field of mKdV equation $u_t + u_{xxx} + 6u^2 u_x = 0$ is as follows:

$$V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial t}, \qquad V_3 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}.$$

This is a basis of Lie algebra for the mKdV equation as we known.

When $a_1 = a_3 = 1$ and $a_2 = a_4 = 0$, we obtain the vector field of KdV equation $u_t + u_{xxx} + uu_x = 0$ is as follows:

$$V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial t}, \qquad V_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}.$$

When $a_1 = -1$, $a_3 = -6$ and $a_2 = a_4 = 0$, we obtain the vector field of classical KdV equation $u_t = u_{xxx} + 6uu_x$ is as follows:

$$V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial t}, \qquad V_3 = 36t\frac{\partial}{\partial x} - 6\frac{\partial}{\partial u}.$$

We know that for the concrete two KdV equations, the bases are $\{V_1, V_2, V_3, V_4\}$. It is a four-dimensional Lie algebra, where V_i $(i = 1, 2, 3)$ are the same as above which we obtained, while $V_4 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}$ can not be obtained by the basis of (1).

Remark 2.1 From our previous discussion, we know that the basis of mKdV equation rather than KdV type equation is obtained in terms of the basis of (1). In view of this, we would rather named (1) the extended mKdV equation.

Next, we will reduce (1) to ordinary differential equations (ODEs) by the Lie group method.

(i) In general, the linear combination of the two generators V_1 and V_2 generates the traveling wave solutions for a PDE (see Sect. 3).
(ii) For the generator of the scaling transformation V_3, we have the following similarity variables

$$\xi = xt^{-\frac{1}{3}} - \frac{4a_2a_4 - a_3^2}{4a_4}t^{\frac{2}{3}}, \qquad \omega = (2a_4u + a_3)t^{\frac{2}{3}},$$

and the group-invariant solution is $\omega = f(\xi)$, that is

$$u = \frac{1}{2a_4}t^{-\frac{1}{3}} f\left(xt^{-\frac{1}{3}} - \frac{4a_2a_4 - a_3^2}{4a_4}t^{\frac{2}{3}}\right) - \frac{a_3}{2a_4}. \tag{2}$$

Substituting (2) into (1), we reduce the equation to the following ODE:

$$12a_1a_4 f''' + 3f^2 f' - 4a_4\xi f' - 4a_4 f = 0, \tag{3}$$

where $f' = \frac{df}{d\xi}$. It implies that if $\omega = f(\xi)$ is a solution of (3), then (2) is a solution of (1).

3 Bifurcations and Exact Traveling Wave Solutions

In this section, we consider the existence and dynamical behavior of the bounded traveling wave solutions of (1) in different regions of the parametric space, by using the method of dynamical systems [9, 13, 14]. We shall give the possible exact explicit parametric representations for these bounded traveling wave solutions of (1) simultaneously.

3.1 Traveling Wave Transformation

Letting $u(x,t)=\phi(x-ct)=\phi(\xi)$, where $\xi=x-ct$ and c is the propagating wave velocity. Then, (1) can becomes

$$a_1\phi''+(a_2-c)\phi+\frac{1}{2}a_3\phi^2+\frac{1}{3}a_4\phi^3=0, \tag{4}$$

where $\phi'=\frac{d\phi}{d\xi}$.

Equation (4) is equivalent to the system

$$\frac{d\phi}{d\xi}=y, \qquad \frac{dy}{d\xi}=-\alpha\phi-\beta\phi^2-\gamma\phi^3, \tag{5}$$

where $\alpha=\frac{a_2-c}{a_1}$, $\beta=\frac{a_3}{2a_1}$, $\gamma=\frac{a_4}{3a_1}$, and $\alpha,\beta,\gamma\in\mathbb{R}$.

This system has the Hamiltonian function

$$H(\phi,y)=\frac{1}{2}y^2+\frac{1}{2}\alpha\phi^2+\frac{1}{3}\beta\phi^3+\frac{1}{4}\gamma\phi^4=h, \tag{6}$$

where $h\in\mathbb{R}$.

In the following sections, we will investigate the profiles of the traveling wave solutions and give all possible exact explicit parametric representations for the bounded traveling wave solutions.

3.2 Bifurcations of the Phase Portraits of System (5)

Now we discuss the bifurcations of the phase portraits of system (5) in the parameter space (α,β,γ). By qualitative analysis, we have the following results.

Firstly, note that there are three equilibrium points $O(0,0)$, $A_1(\phi_1,0)$ and $A_2(\phi_2,0)$ of system (5) on the ϕ-axis, where

$$\phi_1=\frac{1}{2\gamma}(-\beta+\sqrt{\Delta}), \qquad \phi_2=\frac{1}{2\gamma}(-\beta-\sqrt{\Delta}),$$

$\Delta=\beta^2-4\alpha\gamma$.

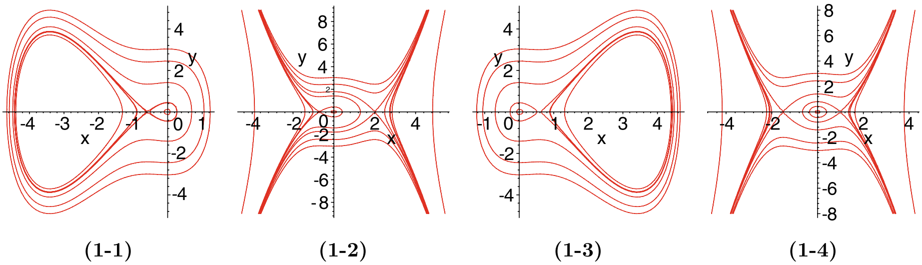
3.2.1 The Case $\alpha>0$

In this case, the equilibrium point $O(0,0)$ is a center point of system (5).

When $\Delta>0$, we have

$$\phi_1+\phi_2=-\frac{\beta}{\gamma}, \qquad \phi_1\phi_2=\frac{\alpha}{\gamma}. \tag{7}$$

Moreover, in view of $\phi_1-\phi_2=\frac{\sqrt{\Delta}}{\gamma}$, then we have $\phi_1>\phi_2$, if $\gamma>0$; and $\phi_1<\phi_2$, if $\gamma<0$ respectively.



(**1-1**) $\beta > 0, \gamma > 0$; (**1-2**) $\beta > 0, \gamma < 0$; (**1-3**) $\beta < 0, \gamma > 0$; (**1-4**) $\beta < 0, \gamma < 0$

Fig. 1 The bifurcations of phase portraits of (5) for $\Delta > 0$, $\alpha > 0$

(i) $\beta > 0, \gamma > 0$, the equilibrium point $A_1(\phi_1, 0)$ is a saddle point, while the equilibrium point $A_2(\phi_2, 0)$ is a center of system (5). And in view of (7), we have $\phi_2 < \phi_1 < 0$.
(ii) $\beta > 0, \gamma < 0$, the equilibrium points $A_1(\phi_1, 0)$ and $A_2(\phi_2, 0)$ are all saddle points of system (5). In view of (7), we have $\phi_1 < 0 < \phi_2$ and $|\phi_1| < |\phi_2|$.
(iii) $\beta < 0, \gamma > 0$, the equilibrium point $A_1(\phi_1, 0)$ is a center, while the equilibrium point $A_2(\phi_2, 0)$ is a saddle point of system (5). And in view of (7),we have $\phi_1 > \phi_2 > 0$.
(iv) $\beta < 0, \gamma < 0$, the equilibrium points $A_1(\phi_1, 0)$ and $A_2(\phi_2, 0)$ are all saddle points of system (5). In view of (7), we have $\phi_1 < 0 < \phi_2$ and $|\phi_1| > |\phi_2|$.

Figure 1 shows the bifurcations of phase portraits of system (5) in different regions of the (α, β, γ)-parameter space.

When $\Delta = 0$, there are two equilibrium points $O(0, 0)$ and $A(-\frac{\beta}{2\gamma}, 0)$ of system (5). In this case, $O(0, 0)$ is a center, while $A(-\frac{\beta}{2\gamma}, 0)$ is a higher order singular point of system (5).

When $\Delta < 0$, there are two imaginary points of system (5) in the (α, β, γ)-parameter space and $O(0, 0)$ is the only real equilibrium point (a center). Here, we are interested in the case $\Delta > 0$ mainly for two reasons. First, it is a general case in the theory of planar dynamical systems. Second, in practical, most of the applied systems satisfy this condition (see Remark 5.1).

3.2.2 *The Case* $\alpha = 0$

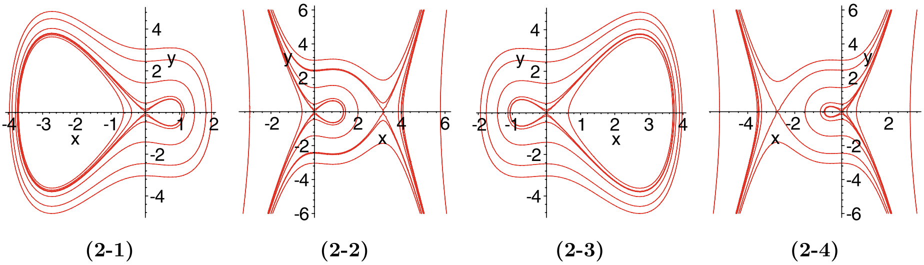
In this case, the equilibrium points of system (5) are higher-order singular points. We don't think to discuss the case in this paper.

3.2.3 *The Case* $\alpha < 0$

In this case, the equilibrium point $O(0, 0)$ is a saddle point of system (5).

When $\Delta > 0$, we have

(i) $\beta > 0, \gamma > 0$, the equilibrium points $A_1(\phi_1, 0)$ and $A_2(\phi_2, 0)$ are all center points of system (5). In view of (7), we have $\phi_2 < 0 < \phi_1$ and $|\phi_1| < |\phi_2|$.
(ii) $\beta > 0, \gamma < 0$, the equilibrium point $A_1(\phi_1, 0)$ is a center, while $A_2(\phi_2, 0)$ is a saddle point of system (5). And in view of (7), we have $\phi_2 > \phi_1 > 0$.
(iii) $\beta < 0, \gamma > 0$, the equilibrium points $A_1(\phi_1, 0)$ and $A_2(\phi_2, 0)$ are all center points of system (5). In view of (7), we have $\phi_2 < 0 < \phi_1$ and $|\phi_1| > |\phi_2|$.
(iv) $\beta < 0, \gamma < 0$, the equilibrium point $A_1(\phi_1, 0)$ is a saddle point, while $A_2(\phi_2, 0)$ is a center of system (5). And in view of (7), we have $\phi_1 < \phi_2 < 0$.



(**2-1**) $\beta > 0, \gamma > 0$; (**2-2**) $\beta > 0, \gamma < 0$; (**2-3**) $\beta < 0, \gamma > 0$; (**2-4**) $\beta < 0, \gamma < 0$

Fig. 2 The bifurcations of phase portraits of (5) for $\Delta > 0$, $\alpha < 0$

Figure 2 shows the bifurcations of phase portraits of system (5) in different regions of the (α, β, γ)-parameter space.

When $\Delta = 0$ and $\Delta < 0$, it is similar to the case $\alpha > 0$.

Clearly, the phase portraits (1-3) and (1-4) are just the reflections of (1-1) and (1-2) with respect to the y-axis in Fig. 1. While the phase portraits (2-3) and (2-4) are just the reflections of (2-1) and (2-2) with respect to the y-axis in Fig. 2.

3.3 The Parametric Representations for the Bounded Traveling Wave Solutions

Next, we use the results given by Sect. 2 to determine the parametric representations of the bounded phase orbits of system (5) in its parameter space. Then, the exact explicit traveling wave solutions for (1) will be given simultaneously by employing the hyperbolic and elliptic functions [15]. From the preceding discussion, we only consider the cases (1-1) and (1-2) in Fig. 1, the other cases are similar to be obtained.

3.3.1 The Case (1-1)

In this case, the parameter region satisfies $\Delta > 0$, $\alpha > 0$, $\beta > 0$ and $\gamma > 0$, see Fig. 1(1-1). The system (5) has the first integral (6).

(i) Corresponding to $H(\phi, y) = h$, $h \in (h_2, h_1)$, where $h_1 = H(\phi_1, 0)$, $h_2 = H(\phi_2, 0)$ (see Remark 5.2), we have two families of periodic orbits of system (5) for which the function (6) can be written as $y^2 = -\frac{1}{2}\gamma\phi^4 - \frac{2}{3}\beta\phi^3 - \alpha\phi^2 + 2h = \frac{1}{2}\gamma(r_4 - \phi)(\phi - r_1)(\phi - r_2)(\phi - r_3)$, and $y^2 = -\frac{1}{2}\gamma\phi^4 - \frac{2}{3}\beta\phi^3 - \alpha\phi^2 + 2h = \frac{1}{2}\gamma(r_4 - \phi)(r_3 - \phi)(r_2 - \phi)(\phi - r_1)$, respectively. By using the first equation of system (5), we obtain the following two parametric representations:

$$\phi(\xi) = r_3 + \frac{(r_3 - r_2)(r_3 - r_1)}{(r_2 - r_1)\mathrm{sn}^2(\omega_1\xi, k_1) - (r_3 - r_1)} \tag{8}$$

and

$$\phi(\xi) = r_1 + \frac{(r_4 - r_1)(r_3 - r_1)}{(r_4 - r_3)\mathrm{sn}^2(\omega_1\xi, k_1) + (r_3 - r_1)}, \tag{9}$$

where $\omega_1 = \sqrt{\frac{(r_4-r_2)(r_3-r_1)\gamma}{8}}$, $k_1 = \sqrt{\frac{(r_4-r_3)(r_2-r_1)}{(r_4-r_2)(r_3-r_1)}} < 1$. Hence, there exist the following periodic traveling solutions of (1):

$$u(x,t) = r_3 + \frac{(r_3 - r_2)(r_3 - r_1)}{(r_2 - r_1)\mathrm{sn}^2(\omega_1(x - ct), k_1) - (r_3 - r_1)}, \tag{10}$$

and

$$u(x,t)=r_1+\frac{(r_4-r_1)(r_3-r_1)}{(r_4-r_3)\mathrm{sn}^2(\omega_1(x-ct),k_1)+(r_3-r_1)}, \tag{11}$$

where r_i $(i=1,\dots,4)$ can be got by solving the following algebraic equation with respect to $\phi: H(\phi,0)=h$, $h\in(h_2,h_1)$. Note that for the concrete parameters, we can get the values r_i $(i=1,\dots,4)$ by solving the algebraic equation $H(\phi,0)=h$, that is $3\gamma r^4+4\beta r^3+6\alpha r^2-12h=0$.

(ii) Corresponding to $H(\phi,y)=h_1$, where $h_1=H(\phi_1,0)$, we have two homoclinic orbits of (5). The function (6) can be written as

$$y^2=2h_1-\alpha\phi^2-\frac{2}{3}\beta\phi^3-\frac{1}{2}\gamma\phi^4=\frac{1}{2}\gamma(\phi-r_2)^2(r_3-\phi)(\phi-r_1).$$

By using the first equation of (5), then we obtain the following two parametric representations:

$$\phi(\xi)=r_2+\frac{2(r_3-r_2)(r_2-r_1)}{(r_3-r_1)\cosh(\omega_2\xi)-(r_3-2r_2+r_1)}, \tag{12}$$

$$\phi(\xi)=r_2+\frac{2(r_3-r_2)(r_2-r_1)}{(r_1-r_3)\cosh(\omega_2\xi)-(r_3-2r_2+r_1)}, \tag{13}$$

where $\omega_2=\sqrt{\frac{(r_2-r_1)(r_3-r_2)\gamma}{2}}$. Therefore, we obtain two solitary wave solutions of (1) with the peak and valley type, respectively, as follows:

$$u(x,t)=r_2+\frac{2(r_3-r_2)(r_2-r_1)}{(r_3-r_1)\cosh(\omega_2(x-ct))-(r_3-2r_2+r_1)}, \tag{14}$$

and

$$u(x,t)=r_2+\frac{2(r_3-r_2)(r_2-r_1)}{(r_1-r_3)\cosh(\omega_2(x-ct))-(r_3-2r_2+r_1)}. \tag{15}$$

(iii) Corresponding to $H(\phi,y)=h$, $h\in(h_1,\infty)$, we have a family of periodic orbits of (5) enclosing three equilibrium points O, A_1 and A_2, for which the function (6) can be written as

$$y^2=2h-\alpha\phi^2-\frac{2}{3}\beta\phi^3-\frac{1}{2}\gamma\phi^4=\frac{1}{2}\gamma(r_2-\phi)(\phi-r_1)[(\phi-g_1)^2+g_2^2].$$

By using this formula and the first equation of (5), we obtain the following parametric representations:

$$\phi(\xi)=\frac{(r_1A-r_2B)\mathrm{cn}(\omega_3\xi,k_3)+r_1A+r_2B}{(A-B)\mathrm{cn}(\omega_3\xi,k_3)+A+B}, \tag{16}$$

where $\omega_3=\sqrt{\frac{AB\gamma}{2}}$, $k_3^2=\frac{(r_2-r_1)^2-(A-B)^2}{4AB}$, $A^2=(r_2-g_1)^2+g_2^2$, $B^2=(r_1-g_1)^2+g_2^2$. Thus, we have the periodic traveling wave solutions of (1):

$$u(x,t)=\frac{(r_1A-r_2B)\mathrm{cn}(\omega_3(x-ct)),k_3)+r_1A+r_2B}{(A-B)\mathrm{cn}(\omega_3(x-ct)),k_3)+A+B}. \tag{17}$$

3.3.2 The Case (1-2)

In this case, the parameter region satisfies $\Delta > 0$, $\alpha > 0$, $\beta > 0$ and $\gamma < 0$, see Fig. 1(1-2). The system (5) has the first integral (6) also.

(i) Corresponding to $H(\phi, y) = h$, $h \in (0, h_1)$, where $h_1 = H(\phi_1, 0)$ (see Remark 5.2), we have a family of periodic orbits of (5) which enclosing the equilibrium point $O(0,0)$. For which, the Hamiltonian function (6) can be written as $y^2 = 2h - \alpha\phi^2 - \frac{2}{3}\beta\phi^3 - \frac{1}{2}\gamma\phi^4 = -\frac{1}{2}\gamma(r_4 - \phi)(r_3 - \phi)(\phi - r_1)(\phi - r_2)$. By using the first equation of (5), we obtain the following parametric representation

$$\phi(\xi) = r_1 - \frac{(r_3 - r_1)(r_2 - r_1)}{(r_3 - r_2)\mathrm{sn}^2(\omega_4\xi, k_4) - (r_3 - r_1)}, \tag{18}$$

where $\omega_4 = \sqrt{\frac{-(r_3-r_1)(r_4-r_2)\gamma}{8}}$, $k_4^2 = \frac{(r_3-r_2)(r_4-r_1)}{(r_3-r_1)(r_4-r_2)} < 1$. Hence, there exists a following periodic traveling wave solutions of (1)

$$u(x,t) = r_1 - \frac{(r_3 - r_1)(r_2 - r_1)}{(r_3 - r_2)\mathrm{sn}^2(\omega_4(x - ct), k_4) - (r_3 - r_1)}. \tag{19}$$

(ii) Corresponding to $H(\phi, y) = h_1$, we have a homoclinic orbit of (5). The function (6) can be written as

$$y^2 = 2h_1 - \alpha\phi^2 - \frac{2}{3}\beta\phi^3 - \frac{1}{2}\gamma\phi^4 = -\frac{1}{2}\gamma(\phi - r_1)^2(r_2 - \phi)(r_3 - \phi),$$

then we can obtain the following the parametric representation

$$\phi(\xi) = r_1 + \frac{2(r_2 - r_1)(r_1 - r_3)}{(r_3 - r_2)\cosh(\omega_5\xi) - (r_2 + r_3 - 2r_1)}, \tag{20}$$

where $\omega_5 = \sqrt{-\frac{(r_2-r_1)(r_3-r_2)\gamma}{2}}$. Thus, we obtain

$$u(x,t) = r_1 + \frac{2(r_2 - r_1)(r_1 - r_3)}{(r_3 - r_2)\cosh(\omega_5(x - ct)) - (r_2 + r_3 - 2r_1)}. \tag{21}$$

This gives rise to a solitary wave solution of (1).

3.3.3 The Case $\beta = 0$

In particular, we discuss the special but important case $\beta = 0$ (see Remark 5.3). In view of (5), we have

$$\frac{d\phi}{d\xi} = y, \qquad \frac{dy}{d\xi} = -\alpha\phi - \gamma\phi^3. \tag{22}$$

This system has the Hamiltonian function

$$H(\phi, y) = \frac{1}{2}y^2 + \frac{1}{2}\alpha\phi^2 + \frac{1}{4}\gamma\phi^4 = h. \tag{23}$$

Clearly, there are three equilibrium points $O(0,0)$, $A_1(\phi_1, 0)$ and $A_2(\phi_2, 0)$ for system (22) on the ϕ-axis, where $\phi_1 = \sqrt{-\frac{\alpha}{\gamma}}$, $\phi_2 = -\sqrt{-\frac{\alpha}{\gamma}}$, and $\alpha\gamma < 0$ (we consider this case only. Otherwise, the system has one equilibrium point $O(0,0)$, it is a trivial case). By qualitative analysis, we have the following conclusions:

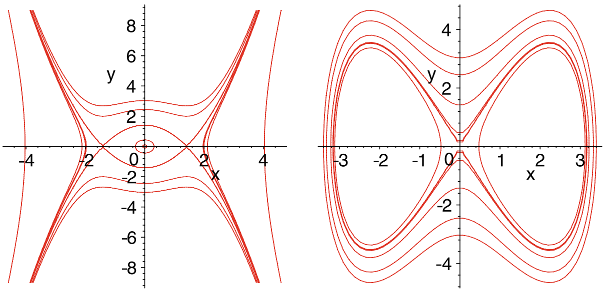


Fig. 3 The bifurcations of phase portraits of (22)

(3-1) **(3-2)**

(**3-1**) $\alpha > 0$, $\gamma < 0$; (**3-2**) $\alpha < 0$, $\gamma > 0$

(i) $\alpha > 0$, $\gamma < 0$, the equilibrium point $O(0,0)$ is a center, while the equilibrium points $A_1(\phi_1, 0)$ and $A_2(\phi_2, 0)$ are saddle points of system (22).

(ii) $\alpha < 0$, $\gamma > 0$, the equilibrium point $O(0,0)$ is a saddle, while the equilibrium points $A_1(\phi_1, 0)$ and $A_2(\phi_2, 0)$ are center points of system (22).

Now, we consider the case $\alpha > 0$, $\gamma < 0$, see Fig. 3(3-1).

(i) Corresponding to $H(\phi, y) = -\frac{\alpha^2}{4\gamma}$, we have two heteroclinic orbits of (22) connecting the equilibrium points A_1 and A_2. The function (23) can be written as

$$y^2 = -\frac{\alpha^2}{2\gamma} - \alpha\phi^2 - \frac{1}{2}\gamma\phi^4 = -\frac{1}{2}\gamma(\phi_1 - \phi)^2(\phi - \phi_2)^2.$$

By using this formula and the first equation of (22), we obtain the following two parametric representations

$$\phi(\xi) = \pm\sqrt{-\frac{\alpha}{\gamma}}\tanh(\omega_6\xi), \tag{24}$$

where $\omega_6 = \sqrt{\frac{\alpha}{2}}$. Hence, we obtain the kink wave solution and anti-kink wave solution of (1) as follows

$$u(x,t) = \pm\sqrt{-\frac{\alpha}{\gamma}}\tanh(\omega_6(x - ct)). \tag{25}$$

(ii) Corresponding to $H(\phi, y) = h$, $h \in (0, -\frac{\alpha^2}{4\gamma})$, we have a family of periodic orbits of (22) enclosing the equilibrium point $O(0,0)$, for which the function (23) can be written as

$$y^2 = 2h - \alpha\phi^2 - \frac{1}{2}\gamma\phi^4 = -\frac{1}{2}\gamma(a^2 - \phi^2)(b^2 - \phi^2),$$

where $a^2 = \frac{1}{-\gamma}(\alpha + \sqrt{\alpha^2 + 4h\gamma})$, $b^2 = \frac{1}{-\gamma}(\alpha - \sqrt{\alpha^2 + 4h\gamma})$. By using this formula and the first equation of (22), we obtain the following parametric representation of the family of periodic orbits

$$\phi(\xi) = b\,\mathrm{sn}(\omega_7\xi, k_7), \tag{26}$$

where $\omega_7 = a\sqrt{-\frac{\gamma}{2}}$, $k_7^2 = \frac{b^2}{a^2} < 1$. It follows that

$$u(x,t) = b\,\mathrm{sn}(\omega_7(x-ct), k_7). \tag{27}$$

This give rise to a family of periodic wave solutions of (1).

While the case (3-2) is similar to (1-1), we omit it here.

4 Exact Analytic Solutions

In general, we can not obtain the exact and explicit solutions for the nonlinear ODEs such as (3) by using the elementary functions. We know that the power series can be used to solve differential equations (DEs), including many complicated differential equations with non-constant coefficients, [16–19]. In this section, we will consider the exact analytic solutions of the reduced equation (3) by using the power series method.

Now, we seek a solution of (3) in a power series of the form

$$f(\xi) = \sum_{n=1}^{\infty} c_n \xi^n. \tag{28}$$

Substituting (28) into (3), we have

$$\begin{aligned} &72a_1a_4c_3 + 288a_1a_4c_4\xi + 12a_1a_4\sum_{n=1}^{\infty}(n+2)(n+3)(n+4)c_{n+4}\xi^{n+1} \\ &+ 3\sum_{n=1}^{\infty}\left[\sum_{k=1}^{n}\sum_{i=1}^{k}(n+1-k)c_i c_{k+1-i} c_{n+1-k}\right]\xi^{n+1} - 8a_4c_1\xi \\ &- 4a_4\sum_{n=1}^{\infty}(n+1)c_{n+1}\xi^{n+1} - 4a_4\sum_{n=1}^{\infty}c_{n+1}\xi^{n+1} = 0. \end{aligned} \tag{29}$$

From (29), comparing coefficients, we obtain

$$c_3 = 0, \qquad c_4 = \frac{c_1}{36a_1}. \tag{30}$$

Generally, for $n \geq 1$, we have

$$\begin{aligned} c_{n+4} = &\frac{1}{12a_1a_4(n+2)(n+3)(n+4)} \\ &\times\left[4a_4(n+2)c_{n+1} - 3\sum_{k=1}^{n}\sum_{i=1}^{k}(n+1-k)c_i c_{k+1-i}c_{n+1-k}\right], \quad n = 1, 2, \ldots. \end{aligned} \tag{31}$$

In view of (31), we can get all the coefficients c_n $(n \geq 5)$ of the power series (28), e.g.,

$$c_5 = \frac{1}{240a_1a_4}(4a_4c_2 - c_1^3) \quad (n=1); \qquad c_6 = -\frac{1}{120a_1a_4}c_1^2c_2 \quad (n=2);$$

and so on. Thus, for arbitrary chosen constant numbers $c_1 = \eta$, $c_2 = \lambda$ and $c_3 \equiv 0$, then the other terms of the sequence $\{c_n\}_{n=4}^{\infty}$ can be determined successively from (30) and (31) in a unique manner. This implies that for (3), there exists a power series solution given by (28) with the coefficients (30) and (31).

In fact, the power series solution of (3) can be written as follows:

$$f(\xi) = c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + \sum_{n=1}^{\infty} c_{n+4}\xi^{n+4} = \eta\xi + \lambda\xi^2 + \frac{\eta}{36a_1}\xi^4 + \sum_{n=1}^{\infty} c_{n+4}\xi^{n+4}. \tag{32}$$

In view of (2) and (32), we can get the power series solution of (1) as following:

$$\begin{aligned} u(x,t) = -\frac{a_3}{2a_4} + \frac{1}{2a_4}t^{-\frac{1}{3}}\Bigg[\eta\left(xt^{-\frac{1}{3}} - \frac{4a_2a_4 - a_3^2}{4a_4}t^{\frac{2}{3}}\right) + \lambda\left(xt^{-\frac{1}{3}} - \frac{4a_2a_4 - a_3^2}{4a_4}t^{\frac{2}{3}}\right)^2 \\ + \frac{\eta}{36a_1}\left(xt^{-\frac{1}{3}} - \frac{4a_2a_4 - a_3^2}{4a_4}t^{\frac{2}{3}}\right)^4 + \sum_{n=1}^{\infty} c_{n+4}\left(xt^{-\frac{1}{3}} - \frac{4a_2a_4 - a_3^2}{4a_4}t^{\frac{2}{3}}\right)^{n+4}\Bigg], \end{aligned} \tag{33}$$

where c_{n+4} $(n = 1, 2, \ldots)$ are given by (31) successively.

5 Conclusion and Remarks

In the present paper, we have considered the solutions of the extended mKdV equation by using Lie symmetry analysis and the method of dynamical systems, the all exact solutions based on the Lie group approach are obtained. By using the qualitative analysis method, we have obtained the bifurcations and exact parametric representations of (1) in different regions of the parametric space. Especially, the exact analytic solutions for the reduced equation are given by using the power series method for the first time in this paper.

Remark 5.1 In view of the condition $\Delta > 0$, that is $12a_1^2 + 16a_2a_4 - 3a_3^2 - 16a_4c < 0$, $a_i \in \mathbb{R}$ $(i = 1, \ldots, 4)$ and $a_1 \neq 0$. It is easy to check that the condition holds for KdV and mKdV equations. In fact, most of the KdV type equations derived from the physical systems satisfy the condition, the results are obtained in this paper can be applied to all these equations, certainly.

Remark 5.2 The conditions $h \in (h_2, h_1)$ and $h \in (0, h_1)$ are necessary in our results. In fact, in view of $\Delta > 0$, $\alpha > 0$, $\beta > 0$ and $\gamma > 0$, we have $h_1 - h_2 = H(\phi_1, 0) - H(\phi_2, 0) = \frac{1}{2}\alpha\phi_1^2 + \frac{1}{3}\beta\phi_1^3 + \frac{1}{4}\gamma\phi_1^4 - (\frac{1}{2}\alpha\phi_2^2 + \frac{1}{3}\beta\phi_2^3 + \frac{1}{4}\gamma\phi_2^4) = \frac{\beta}{12\gamma^3}\Delta^{3/2} > 0$, that is $h_1 > h_2$. On the other hand, we have $H(0,0) = 0$, and $h_1 = H(\phi_1, 0) = \frac{\phi_1^2}{24\gamma}[\sqrt{\Delta}(\beta - \sqrt{\Delta}) + 2\alpha\gamma] > 0$ (note that since $\beta^2 > \beta^2 - 4\alpha\gamma$, that is $\beta > \sqrt{\Delta}$).

Remark 5.3 Corresponding to $\beta = 0$, we have $a_3 = 0$ in (1). The famous mKdV equation belongs to this case. Moreover, we can get that $h_1 = h_2$ from Remark 5.2. It means that this equation has two heteroclinic orbits under this condition (see Fig. 3(3-1)). In general case (see Figs. 1 and 2), (1) has no any heteroclinic orbit, this coincides with the condition $h_1 \neq h_2$ (see Remark 5.2). From this, we can see that (1) differs greatly from the classical mKdV equation in dynamical behavior.

Remark 5.4 We reiterate that the power series solution (33) for the extended mKdV equation is an exact analytic solution, the convergence will be shown in appendix. Moreover, the solution of the power series converges quickly, so it is convenient for computations in both applications and physical systems. For example, based on the above formulaes, we have the approximate form of (33) as following:

$$\begin{aligned} u(x,t) = &-\frac{a_3}{2a_4} + \frac{1}{2a_4} t^{-\frac{1}{3}} \Bigg[\eta\left(xt^{-\frac{1}{3}} - \frac{4a_2a_4 - a_3^2}{4a_4} t^{\frac{2}{3}}\right) + \lambda\left(xt^{-\frac{1}{3}} - \frac{4a_2a_4 - a_3^2}{4a_4} t^{\frac{2}{3}}\right)^2 \\ &+ \frac{\eta}{36a_1}\left(xt^{-\frac{1}{3}} - \frac{4a_2a_4 - a_3^2}{4a_4} t^{\frac{2}{3}}\right)^4 + \frac{4a_4c_2 - c_1^3}{240a_1a_4}\left(xt^{-\frac{1}{3}} - \frac{4a_2a_4 - a_3^2}{4a_4} t^{\frac{2}{3}}\right)^5 \\ &- \frac{c_1^2c_2}{120a_1a_4}\left(xt^{-\frac{1}{3}} - \frac{4a_2a_4 - a_3^2}{4a_4} t^{\frac{2}{3}}\right)^6 + \cdots \Bigg]. \end{aligned} \tag{34}$$

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Appendix

In this appendix, we will show that the convergence of the power series solution (28) of (3) by using the implicit function theorem.

In view of (31), we have

$$|c_{n+4}| \le M\left(|c_{n+1}| + \sum_{k=1}^{n}\sum_{i=1}^{k} |c_i||c_{k+1-i}||c_{n+1-k}|\right), \quad n = 1, 2, \ldots,$$

where $M = \max\{\frac{1}{3|a_1|}, \frac{1}{4|a_1a_4|}\}$. If we define a power series $\mu = P(\xi) = \sum_{n=1}^{\infty} p_n\xi^n$ by

$$p_1 = |c_1| = |\eta|, \qquad p_2 = |c_2| = |\lambda|, \qquad p_3 = |c_3| = 0, \qquad p_4 = |c_4| = \frac{|\eta|}{36|a_1|}$$

and

$$p_{n+4} = M\left(p_{n+1} + \sum_{k=1}^{n}\sum_{i=1}^{k} p_i p_{k+1-i} p_{n+1-k}\right), \quad n = 1, 2, \ldots,$$

then it is easily seen that

$$|c_n| \le p_n, \quad n = 1, 2, \ldots.$$

In other words, the series $\mu = P(\xi) = \sum_{n=1}^{\infty} p_n\xi^n$ is a majorant series of (28). Next, we show that this series $\mu = P(\xi)$ has a positive radius of convergence. Indeed, note that by formal calculation, we have

$$\begin{aligned} P(\xi) &= p_1\xi + p_2\xi^2 + p_4\xi^4 + \sum_{n=1}^{\infty} p_{n+4}\xi^{n+4} = p_1\xi + p_2\xi^2 + p_4\xi^4 \\ &\quad + M\left[\sum_{n=1}^{\infty} p_{n+1}\xi^{n+4} + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\sum_{i=1}^{k} p_i p_{k+1-i} p_{n+1-k}\right)\xi^{n+4}\right] \\ &= |\eta|\xi + |\lambda|\xi^2 + \frac{|\eta|}{36|a_1|}\xi^4 + M[\xi^3(P(\xi) - |\eta|\xi) + \xi^2 P^3(\xi)]. \end{aligned}$$

Consider now the implicit functional equation

$$F(\xi,\mu)=\mu-|\eta|\xi-|\lambda|\xi^2-\frac{|\eta|}{36|a_1|}\xi^4-M\xi^2[\xi(\mu-|\eta|\xi)+\mu^3]=0.$$

Since F is analytic in the plane and $F(0,0)=0$, $F'_\mu(0,0)=1\neq 0$, by the implicit function theorem [20, 21], we see that $\mu=P(\xi)$ is analytic in a neighborhood of the origin of the plane and with a positive radius. This completes the proof.

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