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Zeros of Gegenbauer-Sobolev Orthogonal Polynomials: Beyond Coherent Pairs

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Abstract Iserles et al. (J. Approx. Theory 65:151–175, 1991) introduced the concepts of coherent pairs and symmetrically coherent pairs of measures with the aim of obtaining Sobolev inner products with their respective orthogonal polynomials satisfying a particular type of recurrence relation. Groenevelt (J. Approx. Theory 114:115–140, 2002) considered the special Gegenbauer-Sobolev inner products, covering all possible types of coherent pairs, and proves certain interlacing properties of the zeros of the associated orthogonal polynomials. In this paper we extend the results of Groenevelt, when the pair of measures in the Gegenbauer-Sobolev inner product no longer form a coherent pair.

Keywords Gegenbauer polynomials · Zeros of Gegenbauer-Sobolev orthogonal polynomials · Symmetrically coherent pairs

Mathematics Subject Classification (2000) 33C45 · 33C47 · 42C05

1 Introduction

Let $\{P_n^{\psi}\}_{n=0}^{\infty}$ be the sequence of monic orthogonal polynomials with respect to the inner product

$$\langle f, g\rangle_{\psi} = \int_{-b}^{b} f(x)g(x)d\psi(x)$$

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and let $\rho_n^{\psi} = \langle P_n^{\psi}, P_n^{\psi} \rangle_{\psi}$, where $d\psi$ is a symmetric measure on $[-b, b]$, $0 < b \le \infty$.

In [10], Iserles et al. have introduced the concept of symmetrically coherent pairs of measures (also the concept of coherent pairs of measures), showing that the pair of symmetric measures $\{d\psi_0, d\psi_1\}$ form a symmetrically coherent pair is equivalent to

$$P_n^{\psi_1}(x) = \frac{1}{n+1}[P_{n+1}^{\prime\,\psi_0}(x) + b_{n-1} P_{n-1}^{\prime\,\psi_0}(x)], \quad \text{for } n \ge 1,$$

where b_n are non-zero constants and $\{P_n^{\psi_0}\}$ and $\{P_n^{\psi_1}\}$ are the respective monic orthogonal polynomials. As a consequence of this, as proved in [10], the monic orthogonal polynomials P_n^S associated with the Sobolev inner product

$$\langle f, g \rangle_S = \langle f, g \rangle_{\psi_0} + \kappa \langle f', g' \rangle_{\psi_1},$$

where $\kappa \ge 0$, satisfy

$$P_{n+1}^S(x) + a_{n-1} P_{n-1}^S(x) = P_{n+1}^{\psi_0}(x) + b_{n-1} P_{n-1}^{\psi_0}(x), \quad n \ge 2. \tag{1}$$

Meijer, in [13], showed that in a coherent pair or symmetrically coherent pair one of the measures must be classical. In particular it was established that there are only five types of symmetrically coherent pairs of measures involving the classical measure of Gegenbauer, namely

- Type I $\{(1-x^2)^{\lambda-1/2}dx,\ \frac{(1-x^2)^{\lambda+1/2}}{x^2+\xi^2}dx\},\ \lambda > -1/2,\ \xi \ne 0.$
- Type II $\{(1-x^2)^{\lambda-1/2}dx,\ \frac{(1-x^2)^{\lambda+1/2}}{\xi^2-x^2}dx + M\delta(x+\xi)dx + M\delta(x-\xi)dx\}$, $\lambda > -1/2$, $|\xi| > 1$, $M \ge 0$.
- Type III $\{(x^2+\xi^2)(1-x^2)^{\lambda-1/2}dx,\ (1-x^2)^{\lambda+1/2}dx\},\ \lambda > -1/2.$
- Type IV $\{(\xi^2-x^2)(1-x^2)^{\lambda-1/2}dx,\ (1-x^2)^{\lambda+1/2}dx\},\ \lambda > -1/2,\ |\xi| > 1.$
- Type V $\{(1+M\delta(x+1)+M\delta(x-1))dx,\ dx\},\ M \ge 0.$

Here and in (6), $\delta(x)$ is the Dirac delta function such that $\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$ for f continuous.

Marcellán, Pérez and Piñar [12], in 1994, came up with a very nice idea to study the interlacing properties of the zeros of a special case of Gegenbauer-Sobolev orthogonal polynomials. Meijer and de Bruin [14] and de Bruin, Groenevelt and Meijer [8], both in 2002, extend this idea to study the zeros of the orthogonal polynomials with respect to the Jacobi-Sobolev inner products, Laguerre-Sobolev inner products (both in [14]) and Hermite-Sobolev inner products (in [8]). Using such an idea, also in the same year, Groenevelt [9] gives certain interlacing properties of the zeros of orthogonal polynomials with respect to the Gegenbauer-Sobolev inner products, covering all 5 types of coherent pairs written above. We mention that the papers [14] and [8] are cited in Groenevelt [9]. For an updated survey of recent results concerning zeros and asymptotics of orthogonal polynomials with respect to inner products on unbounded support, specifically Laguerre-Sobolev and Hermite-Sobolev inner products, we refer to [11].

Berti and Sri Ranga [3] and Berti, Bracciali and Sri Ranga [4] introduced an alternative approach to study Sobolev inner products with their orthogonal polynomials satisfying a recurrence relation of the form (1). This approach permits one to extend the pair of measures in the Sobolev inner products beyond coherent pairs and still maintain the required recurrence relation. The results given in the present paper are based on the following theorem proved in [3].

Theorem A *Let $d\phi_0$ and $d\psi_0$ be classical symmetric measures such that the monic orthogonal polynomials $\{P_n^{\phi_0}\}$ and $\{P_n^{\psi_0}\}$ satisfy $P_n^{\prime\,\psi_0}(x) = nP_{n-1}^{\phi_0}(x)$, $n \ge 1$. Let the symmetric measure $d\psi_1$ be such that $(1+qx^2)d\psi_1(x) = d\phi_0(x)$. Then the monic orthogonal polynomials $\{P_n^{S_1}\}$ associated with the inner product*

$$\langle f, g\rangle_{S_1} = \langle f, g\rangle_{\psi_0} + \langle f', g'\rangle_{\kappa_1\phi_0+\kappa_2\psi_1}, \tag{2}$$

satisfy $P_0^{S_1}(x) = P_0^{\psi_0}(x) = 1$, $P_1^{S_1}(x) = P_1^{\psi_0}(x) = x$, $P_2^{S_1}(x) = P_2^{\psi_0}(x)$ and

$$P_{n+1}^{S_1}(x) + a_{n-1}(q,\kappa_1,\kappa_2)P_{n-1}^{S_1}(x) = P_{n+1}^{\psi_0}(x) + b_{n-1}(q)P_{n-1}^{\psi_0}(x), \quad n \ge 2,$$

where

$$a_n(q,\kappa_1,\kappa_2) = b_n(q)\frac{\rho_n^{\psi_0} + \kappa_1 n^2 \rho_{n-1}^{\phi_0}}{\langle P_n^{S_1}, P_n^{S_1}\rangle_{S_1}}, \qquad b_n(q) = \frac{(n+2)\rho_{n+1}^{\psi_1}}{n\rho_{n-1}^{\phi_0}}q, \quad \text{for } n \ge 1. \tag{3}$$

Hence $b_n(0) = a_n(0,\kappa_1,\kappa_2) = 0$, $n \ge 1$. Moreover, when $q \ne 0$, the coefficients $a_{n+1}(q,\kappa_1,\kappa_2)$, $n \ge 2$, can be recursively generated by

$$\begin{aligned} &a_{n+1}(q,\kappa_1,\kappa_2) \\ &\quad= \frac{\nu_{n+1}(\kappa_1)\rho_n^{\phi_0} b_{n+1}(q)}{\nu_{n+1}(\kappa_1)\rho_n^{\phi_0} + b_{n-1}(q)\rho_{n-2}^{\phi_0}\{(n^2-1)q^{-1}\kappa_2 + \nu_{n-1}(\kappa_1)[b_{n-1}(q) - a_{n-1}(q,\kappa_1,\kappa_2)]\}}, \end{aligned}$$

with $a_1(q,\kappa_1,\kappa_2) = \frac{\nu_1(\kappa_1)b_1(q)}{\nu_1(\kappa_1)+\kappa_2\rho_0^{\psi_1}/\rho_0^{\phi_0}}$, $a_2(q,\kappa_1,\kappa_2) = \frac{\nu_2(\kappa_1)b_2(q)}{\nu_2(\kappa_1)+4\kappa_2\rho_1^{\psi_1}/\rho_1^{\phi_0}}$ and $\nu_n(\kappa_1) = n^2\kappa_1 + \rho_n^{\psi_0}/\rho_{n-1}^{\phi_0}$, $n \ge 1$. Here q, κ_1 and κ_2 are such that the inner products $\langle .,.\rangle_{\psi_1}$ and $\langle .,.\rangle_{S_1}$ are positive definite.

In [7], Delgado and Marcellán consider the following problem. Given the inner product $\langle f, g\rangle_S = \langle f, g\rangle_\psi + \langle f', g'\rangle_{\tilde\psi}$, where ψ and $\tilde\psi$ are symmetric measures on the real line, find the relation between these measures so that the recurrence relation of the type $P_{n+1}^S(x) + a_{n-1}P_{n-1}^S(x) = P_{n+1}^\psi(x) + b_{n-1}P_{n-1}^\psi(x)$ holds. It seems that the only known situation, with finite support for the measures, where such a recurrence relation holds is given by the above theorem.

In (2), if one takes $d\psi_0(x) = (1-x^2)^{\lambda-1/2}dx$, then

$$\begin{aligned} \langle f, g\rangle_{S_1} &= \langle f, g\rangle_{S_1(\lambda,q,\kappa_1,\kappa_2)} \\ &= \int_{-1}^1 f(x)g(x)(1-x^2)^{\lambda-1/2}dx \\ &\quad+ \int_{-1}^1 f'(x)g'(x)\left(\kappa_1 + \frac{\kappa_2}{1+qx^2}\right)(1-x^2)^{\lambda+1/2}dx \\ &\quad+ \kappa_2 M_q\left[f'\left(\frac{-1}{\sqrt{-q}}\right)g'\left(\frac{-1}{\sqrt{-q}}\right) + f'\left(\frac{1}{\sqrt{-q}}\right)g'\left(\frac{1}{\sqrt{-q}}\right)\right], \end{aligned} \tag{4}$$

where $\lambda > -1/2$ and $q \ge -1$. Moreover,

$$M_q \ge 0 \quad \text{for } -1 \le q < 0; \qquad M_q = 0 \quad \text{for } q \ge 0.$$

When $\kappa_1 = 0$, $q \neq 0$, $\kappa_2 > 0$ and $\lambda > -1/2$, the inner product (4) covers Type I and Type II of the coherent pairs of measures established by Meijer [13]. In these types, as proved in Groenevelt [9], there are n different real zeros of the corresponding Sobolev orthogonal polynomials $\{P_n^{S_1}\}$, with the exception for type I where only $n-2$ different zeros are guaranteed when n is even. For example, the $\lfloor n/2 \rfloor - 1$ largest positive zeros of $\{P_n^{S_1}\}$ interlace with the largest positive zeros of the Gegenbauer polynomial of degree n associated with the measure $d\psi_0(x) = (1-x^2)^{\lambda-1/2}dx$.

We mention that in [12] the authors have established the interlacing properties of the zeros of the Sobolev orthogonal polynomials with respect to the special case where $\kappa_1 = 0$, $q = -1$, $M_q = 0$, $\kappa_2 > 0$ and $\lambda > -1/2$.

When $\kappa_1 \neq 0$, the pairs of measures that appear in (4) no longer form a coherent pair. However, numerical experiments indicate that the interlacing properties established in [9] for $\kappa_1 = 0$ may still hold even when $\kappa_1 \neq 0$. In this paper we will prove these results for a larger domain of κ_1 and κ_2 such that $\kappa_1 \geq 0$ and $\kappa_2 \geq 0$, which includes

$$\lambda \geq 0, \qquad q \geq -1 \quad \text{and} \quad \kappa_2 \geq \left[\frac{2\lambda+3}{2\lambda+2} + 2(1+q)\right]\kappa_1 > 0. \tag{5}$$

The layout of the paper is as follows. In Sect. 2 the basic facts about the Gegenbauer and Gegenbauer-Sobolev polynomials needed will be given, followed in Sect. 3 by a chain of 5 lemmas from which new results—given in Sect. 4—can be derived. In Sect. 5, some special cases will be treated. Finally, also in Sect. 5, numerical examples are given that shows that the interlacing properties still hold under conditions not covered in this paper.

2 Gegenbauer to Gegenbauer-Sobolev Polynomials

In order to obtain information about the orthogonal polynomials with respect to the Gegenbauer-Sobolev inner product (4) from Theorem A, we set $d\psi_0 = d\Psi^{(\lambda)}$, $d\phi_0 = d\Psi^{(\lambda+1)}$ and $d\psi_1 = d\Psi^{(\lambda,q)}$, where

$$\begin{aligned} d\Psi^{(\lambda)}(x) &= (1-x^2)^{\lambda-1/2}dx \quad \text{and} \\ d\Psi^{(\lambda,q)}(x) &= \frac{d\Psi^{(\lambda+1)}(x)}{1+qx^2} + M_q[\delta(x+1/\sqrt{-q})dx + \delta(x-1/\sqrt{-q})dx]. \end{aligned} \tag{6}$$

Here, $\lambda > -1/2$ and $q \geq -1$. Moreover, $M_q \geq 0$ if $-1 \leq q < 0$ and $M_q = 0$ if $q \geq 0$. The absolutely continuous part of these measures are supported exactly on $[-1, 1]$.

The monic orthogonal polynomials $P_n^{\psi_0}$ and $P_n^{\phi_0}$ are the monic Gegenbauer polynomials $G_n^{(\lambda)}$ and $G_n^{(\lambda+1)}$, respectively. It is well known that the following recurrence relation holds:

$$G_{n+1}^{(\lambda)}(x) = xG_n^{(\lambda)}(x) - \alpha_{n+1}^{(\lambda)}G_{n-1}^{(\lambda)}(x), \quad n \geq 1, \tag{7}$$

with $G_0^{(\lambda)}(x) = 1$, $G_1^{(\lambda)}(x) = x$ and

$$\alpha_{n+1}^{(\lambda)} = \frac{n(n+2\lambda-1)}{4(n+\lambda)(n+\lambda-1)}, \quad n \geq 1.$$

Moreover, the n zeros of $G_n^{(\lambda)}$ are all real, simple and lie symmetrically inside $(-1, 1)$. We denote and arrange the $m = \lfloor n/2 \rfloor$ positive zeros of $G_n^{(\lambda)}$ by

$$0 < x_{n,m}^{(\lambda)} < \cdots < x_{n,2}^{(\lambda)} < x_{n,1}^{(\lambda)} < 1. \tag{8}$$

For more details about these polynomials see, for example, [6, 15].

If we denote the monic orthogonal polynomials $P_n^{\psi_1}$ by $G_n^{(\lambda,q)}$, the following results are also known (see, for example, [5])

$$G_n^{(\lambda,q)}(x) = G_n^{(\lambda+1)}(x) + d_{n-2}G_{n-2}^{(\lambda+1)}(x), \quad n \geq 2,$$

where $d_{n-2} = d_{n-2}(\lambda, q) = q\,\rho_n^{\psi_1}/\rho_{n-2}^{(\lambda+1)}$. Here, $\rho_n^{(\lambda+1)} = \rho_n^{\phi_0}$.

We have $d_{n-2}(\lambda, 0) = 0$, $n \geq 2$, and for $q \neq 0$ these coefficients can also be generated by $d_{n-2} = l_{n-1}l_n/q$, with

$$l_n = -1 + \frac{q\alpha_n^{(\lambda+1)}}{l_{n-1}}, \quad n \geq 2 \quad \text{and} \quad l_1 = q\alpha_2^{\psi_1}.$$

The zeros of $G_n^{(\lambda,q)}$ are also real, simple and lie symmetrically inside the interval $(-\xi_q, \xi_q)$, where

$$\xi_q = \begin{cases} 1/\sqrt{-q}, & \text{if } -1 \leq q < 0 \text{ and } M_q > 0, \\ 1, & \text{otherwise.} \end{cases} \tag{9}$$

To be precise, if we denote the $m = \lfloor n/2 \rfloor$ positive zeros of $G_n^{(\lambda,q)}$ by $x_{n,i}^{(\lambda,q)}$, $i = 1, 2, \ldots, m$, in decreasing order, then (see for example [5, Lemma 2])

$$x_{n,m}^{(\lambda,q)} < x_{n,m}^{(\lambda+1)} < x_{n-2,m-1}^{(\lambda+1)} < x_{n,m-1}^{(\lambda,q)} < \cdots < x_{n-2,1}^{(\lambda+1)} < x_{n,1}^{(\lambda,q)} < x_{n,1}^{(\lambda+1)} < 1,$$

if $q > 0$ and

$$x_{n,m}^{(\lambda+1)} < x_{n,m}^{(\lambda,q)} < x_{n-2,m-1}^{(\lambda+1)} < \cdots < x_{n,2}^{(\lambda,q)} < x_{n-2,1}^{(\lambda+1)} < x_{n,1}^{(\lambda+1)} < x_{n,1}^{(\lambda,q)} < \xi_q,$$

if $-1 \leq q < 0$.

Note that if $-1 \leq q < 0$ and $M_q > 0$ then $\xi_q > 1$. In this case the largest zero $x_{n,1}^{(\lambda,q)}$ and smallest zero $-x_{n,1}^{(\lambda,q)}$ of $G_n^{(\lambda,q)}$ can be outside the interval $(-1, 1)$. However, the remaining zeros stay within the interval $(-1, 1)$.

Substitution in Theorem A of the three measures $d\psi_0$, $d\phi_0$ and $d\psi_1$ given in (6) leads to the inner product (4) and information regarding the associated Gegenbauer-Sobolev orthogonal polynomials. We denote these orthogonal polynomials by $P_n^{S_1} = \mathcal{S}_n^{(\lambda,q,\kappa_1,\kappa_2)}$, or simply by $\mathcal{S}_n$ when no confusion arises.

As it was already shown in [3], $\mathcal{S}_0(x) = G_0^{(\lambda)}(x) = 1$, $\mathcal{S}_1(x) = G_1^{(\lambda)}(x) = x$, $\mathcal{S}_2(x) = G_2^{(\lambda)}(x)$ and

$$\mathcal{S}_{n+1}(x) + a_{n-1}\mathcal{S}_{n-1}(x) = G_{n+1}^{(\lambda)}(x) + b_{n-1}G_{n-1}^{(\lambda)}(x), \quad n \geq 2, \tag{10}$$

where $b_n = b_n(\lambda, q) = d_{n-1}(\lambda, q)(n+2)/n$ and the coefficients $a_n = a_n(\lambda, q, \kappa_1, \kappa_2)$ satisfy $a_n(\lambda, 0, \kappa_1, \kappa_2) = b_n(\lambda, 0) = 0$, $n \geq 1$, and for $q \neq 0$,

$$a_{n+2} = \frac{\nu_{n+2}\,\alpha_{n+1}^{(\lambda+1)}\alpha_{n+2}^{(\lambda+1)}b_{n+2}}{\nu_{n+2}\,\alpha_{n+1}^{(\lambda+1)}\alpha_{n+2}^{(\lambda+1)} + b_n\{n(n+2)\kappa_2 q^{-1} + \nu_n[b_n - a_n]\}} \tag{11}$$

for $n \geq 1$, with

$$a_1 = \frac{\nu_1 b_1}{\nu_1 + \kappa_2\rho_0^{\psi_1}/\rho_0^{\phi_0}}, \qquad a_2 = \frac{\nu_2 b_2}{\nu_2 + 4\kappa_2\rho_1^{\psi_1}/\rho_1^{\phi_0}}. \tag{12}$$

Here $\nu_n = \nu_n(\kappa_1) = n^2\kappa_1 + n/(n+2\lambda)$.

From (3) and the above equations one can also note that

$$\begin{aligned} &|a_n| \le |b_n|, \\ &\operatorname{sgn}(a_n) = \operatorname{sgn}(b_n) = \operatorname{sgn}(q), \quad n \ge 1, \end{aligned} \tag{13}$$

where strict inequality can be assumed if $\kappa_2 > 0$. Here

$$\operatorname{sgn}(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

By looking at the inner product (4) one can observe that it is positive definite if $\kappa_1 + \kappa_2(1+qx^2)^{-1} \ge 0$ for $x \in [-1,1]$ and $\kappa_2 M_q \ge 0$. Hence, if $q \ge 0$ then one must have $\{\kappa_2 \ge 0$ and $\kappa_1 \ge -\kappa_2(1+q)^{-1}\}$ or $\{0 > \kappa_2 \ge -\kappa_1$ and $\kappa_1 \ge 0\}$. While if $-1 \le q < 0$ then one must have $\{\kappa_2 \ge 0$ and $\kappa_1 \ge -\kappa_2\}$ or $\{0 > \kappa_2 \ge -(1+q)\kappa_1,\ \kappa_1 \ge 0$ and $M_q = 0\}$.

3 Preliminary Results

For $m \ge 0$, $i \ge 0$ and $k = 0, 1$ define

$$\mu_{i,m}^{(k)} = \int_{-1}^{1} x^k \mathcal{S}_{2m+k}(x)(1+qx^2)^i d\Psi^{(\lambda)}(x) \tag{14}$$

and

$$\hat{\mu}_{i,m}^{(k)} = \int_{-1}^{1} (qx)^{1-k} \mathcal{S}'_{2m+k}(x)(1+qx^2)^i d\Psi^{(\lambda,q)}(x), \tag{15}$$

where $d\Psi^{(\lambda)}$ and $d\Psi^{(\lambda,q)}$ are as in (6).

If $i \ge 1$ then $\hat{\mu}_{i,m}^{(k)} = \int_{-1}^{1} (qx)^{1-k} \mathcal{S}'_{2m+k}(x)(1+qx^2)^{i-1} d\Psi^{(\lambda+1)}(x)$. Hence, using integration by parts, we get

$$\hat{\mu}_{i,m}^{(0)} = 2(\lambda+i)\mu_{i,m}^{(0)} - [2\lambda + (2i-1)(2+q)]\mu_{i-1,m}^{(0)} + 2(i-1)(1+q)\mu_{i-2,m}^{(0)}, \tag{16}$$

$$\hat{\mu}_{i,m}^{(1)} = (2\lambda + 2i - 1)\mu_{i-1,m}^{(1)} - 2(i-1)(1+q)\mu_{i-2,m}^{(1)}, \tag{17}$$

for $i \ge 1$, with $\mu_{-1,m}^{(k)}$ arbitrary. However, we set $\mu_{-1,m}^{(k)} = 0$.

Lemma 1 *Let $m \ge 3$ and $2 \le i \le m-1$. Then,*

$$\mu_{i,m}^{(k)} = -\xi_i^{(k)} \left\{ \gamma_i^{(k)} \mu_{i-1,m}^{(k)} - \delta_i^{(k)} \mu_{i-2,m}^{(k)} + \sigma_i^{(k)} \mu_{i-3,m}^{(k)} \right\}, \quad k = 0, 1, \tag{18}$$

where

$$\xi_i^{(0)} = \frac{2i}{1 + 4i(i+\lambda)\kappa_1}, \qquad \xi_i^{(1)} = \frac{2}{1 + (2i+1)(2i+2\lambda+1)\kappa_1},$$

$$\gamma_i^{(0)} = 2A_1^{(0)}(i-2) + B_1^{(0)}, \qquad \gamma_i^{(1)} = 2A_1^{(1)}(i-2)^2 + B_1^{(1)}(i-2) + C_1^{(1)}/2,$$

$$\delta_i^{(0)} = 2A_2^{(0)}(i-2) + B_2^{(0)}, \qquad \delta_i^{(1)} = 2A_2^{(1)}(i-2)^2 + B_2^{(1)}(i-2) + C_2^{(1)},$$

$$\sigma_i^{(0)} = 2(i-2)(1+q)\kappa_2 \quad \textit{and} \quad \sigma_i^{(1)} = 2i(i-2)(1+q)\kappa_2.$$

Here

$$\begin{aligned}
&A_1^{(0)} = A_1^{(1)} = \kappa_2 - (2+q)\kappa_1, \qquad A_2^{(0)} = A_2^{(1)} = (2+q)\kappa_2 - (1+q)\kappa_1,\\
&B_1^{(0)} = 2(\lambda+1)\kappa_2 - (2\lambda+6+3q)\kappa_1, \qquad B_2^{(0)} = (2\lambda+2+q)\kappa_2 - 2(1+q)\kappa_1,\\
&B_1^{(1)} = 2(\lambda+4)\kappa_2 - (2\lambda+16+9q)\kappa_1, \qquad B_2^{(1)} = (2\lambda+12+7q)\kappa_2 - 6(1+q)\kappa_1,\\
&C_1^{(1)} = 5(2\lambda+3)\kappa_2 - 4(2\lambda+8+5q)\kappa_1, \qquad C_2^{(1)} = (4\lambda+7+5q)\kappa_2 - 4(1+q)\kappa_1.
\end{aligned} \tag{19}$$

Proof For $2 \le i \le m-1$ we have $\langle \mathcal{S}_{2m+k}, x^k(1+qx^2)^i\rangle_{S_1} = 0$. From this,

$$\mu_{i,m}^{(0)} = -2i\left\{\kappa_1\hat{\mu}_{i,m}^{(0)} + \kappa_2\hat{\mu}_{i-1,m}^{(0)}\right\}, \tag{20}$$

$$\mu_{i,m}^{(1)} = -\left\{(2i+1)\kappa_1\hat{\mu}_{i+1,m}^{(1)} + [(2i+1)\kappa_2 - 2i\kappa_1]\hat{\mu}_{i,m}^{(1)} - 2i\kappa_2\hat{\mu}_{i-1,m}^{(1)}\right\}. \tag{21}$$

Thus, using (16) and (17) in (20) and (21), respectively, we get the results of the lemma. □

Since $\langle \mathcal{S}_{2m}, 1\rangle_{S_1} = 0$ for $m \ge 1$, we have $\mu_{0,m}^{(0)} = 0$, $m \ge 1$. Now, from (10) and (13),

$$\operatorname{sgn}(\mu_{1,2}^{(0)}) = [\operatorname{sgn}(q)]^2 \quad \text{and} \quad \operatorname{sgn}(\mu_{0,1}^{(1)}) = \operatorname{sgn}(q). \tag{22}$$

Also from (10), for $m \ge 2$,

$$\mu_{i,m}^{(k)} = -a_{2m-2+k}\mu_{i,m-1}^{(k)}, \quad 0 \le i \le m-2. \tag{23}$$

Then from (22) and the above, we obtain for $m \ge 2$,

$$\operatorname{sgn}(\mu_{1,m}^{(0)}) = (-1)^m[\operatorname{sgn}(q)]^m \quad \text{and} \quad \operatorname{sgn}(\mu_{0,m-1}^{(1)}) = (-1)^m[\operatorname{sgn}(q)]^{m-1}. \tag{24}$$

Now, since $\langle \mathcal{S}_5, x(1+qx^2)\rangle_{S_1} = 0$ and $\langle \mathcal{S}_5, x\rangle_{S_1} = 0$, from (14) and (15) we obtain

$$\mu_{1,2}^{(1)} + 3\kappa_1\hat{\mu}_{2,2}^{(1)} - (2\kappa_1 - 3\kappa_2)\hat{\mu}_{1,2}^{(1)} - 2\kappa_2\hat{\mu}_{0,2}^{(1)} = 0 \quad \text{and} \quad \mu_{0,2}^{(1)} + \kappa_1\hat{\mu}_{1,2}^{(1)} + \kappa_2\hat{\mu}_{0,2}^{(1)} = 0.$$

These, with (17), lead to

$$\mu_{1,2}^{(1)} = -\frac{D_1^{(1)}}{1+3(2\lambda+3)\kappa_1}\mu_{0,2}^{(1)} \quad \text{and} \quad D_1^{(1)} = 2 - 6(1+q)\kappa_1 + 3(2\lambda+1)\kappa_2. \tag{25}$$

Lemma 2 *If* $\kappa_1 \ge 0$ *and* $\kappa_2 \ge 0$ *are such that*

$$\begin{aligned}
&A_1^{(0)} > 0, \quad A_2^{(0)} > 0, \quad B_1^{(0)} > 0, \quad B_2^{(0)} > 0, \quad B_1^{(1)} > 0, \quad B_2^{(1)} > 0,\\
&C_1^{(1)} > 0, \quad C_2^{(1)} > 0 \quad \text{and} \quad D_1^{(1)} > 0,
\end{aligned} \tag{26}$$

where the above coefficients are as given in (19) *and* (25), *then*

$$\begin{aligned}
&\mu_{0,m}^{(0)} = 0 \quad \text{and} \quad \operatorname{sgn}(\mu_{0,m}^{(1)}) = (-1)^{m-1}[\operatorname{sgn}(q)]^m \quad \text{for } m \ge 1,\\
&\operatorname{sgn}(\mu_{i,m}^{(k)}) = (-1)^{m+i-1}[\operatorname{sgn}(q)]^m \quad \text{for } k = 0, 1,\ i = 1, 2, \ldots, m-1 \text{ and } m \ge 2.
\end{aligned}$$

Proof With the condition $D_1^{(1)} > 0$ we obtain from (24) and (25) that $\mathrm{sgn}(\mu_{1,2}^{(1)}) = -\mathrm{sgn}(\mu_{0,2}^{(1)}) = [\mathrm{sgn}(q)]^2$. Hence, from (13) and (23),

$$\mathrm{sgn}(\mu_{1,m}^{(1)}) = (-1)^m [\mathrm{sgn}(q)]^m, \quad m \geq 2.$$

Moreover, in Lemma 1, the remaining conditions of (26) assure $\xi_i^{(k)} > 0$, $\gamma_i^{(k)} > 0$, $\delta_i^{(k)} > 0$ and $\sigma_{i+1}^{(k)} > 0$ for $k = 0, 1$ and $i \geq 2$. Hence, Lemma 2 can be concluded from the signs of $\mu_{0,m}^{(k)}$ and $\mu_{1,m}^{(k)}$ and the recurrence relation (18) in Lemma 1. □

Clearly, the conditions in (26) hold if $\lambda > -1/2$, $q \geq -1$, $\kappa_2 > 0$ and $\kappa_1 = 0$, thus establishing results obtained in Groenevelt [9]. Furthermore, one can also easily verify that another sufficient condition for (26) to hold is given by (5).

Lemma 3 *With* $m \geq 1$ *and* $k = 0, 1$, *let* π *be an even monic polynomial of degree* $2r$, $0 \leq r \leq m-1$, *with real zeros in* $(-1, 1)$. *If we define*

$$I_{r,m}^{(k)} = \int_{-1}^{1} x^k \mathcal{S}_{2m+k}(x) \pi(x) d\Psi^{(\lambda)}(x), \quad k = 0, 1, \tag{27}$$

then $I_{0,m}^{(0)} = 0$ *and otherwise* $\mathrm{sgn}(I_{r,m}^{(k)}) = (-1)^{m+r-1} [\mathrm{sgn}(q)]^{m+r}$, *provided that* (26) *holds or, at least*, (5) *holds.*

Proof If $q = 0$, this result is clearly true since $\mathcal{S}_{2m+k} = \mathcal{S}_{2m+k}^{(\lambda,0,\kappa_1,\kappa_2)} = G_{2m+k}^{(\lambda)}$ and, consequently, $I_{r,m}^{(k)} = 0$ follows from orthogonality. Thus we assume in the remaining of the proof that $q \neq 0$. Since π is a even monic polynomial of degree $2r$ with all its zeros in $(-1, 1)$, one can write

$$\pi(x) = \prod_{j=1}^{r} \left(x^2 - y_j^2\right) = \prod_{j=1}^{r} \left((x^2 + q^{-1}) - (q^{-1} + y_j^2)\right),$$

where $0 \leq y_j < 1$, $j = 1, 2, \ldots, r$, are the positive zeros of π. Noting that $(q^{-1} + y_j^2) < 0$ if $-1 < q < 0$ and $(q^{-1} + y_j^2) > 0$ if $q > 0$, we then obtain

$$\pi(x) = \sum_{i=0}^{r} \frac{\varepsilon_i}{q^i} (1 + qx^2)^i,$$

with $\varepsilon_r = 1$ and $\mathrm{sgn}(\varepsilon_i) = (-1)^{r-i} \mathrm{sgn}(q^{r-i})$, $i = 0, 1, \ldots, r$. Substituting the above expression for π in (27), we obtain

$$I_{r,m}^{(k)} = \sum_{i=0}^{r} \left[\frac{\varepsilon_i}{q^i} \int_{-1}^{1} x^k \mathcal{S}_{2m+k}(x) (1 + qx^2)^i d\Psi^{(\lambda)}(x) \right] = \sum_{i=0}^{r} \frac{\varepsilon_i}{q^i} \mu_{i,m}^{(k)}.$$

Since $\varepsilon_0 \mu_{0,m}^{(k)} = 0$ and otherwise $\mathrm{sgn}(\frac{\varepsilon_i}{q^i} \mu_{i,m}^{(k)}) = (-1)^{m+r-1} [\mathrm{sgn}(q)]^{m+r}$, we conclude the lemma. □

Now we look at the signs of $\hat{\mu}_{i,m}^{(k)}$. We can state the following lemma.

Lemma 4 *If* (26) *holds or, in particular, if* (5) *holds then*

$$\operatorname{sgn}(\hat{\mu}_{0,0}^{(1)}) > 0, \qquad \operatorname{sgn}(\hat{\mu}_{0,m}^{(1)}) = (-1)^m[\operatorname{sgn}(q)]^m \quad \text{for } m \geq 1,$$
$$\operatorname{sgn}(\hat{\mu}_{i,m}^{(k)}) = (-1)^{m+i+1-k}[\operatorname{sgn}(q)]^m \quad \text{for } k = 0, 1,\ k \leq i \leq m+k-1 \text{ and } m \geq 2.$$

Proof The results of $\hat{\mu}_{i,m}^{(0)}$ for $1 \leq i \leq m-1$ and of $\hat{\mu}_{i,m}^{(1)}$ for $1 \leq i \leq m$ follow from (16), (17) and Lemma 2.

To look at the signs of $\hat{\mu}_{0,m}^{(0)}$, we have

$$\hat{\mu}_{0,m}^{(0)} = q\left[\int_{-1}^{1} x\mathcal{S}_{2m}'(x)\frac{d\Psi^{(\lambda+1)}(x)}{1+qx^2} + 2M_q\frac{1}{\sqrt{-q}}\mathcal{S}_{2m}'\left(\frac{1}{\sqrt{-q}}\right)\right]. \tag{28}$$

We recall that one must take $M_q = 0$ if $q \geq 0$ and that one can take $M_q \geq 0$ if $-1 \leq q < 0$.

Since $\mathcal{S}_2'(x) = 2x$, we immediately obtain $\operatorname{sgn}(\hat{\mu}_{0,1}^{(0)}) = \operatorname{sgn}(q)$.

Now, from (28) and $\langle \mathcal{S}_{2m}, 1+qx^2\rangle_{S_1} = 0$, for $m \geq 2$,

$$\mu_{1,m}^{(0)} + 2\kappa_1\hat{\mu}_{1,m}^{(0)} + 2\kappa_2\hat{\mu}_{0,m}^{(0)} = 0, \quad m \geq 2.$$

Hence, from (16) and Lemma 2, $\operatorname{sgn}(\hat{\mu}_{0,m}^{(0)}) = (-1)^{m+1}[\operatorname{sgn}(q)]^m$, $m \geq 2$.

Now we look at the signs of $\hat{\mu}_{0,m}^{(1)}$. We know that

$$\hat{\mu}_{0,m}^{(1)} = \int_{-1}^{1} \mathcal{S}_{2m+1}'(x)\frac{d\Psi^{(\lambda+1)}(x)}{1+qx^2} + 2M_q\mathcal{S}_{2m+1}'\left(\frac{1}{\sqrt{-q}}\right).$$

Since $\mathcal{S}_1(x) = x$, we obtain $\hat{\mu}_{0,0}^{(1)} = \int_{-1}^{1}\frac{d\Psi^{(\lambda+1)}(x)}{1+qx^2} + 2M_q > 0$. From $\langle \mathcal{S}_{2m+1}, x\rangle_{S_1} = 0$ for $m \geq 1$, we also obtain

$$\mu_{0,m}^{(1)} + \kappa_1\hat{\mu}_{1,m}^{(1)} + \kappa_2\hat{\mu}_{0,m}^{(1)} = 0, \quad m \geq 1.$$

Using this with (17) and Lemma 2 we have $\operatorname{sgn}(\hat{\mu}_{0,m}^{(1)}) = (-1)^m[\operatorname{sgn}(q)]^m$, $m \geq 1$. This completes the proof of the lemma. □

Lemma 5 *For $m \geq 1$ and $k = 0, 1$, let $\pi(x)$ denote a symmetric monic polynomial of degree $2r+k-1$, $1 \leq r \leq m$, with real zeros in $(-\xi_q, \xi_q)$, where ξ_q is as in* (9). *Define*

$$J_{r,m}^{(k)} = \int_{-1}^{1} \mathcal{S}_{2m+k}'(x)\pi(x)d\Psi^{(\lambda,q)}(x). \tag{29}$$

Then $\operatorname{sgn}(J_{r,m}^{(k)}) = (-1)^{m+r}\operatorname{sgn}(q^{m+r})$, $r = 1, 2, \ldots, m-1$ *and* $\operatorname{sgn}(J_{m,m}^{(k)}) = 1$, *provided that* (26) *holds or, at least,* (5) *holds.*

Proof If $q = 0$, the results is clearly true since $\mathcal{S}_{2m+k}' = G_{2m+k}^{\prime(\lambda)} = (2m+k)G_{2m+k-1}^{(\lambda+1)}$ and, consequently, $J_{r,m}^{(k)} = 0$, $1 \leq r < m$ and $J_{m,m}^{(k)} > 0$ follows from orthogonality. Let $q \neq 0$. Since all the zeros of π are within $(-\xi_q, \xi_q)$, if we write (as in Lemma 3)

$$\pi(x) = x^{1-k}\sum_{i=0}^{r+k-1}\frac{\varepsilon_i}{q^i}(1+qx^2)^i,$$

then $\varepsilon_{r+k-1} = 1$ and $\mathrm{sgn}(\varepsilon_i) = (-1)^{r-i+k-1}\mathrm{sgn}(q^{r-i+k-1})$, $i = 0, 1, \ldots, r+k-1$. Hence, from (29),

$$J_{r,m}^{(k)} = \sum_{i=0}^{r+k-1} \frac{\varepsilon_i}{q^i} \int_{-1}^{1} x^{1-k} \mathcal{S}'_{2m+k}(x)(1+qx^2)^i d\Psi^{(\lambda,q)}(x) = q^{k-1} \sum_{i=0}^{r+k-1} \frac{\varepsilon_i}{q^i} \hat{\mu}_{i,m}^{(k)}$$

and the result follows from Lemma 4. □

4 Zeros and Extremal Points

Throughout this section we assume (26) holds or, at least, (5) holds. Let $s_{n,j}$, $j = 1, 2, \ldots, n$, in decreasing order, be the positive zeros of the orthogonal polynomial $\mathcal{S}_n$ of degree n with respect to the Gegenbauer-Sobolev inner product (4).

Theorem 1 *Let $m = \lfloor n/2 \rfloor$. Then the following statements hold*:
(i) *For $q > 0$, the polynomial $\mathcal{S}_n$ for any $n \geq 3$ has at least $n-2$ different real zeros. To be precise, if $n \geq 4$, the $m-1$ largest positive zeros of $\mathcal{S}_n$ satisfy*

$$x_{n,m}^{(\lambda)} < x_{n-2,m-1}^{(\lambda)} < s_{n,m-1} < x_{n,m-1}^{(\lambda)} < \cdots < x_{n,2}^{(\lambda)} < x_{n-2,1}^{(\lambda)} < s_{n,1} < x_{n,1}^{(\lambda)}.$$

(ii) *For $-1 \leq q < 0$, the polynomial $\mathcal{S}_n$ for any $n \geq 1$ has n different real zeros. To be precise, $x_{2,1}^{(\lambda)} = s_{2,1}$, $x_{3,1}^{(\lambda)} < s_{3,1}$ and, if $n \geq 4$, the positive zeros of $\mathcal{S}_n$ satisfy*

$$x_{n,m}^{(\lambda)} < s_{n,m} < x_{n-2,m-1}^{(\lambda)} < x_{n,m-1}^{(\lambda)} < \cdots < s_{n,2} < x_{n-2,1}^{(\lambda)} < x_{n,1}^{(\lambda)} < s_{n,1}.$$

Here $x_{n,j}^{(\lambda)}$ are the zeros of the nth degree Gegenbauer polynomial $G_n^{(\lambda)}$ as given in (8).

Proof The interlacing behaviour between the zeros of $G_n^{(\lambda)}$ and $G_{n-2}^{(\lambda)}$, which follows from orthogonality, is well known. See, for example, Szegő [15].

Let $n = 2m + k$, where $k = 0, 1$ and $m \geq 1$. For $1 \leq i \leq m$ consider

$$\pi_i(x) = \frac{G_{2m+k}^{(\lambda)}(x)}{x^k[x^2 - (x_{2m+k,i}^{(\lambda)})^2]},$$

symmetric monic polynomials of exact degree $2m-2$. Then from the Gaussian quadrature formula based on the zeros of $G_{2m+k}^{(\lambda)}$,

$$\int_{-1}^{1} x^k \mathcal{S}_{2m+k}(x)\pi_i(x) d\Psi^{(\lambda)}(x) = \omega_{2m+k,i}^{(\lambda)} \mathcal{S}_{2m+k}(x_{2m+k,i}^{(\lambda)}) \frac{G'^{(\lambda)}_{2m+k}(x_{2m+k,i}^{(\lambda)})}{x_{2m+k,i}^{(\lambda)}}.$$

Note that we have used the information that the nodes (zeros) and weights of this quadrature rule satisfy $\omega_{2m+k,2m+k+1-i}^{(\lambda)} = \omega_{2m+k,i}^{(\lambda)}$ and $x_{2m+k,2m+k+1-i}^{(\lambda)} = -x_{2m+k,i}^{(\lambda)}$. Thus, from Lemma 3 (with $r = m-1$),

$$\mathrm{sgn}(\mathcal{S}_{2m+k}(x_{2m+k,i}^{(\lambda)})) = (-1)^{i-1}\mathrm{sgn}(q), \quad i = 1, 2, \ldots, m. \tag{30}$$

Hence, there are at least $m-1$ positive zeros of $\mathcal{S}_n$. By symmetry, then there are at least $2m-2$ real zeros of $\mathcal{S}_n$. When $q < 0$, since $\mathrm{sgn}(\mathcal{S}_{2m+k}(x_{2m+k,1}^{(\lambda)})) < 0$, there are also zeros of $\mathcal{S}_n$ outside the interval $[-x_{2m+k,1}^{(\lambda)}, x_{2m+k,1}^{(\lambda)}]$. Hence, with the additional observation that when n is odd $\mathcal{S}_n$ has a zero at the origin, we arrive at the interlacing behaviour of the zeros of $\mathcal{S}_n$ and $G_n^{(\lambda)}$.

To obtain the required interlacing behaviour of the zeros of $\mathcal{S}_n$ and $G_{n-2}^{(\lambda)}$, we consider

$$\pi_i(x) = \frac{G_{2m+k-2}^{(\lambda)}(x)}{x^k[x^2 - (x_{2m+k-2,i}^{(\lambda)})^2]}, \quad i = 1, 2, \dots, m-1,$$

symmetric monic polynomials of degree $2m - 4$.

For the integrals $\int_{-1}^{1} x^k \mathcal{S}_{2m+k}(x)\pi_i(x)d\Psi^{(\lambda)}(x)$, applying Lemma 3 (with $r = m-2$) and the Gaussian quadrature rule on the zeros of $G_{2m+k-2}^{(\lambda)}$ (now also with the error term), give

$$\omega_{2m+k-2,i}^{(\lambda)} \mathcal{S}_{2m+k}(x_{2m+k-2,i}^{(\lambda)}) \frac{G_{2m+k-2}^{\prime(\lambda)}(x_{2m+k-2,i}^{(\lambda)})}{x_{2m+k-2,i}^{(\lambda)}} + \rho_{2m+k-2,i}^{(\lambda)} < 0.$$

Thus $\text{sgn}(\mathcal{S}_{2m+k}(x_{2m+k-2,i}^{(\lambda)})) = \text{sgn}(\mathcal{S}_n(x_{n-2,i}^{(\lambda)})) = (-1)^i$, $i = 1, 2, \dots, m-1$, which leads to the required results. □

Now we analyse the extremal points of $\mathcal{S}_n$. We denote the extremal points of $\mathcal{S}_n$, in decreasing order, by $\hat{s}_{n,j}$, $j = 1, 2, \dots$.

Theorem 2 *For any $n \geq 3$, the polynomial $\mathcal{S}_n$ has $n-1$ extremal points in the interval $(-\xi_q, \xi_q)$. Furthermore, if $m = \lfloor n/2 \rfloor$ and $k = n - 2m$, then*
(i) *for $q > 0$, the $m + k - 1$ largest positive extremal points of $\mathcal{S}_n$ satisfy*

$$x_{n+1,m+k}^{(\lambda,q)} < x_{n-1,m+k-1}^{(\lambda,q)} < \hat{s}_{n,m+k-1} < \cdots < x_{n+1,2}^{(\lambda,q)} < x_{n-1,1}^{(\lambda,q)} < \hat{s}_{n,1} < x_{n+1,1}^{(\lambda,q)};$$

(ii) *for $-1 < q < 0$, the $m + k - 1$ largest positive extremal points of $\mathcal{S}_n$ satisfy*

$$x_{n+1,m+k}^{(\lambda,q)} < \hat{s}_{n,m+k-1} < \cdots < \hat{s}_{n,2} < x_{n-1,2}^{(\lambda,q)} < x_{n+1,2}^{(\lambda,q)} < \hat{s}_{n,1} < x_{n-1,1}^{(\lambda,q)} < x_{n+1,1}^{(\lambda,q)}.$$

Here ξ_q and $x_{n,j}^{(\lambda,q)}$ are given as in Sect. 2.

Proof The interlacing properties of the zeros of $G_{n+1}^{(\lambda,q)}$ and $G_{n-1}^{(\lambda,q)}$ follow from orthogonality.

For $k = 0, 1$, $m \geq 1$ and $n = 2m + k$ consider

$$\pi_i(x) = \frac{G_{2m+k+1}^{(\lambda,q)}(x)}{x^2 - (x_{2m+k+1,i}^{(\lambda,q)})^2}, \quad i = 1, 2, \dots, m+k,$$

symmetric monic polynomials of degree $2m + k - 1$.

The Gaussian quadrature formula based on the zeros of $G_{2m+k+1}^{(\lambda,q)}$ gives

$$\int_{-1}^{1} \mathcal{S}_{2m+k}'(x)\pi_i(x)d\Psi^{(\lambda,q)}(x) = \omega_{2m+k+1,i}^{(\lambda,q)} \mathcal{S}_{2m+k}'(x_{2m+k+1,i}^{(\lambda,q)}) \frac{G_{2m+k+1}^{\prime(\lambda,q)}(x_{2m+k+1,i}^{(\lambda,q)})}{x_{2m+k+1,i}^{(\lambda,q)}}.$$

Again we have used the fact that the nodes (zeros) and weights of the quadrature rule satisfy $\omega_{2m+k+1,2m+k+2-i}^{(\lambda,q)} = \omega_{2m+k+1,i}^{(\lambda,q)}$ and $x_{2m+k+1,2m+k+2-i}^{(\lambda,q)} = -x_{2m+k+1,i}^{(\lambda,q)}$. Hence, from Lemma 5 (with $r = m$), we obtain

$$\text{sgn}(\mathcal{S}_{2m+k}'(x_{2m+k+1,i}^{(\lambda,q)})) = (-1)^{i+1}, \quad i = 1, 2, \dots, m+k. \tag{31}$$

This proves the interlacing behaviour between $\hat{s}_{n,i}$ and $x_{n+1,i}^{(\lambda,q)}$ and, hence, also why there are $n-1$ extremal points in the interval $(-\xi_q, \xi_q)$. Note that we have also included here the extremal point of $\mathcal{S}_n$ at the origin when $n = 2m$ is even.

Now consider the following symmetric monic polynomials of degree $n-3 = 2m+k-3$,

$$\pi_i(x) = \frac{G_{2m+k-1}^{(\lambda,q)}(x)}{x^2 - (x_{2m+k-1,i}^{(\lambda,q)})^2}, \quad i = 1, 2, \ldots, m+k-1.$$

From Lemma 5 (with $r = m-1$), we obtain

$$\operatorname{sgn}\left(\int_{-1}^{1} \mathcal{S}'_{2m+k}(x)\pi_i(x)d\Psi^{(\lambda,q)}(x)\right) = -\operatorname{sgn}(q), \quad i = 1, 2, \ldots, m+k-1.$$

The Gaussian quadrature formula based on the zeros of $G_{2m+k-1}^{(\lambda,q)}$ gives

$$\int_{-1}^{1} \mathcal{S}'_{2m+k}(x)\pi_i(x)d\Psi^{(\lambda,q)}(x) = \omega_{2m+k-1,i}^{(\lambda,q)} \mathcal{S}'_{2m+k}(x_{2m+k-1,i}^{(\lambda,q)}) \frac{G_{2m+k-1}^{\prime(\lambda,q)}(x_{2m+k-1,i}^{(\lambda,q)})}{x_{2m+k-1,i}^{(\lambda,q)}}.$$

Hence

$$\operatorname{sgn}(\mathcal{S}'_{2m+k}(x_{2m+k-1,i}^{(\lambda,q)})) = (-1)^i \operatorname{sgn}(q), \quad i = 1, 2, \ldots, r+k-1.$$

This concludes the interlacing behaviour between $\hat{s}_{n,i}$ and $x_{n-1,i}^{(\lambda,q)}$. □

In Theorem 1 it was shown that $\mathcal{S}_{2m+1}$, for any $m \geq 0$, has $2m+1$ distinct real zeros, provided that $-1 \leq q < 0$. This is also true if $q = 0$, since in this case, $\mathcal{S}_{2m+1} = \mathcal{S}_{2m+1}^{(\lambda,0,\kappa_1,\kappa_2)} = G_{2m+1}^{(\lambda)}$. Using Theorem 2, we can extend this result for $q > 0$ and odd values of n.

Theorem 3 *For any $m \geq 0$, the polynomial $\mathcal{S}_{2m+1}$ has $2m+1$ different real zeros.*

Proof From Theorem 1, since $\mathcal{S}_{2m+1}$ has at least $2m-1$ different real zeros (including the one at the origin), if we can show that $\operatorname{sgn}(\mathcal{S}'_{2m+1}(0)) = (-1)^m$, then $\mathcal{S}_{2m+1}$ must have exactly $2m+1$ different real zeros.

$\mathcal{S}'_{2m+1}$ is a polynomial of exact degree $2m$. From Theorem 2, all m positive zeros of $\mathcal{S}'_{2m+1}$ lie inside the interval $(x_{2m+2,m+1}^{(\lambda,q)}, x_{2m+2,1}^{(\lambda,q)})$. By symmetry, all m negative zeros lie inside the interval $(-x_{2m+2,1}^{(\lambda,q)}, -x_{2m+2,m+1}^{(\lambda,q)})$. Hence, in the closed interval $[-x_{2m+2,m+1}^{(\lambda,q)}, x_{2m+2,m+1}^{(\lambda,q)}]$, the polynomial $\mathcal{S}'_{2m+1}$ does not change sign. From (31) one can observe that $\operatorname{sgn}(\mathcal{S}'_{2m+1}(x_{2m+2,m+1}^{(\lambda,q)})) = (-1)^m$. This means $\operatorname{sgn}(\mathcal{S}'_{2m+1}(0)) = (-1)^m$, thus concluding the proof of the theorem. □

Again from Theorem 1, the zeros of the polynomials $\mathcal{S}_{2m}$, for $-1 \leq q \leq 0$, are all real and simple. However, this is not the case for $q > 0$, as we can observe from Fig. 1, where we have given the graph of $\mathcal{S}_{12}^{(1,500,2,10000)}$. The polynomial $\mathcal{S}_{12}^{(1,500,2,10000)}$ has two imaginary zeros.

The following theorem is an improvement from Theorem 1 for the zeros of $\mathcal{S}_{2m}$.

Theorem 4 *The polynomial S_2 has two distinct real zeros and if q is such that*

$$b_{2m}(\lambda, q) < \alpha_{2m+2}^{(\lambda)} = \frac{(2m+1)(m+\lambda)}{2(2m+1+\lambda)(2m+\lambda)}, \quad m \geq 1,$$

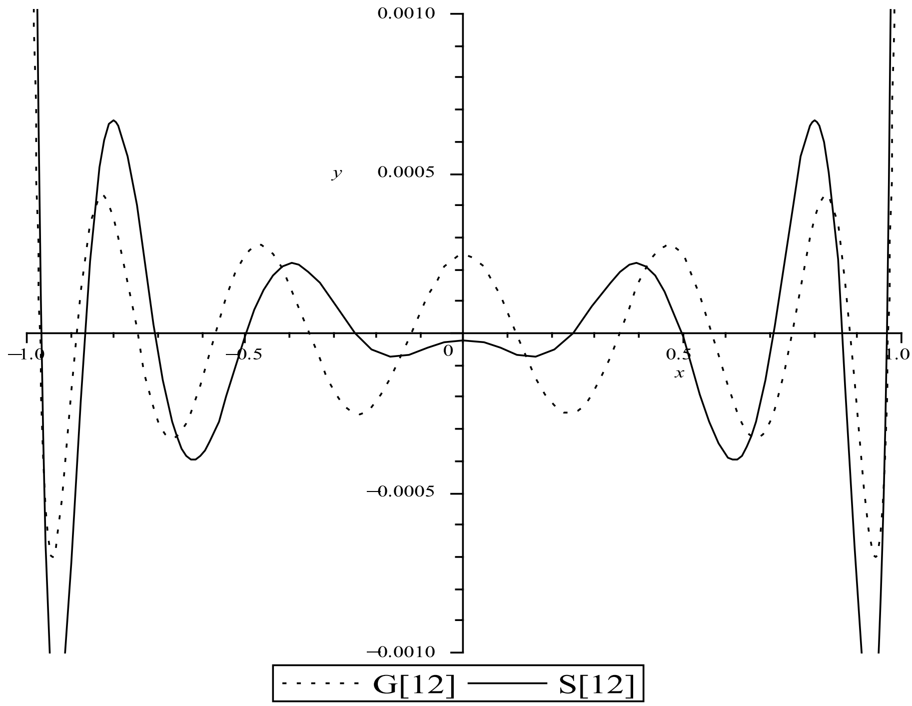


Fig. 1 Graph of $y = G_{12}^{(\lambda)}(x)$ and $y = \mathcal{S}_{12}^{(\lambda,q,\kappa_1,\kappa_2)}(x)$ on $[-1, 1]$ when $\lambda = 1$, $q = 500$, $\kappa_1 = 2$ and $\kappa_2 = 10^4$

then, for any $m \geq 2$, the polynomial $\mathcal{S}_{2m}$ also has $2m$ different real zeros.

Proof Since $\mathrm{sgn}(b_{2m}) = \mathrm{sgn}(q)$ for all $m \geq 1$, the validity of this theorem for $-1 \leq q \leq 0$ follows from Theorem 1. Thus we restrict ourselves to $q > 0$. From (30), if $\mathcal{S}_{2m}$ has m positive zeros, then the smallest positive zero $s_{2m,m}$ should lie between $x_{2m,m}^{(\lambda)}$ and the origin, and this is true only if $(-1)^m \mathcal{S}_{2m}(0) > 0$. Thus, we look for a condition such that $(-1)^m \mathcal{S}_{2m}(0) > 0$ for $m \geq 1$. From (7) and (10) we have $-\mathcal{S}_2(0) = -G_2(0) > 0$ and, for $m \geq 2$,

$$(-1)^m \mathcal{S}_{2m}(0) = a_{2m-2}(-1)^{m-1}\mathcal{S}_{2m-2}(0) + (-1)^m G_{2m}^{(\lambda)}(0)[1 - b_{2m-2}/\alpha_{2m}^{(\lambda)}].$$

If $1 - b_{2m-2}/\alpha_{2m}^{(\lambda)} > 0$, $m \geq 2$, we can verify by induction that $(-1)^m \mathcal{S}_{2m}(0) > 0$ for all $m \geq 1$. This completes the proof of the theorem. □

5 Some Special Cases and Examples

First we look at the behaviour of the polynomials $\mathcal{S}_n = \mathcal{S}_n^{(\lambda,q,\kappa_1,\kappa_2)}$ for some extremal situations.

By letting κ_1 tends to ∞ in $\frac{1}{\kappa_1}\langle f,g\rangle_{S_1}$, we arrive at the monic polynomials $\mathcal{S}_n^{(\lambda,q,\infty,\kappa_2)}$, which must satisfy

$$\int_{-1}^{1}\mathcal{S}_m'^{(\lambda,q,\infty,\kappa_2)}(x)\mathcal{S}_n'^{(\lambda,q,\infty,\kappa_2)}d\Psi^{(\lambda+1)}(x)=0,\quad \text{if } m\neq n.$$

This means $\mathcal{S}_n^{(\lambda,q,\infty,\kappa_2)}(x)=G_n^{(\lambda)}(x)+c_n$, where c_n is any constant. Since it can be verified from (11) and (12) that $a_n(\lambda,q,\infty,\kappa_2)=b_n(\lambda,q)$, we can conclude that $\mathcal{S}_n^{(\lambda,q,\infty,\kappa_2)}=G_n^{(\lambda)}$, $n\geq 0$. This is a situation where the zeros of the orthogonal polynomials are real and simple, even though the conditions in (26) do not hold.

If condition (26) is our goal, then we could look at the monic polynomial $\mathcal{S}_n^{(\lambda,q,\kappa_1,\infty)}$ which should satisfy

$$\int_{-1}^{1}\mathcal{S}_m'^{(\lambda,q,\kappa_1,\infty)}(x)\mathcal{S}_n'^{(\lambda,q,\kappa_1,\infty)}(x)d\Psi^{(\lambda,q)}(x)=0,\quad \text{if } m\neq n.$$

By verifying $a_n(\lambda,q,\kappa_1,\infty)=0$, we can conclude that $\mathcal{S}_n^{(\lambda,q,\kappa_1,\infty)}(x)=G_n^{(\lambda)}(x)+b_{n-2}\times G_{n-2}^{(\lambda)}(x)$, $n\geq 0$, with $b_{-2}=b_{-1}=b_0=0$.

Hence, from [5, Lemma 2], $\mathcal{S}_n^{(\lambda,q,\kappa_1,\infty)}$ has n distinct real zeros and the behaviour of these zeros can be given by:

Theorem 5 *Let $\lambda\geq -1/2$ and $q\geq -1$. Then, for $n\geq 3$, the $m=\lfloor n/2\rfloor$ positive zeros of $\mathcal{S}_n^{(\lambda,q,\kappa_1,\infty)}$ satisfy*

$$s_{n,m}<x_{n,m}^{(\lambda)}<x_{n-2,m-1}^{(\lambda)}<s_{n,m-1}<\cdots<x_{n-2,1}^{(\lambda)}<s_{n,1}<x_{n,1}^{(\lambda)}<1,$$

if $q>0$ and

$$x_{n,m}^{(\lambda)}<s_{n,m}<x_{n-2,m-1}^{(\lambda)}<\cdots<s_{n,2}<x_{n-2,1}^{(\lambda)}<x_{n,1}^{(\lambda)}<s_{n,1}<\xi_q,$$

if $-1\leq q<0$.

Now let $-1\leq q<0$, $M_q>0$ and consider the inner product $\langle f,g\rangle_{S_{1,1}}$ given by

$$\begin{aligned}\langle f,g\rangle_{S_{1,1}}&=\lim_{M_q\to\infty}\langle f,g\rangle_{S_1(\lambda,q,0,\kappa/M_q)}\\&=\int_{-1}^{1}f(x)g(x)d\Psi^{(\lambda)}(x)+\kappa\left[f'\left(\frac{-1}{\sqrt{-q}}\right)g'\left(\frac{-1}{\sqrt{-q}}\right)+f'\left(\frac{1}{\sqrt{-q}}\right)g'\left(\frac{1}{\sqrt{-q}}\right)\right],\end{aligned}$$

where $\kappa>0$. When $q=-1$, this is a special case of an inner product considered in [1] and [2].

For $\mu_{i,m}^{(k)}=\int_{-1}^{1}x^k\tilde{\mathcal{S}}_{2m+k}(x)(1+qx^2)^i d\Psi^{(\lambda)}(x)$, where $\tilde{\mathcal{S}}_{2m+k}=\lim_{M_q\to\infty}\mathcal{S}_{2m+k}^{(\lambda,q,0,\kappa/M_q)}$ are the associated orthogonal polynomials, we have

$$\begin{aligned}&\mu_{0,m}^{(0)}=0,\qquad \mu_{0,m}^{(1)}=-2\kappa\tilde{\mathcal{S}}_{2m+1}'(1/\sqrt{-q}),\quad m\geq 1,\\&\mu_{1,m}^{(0)}=4\kappa\sqrt{-q}\,\tilde{\mathcal{S}}_{2m}'(1/\sqrt{-q}),\qquad \mu_{1,m}^{(1)}=4\kappa\tilde{\mathcal{S}}_{2m+1}'(1/\sqrt{-q}),\quad m\geq 2,\\&\mu_{i,m}^{(k)}=0,\quad k=0,1,\ 2\leq i\leq m-1,\ m\geq 3.\end{aligned}\tag{32}$$

From part (ii) of Theorem 2, the largest extremal point, say $\hat{s}_{n,1}^{(\lambda,q,0,\kappa/M_q)}$, of the polynomial $\mathcal{S}_n^{(\lambda,q,0,\kappa/M_q)}$ is such that

$$\hat{s}_{n,1}^{(\lambda,q,0,\kappa/M_q)} < x_{n+1,1}^{(\lambda,q)} < 1/\sqrt{-q}.$$

Since $\tilde{\mathcal{S}}_n = \lim_{M_q\to\infty} \mathcal{S}_n^{(\lambda,q,0,\kappa/M_q)}$, by continuity what one could conclude is that the largest extremal point $\tilde{s}_{n,1}$ of $\tilde{\mathcal{S}}_n$ satisfies

$$\tilde{s}_{n,1} \le x_{n+1,1}^{(\lambda,q)} < 1/\sqrt{-q}.$$

This means $\mathcal{S}_n'^{(\lambda,q,0,\kappa/M_q)}(1/\sqrt{-q}) > 0$ and also $\tilde{\mathcal{S}}_n'(1/\sqrt{-q}) > 0$ for all $n \ge 1$. Consequently, in (32),

$$\mu_{0,m}^{(1)} < 0, \qquad \mu_{1,m}^{(0)} > 0 \quad \text{and} \quad \mu_{1,m}^{(1)} > 0.$$

Hence, if $\pi(x) = \sum_{i=0}^{r} \varepsilon_i q^{-i}(1+qx^2)^i$ is an even polynomial of degree $2r$, $1 \le r \le m-1$, $m \ge 2$, with real zeros in $(-1, 1)$ then, as in Lemma 3,

$$I_{r,m}^{(k)} = \int_{-1}^{1} x^k \tilde{\mathcal{S}}_{2m+k}(x)\pi(x)d\Psi^{(\lambda)}(x) = \sum_{i=0}^{r} \frac{\varepsilon_i}{q^i}\mu_{i,m}^{(k)} = \sum_{i=0}^{1} \frac{\varepsilon_i}{q^i}\mu_{i,m}^{(k)} < 0.$$

Therefore, successively choosing π to be the polynomials

$$\pi_i(x) = \frac{G_{2m+k}^{(\lambda)}(x)}{x^k[x^2 - (x_{2m+k,i}^{(\lambda)})^2]}, \quad i = 1, 2, \ldots, m,$$

and also the polynomials

$$\pi_i(x) = \frac{G_{2m+k-2}^{(\lambda)}(x)}{x^k[x^2 - (x_{2m+k-2,i}^{(\lambda)})^2]}, \quad i = 1, 2, \ldots, m-1,$$

we obtain, as in Theorem 1, the following.

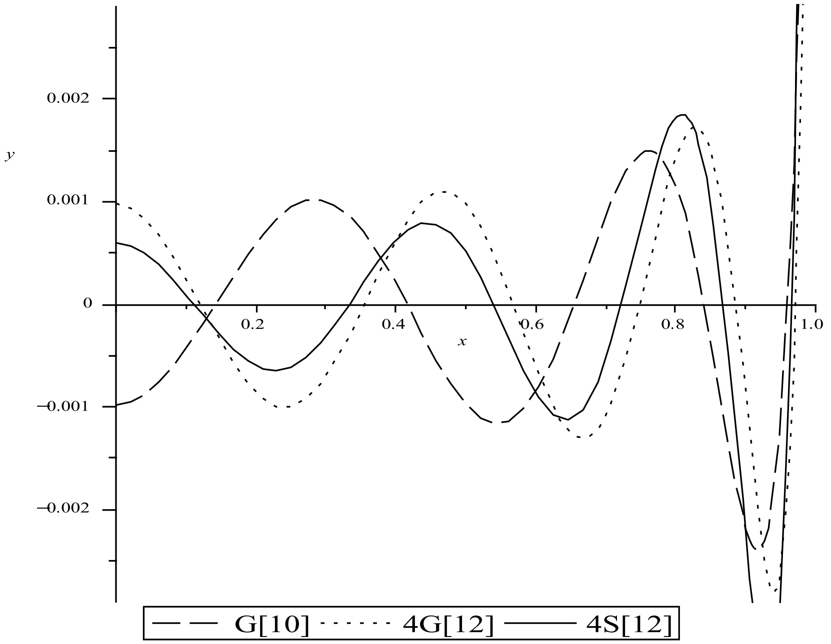
Theorem 6 *Let $m = \lfloor n/2 \rfloor$. Then the polynomial $\tilde{\mathcal{S}}_n$ for any $n \ge 1$ has n different real zeros. To be precise, $x_{2,1}^{(\lambda)} = \tilde{s}_{2,1}$, $x_{3,1}^{(\lambda)} < \tilde{s}_{3,1}$ and, if $n \ge 4$, the positive zeros of $\mathcal{S}_n$ satisfy*

$$x_{n,m}^{(\lambda)} < \tilde{s}_{n,m} < x_{n-2,m-1}^{(\lambda)} < x_{n,m-1}^{(\lambda)} < \cdots < \tilde{s}_{n,2} < x_{n-2,1}^{(\lambda)} < x_{n,1}^{(\lambda)} < \tilde{s}_{n,1}.$$

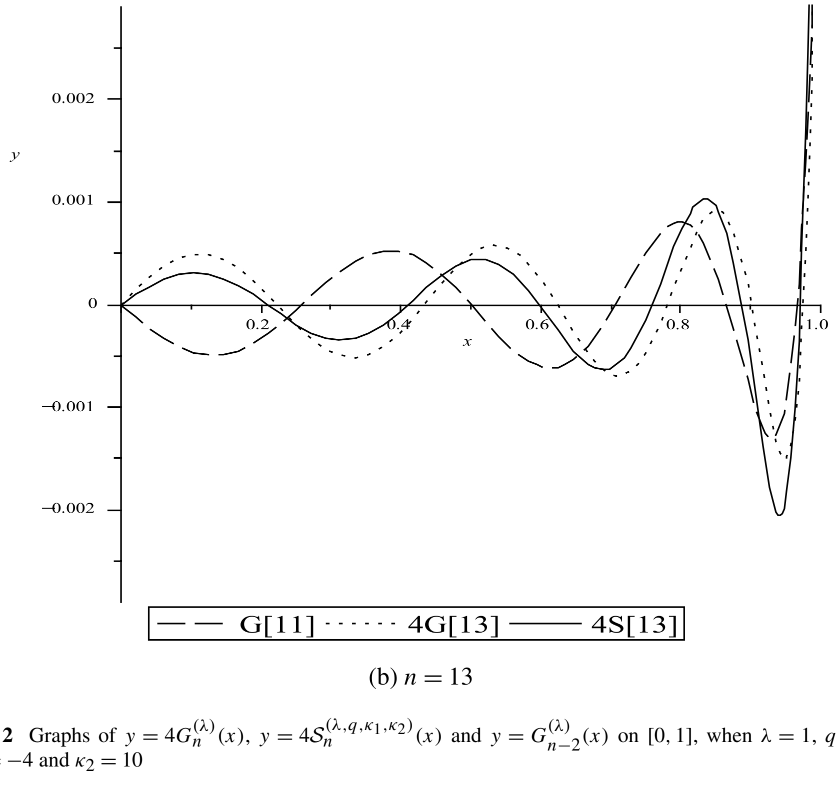
Here $x_{n,j}^{(\lambda)}$ are the zeros of the nth degree Gegenbauer polynomial $G_n^{(\lambda)}$ as given in (8).

Finally, numerical experiments as the one given below and the limit case $\mathcal{S}_n^{(\lambda,q,\infty,\kappa_2)}$ suggest that the domain of κ_1 and κ_2 can be extended beyond the conditions in (26) and still the same interlacing behaviour of the zeros is maintained.

Example Let $\kappa_1 = -4$ and $\kappa_2 = 10$. Note that for these values of κ_1 and κ_2 the conditions (26) do not hold. However, as we have given at the end of Sect. 2, the inner products $\langle f, g\rangle_{S_1(1,1,-4,10)}$ and $\langle f, g\rangle_{S_1(1,-0.8,-4,10)}$ are still positive definite, thus the existence of the sequence of orthogonal polynomials is guaranteed. What can we say about the zeros of these polynomials?

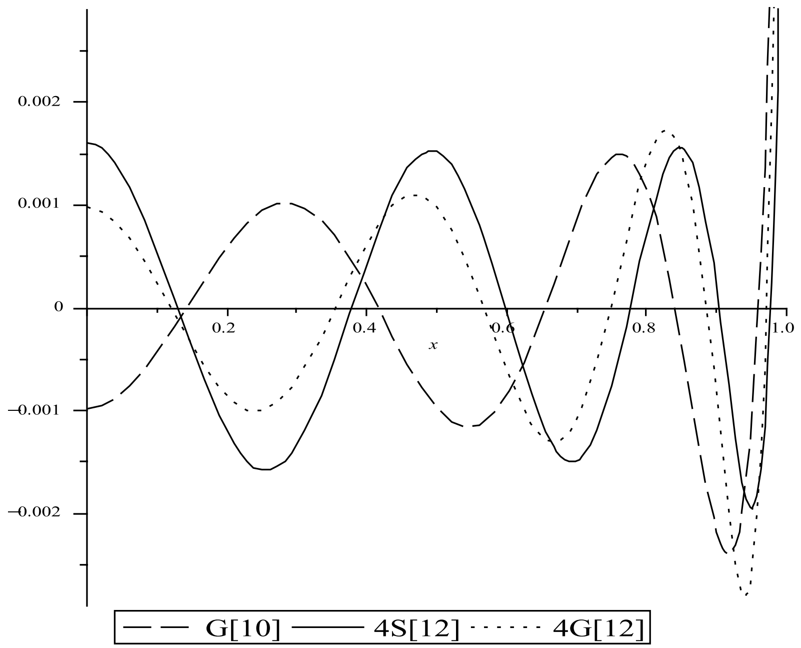


(a) $n = 12$

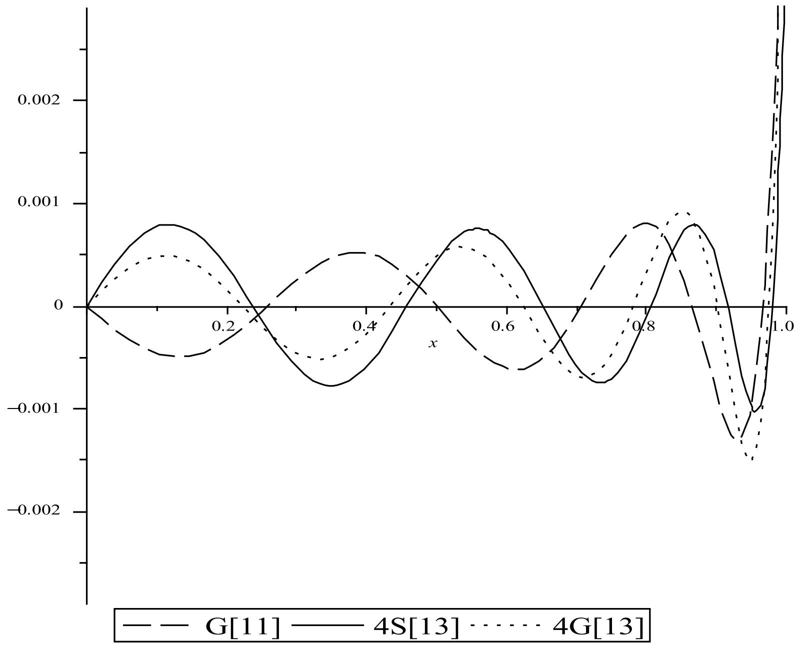


(b) $n = 13$

Fig. 2 Graphs of $y = 4G_n^{(\lambda)}(x)$, $y = 4\mathcal{S}_n^{(\lambda,q,\kappa_1,\kappa_2)}(x)$ and $y = G_{n-2}^{(\lambda)}(x)$ on $[0, 1]$, when $\lambda = 1$, $q = 1$, $\kappa_1 = -4$ and $\kappa_2 = 10$



(a) $n = 12$



(b) $n = 13$

Fig. 3 Graphs of $y = 4G_n^{(\lambda)}(x)$, $y = 4\mathcal{S}_n^{(\lambda,q,\kappa_1,\kappa_2)}(x)$ and $y = G_{n-2}^{(\lambda)}(x)$ on [0, 1], when $\lambda = 1$, $q = -0.8$, $M_q = 0$, $\kappa_1 = -4$ and $\kappa_2 = 10$

Figure 2 shows the interlacing behaviour of the zeros and extremal points of the polynomials $\mathcal{S}_{12}^{(1,1,-4,10)}$ and $\mathcal{S}_{13}^{(1,1,-4,10)}$. Likewise, Fig. 3 shows the interlacing behaviour of the zeros and extremal points of the polynomials $\mathcal{S}_{12}^{(1,-0.8,-4,10)}$ and $\mathcal{S}_{13}^{(1,-0.8,-4,10)}$. The interlacing behaviour of these zeros and extremal points are the same as in Theorems 1 and 2. In fact, in both cases the respective polynomials $\mathcal{S}_{12}$ have a complete set of distinct real zeros.

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