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What is Q-Curvature?

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Abstract Branson's Q-curvature is now recognized as a fundamental quantity in conformal geometry. We outline its construction and present its basic properties.

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1 Introduction

Throughout his distinguished research career, Tom Branson was fascinated by conformal differential geometry and made several substantial contributions to this field. There is no doubt, however, that his favorite was the notion of Q-curvature. In this article we outline the construction and basic properties of Branson's Q-curvature. As a Riemannian invariant, defined on even-dimensional manifolds, there is apparently nothing special about Q. On a

In memory of Thomas P. Branson (1953–2006).

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surface Q is essentially the Gaussian curvature. In 4 dimensions there is a simple but unrevealing formula (4.1) for Q. In 6 dimensions an explicit formula is already quite difficult. What is truly remarkable, however, is how Q interacts with conformal, i.e. angle-preserving, transformations.

We shall suppose that the reader is familiar with the basics of Riemannian differential geometry but a few remarks on notation are in order. Sometimes, we shall write g_{ab} for a metric and ∇_a for the corresponding connection. Let us write R_{ab} and R for the Ricci and scalar curvatures, respectively. We shall use the metric to 'raise and lower' indices in the usual fashion and adopt the summation convention whereby one implicitly sums over repeated indices. Using these conventions, the Laplacian is the differential operator $\Delta \equiv g^{ab}\nabla_a\nabla_b = \nabla^a\nabla_a$.

A conformal structure on a smooth manifold is a metric defined only up to smoothly varying scale. Thus, one can measure angles between vectors but not their lengths. For simplicity, let us suppose that all manifolds in this article are oriented.

2 The Functional Determinant

Spectral geometry is concerned with the interplay between the geometry of a compact Riemannian manifold M and objects in the analysis of natural differential and pseudodifferential operators on M; for the Laplacian Δ on functions for example, we may study its eigenvalues (actually, those of $-\Delta$)

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$$

and ask how they carry information about the curvature of M and other similar invariants. A good tool here is the heat operator $\exp(t\Delta)$, which for positive t is a kernel operator of trace-class with

$$\mathrm{Tr}(\exp(t\Delta)) = \sum_{n=0}^{\infty} e^{-\lambda_n t}.$$

The main point is that this trace admits an asymptotic expansion in powers of t, for small t, and that the coefficients in this expansion are integrals of local geometric invariants over M. The coefficient to t^0 is of particular interest, and for elliptic complexes, where we consider certain families of operators like Δ above, it leads to Atiyah-Singer index theorems. There is also a connection to the so-called functional determinants, that arise by considering the 'zeta function'

$$\zeta(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}$$

say for the Δ above, or more generally over the spectrum of a non-negative elliptic operator A, disregarding the kernel. This function captures (as does the heat kernel) in a marvelous way the spectrum, and it is meromorphic in s with a removable singularity at $s = 0$. One may define the determinant of A (of use for example in string theory) via $\zeta'(0)$, which in some cases, notably in connection with conformal geometry, is a quantity one may compute. It is at this point that the Q-curvature appears in a natural way. It is remarkable, that the same quantity also appears naturally in quite different contexts, as we shall see later.

Consider an elliptic differential operator A of order 2ℓ on M, i.e. possibly acting on sections of a vector bundle; we wish to make a number of technical assumptions, e.g. that

A is naturally built from the metric and transforms covariantly with respect to conformal change of the metric (or is a power of such an operator)—suffice it to say, that A could be the Yamabe operator $Y = -\Delta + \frac{n-2}{4(n-1)}R$, where n is the dimension (which we assume to be even) of M, or D^2, where D is the Dirac operator (for M spin). Suppose for simplicity that A has only positive spectrum, and consider its heat expansion, zeta function, and $\zeta'(0)$. Now the value $\zeta(0)$ coincides with the coefficient of t^0 in the heat expansion, and it is a conformal invariant expressed as an integral over M:

$$\zeta(0) = \int_M Q$$

and for the derivative of $\zeta'(0)$ along a curve of conformal metrics, viz. $e^{2\epsilon\omega} g_{ab}$ we get

$$\frac{d}{d\epsilon}\zeta'(0)|_{\epsilon=0} = 2\ell \int_M \omega\, Q.$$

Here ω is a smooth real function on M and Q is the Q-curvature, modulo some divergence terms, described explicitly below. With more work on the local invariants, based on the formulas above and Sobolev inequalities, one derives the fact, that on the standard spheres of dimensions 2, 4, and 6, the determinant of the Yamabe operator has a maximum, resp. minimum, resp. maximum within the standard conformal class (fixing the total volume); for the square of the Dirac operator the same is true, except now we have a minimum, resp. maximum, resp. minimum. The case of dimension 6 is particularly demanding, and it is due to Branson, see [1]. It would be very interesting to see if this pattern continues, and probably it does, owing to some deep properties of the Q-curvature. In fact, we have been advised by Okikiolu that she has been able to extend [12] to provide evidence for this pattern regarding the Yamabe operator: the standard metrics on even-dimensional spheres provide local extrema of the predicted type at least when restricted to a suitable finite-codimensional submanifold of metrics.

3 Q-Curvature in 2 Dimensions

In 2 dimensions, scalar curvature R or, equivalently, Gaussian curvature $K = R/2$ are fundamental. These quantities enjoy a well-known rôle in controlling the topology of a compact surface M via the Gauss-Bonnet theorem:

$$\int_M K\, d\mathrm{vol} = \frac{1}{2}\int_M R\, d\mathrm{vol} = 2\pi\chi(M),$$

where $\chi(M)$ is the Euler characteristic of M. Furthermore, R transforms in a simple fashion under conformal rescaling of the metric. Specifically, if $\hat{g}_{ab} = e^{2f} g_{ab}$ for some smooth function f, then

$$\hat{R} = e^{-2f}(R - 2\Delta f),$$

where $\hat{R}$ denotes the scalar curvature of the metric $\hat{g}_{ab}$. A neater viewpoint on these fundamental properties of scalar curvature is provided by introducing

$$Q \equiv -\frac{1}{2}R\, d\mathrm{vol} \quad \text{and} \quad Pf \equiv \Delta f\, d\mathrm{vol} \tag{3.1}$$

for then

$$\int_M Q = -2\pi\chi(M) \quad \text{and} \quad \hat{Q} = Q + Pf,$$

where P is regarded as a differential operator from functions to 2-forms. It follows that P is conformally invariant. It is also clear, without the Gauss-Bonnet theorem, that $\int_M Q$ is a conformal invariant. This is because Pf is a divergence for any f.

4 Q-Curvature in 4 Dimensions

In 4 dimensions, the concept of Q-curvature is due to Branson and Ørsted [3] (with a similar quantity being introduced by Riegert [14]):

$$Q \equiv \left[-\frac{1}{6}\Delta R - \frac{1}{2}R^{ab}R_{ab} + \frac{1}{6}R^2 \right] d\text{vol}. \tag{4.1}$$

Its properties are very much akin to those of scalar curvature in 2 dimensions. Under conformal rescaling of the metric $\hat{g}_{ab} = e^{2f}g_{ab}$, we find

$$\hat{Q} = Q + Pf \tag{4.2}$$

where P is the 'Paneitz operator' [13] from functions to 4-forms given by

$$Pf = \nabla_a \left[\nabla^a \nabla^b + 2R^{ab} - \frac{2}{3}Rg^{ab} \right] \nabla_b f \, d\text{vol}.$$

Riemannian invariants generally transform under conformal rescaling $g_{ab} \mapsto e^{2f}g_{ab}$ by differential operators that are polynomial in f and its derivatives. The transformation law (4.2) is remarkable in being linear. As in 2 dimensions, this law has two immediate consequences. The first is that P is a conformally invariant differential operator. The second is that for any compact Riemannian 4-manifold M, the integral $\int_M Q$ is an invariant of the conformal structure. It is not just a topological invariant. Rather,

$$Q = \frac{1}{16}\text{Pfaff} - \left[\frac{1}{4}W^{abcd}W_{abcd} + \frac{1}{6}\Delta R \right] d\text{vol},$$

where Pfaff is the Pfaffian 4-form and W_{abcd} is the Weyl curvature tensor, a well-known local conformal invariant. It follows that

$$\int_M Q = 8\pi^2\chi(M) - \frac{1}{4}\int_M W^{abcd}W_{abcd} \, d\text{vol}.$$

In particular, on a conformally flat 4-manifold, we have $\int_M Q = 8\pi^2\chi(M)$.

5 Q-Curvature in $2m$ Dimensions

The notion of Q-curvature was extended to arbitrary even dimensions by Tom Branson [1, p. 11]. Its construction and properties have since been studied by many authors besides

Branson himself, including S. Alexakis, S.-Y.A. Chang, Z. Djadli, K. El Mehdi, C. Fefferman, A.R. Gover, C.R. Graham, M. Gursky, F. Hang, E. Hebey, K. Hirachi, A. Malchiodi, B. Ørsted, L. Peterson, J. Qing, F. Robert, M. Struwe, J. Viaclovsky, P. Yang, and M. Zworski. Even this list is surely incomplete and is rapidly growing. The following construction of Q is due to Fefferman and Hirachi [7].

Let us consider a Lorentzian metric $\tilde{g}$ given in local coordinates (t, x^i, ρ) by

$$\tilde{g} = 2\rho\, dt^2 + 2t\, dt\, d\rho + t^2 g_{ij}(x,\rho)\, dx^i\, dx^j, \tag{5.1}$$

where $g_{ij}(x,\rho)\, dx^i\, dx^j$ is a Riemannian metric in the coordinates $x = (x^1, \dots, x^{2m})$ for each fixed ρ. Let us write $\tilde{\Delta}$ for the 'Laplacian', more correctly the wave-operator, of $\tilde{g}$. Then, associated to $\tilde{g}$ is the quantity

$$Q \equiv -\tilde{\Delta}^m \log t|_{\rho=0,t=1}\, d\mathrm{vol}_{g(x,0)}, \tag{5.2}$$

depending on x alone. Now suppose that $\tilde{g}$ is Ricci-flat. Then it turns out that Q depends only on $g_{ij}(x) \equiv g_{ij}(x,0)$. In other words, we have associated to the metric $g_{ij}(x)\, dx^i\, dx^j$ what turns out to be a Riemannian invariant volume-form Q. The only problem with this recipe is that Ricci-flat metrics of the form (5.1) generally do not exist for a given 'initial' metric $g_{ij}(x)\, dx^i\, dx^j$. However, for the purposes of defining Q, this simply does not matter. As a formal power series in ρ, one can show following [9], that $g_{ij}(x,\rho)$ is sufficiently well-defined to a high enough order that the formula (5.2) makes perfect sense. (This definition of Q is a consequence of the Fefferman-Graham 'ambient metric' construction.)

It is a simple enough exercise to check that (5.2) gives (4.1) in 4 dimensions. The basic transformation law (4.2) holds in general, where in dimension $2m$ the conformally invariant operator P is due to Graham, Jenne, Mason, and Sparling [9]. Explicit formulae for P and Q, however, are only known in low dimensions. Certainly,

$$Q = \left[-\frac{1}{2(2m-1)}\Delta^{m-1}R + \cdots\right] d\mathrm{vol} \quad \text{and} \quad Pf = [\Delta^m f + \cdots]\, d\mathrm{vol}$$

but the lower order terms represented by the ellipses $\cdots$ are difficult to determine from the recipe (5.2). In particular, a link with the Pfaffian is only known in the conformally flat case, from which:

$$\int_M Q = (-1)^m (m-1)!\, 2^{2m-1} \pi^m \chi(M)$$

on a conformally flat $2m$-manifold. In general, it is clear only that $\int_M Q$ is a conformal invariant. Though much is known about Q in general dimensions, a characterization in terms of its many special properties remains elusive. An complete analysis of Q in dimension 6 is given in [2].

6 Q-Curvature and Geometric Analysis

The analytic and geometric significance of Q-curvature comes from its close relation to the Pfaffian. The integral of Q-curvature is clearly a conformal invariant, since the transformation rule (4.2) and its analogue in general dimensions displays the difference of the Q-curvature forms under conformal change of metric as a divergence. In fact, in low dimensions its relation to the Pfaffian is explicit, and thus its sign plays an important rôle in controlling the geometry and topology of the underlying manifold. For example, in dimension four the positivity of the conformal Laplacian and positivity of the conformal invariant $\int Q$ implies the positivity of the Paneitz operator P; and under these assumptions we have the vanishing of the first Betti number [11]; also $\int Q \le 16\pi^2$ with equality holding only if the manifold is conformally equivalent to the standard 4-sphere—which can be viewed as an effective version of the positive mass theorem. The study of the "prescribing Q-curvature" equation also leads to the recent development involving fully non-linear equations related to the symmetric functions of the Schouten tensor; with stronger topological results. To see this, we recall that the natural curvature invariants in conformal geometry are the Weyl tensor W and the Schouten tensor $A_g = \frac{1}{n-2}[\mathrm{Ric} - \frac{R}{2(n-1)} g]$ that occur in the decomposition of the Riemann curvature tensor:

$$\mathrm{Riem} = W \oplus A \owedge g,$$

where $\owedge$ denotes the Kulkarni-Nomizu product. Since the Weyl tensor W transforms by scaling under conformal change $\hat{g} = e^{2w} g$, only the Schouten tensor depends on the derivatives of the conformal factor. It is thus natural to consider $\sigma_k(A_g)$ the k-th symmetric function of the eigenvalues of the Schouten tensor A_g as a curvature invariant of the conformal metric. In general, $\sigma_1 = \mathrm{trace}\, A = \frac{1}{2(n-1)} R$. In dimension 4, a simple computation yields

$$\sigma_2(A)\, d\mathrm{vol} = Q + \frac{1}{6} \Delta R\, d\mathrm{vol}.$$

Thus on closed 4-manifolds, $\int \sigma_2(A)\, d\mathrm{vol} = \int Q$ is again conformally invariant. The study of the fully non-linear equations

$$\sigma_k(A_{\hat{g}}) = \mathrm{constant}$$

is a fully non-linear version of the Yamabe problem. This is a research direction currently under intensive study. The interested reader is referred to the survey [4].

In general even dimensions the study of Q-curvature as a local curvature invariant can also be approached as part of the general program of Fefferman-Graham as introduced in the article [6]. This approach has led to a general formulation of the Paneitz operator P, its generalization to higher dimensions, and associated Q-curvature in terms of the ambient metric construction explained in the previous section. An equivalent formulation of the ambient metric construction is to view the conformal structure on M^n as the boundary at infinity of an asymptotically hyperbolic Einstein manifold (X^{n+1}, g). This latter alternative approach has led to the intrinsic realization of the operators P in all even dimensions as residues of scattering operators studied by Graham and Zworski [10]. Q-curvature is also related to the concept of "renormalized volume" on (X^{n+1}, g) in conformal field theory. When n is odd, one can relate the renormalized volume as the integral of the Q-curvature on (X^{n+1}, g) via the Gauss-Bonnet formula; when n is even, one can relate the renormalized volume to the conformal primitive of the Q-curvature on M^n (see [8] and [5]).

This circle of ideas has clearly placed the notion of Q-curvature at the intersection of several interacting areas of research. It can be expected that Q-curvature will continue to play a crucial rôle in the study of conformal geometry.

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