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Periodic Solution for Strongly Nonlinear Vibration Systems by He's Energy Balance Method

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Abstract This paper applies He's Energy balance method (EBM) to study periodic solutions of strongly nonlinear systems such as nonlinear vibrations and oscillations. The method is applied to two nonlinear differential equations. Some examples are given to illustrate the effectiveness and convenience of the method. The results are compared with the exact solution and the comparison showed a proper accuracy of this method. The method can be easily extended to other nonlinear systems and can therefore be found widely applicable in engineering and other science.

Keywords Energy balance method · Nonlinear oscillation · Periodic solutions

1 Introduction

Nonlinear oscillations systems are such phenomena that mostly occur nonlinearly. These systems are important in engineering because many practical engineering components consist of vibrating systems that can be modeled using oscillator systems such as elastic beams supported by two springs or mass-on-moving belt or nonlinear pendulum and vibration of a milling machine [1, 2]. Hence solving of governing equations and due to limitation of existing exact solutions have been one of the most time-consuming and difficult affairs among researchers of vibrations. If there is no small parameter in the equation, the traditional perturbation methods cannot be applied directly. Recently, considerable attention has been directed towards the analytical solutions for nonlinear equations without possible small parameters. The traditional perturbation methods have many shortcomings, and they are not valid for strongly nonlinear equations. To overcome the shortcomings, many new techniques have appeared in open literature [3–14], such as non-perturbative methods [3], homotopy perturbation method [4–7], perturbation techniques [8], Lindstedt–Poincaré method [9, 10], parameter–expansion method [11, 12] and parameterized perturbation method [13, 14].

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Recently, some approximate variational methods, including approximate energy method [15–17, 32], variational iteration method [18–22] and variational approach [23–31], to solution, bifurcation, limit cycle and period solutions of nonlinear equations have been given much attention.

This paper presents Energy balance method (EBM) to study periodic solutions of strongly nonlinear systems. In this method according to basic idea of the energy balance method, if $\theta = 0$, it shows the whole energy is in form of kinetic energy and if $\theta = \pi/2$, it shows the whole energy is in form of potential energy, in $\theta = \pi/4$ there is a balance between the potential energy and kinetic energy so we can benefit from this point. Then a Hamiltonian is constructed, from which the angular frequency can be readily obtained by collocation method. The results are valid not only for weakly nonlinear systems, but also for strongly nonlinear ones. Some examples reveal that even the lowest order approximations are of high accuracy.

2 Energy Balance Method

In the present paper, we consider a general nonlinear oscillator in the form [23]:

$$u'' + f(u(t)) = 0 \tag{1}$$

in which u and t are generalized dimensionless displacement and time variables, respectively.

Its variational principle can be easily obtained:

$$J(u) = \int_0^t \left(-\frac{1}{2}u'^2 + F(u)\right) dt \tag{2}$$

where $T = 2\pi/\omega$ is period of the nonlinear oscillator, $F(u) = \int f(u)du$.

Its Hamiltonian, therefore, can be written in the form:

$$H = \frac{1}{2}u'^2 + F(u) = F(A) \tag{3}$$

or

$$R(t) = \frac{1}{2}u'^2 + F(u) - F(A) = 0 \tag{4}$$

Oscillatory systems contain two important physical parameters, i.e. the frequency ω and the amplitude of oscillation, A. So let us consider such initial conditions:

$$u(0) = A, \qquad u'(0) = 0 \tag{5}$$

Assume that its initial approximate guess can be expressed as:

$$u(t) = A\cos(\omega t) \tag{6}$$

Substituting (6) into u term of (4), yield:

$$R(t) = \frac{1}{2}\omega^2 A^2 \sin^2 \omega t + F(A\cos\omega t) - F(A) = 0 \tag{7}$$

If, by chance, the exact solution had been chosen as the trial function, then it would be possible to make R zero for all values of t by appropriate choice of ω. Since (5) is only an approximation to the exact solution, R cannot be made zero everywhere. Collocation at $\omega t = \pi/4$ gives:

$$\omega = \sqrt{\frac{2(F(A) - F(A\cos\omega t))}{A^2 \sin^2 \omega t}} \tag{8}$$

Its period can be written in the form:

$$T = \frac{2\pi}{\sqrt{\frac{2(F(A)-F(A\cos\omega t))}{A^2\sin^2\omega t}}} \tag{9}$$

3 Applications of Strongly Nonlinear Vibration Systems

In this section, some practical examples for some strongly nonlinear vibration systems are illustrated to show the applicability, accuracy and effectiveness of the proposed approach.

Example 1 As a first example, let us consider a family of nonlinear differential equations [30]:

$$u'' + \alpha u + \gamma u^{2m+1} = 0, \quad \alpha \geq 0,\ \gamma > 0,\ m = 1, 2, 3, K \tag{10}$$

where α, γ and m are constant values. With the initial conditions

$$u(0) = A, \qquad u'(0) = 0 \tag{11}$$

For this problem,

$$f(u) = \alpha u + \gamma u^{2m+1} \quad \text{and} \quad F(u) = \frac{1}{2}\alpha u^2 + \frac{\gamma u^{2m+2}}{2m+2}.$$

Its variational and Hamiltonian formulations can be readily obtained as follows:

$$J(u) = \int_0^t \left(-\frac{1}{2}u'^2 + \frac{1}{2}\alpha u^2 + \frac{\gamma u^{2m+2}}{2m+2}\right) dt \tag{12}$$

$$H = \frac{1}{2}u'^2 + \frac{1}{2}\alpha u'^2 + \frac{\gamma u^{2m+2}}{2m+2} = \frac{1}{2}\alpha A^2 + \frac{\gamma A^{2m+2}}{2m+2} \tag{13}$$

$$R(t) = \frac{1}{2}u'^2 + \frac{1}{2}\alpha u^2 + \frac{\gamma u^{2m+2}}{2m+2} - \frac{1}{2}\alpha A^2 - \frac{\gamma A^{2m+2}}{2m+2} = 0 \tag{14}$$

Substituting (6) into (14), we obtain:

$$R(t) = \frac{1}{2}A^2\omega^2 \sin^2 \omega t + \frac{1}{2}\alpha A^2 \cos^2 \omega t + \frac{\gamma (A\cos\omega t)^{(2m+1)}}{2m+1} - \frac{\alpha A^2}{2} - \frac{\gamma A^{(2m+2)}}{2m+2} = 0 \tag{15}$$

We obtain the following result:

$$\omega = \frac{\sqrt{\left(-(m+1)\left(\begin{array}{c}-\alpha A^2 m + \alpha A^2 m \cos^2 \omega t + \gamma (A \cos \omega t)^{2m+2} - \gamma A^{2m+2} \\ + \alpha A^2 (1 - \cos^2 \omega t)\end{array}\right)\right)}}{A(m+1) \sin \omega t} \tag{16}$$

with $T = \frac{2\pi}{\omega}$, yields:

$$T = \frac{2\pi A(m+1) \sin \omega t}{\sqrt{\left(-(m+1)\left(\begin{array}{c}-\alpha A^2 m + \alpha A^2 m \cos^2 \omega t + \gamma (A \cos \omega t)^{2m+2} - \gamma A^{2m+2} \\ + \alpha A^2 (\cos^2 \omega t - 1)\end{array}\right)\right)}} \tag{17}$$

If we collocate at $\omega t = \pi/4$, we obtain:

$$\omega = \frac{\sqrt{((m+1)A^2(\alpha(1+m) + 2\gamma A^{2m}(1 - 2^{m-1}4^{-m})))}}{(m+1)A} \tag{18}$$

with $T = \frac{2\pi}{\omega}$, yields:

$$T = \frac{2\pi A(m+1)}{\sqrt{((m+1)A^2(\alpha(1+m) + 2\gamma A^{2m}(1 - 2^{m-1}4^{-m})))}} \tag{19}$$

The above results are in good agreement with the results obtained by the exact solution in [30] as illustrated in Fig. 1(a–b).

Example 2 In dimensionless form, a mass attached to the center of a stretched elastic wire has the equation of motion [31]:

$$u'' + u - \frac{\lambda u}{\sqrt{1+u^2}} = 0, \quad u(0) = A,\ u'(0) = 0 \tag{20}$$

This is an example of a conservative nonlinear oscillatory system having an irrational elastic item. All the motions corresponding to (20) are periodic [31], the system will oscillate between symmetric bounds $[-A, A]$, and its angular frequency and corresponding periodic solution are dependent on the amplitude A.

Its variational formulation can be readily obtained as follows:

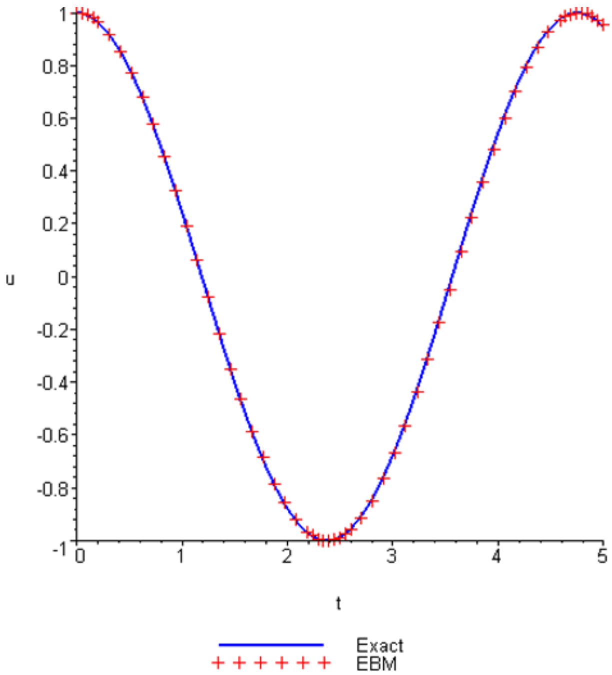
$$J(u) = \int_0^t \left(-\frac{1}{2}u'^2 + \frac{1}{2}u^2 + \lambda\sqrt{1+u^2}\right) dt \tag{21}$$

By a similar manipulation as illustrated in previous example by using (6) and with $T = \frac{2\pi}{\omega}$, we obtain the following result:

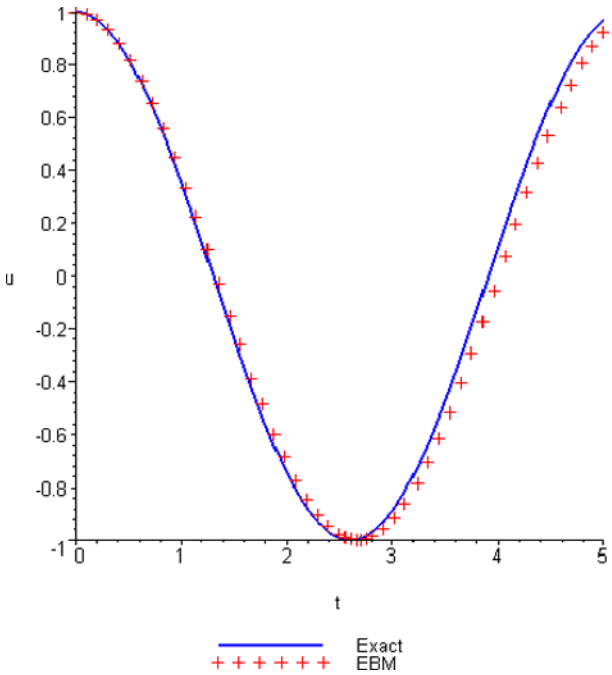
$$R(t) = \frac{1}{2}A^2\omega^2 \sin^2 \omega t + \frac{1}{2}A^2 \cos^2 \omega t + \gamma\sqrt{1+A^2} - \frac{A^2}{2} - \frac{\gamma A^{(2m+2)}}{2m+2} - \gamma\sqrt{1 + A^2 \cos^2 \omega t} = 0 \tag{22}$$

$$\omega = \frac{\sqrt{A^2 \sin^2 \omega t + 2\gamma(\sqrt{1 - A^2 \sin^2 \omega t + A^2} + \sqrt{1+A^2})}}{A \sin \omega t} \tag{23}$$

Fig. 1 (**a**) $\alpha = \gamma = A = m = 1$; (**b**) $\alpha = \gamma = A = 1$, $m = 4$



(a)



(b)

$$T = \frac{2\pi A \sin \omega t}{\sqrt{A^2 \sin^2 \omega t + 2\gamma(\sqrt{1 - A^2 \sin^2 \omega t + A^2} + \sqrt{1 + A^2})}} \tag{24}$$

Substituting $\omega t = \pi/4$ into (25), (26), we have:

$$\omega = \frac{\sqrt{A^2 + 2\gamma(\sqrt{4 + 2A^2} - 2\sqrt{1 + A^2})}}{A} \tag{25}$$

$$T = \frac{2\pi A}{\sqrt{A^2 + 2\gamma(\sqrt{4 + 2A^2} - 2\sqrt{1 + A^2})}} \tag{26}$$

The above results are in good agreement with the results obtained by the exact solution in [31] as illustrated in Fig. 2(a–d).

Example 3 In this example, we consider the following nonlinear Duffing-harmonic oscillation [33]:

$$u'' + \frac{u^3}{1 + u^2} = 0, \quad u(0) = A,\ u'(0) = 0 \tag{27}$$

Which, $f(u) = \frac{u^3}{1+u^2}$. Its variational formulation is:

$$J(u) = \int_0^t \left(-\frac{1}{2}u'^2 + \frac{1}{2}u^2 - \frac{1}{2}\ln(1 + u^2)\right) dt \tag{28}$$

Similar of previous examples, we have:

$$\begin{aligned} R(t) = {} & \frac{1}{2}A^2\omega^2 \sin^2 \omega t + \frac{1}{2}A^2 \cos^2 \omega t - \frac{1}{2}\ln(1 + A^2 \cos^2 \omega t) \\ & - \frac{A^2}{2} + \frac{1}{2}\ln(1 + A^2) = 0 \end{aligned} \tag{29}$$

From (29) and with $\omega t = \pi/4$, we have:

$$\omega = \frac{\sqrt{A^2 + 2\ln\left(1 + \frac{A^2}{2}\right) - 2\ln(1 + A^2)}}{A} \tag{30}$$

$$T = \frac{2\pi A}{\sqrt{A^2 + 2\ln\left(1 + \frac{A^2}{2}\right) - 2\ln(1 + A^2)}} \tag{31}$$

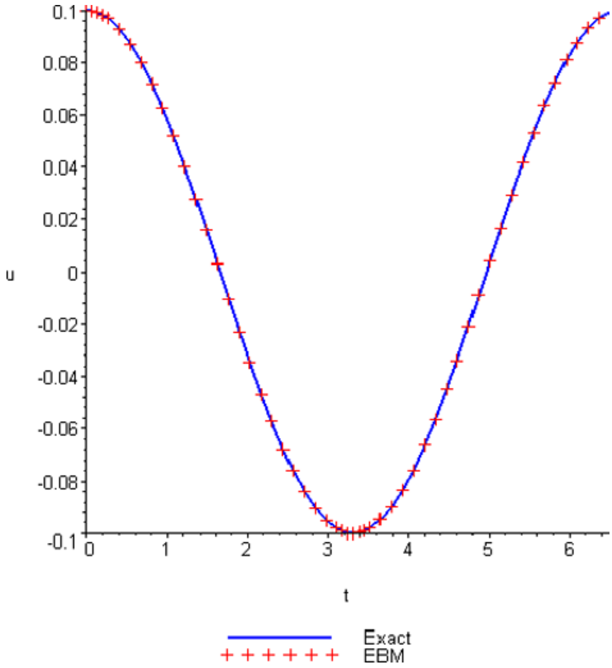
Example 4 As a last example, we consider the following nonlinear the relativistic oscillator [34]:

$$u'' + \frac{u}{\sqrt{1 + u^2}} = 0, \quad u(0) = B,\ u'(0) = 0. \tag{32}$$

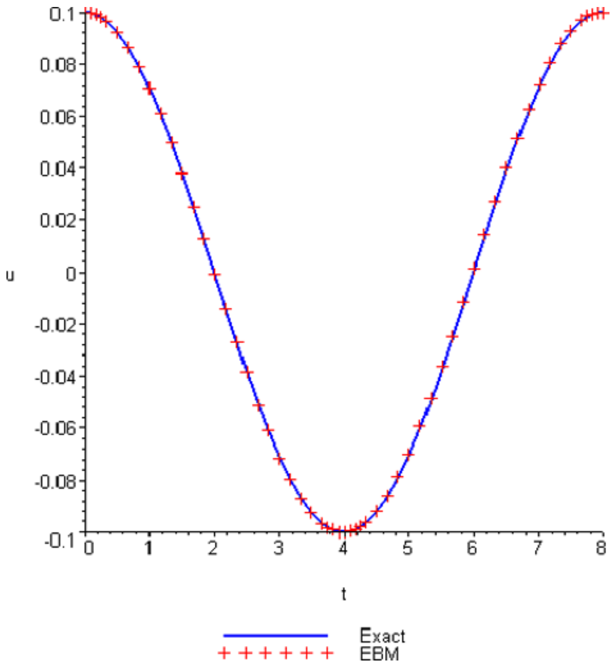
Which, $f(u) = \frac{u}{\sqrt{1+u^2}}$. Its variational formulation is:

$$J(u) = \int_0^t \left(-\frac{1}{2}u'^2 + \frac{1}{2}u^2 - (1 + u^2)\right) dt \tag{33}$$

Fig. 2 Comparison of the approximate solution (EBM) with the exact solution for Example 2. (**a**) $\lambda = 0.1$, $A = 0.1$; (**b**) $\lambda = 0.5$, $A = 1$ (vertical axes must multiply by 10^1); (**c**) $\lambda = 0.75$, $A = 10.0$ (vertical axes must multiply by 10^2); (**d**) $\lambda = 0.95$, $A = 100.0$ (vertical axes must multiply by 10^3)

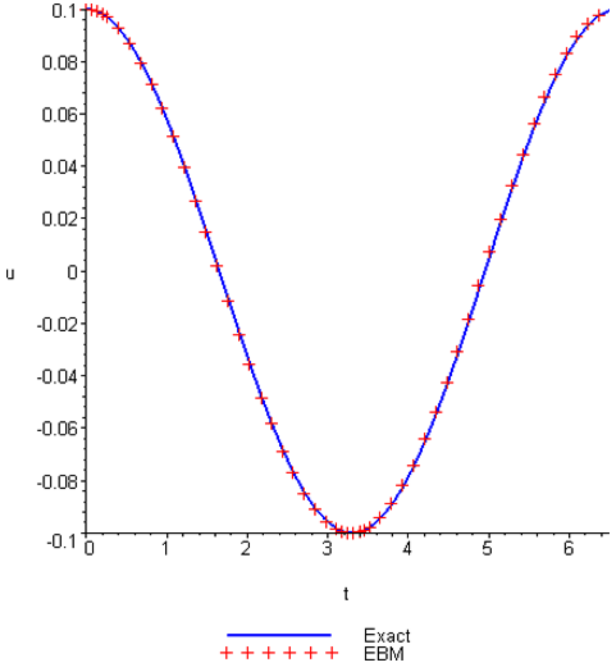


(a)

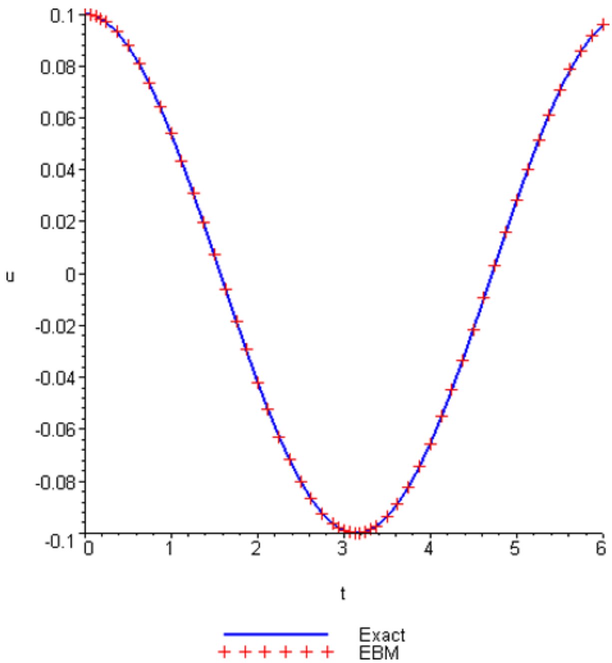


(b)

Fig. 2 (*Continued*)



(c)



(d)

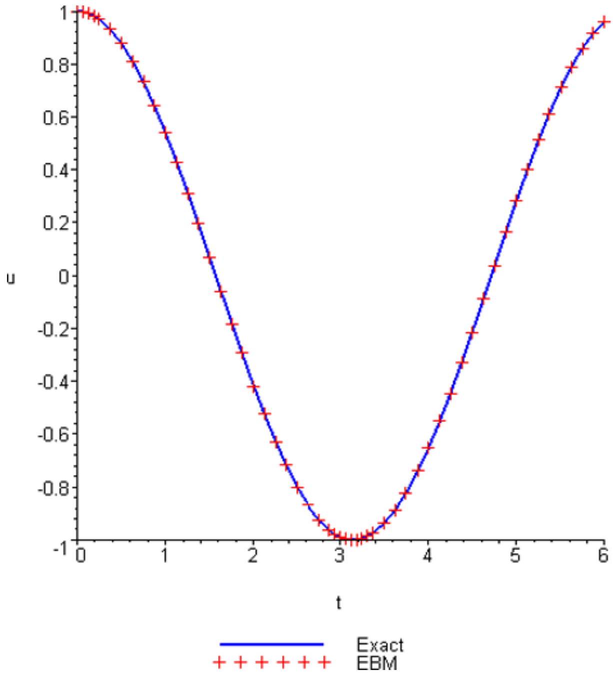


Fig. 3 Comparison of the approximate solution (EBM) with the exact solution $A = 100.0$ for Example 3 (vertical axes must multiply by 10^2)

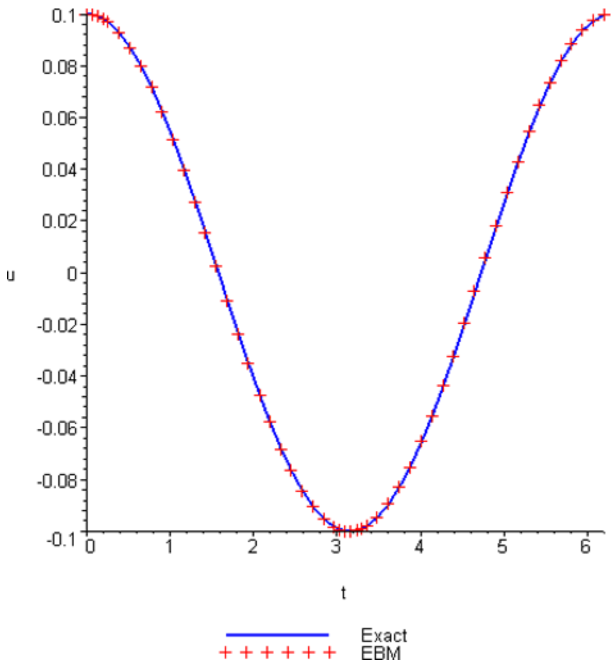


Fig. 4 Comparison of the approximate solution (EBM) with the exact solution $A = 0.1$ for Example 4

We have:

$$R(t) = \frac{1}{2}A^2\omega^2 \sin^2 \omega t + \sqrt{1 + A^2 \cos^2 \omega t} - \sqrt{1 + A^2} = 0 \tag{34}$$

From (29) and with $\omega t = \pi/4$, we have:

$$\omega = \frac{\sqrt{-2\sqrt{4 + 2A^2} + 4\sqrt{1 + A^2}}}{A} \tag{35}$$

$$T = \frac{2\pi A}{\sqrt{-2\sqrt{4 + 2A^2} + 4\sqrt{1 + A^2}}} \tag{36}$$

4 Discussion of Examples

To illustrate and verify the accuracy of the Energy Balance Method (EBM), the comparison with some published data and the corresponding exact solution is presented. The exact period T_{exa} for a dynamic system is governed by (10), (20) and (32) can be derived as shown in (37), (38) and (39), respectively [34, 36–38].

The exact period T_{exa}[30] for Example 1 is:

$$T_{exa} = 4\int_0^{\pi/2} \frac{dt}{\sqrt{\alpha + \frac{\gamma}{m+1}A^{2m}(1 + \sin^2 t + L + \sin^{2m} t)}}. \tag{37}$$

and the exact period T_{exa} [31] for Example 2 is:

$$T_{exa} = 4\int_0^{\pi/2} \left[\frac{1 - 2\lambda}{(\sqrt{1 + A^2 \sin^2 t} + \sqrt{1 + A^2})}\right]^{-1/2} \tag{38}$$

and the exact period T_{exa} [34] for Example 4 is:

$$T_{exa}(A) = \left[4\sqrt{4 + A^2}E\left(\frac{A^2}{4 + A^2}\right) - \frac{8}{\sqrt{4 + A^2}}K\left(\frac{A^2}{4 + A^2}\right)\right]^{-1} \tag{39}$$

The corresponding analytical approximation results are tabulated in Tables 1–6 for different parameters α, γ, m, λ and A.

From the Tables 1 and 2, the error percentage of the EBM is 0.39% for $\alpha = \gamma = 1$, $m = 1$ and $A = 1$ and for $\alpha = \gamma = 1$, $m = 4$ and $A = 1$ is 0.347%, in Example 1.

In Example 2, it is seen from the Table 3 that the error percentage of EBM is 0.00001510% for $\lambda = 0.1$ and $A = 1$. So from the Table 4, the error percentage of the EBM is 0.0% with $\lambda = 0.75$ and $A = 0.1$, 100.0.

In Example 3, we assume $A = 0.01$, 0.05, 0.1, 0.5, 1.0, 5.0, 10.0, 50.0, 100.0. The obtained exact results are expressed in [35].

The computed results for the EBM frequency ω with exact frequency ω_{exa} and CVM frequency ω_{CVM} listed in Table 5. From listed results we see that the maximum error is 2.243%. Hence, it is concluded provide excellent agreement with the exact solutions for the nonlinear systems. Also, the comparisons between the analytical and exact solutions are given in Table 6.

Table 1 Comparison between analytical (EBM and CVM) and Exact r solutions for Example 1, when $m = 1$, 3 and $\alpha, \gamma = 1$

A	$m = 1$				$m = 3$			
	T_{VCM}	T_{EBM}	T_{exa}	Error percentage	T_{VCM}	T_{EBM}	T_{exa}	Error percentage
1	4.7497	4.7496	4.7682	0.390	5.052	5.184	5.106	1.53
10	0.7208	0.7207	0.7363	2.119	8.496 E-3	9.0177 E-3	9.309 E-3	3.13
100	0.0725	0.0725	0.0742	2.291	8.496 E-6	9.0177 E-6	9.309 E-6	3.13
200	0.0368	0.0363	0.0371	2.156	1.062 E-6	1.147 E-6	1.164 E-6	1.46

Table 2 Comparison between analytical (EBM and CVM) and Exact r solutions for Example 1, when $m = 4$, 6 and $\alpha, \gamma = 1$

A	$m = 4$				$m = 6$			
	T_{VCM}	T_{EBM}	T_{exa}	Error percentage	T_{VCM}	T_{EBM}	T_{exa}	Error percentage
1	5.1436	5.3341	5.2080	2.421	5.2749	5.5461	5.1480	7.733
10	8.956 E-4	1.009 E-4	1.013 E-3	0.395	9.7079 E-6	1.1801 E -5	1.1758 E-5	0.366
100	8.956 E-8	1.009 E-8	1.013 E-7	0.395	9.7079 E-12	1.1801 E-11	1.1758 E-11	0.366
200	5.598 E-9	6.308 E-9	6.330 E-9	0.347	1.5168 E-13	1.8439 E-13	1.8373 E-13	0.359

Table 3 Comparison between analytical (EBM and CVM) and Exact r solutions for Example 2, when $\lambda = 0.1$ and $\lambda = 0.5$

$A \times 10^{-1}$	$\lambda = 0.1$				$\lambda = 0.5$			
	T_{VCM}	T_{EBM}	T_{exa}	Error percentage	T_{VCM}	T_{EBM}	T_{exa}	Error percentage
1	6.621688	6.621687	6.621688	1.510 E-5.0	8.869255	8.869247	8.869257	1.127 E-4.0
10	6.537455	6.535728	6.537507	2.721 E-2.0	7.988548	7.972824	7.992134	2.416 E-1.0
100	6.322923	6.320056	6.322938	4.550 E-2.0	6.489765	6.474308	6.490208	2.450 E-1.0
1000	6.287188	6.286869	6.287188	5.074 E-3.0	6.303276	6.301668	6.303281	2.560 E-2.0

Table 4 Comparison between analytical (EBM and CVM) and Exact r solutions for Example 2, when $\lambda = 0.75$ and $\lambda = 0.95$

$A \times 10^{-1}$	$\lambda = 0.75$				$\lambda = 0.95$			
	T_{VCM}	T_{EBM}	T_{exa}	Error percentage	T_{VCM}	T_{EBM}	T_{exa}	Error percentage
1	12.4967	12.4967	12.4967	0.00	27.1544	27.15395	27.15678	1.042 E-2.0
10	9.6049	9.5641	9.6254	0.637	11.9733	11.8737	12.0753	1.67 E -2.0
100	6.6010	6.5766	6.6021	0.386	6.6942	6.6621	6.6961	5.078 E-1.0
1000	6.3134	6.3110	6.3134	0.038	6.3215	6.3184	6.3215	4.904 E-2.0

Table 5 Comparison results for the angular frequencies of approximation (EBM, CVM) and exact solution with various A for Example 3

A	0.01	0.05	0.1	0.5	1.0	5.0	10.0	50.0	100.0
ω_{EBM}	0.00866	0.04326	0.08627	0.39638	0.65164	0.97343	0.99314	0.99972	0.99993
ω_{CVM}	0.00837	0.04325	0.08624	0.39423	0.64360	0.96731	0.99099	0.99952	0.99980
ω_{exa}	0.00847	0.04232	0.08439	0.38737	0.63678	0.96698	0.99092	0.99961	0.99990
Error percentage	2.243	2.22	2.22	2.33	2.33	0.667	0.224	0.011	0.003

Table 6 Comparison results for the angular frequencies of approximation (EBM) and exact solution with various A for Example 4

A	T_{EBM}	T_{exa}	Error percentage
0.1	6.294931	6.306635	0.18558
1.0	7.217406	7.942672	9.1313
10.0	18.421258	20.209038	8.84644
100.0	58.051115	57.974057	0.01329
1000.0	183.5673202	179.537534	2.24454

To further illustrate and verify the accuracy of this approximate analytical approach, a comparison of the time history oscillatory displacement responses with exact solution is presented in Figs. 1, 2, 3 and 4.

Figures 1(a–b) represent the displacement $u(t)$ for the various values of constants with different initial conditions in Example 1 while Figs. 2(a–d) represent the corresponding displacement $u(t)$ in Example 2. Apparently, it is confirmed that the analytical approximations shows excellent agreement with the exact solutions.

5 Conclusion

We used a simple but efficient method (EBM) for nonlinear oscillators. This method was applied for approaching frequency of the system. These examples have been shown that the approximate analytical solutions are in excellent agreement with the corresponding exact solutions. The method can be easily extended to any nonlinear oscillator without any difficulty. Moreover, the present work can be used as paradigms for many other applications in searching for periodic solutions of nonlinear oscillations and so can be found widely applicable in engineering and science.

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