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Solution and Asymptotic Behaviour for a Nonlocal Coupled System of Reaction-Diffusion

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Abstract This paper concerns with the existence, uniqueness and asymptotic behaviour of the solutions for a nonlocal coupled system of reaction-diffusion. We prove the existence and uniqueness of weak solutions by the Faedo-Galerkin method and exponential decay of solutions by the classic energy method. We improve the results obtained by Chipot-Lovato and Menezes for coupled systems. A numerical scheme is presented.

Keywords Coupled system of reaction-diffusion · The Faedo-Galerkin method · Asymptotic behaviour · Numerical methods

Mathematics Subject Classification (2000) 35K57 · 35B35

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1 Introduction

We consider the reaction-diffusion coupled system in parallel way, via a parameter $\alpha > 0$, of the form

$$u_t - a(l(u))\Delta u + f(u-v) = \alpha(u-v) \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

$$v_t - a(l(v))\Delta v - f(u-v) = \alpha(v-u) \quad \text{in } \Omega \times (0,T), \tag{1.2}$$

$$u = v = 0 \quad \text{in } \partial\Omega \times (0,T), \tag{1.3}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega, \tag{1.4}$$

$$v(x,0) = v_0(x) \quad \text{in } \Omega, \tag{1.5}$$

where $u = u(x,t)$ and $v = v(x,t)$ are real valued functions. Ω is a bounded domain of $\mathbb{R}^n$, $\partial\Omega$ is the boundary of Ω of class C^2. $f : \mathbb{R} \to \mathbb{R}$ and $a : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions with $a(\xi) \geqslant m > 0$. $l : L^2(\Omega) \to \mathbb{R}$ is a continuous linear form.

For the last several decades, various types of equations have been employed as some mathematical model describing physical, chemical, biological and ecological systems. Among them, the most successful and crucial one is the following model of semilinear parabolic partial differential equation, called the reaction-diffusion system

$$\frac{\partial u}{\partial t} - D\Delta u - f(u) = 0, \tag{1.6}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function, and D is an $n \times n$ real matrix of diffusion. This reaction-diffusion model is obtained by combining the system of nonlinear ordinary differential equations called the reaction system

$$\frac{du}{dt} - f(u) = 0, \tag{1.7}$$

and the system of linear partial differential equation called the diffusion system

$$\frac{\partial u}{\partial t} - D\Delta u = 0. \tag{1.8}$$

In 1998, L.A.F. Oliveira [10] considered the reaction-diffusion system where D is an $n \times n$ real matrix and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^2 function. He studied the exponential decay for some cases. Except for some publications on the subject, such as the searching for traveling waves solutions and some problem in ecology and epidemic theory, most authors assume that the diffusion matrix D is diagonal, so that the coupling between the equations are present only on the nonlinearity of the reaction term f. However, cross-diffusion phenomena are not uncommon (see [4] and references therein) and can be treated as equations in which D is not even diagonalizable. In 1997, M. Chipot and B. Lovat [5] studied the existence and uniqueness of the solutions for non local problems

$$u_t - a\left(\int_\Omega u(x,t)dx\right)\Delta u = f(x,t) \quad \text{in } \Omega \times (0,T), \tag{1.9}$$

$$u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.10}$$

$$u(x,0) = u_0(x) \quad \text{on } \Omega, \tag{1.11}$$

where Ω is a bounded open subset in $\mathbb{R}^n$, $n \geqslant 1$ with smooth boundary $\partial\Omega$. T is some arbitrary time, and a is some function from $\mathbb{R}$ into $(0, +\infty)$. This problem arises in various situations, for instance u could describe the density of a population (for instance of bacteria) subject to spreading. The diffusion coefficient a is then supposed to depend on the entire population in the domain rather than on the local density, i.e., moves are guided by considering the global state of the medium. They proved the following result:

Theorem 1.1 *Let $T_0 > 0$, $u_0 \in L^2(\Omega)$, $u_0 \geqslant 0$, $u_0 \not\equiv 0$. Let a be a continuous function positive in a neighborhood of $\int_\Omega u_0 dx$. Then for $f \in L^2([0,T] : H^{-1}(\Omega))$ there exists $0 < T \leqslant T_0$ and u solution to (1.9–1.11) such that*

$$u \in L^2([0,T] : H_0^1(\Omega)) \cap C^0([0,T] : L^2(\Omega)),$$

$$u_t \in L^2([0,T] : H^{-1}(\Omega)),$$

$$\langle u_t, v\rangle + a\left(\int_\Omega u dx\right)(\nabla u \cdot \nabla v) = \langle f, v\rangle, \quad \forall v \in H_0^1(\Omega),\ a.e.\ t \in [0, T_0],$$

where $(\nabla u \cdot \nabla v) = \int_\Omega \nabla u \cdot \nabla v dx$.

In 2005, S.D. Menezes [8], gave a simple extension of the result obtained by M. Chipot and B. Lovat [5], considering $a = a(l(u))$, $f = f(u)$ a continuous functions. Indeed, they studied the existence, uniqueness and periodic solution for the following parabolic problem

$$u_t - a(l(u))\Delta u + f(u) = h \quad \text{in } \Omega \times (0, T), \tag{1.12}$$

$$u(x,t) = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{1.13}$$

$$u(x,0) = u_0(x) \quad \text{on } \Omega, \tag{1.14}$$

where Ω is a bounded open subset in $\mathbb{R}^n$, $n \geqslant 1$ with smooth boundary $\partial\Omega$, T is some arbitrary time, $l : L^2(\Omega) \to \mathbb{R}$ is a nonlinear form, $h \in L^2(0, T : H^{-1}(\Omega))$ and $T > 0$ is some fixed time. This problem is nonlocal in the sense that the diffusion coefficient is determined by a global quantity. This kind of problem, besides its mathematical motivation because of the presence of the nonlocal term $a(l(u))$, arises from physical situations related to migration of a population of bacteria in a container in which the velocity of migration $\vec{v} = a\nabla u$ depends on the global population in a subdomain $\Omega' \subset \Omega$ given by $a = a(\int_{\Omega'} u dx)$. For more information see [5] and reference therein.

This article is concerned with the proof of the existence, uniqueness and the exponential decay of the system (1.1–1.5) using the energy method. The method of energy consists in the use of appropriate multipliers to build a functional of Lyapunov. In this direction, we prove that for this model where the energy, that can flow from one part to another, is strong enough to produce exponential decay for the solution of the coupled system.

The differential equations are very helpful in many areas of science, but most interesting real life problems involve more than one unknown function. As a motivation let us consider an island with two types of species: Rabbits and Foxes. Clearly one plays the role of predator while the other the role of a prey. If we are interested to model the population growth of both species, then we have to keep in mind that if, for example, the population of the foxes increases, then the rabbit population will be affected. So, the rate of change of the population of one type will depend on the actual population of the other type. For example, in the absence of the rabbit population, the fox population will decrease (and fast) to face a certain extinction. Something that most of us would like to avoid. In this case, if the difference among the two populations tends to zero, then there is one natural control for the species. In our case u and v could describe the densities of two population that interact through a parameter alpha in an atmosphere common of global form. Does this system type have a solution? If affirmative, is the solution stabilized or not controllable? In case that it is stabilized, at which the rate? We intend to answer these questions. To the best of our knowledge, that result is the first one in this direction, for nonlocal coupled system of reaction-diffusion.

This paper is organized as follows. Before the main result, in Sect. 2 we briefly outline the notation and terminology to be used subsequently. In Sect. 3 three we prove the existence and uniqueness of the solution, in the Sect. 4 four we prove the exponential decay of the solution of the system. Finally, numerical evidence corroborating our theoretical results is given in Sect. 5. In this paper, we prove the following two theorems:

Theorem 1.2 *Let* $(u_0, v_0) \in L^2(\Omega) \times L^2(\Omega)$, *if conditions* (2.6–2.7) *are satisfied, then there exists* $T > 0$ *depending on* (u_0, v_0) *such that there is at most one solution of* (1.1–1.5) *in* $L^2(0, T : H_0^1(\Omega)) \cap C([0, T) : L^2(\Omega)) \times L^2(0, T : H_0^1(\Omega)) \cap C([0, T) : L^2(\Omega))$ *with initial data* $(u(x, 0), v(x, 0)) = (u_0, v_0)$.

Theorem 1.3 *Let* (u, v) *be a solution of the system* (1.1–1.5) *given by Theorem* 1.2, *then there exist positives constants* C *and* η, *such that*

$$E(t) \leqslant C E(0) e^{-\eta t}. \tag{1.15}$$

2 Preliminaries

In this work we consider the reaction-diffusion coupled system in parallel way via parameter $\alpha > 0$ as

$$u_t - a(l(u))\Delta u + f(u - v) = \alpha(u - v) \quad \text{in } \Omega \times (0, T), \tag{2.1}$$

$$v_t - a(l(v))\Delta v - f(u - v) = \alpha(v - u) \quad \text{in } \Omega \times (0, T), \tag{2.2}$$

$$u = v = 0 \quad \text{in } \partial\Omega \times (0, T), \tag{2.3}$$

$$u(x, 0) = u_0(x) \quad \text{in } \Omega, \tag{2.4}$$

$$v(x, 0) = v_0(x) \quad \text{in } \Omega, \tag{2.5}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $\partial\Omega$ is the boundary of Ω of class C^2. $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, that is, there exists $M_1 > 0$ such that

$$|f(s) - f(t)| \leqslant M_1|s - t|, \quad \forall s, t \in \mathbb{R}. \tag{2.6}$$

$a : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, that is, there exists $M_2 > 0$ such that

$$|a(s) - a(t)| \leqslant M_2|s - t|, \quad \forall s, t \in \mathbb{R} \tag{2.7}$$

with

$$a(\xi) \geqslant m > 0, \quad \forall \xi \in \mathbb{R} \tag{2.8}$$

and

$$l : L^2(\Omega) \to \mathbb{R} \quad \text{is a continuous linear form.} \tag{2.9}$$

In the system, the distributed spring coefficient is coupled by the terms $\alpha(u - v)$ and $\alpha(v - u)$. In this sense the Energy can flow from one part to another through this parameter α.

By $\langle \cdot, \cdot \rangle$ we will represent the duality pairing between X and X', X' being the topological dual of the space X. We represent by $H^m(\Omega)$ the usual Sobolev space of order m, by $H_0^m(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$, and by $L^2(\Omega)$ the class of square Lebesgue integrable real functions. In particular, $H_0^1(\Omega)$ has inner product $((\cdot, \cdot))$ and norm $\|\cdot\|$ given by

$$((u, v)) = \int_\Omega \nabla u \cdot \nabla v dx \quad \text{and} \quad \|u\|^2 = \int_\Omega |\nabla u|^2 dx.$$

For the Hilbert space $L^2(\Omega)$ we represent its inner and norm, respectively, by $(\cdot, \cdot)$ and $|\cdot|$, defined by

$$(u, v) = \int_\Omega uv dx \quad \text{and} \quad |u|^2 = \int_\Omega u^2 dx.$$

Throughout this paper c is a generic constant, not necessarily the same at each occasion (it will change from line to line), which depends in an increasing way on the indicated quantities.

We take the initial conditions as following

$$(u_0(x), v_0(x)) \in L^2(\Omega) \times L^2(\Omega). \tag{2.10}$$

We denote the potential energy associated to this system by

$$E(t) = \frac{1}{2} \int_\Omega [|u|^2 + |v|^2] dx. \tag{2.11}$$

3 Existence and Uniqueness of a Local Solution

In this section, we will prove that for $(u_0, v_0) \in L^2(\Omega) \times L^2(\Omega)$ there exists a unique solution of (2.6–2.9) in $L^2(0, T : H_0^1(\Omega)) \cap C([0, T) : L^2(\Omega)) \times L^2(0, T : H_0^1(\Omega)) \cap C([0, T) : L^2(\Omega))$ where the time T depends only $|u_0|_{L^2(\Omega)}$ and $|v_0|_{L^2(\Omega)}$. We make use of Faedo-Galerkin approximation for to prove the existence of weakly solutions. We write the system (2.1–2.5) in the following form,

$$u_t - a(l(u))\Delta u = -f(u, v) + g(u, v) \quad \text{in } Q = \Omega \times (0, T), \tag{3.1}$$

$$v_t - a(l(v))\Delta v = f(u, v) - g(u, v) \quad \text{in } Q = \Omega \times (0, T), \tag{3.2}$$

$$u = v = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{3.3}$$

$$u(x, 0) = u_0(x) \quad \text{in } \Omega, \tag{3.4}$$

$$v(x, 0) = v_0(x) \quad \text{in } \Omega, \tag{3.5}$$

where we denote $f(u, v) \equiv f(u - v)$ and $g(u, v) \equiv \alpha(u - v)$.

Theorem 3.1 (Existence) *Let* $(u_0, v_0) \in L^2(\Omega) \times L^2(\Omega)$ *and* $0 < T < +\infty$. *If* (2.6–2.9) *holds, then there exists* (u, v) *solution of* (3.1–3.5) *such that*

$$(u, v) \in L^2(0, T : H_0^1(\Omega)) \cap C([0, T) : L^2(\Omega)) \times L^2(0, T : H_0^1(\Omega)) \cap C([0, T) : L^2(\Omega)), \tag{3.6}$$

$$(u_t, v_t) \in L^2(0, T : H^{-1}(\Omega)) \times L^2(0, T : H^{-1}(\Omega)), \tag{3.7}$$

$$\frac{d}{dt}(u, h_1) + a(l(u)) \int_\Omega \nabla u \cdot \nabla h_1 dx = -\int_\Omega f(u, v) h_1 dx + \int_\Omega g(u, v) h_1 dx \tag{3.8}$$

for all $h_1 \in H_0^1(\Omega)$, *where* (3.8) *must be understood as an equality in* $\mathcal{D}'(0, T)$.

$$\frac{d}{dt}(v, h_2) + a(l(v)) \int_\Omega \nabla v \cdot \nabla h_2 dx = \int_\Omega f(u, v) h_2 dx - \int_\Omega g(u, v) h_2 dx \tag{3.9}$$

for all $h_2 \in H_0^1(\Omega)$, *where* (3.9) *must be understood as an equality in* $\mathcal{D}'(0, T)$.

Proof (i) *Approximate problem*: Let $\{w_j\}_{j \in \mathbb{N}}$ be a Hilbertian basis of $H_0^1(\Omega)$ (cf. H. Brezis, [3]). For each $j = 1, 2, 3, \ldots$ represent by $\mathbb{V}_j$, the subspace of $H_0^1(\Omega)$ generated by $\{w_1, w_2, \ldots, w_j\}$. The approximate problem, associated with (3.1–3.5), consists of to find $u_j, v_j \in \mathbb{V}_j$ such that

$$(u_j', h_1) - a(l(u_j))(\Delta u_j, h_1) = -(f(u_j, v_j), h_1) + (g(u_j, v_j), h_1), \quad \forall h_1 \in \mathbb{V}_j, \tag{3.10}$$

$$(v_j', h_2) - a(l(v_j))(\Delta v_j, h_2) = (f(u_j, v_j), h_2) - (g(u_j, v_j), h_2), \quad \forall h_2 \in \mathbb{V}_j, \tag{3.11}$$

$$u_j(0) = u_{0j} \to u_0 \quad \text{strongly in } L^2(\Omega), \tag{3.12}$$

$$v_j(0) = v_{0j} \to v_0 \quad \text{strongly in } L^2(\Omega). \tag{3.13}$$

Let $h_1 = w_i(x)$ and $h_2 = w_i(x)$ for $i = 1, \ldots, j$, then in (3.10–3.13) we have for θ_{kj}, $\phi_{kj} \in C^\infty(\Omega)$

$$\begin{aligned}
&\left(\sum_{k=1}^{j} \theta_{kj}'(t) w_k(x), w_i(x)\right) - a\left(l\left(\sum_{k=1}^{j} \theta_{kj}(t) w_k(x)\right)\right)\left(\Delta \sum_{k=1}^{j} \theta_{kj}(t) w_k(x), w_i(x)\right) \\
&\quad = -\left(f\left(\sum_{k=1}^{j} \theta_{kj}(t) w_k(x), \sum_{k=1}^{j} \phi_{kj}(t) w_k(x)\right), w_i(x)\right) \\
&\qquad + \left(g\left(\sum_{k=1}^{j} \theta_{kj}(t) w_k(x), \sum_{k=1}^{j} \phi_{kj}(t) w_k(x)\right), w_i(x)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\sum_{k=1}^{j} \phi_{kj}'(t) w_k(x), w_i(x)\right) - a\left(l\left(\sum_{k=1}^{j} \phi_{kj}(t) w_k(x)\right)\right)\left(\Delta \sum_{k=1}^{j} \phi_{kj}(t) w_k(x), w_i(x)\right) \\
&\quad = \left(f\left(\sum_{k=1}^{j} \theta_{kj}(t) w_k(x), \sum_{k=1}^{j} \phi_{kj}(t) w_k(x)\right), w_i(x)\right) \\
&\qquad - \left(g\left(\sum_{k=1}^{j} \theta_{kj}(t) w_k(x), \sum_{k=1}^{j} \phi_{kj}(t) w_k(x)\right), w_i(x)\right)
\end{aligned}$$

that is,

$$\theta'_{kj}(t) - \lambda_k a(l(u_j))\theta_{kj}(t) = -(f(u_j, v_j), w_i) + (g(u_j, v_j), w_i), \tag{3.14}$$

$$\phi'_{kj}(t) - \lambda_k a(l(v_j))\phi_{kj}(t) = (f(u_j, v_j), w_i) - (g(u_j, v_j), w_i). \tag{3.15}$$

(ii) *Approximate solutions*: We will just work with (3.14). For (3.15) the result is similar. For $i, k = 1, \ldots, j$ in (3.14), we have the following system

$$\begin{bmatrix} \theta'_{1j} \\ \theta'_{2j} \\ \vdots \\ \theta'_{jj} \end{bmatrix} = \begin{bmatrix} \lambda_1 a(l(u_j)) & & 0 \\ & \ddots & \\ 0 & & \lambda_j a(l(u_j)) \end{bmatrix} \begin{bmatrix} \theta_{1j} \\ \theta_{2j} \\ \vdots \\ \theta_{jj} \end{bmatrix} - \begin{bmatrix} (f, w_1) \\ (f, w_2) \\ \vdots \\ (f, w_j) \end{bmatrix} + \begin{bmatrix} (g, w_1) \\ (g, w_2) \\ \vdots \\ (g, w_j) \end{bmatrix}$$

that is,

$$X' = F(X, t), \tag{3.16}$$

$$X(0) = X_0, \tag{3.17}$$

where $F(X, t) = AX + B$ and $X_0 = [\alpha_{1j}, \alpha_{2j}, \ldots, \alpha_{jj}]^T$. The system (3.16–3.17) is equivalent to system of ordinary differential equations of first order. Let us show that the system (3.16–3.17) is in the conditions of Carathéodory's theorem.

Claim For fixed X, we will show that A and B are measurable in t.

In fact, we observed that the matrix A is formed for the elements $\lambda_k a(l(u_j))$ with $k = 1, 2, \ldots, j$. Since l is a lineal and continuous form and the operator a is continuous, then the composition $a(l(u_j))$ is also continuous; therefore $\lambda_k a(l(u_j))$ is continuous for $k = 1, 2, \ldots, j$ and then A is measurable in t. On the other hand, let us observe that B is formed by the elements $(f(u_j, v_j), w_i)$ and $(g(u_j, v_j), w_i)$, with $i = 1, 2, \ldots, j$. Since f and g are continuous and $w_i \in H_0^1(\Omega)$, we concludes that B is continuous and therefore measurable.

Claim For fixed t, we will show that F is continuous in X.

In fact, notice that B is continuous in X, because B is constant in relation to X. For continuity of AX, is enough we verify that A is continuous in X. Let $\prod_k X = \theta_{kj}$ $(k = 1, 2, \ldots, j)$ be the projection $\mathbb{R}^j \longrightarrow \mathbb{R}$ and $\sigma(X) = \prod_k X w_k$. For each t fixed, as

$$u_j(t) = \sum_{k=1}^{j} \theta_{kj}(t) w_k,$$

we can consider the function

$$X \longrightarrow a(l(u_j)) = a\left(l\left(\sum_{k=1}^{j} \theta_{kj}(t) w_k\right)\right) = a\left(l\left(\sum_{k=1}^{j} \prod_k X w_k\right)\right).$$

Since A is lineal combination of continuous functions, proceeds that A is continuous in X, hence the function $F(X, t)$ is continuous in X.

Claim Let K be a compact of $\mathbb{D} = \mathbb{E} \times [0, T]$, where $\mathbb{E} = \{X \in \mathbb{R}^{j\times 1} : \|X\|_{\mathbb{R}^{j\times 1}} \leqslant \delta, \delta > 0\}$. We will show that exists a real function $m_r(t)$, integrable in $[0, T]$, so that

$$\|F(X,t)\|_{\mathbb{R}^{j\times 1}} \leqslant m_r(t), \quad \forall (X,t) \in D.$$

In fact, we denote by $\|\cdot\|_{pq}$ the norm of maximum in $\mathbb{R}^{pq}$. But $F(X,t) = AX + B$, then

$$\|F(X,t)\|_{j\times 1} \leqslant \|A\|_{j\times j}\|X\|_{j\times 1} + \|B\|_{j\times 1}.$$

Since $X \in \mathbb{E}$, we have $\|X\|_{j\times 1} \leqslant \delta$. Then

$$\|F(X,t)\|_{j\times 1} \leqslant \delta\|A\|_{j\times j} + \|B\|_{j\times 1}.$$

Notice that $\lambda_k a(l(u_j))$ are continuous functions, hence $\|A\|_{j\times j} \leqslant C$ $(C > 0)$.

On the other hand, for the matrix B we have

$$|(f(u_j,v_j),w_i)| \leqslant |f(u_j,v_j)||w_i| = |f(u_j,v_j)|,$$
$$|(g(u_j,v_j),w_i)| \leqslant |g(u_j,v_j)||w_i| = |g(u_j,v_j)|.$$

Thus $\|F(X,t)\|_{j\times 1} \leqslant \delta C + |f(u_j,v_j)| + |g(u_j,v_j)| \equiv m_r(t)$, where $m_r(t)$ is integrable in $[0, T]$. Hence, the system (3.16–3.17) satisfies the conditions of Carathéodory, and then exists $\{u_j(t), v_j(t)\} \in [0, t_j) \times [0, t_j)$, $t_j < T_0$.

We now have to establish an estimate that permits to extend the solution $\{u_j(t), v_j(t)\}$ to the whole interval $[0, T]$.

From now on, $\{C_i\}_{i=1\ldots 7}$, will denote positive constants, independents of j and t.

(iii) *A priori estimates*: We put $h_1 = u_j$ and $h_2 = v_j$ in (3.10) and (3.11) respectively, we have

$$(u_j', u_j) - a(l(u_j))(\Delta u_j, u_j) = -(f(u_j,v_j),u_j) + (g(u_j,v_j),u_j), \tag{3.18}$$
$$(v_j', v_j) - a(l(v_j))(\Delta v_j, v_j) = (f(u_j,v_j),v_j) - (g(u_j,v_j),v_j). \tag{3.19}$$

Using the boundary condition and the first Green's identity we have

$$(-\Delta u_j, u_j) = \int_\Omega (-\Delta u_j) u_j dx = \int_\Omega \nabla u_j \cdot \nabla u_j dx = |\nabla u_j|^2 = \|u_j\|^2.$$

Then, we can write (3.18) as

$$\frac{1}{2}\frac{d}{dt}|u_j|^2 + a(l(u_j))\|u_j\|^2 = -(f(u_j,v_j),u_j) + (g(u_j,v_j),u_j). \tag{3.20}$$

In a similar way we can write (3.19) as

$$\frac{1}{2}\frac{d}{dt}|v_j|^2 + a(l(v_j))\|v_j\|^2 = (f(u_j,v_j),v_j) - (g(u_j,v_j),v_j). \tag{3.21}$$

Adding (3.20) with (3.21) and using that $g(u_j,v_j) = \alpha(u_j - v_j)$ and (2.8) we obtain

$$\frac{d}{dt}[|u_j|^2 + |v_j|^2] + 2m\|u_j\|^2 + 2m\|v_j\|^2 = -2(f(u_j,v_j),u_j - v_j) + 2\alpha|u_j - v_j|^2.$$

For simplicity, we suppose first that $f(0)=0$. Since (2.6), we have $|f(s)| \leqslant M|s|$. Thus

$$\frac{d}{dt}[|u_j|^2+|v_j|^2] \leqslant 2M|u_j-v_j|^2+2\alpha|u_j-v_j|^2 = 2(M+\alpha)|u_j-v_j|^2.$$

Using the Minskowsky inequality,

$$\frac{d}{dt}[|u_j|^2+|v_j|^2] \leqslant c(|u_j|^2+|v_j|^2). \tag{3.22}$$

Integrating over $t \in [0,T]$, using that $u_j(0) \to u_0$ and $v_j(0) \to v_0$ strongly in $L^2(\Omega)$ and the Gronwall inequality, follows that $|u_j(0)|^2+|v_j(0)|^2 \leqslant C$ and

$$|u_j(t)|^2+|v_j(t)|^2 \leqslant C.$$

From where follows that $u_j(t)$ and $v_j(t)$ are bounded in $L^\infty(0,T:L^2(\Omega))$. Thus,

$$\int_0^t (\|u_j\|^2+\|v_j\|^2)ds \leqslant C,$$

then $u_j(t)$ and $v_j(t)$ are limited in $L^2(0,T:H_0^1(\Omega))$.

Now, if $f(0) \neq 0$, then we make $f(s)=\widetilde{f}(s)+f(0)$ and we reproduce the estimates above for $\widetilde{f}$. We obtain the same estimate of (3.22) with the extra term $|f(0)|\int_\Omega |u_j-v_j|dx$ added on the right hand side of this inequality. On the other hand, since Ω is bounded, we can apply the Young and Cauchy-Schwarz inequalities:

$$\left|f(0)\int_\Omega |u_j-v_j|dx\right| \leqslant |f(0)\|\,\Omega|^{1/2}\|u_j-v_j\| \leqslant C(1+\|u_j\|^2+\|v_j\|^2)$$

and then

$$\frac{d}{dt}[|u_j|^2+|v_j|^2] \leqslant c(1+|u_j|^2+|v_j|^2).$$

Using a generalized Gronwall type inequality including the extra term "1" on the right hand side of this last inequality (see [2] or [6, Lemma 2.1]) we conclude in a similar way that $u_j(t)$ and $v_j(t)$ are bounded in $L^\infty(0,T:L^2(\Omega))$, and then, they are also limited in $L^2(0,T:H_0^1(\Omega))$, when $f(0) \neq 0$.

From (3.10–3.11), we have that

$$u_j' = a(l(u_j))\Delta u_j - f(u_j,v_j)+g(u_j,v_j) \in H^{-1}(\Omega),$$
$$v_j' = a(l(v_j))\Delta v_j + f(u_j,v_j)-g(u_j,v_j) \in H^{-1}(\Omega).$$

Notice that $-a(l(u_j))\Delta u_j$ defines an element of $H^{-1}(\Omega)$, given by the duality

$$\langle -a(l(u_j))\Delta u_j, h_1\rangle = a(l(u_j))\int_\Omega \nabla u_j\cdot\nabla h_1 dx, \quad \forall h_1 \in H_0^1(\Omega).$$

In a similar way we have

$$\langle -a(l(v_j))\Delta v_j, h_2\rangle = a(l(v_j))\int_\Omega \nabla v_j\cdot\nabla h_2 dx, \quad \forall h_2 \in H_0^1(\Omega).$$

Using the fact that $-a(l(u_j))\Delta u_j$, $-a(l(v_j))\Delta v_j \in H^{-1}(\Omega)$, the dual of $H_0^1(\Omega)$, then they are lineal and continuous forms and therefore both are bounded.

Since $u_j, v_j \in L^2(0, T : L^2(\Omega))$, then

$$\begin{aligned}\int_\Omega |f(u_j, v_j)|dx &\leqslant \int_\Omega \beta|u_j - v_j|dx \leqslant \int_\Omega \beta(|u_j| + |v_j|)dx\\ &\leqslant C\left[\left(\int_\Omega |u_j|^2 dx\right)^{1/2} + \left(\int_\Omega |v_j|^2 dx\right)^{1/2}\right]\end{aligned}$$

and

$$\begin{aligned}\int_\Omega |g(u_j, v_j)|dx &= \int_\Omega \alpha|u_j - v_j|dx \leqslant \int_\Omega \alpha(|u_j| + |v_j|)dx\\ &\leqslant C\left[\left(\int_\Omega |u_j|^2 dx\right)^{1/2} + \left(\int_\Omega |v_j|^2 dx\right)^{1/2}\right].\end{aligned}$$

Therefore

$$f(u_j, v_j), g(u_j, v_j) \in L^2(0, T : L^2(\Omega)) \hookrightarrow L^1(0, T : L^2(\Omega))$$

and we concludes that u'_j, v'_j are bounded in $L^2(0, T : H^{-1}(\Omega))$.

(iv) *Passage to the limit*: We have that

$$u_j, v_j \quad \text{are bounded in } L^\infty(0, T : L^2(\Omega)) \cap L^2(0, T : H_0^1(\Omega)), \tag{3.23}$$

$$u'_j, v'_j \quad \text{are bounded in } L^2(0, T : H^{-1}(\Omega)). \tag{3.24}$$

Now, due the corollary of Banach-Alouglu (see [11], p. 66), we can extract subsequences of $u_{j_k} \equiv u_j$ and $v_{j_k} \equiv v_j$ (which we denote with the same symbol) so that

$$u_j \overset{\star}{\rightharpoonup} u \quad \text{weak star in } L^\infty(0, T : L^2(\Omega)), \tag{3.25}$$

$$v_j \overset{\star}{\rightharpoonup} v \quad \text{weak star in } L^\infty(0, T : L^2(\Omega)), \tag{3.26}$$

$$u_j \rightharpoonup u \quad \text{weak in } L^2(0, T : H_0^1(\Omega)), \tag{3.27}$$

$$v_j \rightharpoonup v \quad \text{weak in } L^2(0, T : H_0^1(\Omega)). \tag{3.28}$$

Consequently

$$\int_0^T (u_j, h_1)dt \longrightarrow \int_0^T (u, h_1)dt, \quad \forall h_1 \in L^\infty(0, T : L^2(\Omega)), \tag{3.29}$$

$$\int_0^T (u_j, h_1)dt \longrightarrow \int_0^T (u, h_1)dt, \quad \forall h_1 \in L^2(0, T : H_0^1(\Omega)), \tag{3.30}$$

$$\int_0^T (v_j, h_2)dt \longrightarrow \int_0^T (v, h_2)dt, \quad \forall h_2 \in L^\infty(0, T : L^2(\Omega)), \tag{3.31}$$

$$\int_0^T (v_j, h_2)dt \longrightarrow \int_0^T (v, h_2)dt, \quad \forall h_2 \in L^2(0, T : H_0^1(\Omega)). \tag{3.32}$$

For (3.24) it proceeds

$$u'_j \rightharpoonup u' \quad \text{weakly in } L^2(0, T : H^{-1}(\Omega)), \tag{3.33}$$

$$v'_j \rightharpoonup v' \quad \text{weakly in } L^2(0, T : H^{-1}(\Omega)). \tag{3.34}$$

On the other hand, $H_0^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$. By Lions-Aubin's compactness Theorem [9] follows that

$$u_j \longrightarrow u \quad \text{strongly in } L^2(0, T : L^2(\Omega)), \tag{3.35}$$

$$v_j \longrightarrow v \quad \text{strongly in } L^2(0, T : L^2(\Omega)). \tag{3.36}$$

The convergence (3.25–3.26) means that

$$\int_0^T (u_j(t), w(t))dt \longrightarrow \int_0^T (u(t), w(t))dt, \quad \forall w \in L^1(0, T : L^2(\Omega)),$$

$$\int_0^T (v_j(t), w(t))dt \longrightarrow \int_0^T (v(t), w(t))dt, \quad \forall w \in L^1(0, T : L^2(\Omega)).$$

We choose $w = \theta h_1$ with $\theta \in \mathcal{D}(0, T)$, $h_1 \in L^2(\Omega)$ and we will show that for all $\theta \in \mathcal{D}(0, T)$ and for all $h_1 \in L^2(\Omega)$,

$$\int_0^T [(g(u_j, v_j), h_1) - (g(u, v), h_1)]\theta(t)dt \longrightarrow 0.$$

Let T be a positive number such that *supp*$(\theta) \subset [0, T]$, then

$$\int_0^T [(g(u_j, v_j), h_1) - (g(u, v), h_1)]\theta(t)dt = \int_0^T (g(u_j, v_j) - g(u, v), h_1)\theta(t)dt.$$

Hence, by straightforward calculations

$$\begin{aligned}\int_0^T (g(u_j, v_j) - g(u, v), h_1)\theta(t)dt &\leqslant \int_0^T\int_\Omega |g(u_j, v_j) - g(u, v)||h_1||\theta(t)|dxdt \\ &\leqslant \int_0^T\int_\Omega (\alpha|u_j - u| + \alpha|v_j - v|)|h_1||\theta(t)|dxdt.\end{aligned}$$

Using $L^2(0, T : L^2(\Omega)) \hookrightarrow L^1(0, T : L^2(\Omega))$ and the Cauchy-Schwartz inequality we obtain

$$\begin{aligned}\int_0^T (g(u_j, v_j) - g(u, v), h_1)\theta(t)dt \leqslant C\int_0^T \left(\int_\Omega |u_j - u|^2 dx\right)^{1/2} \left(\int_\Omega |h_1|^2 dx\right)^{1/2} dt \\ + C\int_0^T \left(\int_\Omega |v_j - v|^2 dx\right)^{1/2} \left(\int_\Omega |h_1|^2 dx\right)^{1/2} dt.\end{aligned}$$

Applying the Cauchy-Schwartz inequality and considering the convergence (3.35) we obtain

$$\begin{aligned}&C\int_0^T \left(\int_\Omega |u_j - u|^2 dx\right)^{1/2} \left(\int_\Omega |h_1|^2 dx\right)^{1/2} dt \\ &\leqslant C\left(\int_0^T\int_\Omega |u_j - u|^2 dxdt\right)^{1/2} \left(\int_0^T\int_\Omega |h_1|^2 dxdt\right)^{1/2} < \varepsilon.\end{aligned}$$

In a similar way using the convergence (3.36) we have

$$C\int_0^T\left(\int_\Omega |v_j-v|^2dx\right)^{1/2}\left(\int_\Omega |h_1|^2dx\right)^{1/2}dt$$
$$\leqslant C\left(\int_0^T\int_\Omega |v_j-v|^2dxdt\right)^{1/2}\left(\int_0^T\int_\Omega |h_1|^2dxdt\right)^{1/2}<\varepsilon.$$

Therefore we have

$$\int_0^T (g(u_j,v_j)-g(u,v),h_1)\theta(t)dt<\varepsilon.$$

Performing similar calculations we can to prove that

$$\int_0^T (f(u_j,v_j)-f(u,v),h_1)\theta(t)dt<\varepsilon.$$

We will show now, that for every $\theta\in\mathcal{D}([0,T])$ and for every $h_1\in L^2(\Omega)$

$$a(l(u_j))\int_0^T\int_\Omega \nabla u_j\cdot\nabla h_1\theta(t)dt\longrightarrow a(l(u))\int_0^T\int_\Omega \nabla u\cdot\nabla h_1\theta(t)dt. \tag{3.37}$$

It is enough we prove that

$$a(l(u_j))\longrightarrow a(l(u))\quad \text{in } L^2(0,T),\ \forall T>0. \tag{3.38}$$

Since a is continuous, we will show that

$$l(u_j)\longrightarrow l(u)\quad \text{strongly in } L^2(0,T). \tag{3.39}$$

In fact, because

$$\int_0^T |l(u_j)-l(u))|^2dt=\int_0^T |l(u_j-u)|^2dt\leqslant C_6\int_0^T |u_j-u|^2dt<\varepsilon.$$

This last one result, is consequence of the convergence (3.35). □

These convergence implies that we can take limits in the approximate problem (3.11–3.15), and then to verify the conditions (i), (ii), (iii) and (iv) of the Theorem.

Now, we will make verify of the initial data and we prove the uniqueness of solutions. Using the result of regularity we have that

$$u,v\in C^0(0,T:L^2(\Omega)). \tag{3.40}$$

In this form, makes sense we calculate $u(0)$ and $v(0)$. Let us consider $\theta\in C^1(0,T:\mathbb{R})$, with $\theta(0)=1$ and $\theta(T)=0$. Since the convergence (3.29) we have

$$\int_0^T (u_j',z)\theta dt\longrightarrow\int_0^T (u',z)\theta dt,\quad z\in L^2(\Omega). \tag{3.41}$$

Performing integration by parts in (3.41) we have

$$-(u_j(0),z)-\int_0^T (u_j,z)\theta' dt\longrightarrow -(u(0),z)-\int_0^T (u,z)\theta' dt. \tag{3.42}$$

Using the convergence (3.29) in (3.42) we have $(u_j(0), z) \longrightarrow (u(0), z)$, for all $z \in H_0^1(\Omega)$. But $u_j(0)$ converges strong for u_0 in $L^2(\Omega)$, consequently weak in $L^2(\Omega)$. Therefore $(u_j(0), z) \longrightarrow (u_0, z)$, for all $z \in H_0^1(\Omega)$. From uniqueness of the limit we have $(u(0), z) \longrightarrow (u_0, z)$, for all $z \in H_0^1(\Omega)$. Thus, $u(0) = u_0$. In a similar way we can prove that $v(0) = v_0$.

To finish this section we will show the uniqueness of solution.

Theorem 3.2 (Uniqueness) *Let* $(u_0, v_0) \in L^2(\Omega) \times L^2(\Omega)$ *and* $0 < T < +\infty$, *where the time* T *depends only* $|u_0|_{L^2(\Omega)}$ *and* $|v_0|_{L^2(\Omega)}$. *If* (2.6–2.9) *holds, then there is at most one solution of* (3.1–3.5) *in* $L^2(0, T : H_0^1(\Omega)) \cap C([0, T) : L^2(\Omega)) \times L^2(0, T : H_0^1(\Omega)) \cap C([0, T) : L^2(\Omega))$ *with initial data* $(u(x, 0), v(x, 0)) = (u_0, v_0)$.

Proof Assume that (u_1, v_1), (u_2, v_2) in $L^2(0, T : H_0^1(\Omega)) \cap C([0, T) : L^2(\Omega)) \times L^2(0, T : H_0^1(\Omega)) \cap C([0, T) : L^2(\Omega))$ are two solutions of (3.1–3.5) with u_t, v_t in $L^2(0, T : H^{-1}(\Omega)) \times L^2(0, T : H^{-1}(\Omega))$, so all integrations below are justified and with the same initial data, in fact $(u_1 - u_2)(x, 0) \equiv 0$ and $(u_2 - u_2)(x, 0) \equiv 0$. Then

$$\begin{aligned}
&\frac{d}{dt}u_1 - a(l(u_1))\triangle u_1 = -f(u_1, v_1) + g(u_1, v_1),\\
&\frac{d}{dt}v_1 - a(l(v_1))\triangle v_1 = f(u_1, v_1) - g(u_1, v_1),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{dt}u_2 - a(l(u_2))\triangle u_2 = -f(u_2, v_2) + g(u_2, v_2),\\
&\frac{d}{dt}v_2 - a(l(v_2))\triangle v_2 = f(u_2, v_2) - g(u_2, v_2).
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{d}{dt}(u_1 - u_2, h_1) + a(l(u_1))\int_\Omega \nabla u_1 \cdot \nabla h_1 dx - a(l(u_2))\int_\Omega \nabla u_2 \cdot \nabla h_1 dx\\
&\quad = -(f(u_1, v_1) - f(u_2, v_2), h_1) + (g(u_1, v_1) - g(u_2, v_2), h_1), \quad \forall h_1 \in H_0^1(\Omega),\\
&\frac{d}{dt}(v_1 - v_2, h_2) + a(l(v_1))\int_\Omega \nabla v_1 \cdot \nabla h_2 dx - a(l(v_2))\int_\Omega \nabla v_2 \cdot \nabla h_2 dx\\
&\quad = (f(u_1, v_1) - f(u_2, v_2), h_2) - (g(u_1, v_1) - g(u_2, v_2), h_2), \quad \forall h_2 \in H_0^1(\Omega).
\end{aligned}$$

Using that $g(u_1, v_1) - g(u_2, v_2) = \alpha(u_1 - u_2) - \alpha(v_1 - v_2)$ and that $f(u, v) = f(u - v)$ we have

$$\begin{aligned}
&\frac{d}{dt}(u_1 - u_2, h_1) + a(l(u_1))\int_\Omega \nabla u_1 \cdot \nabla h_1 dx - a(l(u_2))\int_\Omega \nabla u_2 \cdot \nabla h_1 dx\\
&\quad = -(f(u_1 - v_1) - f(u_2 - v_2), h_1) + \alpha(u_1 - u_2, h_1) - \alpha(v_1 - v_2, h_1), \quad \forall h_1 \in H_0^1(\Omega),\\
&\frac{d}{dt}(v_1 - v_2, h_2) + a(l(v_1))\int_\Omega \nabla v_1 \cdot \nabla h_2 dx - a(l(v_2))\int_\Omega \nabla v_2 \cdot \nabla h_2 dx\\
&\quad = (f(u_1 - v_1) - f(u_2 - v_2), h_2) - \alpha(u_1 - u_2, h_2) + \alpha(v_1 - v_2, h_2), \quad \forall h_2 \in H_0^1(\Omega).
\end{aligned}$$

On the other hand, let $h_1 = u_1 - u_2$ and $h_2 = v_1 - v_2$ we obtain

$$\frac{d}{dt}|u_1 - u_2|^2 + a(l(u_1))\int_\Omega \nabla u_1 \cdot \nabla(u_1 - u_2)dx - a(l(u_2))\int_\Omega \nabla u_2 \cdot \nabla(u_1 - u_2)dx$$
$$= -(f(u_1 - v_1) - f(u_2 - v_2), u_1 - u_2) + \alpha|u_1 - u_2|^2 - \alpha(v_1 - v_2, u_1 - u_2)$$

and

$$\frac{d}{dt}|v_1 - v_2|^2 + a(l(v_1))\int_\Omega \nabla v_1 \cdot \nabla(v_1 - v_2)dx - a(l(v_2))\int_\Omega \nabla v_2 \cdot \nabla(v_1 - v_2)dx$$
$$= (f(u_1 - v_1) - f(u_2 - v_2), v_1 - v_2) - \alpha(u_1 - u_2, v_1 - v_2) + \alpha|v_1 - v_2|^2.$$

Hence

$$\frac{d}{dt}|u_1 - u_2|^2 + a(l(u_1))\|u_1\|^2 + a(l(u_2))\|u_2\|^2 + [a(l(u_1)) - a(l(u_2)]\int_\Omega \nabla u_1 \cdot \nabla u_2 dx$$
$$\leqslant |f(u_1 - v_1) - f(u_2 - v_2)||u_1 - u_2| + \alpha|u_1 - u_2|^2 + \alpha|v_1 - v_2||u_1 - u_2|$$

and

$$\frac{d}{dt}|v_1 - v_2|^2 + a(l(v_1))\|v_1\|^2 + a(l(v_2))\|v_2\|^2 + [a(l(v_1)) - a(l(v_2)]\int_\Omega \nabla v_1 \cdot \nabla v_2 dx$$
$$\leqslant |f(u_1 - v_1) - f(u_2 - v_2)||v_1 - v_2| + \alpha|u_1 - u_2||v_1 - v_2| + \alpha|v_1 - v_2|^2.$$

Using (2.6–2.9) and the Young inequality we have

$$\begin{aligned}
&\frac{d}{dt}|u_1 - u_2|^2 + m\|u_1\|^2 + m\|u_2\|^2 \\
&\quad\leqslant M_1|l(u_1) - l(u_2)|\|u_1\|\|u_2\| + M_3|(u_1 - u_2) - (v_1 - v_2)||u_1 - u_2| \\
&\qquad + \alpha|u_1 - u_2|^2 + \frac{\alpha}{2}|v_1 - v_2|^2 + \frac{\alpha}{2}|u_1 - u_2|^2 \\
&\quad\leqslant M_1C_1|u_1 - u_2|\|u_1\|\|u_2\| + M_3|u_1 - u_2|^2 + M_3|u_1 - u_2||v_1 - v_2| \\
&\qquad + \frac{3}{2}\alpha|u_1 - u_2|^2 + \frac{\alpha}{2}|v_1 - v_2|^2 \\
&\quad\leqslant \frac{m}{2}\|u_1\|^2 + \frac{M_1^2C_1^2}{2m}\|u_2\|^2|u_1 - u_2|^2 + M_3|u_1 - u_2|^2 + \frac{M_3}{2}|u_1 - u_2|^2 \\
&\qquad + \frac{M_3}{2}|v_1 - v_2|^2 + \frac{3}{2}\alpha|u_1 - u_2|^2 + \frac{\alpha}{2}|v_1 - v_2|^2 \\
&\quad\leqslant \frac{m}{2}\|u_1\|^2 + \frac{M_1^2C_1^2}{2m}\|u_2\|^2|u_1 - u_2|^2 + \frac{3}{2}(M_3 + \alpha)|u_1 - u_2|^2 + \frac{1}{2}(M_3 + \alpha)|v_1 - v_2|^2
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{dt}|v_1 - v_2|^2 + m\|v_1\|^2 + m\|v_2\|^2 \\
&\quad\leqslant \frac{m}{2}\|v_1\|^2 + \frac{M_2^2C_2^2}{2m}\|v_2\|^2|v_1 - v_2|^2 + \frac{3}{2}(M_3 + \alpha)|v_1 - v_2|^2 + \frac{1}{2}(M_3 + \alpha)|u_1 - u_2|^2.
\end{aligned}$$

Then

$$\frac{d}{dt}|u_1 - u_2|^2 + \frac{m}{2}\|u_1\|^2 + m\|u_2\|^2$$
$$\leqslant \frac{M_1^2 C_1^2}{2m}\|u_2\|^2|u_1 - u_2|^2 + \frac{3}{2}(M_3 + \alpha)|u_1 - u_2|^2 + \frac{1}{2}(M_3 + \alpha)|v_1 - v_2|^2 \quad (3.43)$$

and

$$\frac{d}{dt}|v_1 - v_2|^2 + \frac{m}{2}\|v_1\|^2 + m\|v_2\|^2$$
$$\leqslant \frac{M_2^2 C_2^2}{2m}\|v_2\|^2|v_1 - v_2|^2 + \frac{3}{2}(M_3 + \alpha)|v_1 - v_2|^2 + \frac{1}{2}(M_3 + \alpha)|u_1 - u_2|^2. \quad (3.44)$$

Adding (3.43) with (3.44) we obtain

$$\frac{d}{dt}(|u_1 - u_2|^2 + |v_1 - v_2|^2) + \frac{m}{2}\|u_1\|^2 + m\|u_2\|^2 + \frac{m}{2}\|v_1\|^2 + m\|v_1\|^2$$
$$\leqslant \frac{M_1^2 C_1^2}{2m}\|u_2\|^2|u_1 - u_2|^2 + \frac{M_2^2 C_2^2}{2m}\|v_2\|^2|v_1 - v_2|^2$$
$$+ 2(M_3 + \alpha)|u_1 - u_2|^2 + 2(M_3 + \alpha)|v_1 - v_2|^2$$
$$= \left[\frac{M_1^2 C_1^2}{2m}\|u_2\|^2 + 2(M_3 + \alpha)\right]|u_1 - u_2|^2 + \left[\frac{M_2^2 C_2^2}{2m}\|v_2\|^2 + 2(M_3 + \alpha)\right]|v_1 - v_2|^2.$$

We define

$$\varphi(t) = \frac{M_1^2 C_1^2}{2m}\|u_2\|^2 + 2(M_3 + \alpha), \qquad \xi(t) = \frac{M_2^2 C_2^2}{2m}\|v_2\|^2 + 2(M_3 + \alpha).$$

Thus,

$$\frac{d}{dt}[|u_1 - u_2|^2 + |v_1 - v_2|^2] \leqslant \varphi(t)|u_1 - u_2|^2 + \xi(t)|v_1 - v_2|^2.$$

Let $\mathcal{R}(t) = \sup\{\varphi(t), \xi(t)\}$, then $\mathcal{R} > 0$ and

$$\frac{d}{dt}(|u_1 - u_2|^2 + |v_1 - v_2|^2) \leqslant \mathcal{R}(t)(|u_1 - u_2|^2 + |v_1 - v_2|^2). \quad (3.45)$$

Integrating (3.45) over $t \in [0, T]$ and using that $u_1(0) = u_2(0)$ and $v_1(0) = v_2(0)$, we obtain

$$|u_1 - u_2|^2 + |v_1 - v_2|^2 \leqslant \int_0^t \mathcal{R}(t)(|u_1 - u_2|^2 + |v_1 - v_2|^2)dx.$$

Let $\rho(t) = |u_1 - u_2|^2 + |v_1 - v_2|^2$, then

$$\rho(t) \leqslant \int_0^t \mathcal{R}(s)\rho(s)ds. \quad (3.46)$$

Applying Gronwall's inequality, we obtain

$$\rho(t) \leqslant 0.$$

Therefore $\rho \equiv 0$, i.e.,

$$|u_1 - u_2|^2 + |v_1 - v_2|^2 = 0.$$

Using the regularity of the solutions, the uniqueness follows. □

4 Exponential Stability

In this section we show that the total energy (2.11) associated to system (2.1–2.5) decay exponentially to zero as t tends to infinity. In what follows we will prove our main result:

Theorem 4.1 *Let* (u, v) *be a solution of system* (1.1–1.5) *given by the Theorems* 3.1 *and* 3.2. *We suppose that* $m > 2c_p(M_1 + \alpha) > 0$, *where* c_p *corresponds to the constant of the Poincaré inequality. Then there exist positives constants* C *and* η, *such that*

$$E(t) \leqslant CE(0)e^{-\eta t}. \tag{4.1}$$

Proof Multiplying (2.1) by u and integrating over $x \in \Omega$ we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + a(\ell(u))\int_\Omega |\nabla u|^2 dx = -\int_\Omega f(u-v)u dx + \alpha\int_\Omega (u-v)u dx.$$

Multiplying (2.2) by v and integrating over $x \in \Omega$ we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |v|^2 dx + a(\ell(v))\int_\Omega |\nabla v|^2 dx = \int_\Omega f(u-v)v dx - \alpha\int_\Omega (u-v)v dx.$$

Adding the expressions above and using (2.8) we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega [|u|^2 + |v|^2] dx + m\int_\Omega [|\nabla u|^2 + |\nabla v|^2] dx \\ \leqslant \int_\Omega |f(u-v)||u-v| dx + \alpha\int_\Omega |u-v|^2 dx.\end{aligned}$$

Using (2.6) follows that

$$\frac{d}{dt}E(t) \leqslant -m\int_\Omega [|\nabla u|^2 + |\nabla v|^2] dx + (M_1 + \alpha)\int_\Omega |u-v|^2 dx.$$

Applying the Poincaré inequality and the Young inequality, we obtain

$$\frac{d}{dt}E(t) \leqslant -\frac{2m}{c_p}E(t) + 4(M_1 + \alpha)E(t).$$

Now choosing $m > 2c_p(M_1 + \alpha) > 0$ follows that there exists $C > 0$ such that

$$\frac{d}{dt}E(t) \leqslant -CE(t).$$

From we concludes that there exists $\eta > 0$ such that

$$E(t) \leqslant CE(0)e^{-\eta t}.$$

The proof follows. □

5 Numerical Results

5.1 Example 1. One Dimensional Space

In this example we consider a particular case of the nonlocal reaction diffusion (2.1–2.5) in one dimensional space ($n = 1$), $\Omega = (0, 1)$, and $l : L^2(\Omega) \to \mathbb{R}$ defined by $l(u) = \int_0^1 u(x)dx$. Then we approximate the solution of the system using implicit finite differences. The numerical scheme reads as follows

$$\frac{u_i^{k+1} - u_i^k}{\delta x} - a\left(\sum_{j=1}^{J} \delta x u_j^k\right) \frac{u_{i+1}^{k+1} - 2u_i^{k+1} + u_{i-1}^{k+1}}{\delta x^2} + f(u_i^k - v_i^k) = \alpha(u_i^k - v_i^k), \quad (5.1)$$

$$\frac{v_i^{k+1} - v_i^k}{\delta x} - a\left(\sum_{j=1}^{J} \delta x v_j^k\right) \frac{v_{i+1}^{k+1} - 2v_i^{k+1} + v_{i-1}^{k+1}}{\delta x^2} - f(u_i^k - v_i^k) = \alpha(v_i^k - u_i^k), \quad (5.2)$$

$$u_0^{k+1} = u_J^{k+1} = v_0^{k+1} = v_J^{k+1} = 0 \quad (5.3)$$

for $i = 1, \dots, J - 1$ and $k = 0, \dots, K$, where $\delta t = T/K$, $\delta x = 1/J$, $x_i = i\delta x$, $i = 0, \dots, J$ and $t_k = k\delta t$, $k = 0, \dots, K$. In (5.1–5.3), u_i^k denotes the approximation of $u(t_k, x_i)$. In order to solve the system (5.1–5.3), we consider the initial condition approximated by

$$u_i^0 = u_0(x_i) \quad \text{and} \quad v_i^0 = v_0(x_i) \text{ for } i = 0, \dots, J. \quad (5.4)$$

For each $k = 0, \dots, K$, the scheme (5.1–5.3) is equivalent to a linear system with a tridiagonal matrix of $\mathbb{R}^{(J-1)\times(J-1)}$ which is positive definite and then there exists a unique solution of (5.1–5.4).

In order to compare the numerical behaviour of the solution with the theoretical and numerical behaviour of the solution of one equation (scalar case), we take the same nonlinear reaction terms with similar parameters and initial conditions of Ackle and Ke [1]. Nonlinear reaction and nonlocal diffusion are given by

$$a(\xi) := \max\left\{\varepsilon, \frac{1}{|\xi|}\right\} + m_0 \quad \text{for all } \xi, \quad (5.5)$$

$$f(w) - \alpha w := rw(\kappa - w) \quad \text{for all } w, \quad (5.6)$$

where ε, m_0, r and κ are constant and positive parameters. We remark that in the numerical example of Ackle and Ke [1], the authors consider a nonlocal diffusion given by the expression $\frac{1}{\int_0^1 u(x)dx}$. In our case the parameter ε is a very small parameter and it plays a practical computational role to avoid the numerical overflow on the diffusion when the extinction of the population occurs, that is, when $u \approx 0$ or $v \approx 0$. We consider a parameter m_0 to the numerical study of an extinction case of population and for a persistence case of population. In fact, the exponential decaying of the energy (4.1), can be interpreted as the extinction of two populations u and v. That occurs when the hypothesis $m > c_p(M_1 + \alpha)$ of the Theorem 4.1, is verified. If m is too small, the decaying of the energy is not guaranteed and a population persistence can occur as we see in Fig. 2.

The initial condition is given by $u_0(x) = \delta \sin(\pi x)$, with $\delta = 1.95$ (see [1]), and $v_0(x) = -u_0(x) = -\delta \sin(\pi x)$. In this example, a negative v_0 has not a physical or biological sense, if u and v represent densities of population, but we want to focus in the importance of the hypothesis of the Theorem 4.1, showing numerically that the exponential decaying of

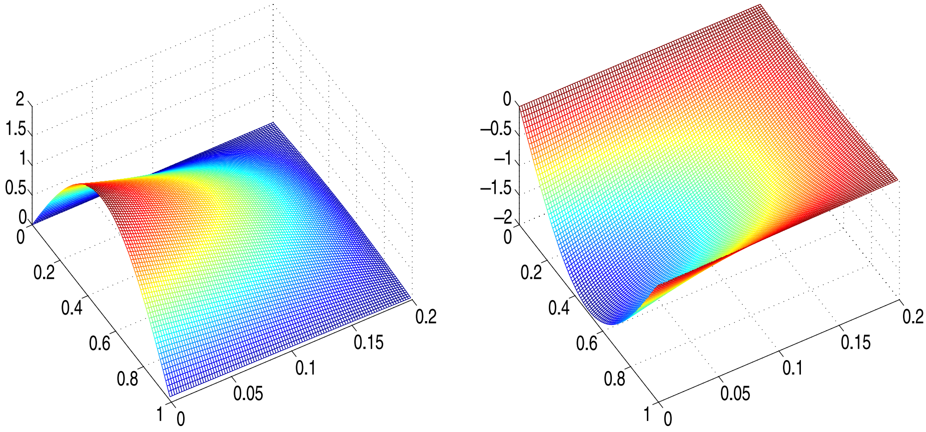

Fig. 1 Extinction of the population density $u(x,t)$ (*left*) and $v(x,t)$ (*right*) at time $T = 0.2$

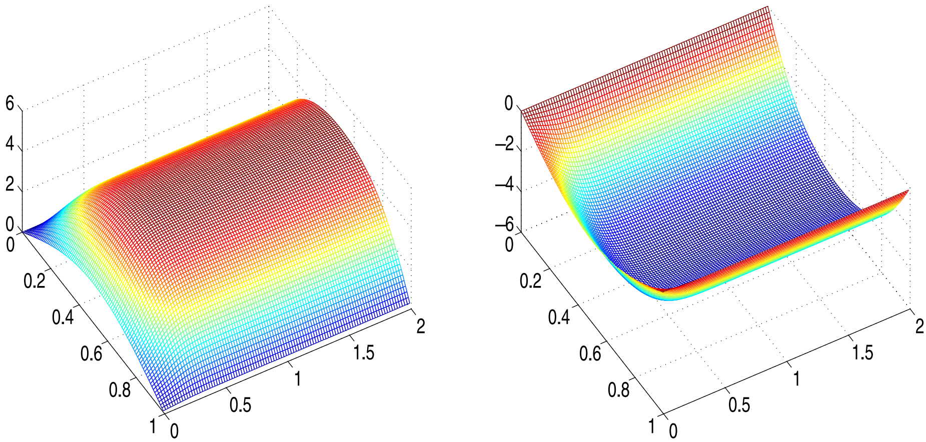

Fig. 2 Persistence of the population density $u(x,t)$ (*left*) and $v(x,t)$ (*right*)

the energy does no occur when the hypothesis is not verified. We choose the parameter $\varepsilon = 10^{-6}$ for the nonlocal diffusion function (5.5), and we choose $r = 1.0$ and $\kappa = 10.0$ for the reaction function (5.6). The discretization is given by $J = 10^4$ and $K = 10^4$; we solve the linear system (5.1–5.3) using Thomas algorithm programming in Fortran90.

We consider here 2 simulations, the first one with $m_0 = 1.0$ (see Fig. 1), and the second one with $m_0 = 0.1$ (see Fig. 2). The population persistence phenomenon does not occur with a choice of a big amplitude δ for the initial condition as it occurs in [1]. In fact we made simulations with different δ values and the exponential decaying is always observed.

5.2 Example 2. Two Dimensional Space

In this example we consider the reaction diffusion equations (2.1–2.5) in the two dimensional space ($n = 2$). This example is close to the model of rabbits and foxes described at the end of the introduction. We consider a square island $\Omega = (0, 1) \times (0, 1)$, and $l : L^2(\Omega) \to \mathbb{R}$ defined by the total mass $l(u) = \frac{1}{\text{meas}(\Omega)} \int_0^1 u(x)dx$. Then we approximate the solution of the

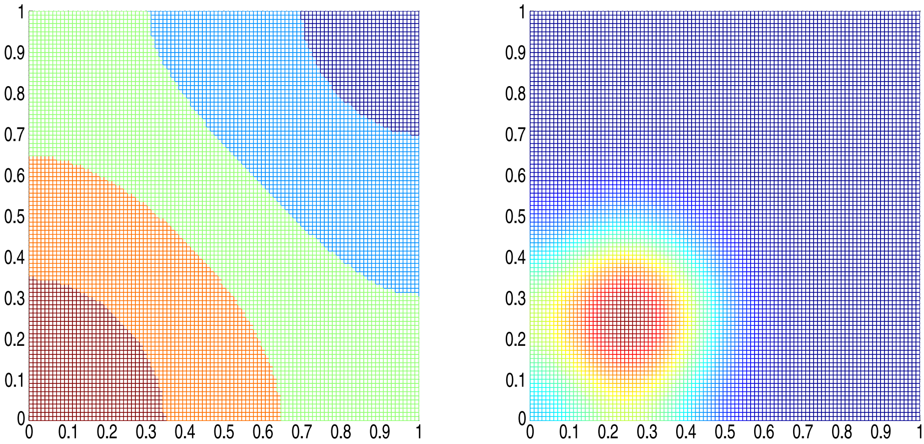

Fig. 3 Rabbits $u(x,t)$ and foxes $v(x,t)$ model; *Left*: simulation with constant diffusion; *Right*: simulation with nonlocal diffusion proportional to the total mass of population

system using implicit finite differences. The numerical scheme is the natural generalization of (5.1–5.3) in two dimensional space and it is equivalent to a Finite Volume discretization in a uniform mesh (see [7]) given by a Cartesian grid with $N_x \times N_y$ control volumes and choosing $N_x = N_y = 100$ for all simulations. It is possible to consider also unstructured meshes, but we will confine here to a uniform mesh for simplicity of the simulated models. The discretization in time is given by $N_t = 1000$ time steps for $T = 2.5$. That is, $\delta t = T/N_t$. We choose the same nonlinear reaction of Example 1 (5.6) with the parameter $r = 1.0$ and $\kappa = 10.0$.

The idea is to compare here the behaviour of the numerical solution between a nonlocal diffusion and a constant diffusion. For that we consider first the case with constant diffusion $a(\xi) = 0.1 = constant$ (see the left picture of Fig. 3) and secondly a case with a nonlocal diffusion $a(\xi) = 0.1\xi$ (see the right picture of Fig. 3).

In both cases the initial condition are given by

$$\begin{cases} u(x,y) = A; \\ v(x,y) = B\operatorname{sech}(\beta(x-x_0))\operatorname{sech}(\beta(y-y_0)). \end{cases}$$

with $A = 0.01$, $B = 5000$, $\beta = 2000$ and $(x_0, y_0) = (1/4, 1/4)$. These initial conditions represent to a rabbit population initially with a low density and constant throughout the domain. On the other hand, a high density fox population is located in a small surface at the center of the quadrant $[0, 1/2] \times [0, 1/2]$ which will diffuse on the rest of the domain.

In Fig. 3, we represent the density of the rabbit population $u(x,t)$. On the other hand, the behaviour of the fox population $v(x,t)$ is symmetric and very similar to the behaviour of $u(x,t)$ and it is really not necessary to show it. We observed the extinction of both species for these simulations which is consistent with the exponential decaying of the solution. Before the extinction of the species, the behaviour of the solution with local diffusion (left) and the behaviour of the solution with nonlocal diffusion (right) are very different: solutions with local diffusion tend very fast to a constant, and solution with a nonlocal diffusion and proportional to the total mass of the population, remains localized in a region during a moment.

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