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Ternary Derivations, Stability and Physical Aspects

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Abstract Ternary algebras and modules are vector spaces on which products of three factors are defined. In this paper, we present several physical applications of ternary structures. Some recent results on the stability of ternary derivations are reviewed. Using a fixed point method, we also establish the generalized Hyers–Ulam–Rassias stability of ternary derivations from a normed ternary algebra into a Banach tri-module associated to the generalized Jensen functional equations and prove a superstability result.

Keywords Generalized Hyers–Ulam–Rassias stability · Superstability · Ternary algebra · Tri-module · Ternary derivation · Generalized Jensen functional equation

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1 Introduction

A classical question in the theory of functional equations is the following: "When is it true that a function, which approximately satisfies a functional equation $\mathcal{E}$ must be close to an exact solution of $\mathcal{E}$?"

If the problem accepts a solution, we say that the equation $\mathcal{E}$ is stable. The first stability problem concerning group homomorphisms was raised from a famous talk given by S.M. Ulam [40] in 1940.

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We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a number $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

In the next year 1941, Ulam's problem was affirmatively solved by D.H. Hyers [16] for Banach spaces:

Suppose that E_1 is a normed space, E_2 is a Banach space and a mapping $f : E_1 \to E_2$ satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon \quad (x, y \in E_1),$$

where $\epsilon > 0$ is a constant. Then the limit $T(x) := \lim 2^{-n} f(2^n x)$ exists for each $x \in E_1$ and T is the unique additive mapping satisfying

$$\|f(x) - T(x)\| \le \epsilon \quad (x \in E_1). \tag{1.1}$$

If f is continuous at a single point of E_1, then T is continuous everywhere in E_1. Moreover (1.1) is sharp.

In 1950, T. Aoki [3] generalized Hyers' theorem for approximately additive mappings. In 1978, Th.M. Rassias [34] extended Hyers' theorem by obtaining a unique linear mapping under certain continuity assumption when the Cauchy difference is allowed to be unbounded (see [28]):

Suppose that E is a real normed space, F is a real Banach space and $f : E \to F$ is a mapping such that for each fixed $x \in E$ the mapping $t \mapsto f(tx)$ is continuous on $\mathbb{R}$. Let there exist $\varepsilon \ge 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p) \quad (x, y \in E). \tag{1.2}$$

Then there exists a unique linear mapping $T : E \to F$ such that

$$\|f(x) - T(x)\| \le \varepsilon\|x\|^p/(1 - 2^{p-1}) \quad (x \in E).$$

In 1990, Th.M. Rassias [35] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \ge 1$. In 1991, Z. Gajda [14] gave an affirmative solution to this question for $p > 1$ by following the same approach as in Rassias' paper [34]. It was proved by Z. Gajda [14], as well as by Th.M. Rassias and P. Šemrl [37] that one cannot prove a Rassias type theorem when $p = 1$. In 1994, P. Găvruta [15] provided a generalization of Rassias theorem in which he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ in (1.2) by a general control function $\varphi(x, y)$. The paper of Th.M. Rassias [34] has provided a lot of influence in the development of what we now call Hyers–Ulam–Rassias stability of functional equations. During the last decades several stability problems for various functional equations have been investigated by many mathematicians; we refer the reader to the monographs [12, 17, 19, 36].

The first result on the stability of the classical Jensen equation $f(\frac{x+y}{2}) = \frac{f(x)+f(y)}{2}$ was given by Z. Kominek [23] and then the stability of this equation and its generalizations were studied by numerous researchers, cf. [13, 18, 24, 27, 30, 38] and references therein. Recently, the authors of [29] examined the stability of homomorphisms of ternary algebras by using the "direct method". In this paper, we use the fixed point method developed by V. Radu [33] to establish the generalized Hyers–Ulam–Rassias stability of ternary derivations from a normed ternary algebra into a Banach tri-module associated to the generalized Jensen functional equations

$$rf\left(\frac{sx+ty}{r}\right)=sf(x)+tf(y),$$

where r, s, t are given constant values. It is easy to see that a mapping $f: X \to Y$ between linear spaces with $f(0)=0$ satisfies the generalized Jensen equation if and only if it is additive; cf. [8]. In addition, we introduce the notion of ε-approximate ternary derivation and prove the superstability of ternary derivations. Our results seem to be of special interest on their own right and to raise certain hopes in view of their possible applications in physics.

2 Physical Aspects of Ternary Structures

Ternary operations were introduced in the nineteenth century by A. Cayley [11] and then various types of ternary structures investigated. The simplest example of a non-trivial ternary operation is given by the following composition rule on "cubic matrices":

$$\{a,b,c\}_{ijk}=\sum_{l,m,n} a_{nil}b_{ljm}c_{mkn}, \quad i,j,k,\ldots=1,2,\ldots,N.$$

Several applications, although still hypothetical, may be found in physics:

(i) *Quark fields*: Replacing the binary relation $AB=\pm BA$ by a ternary relation of the type $ABC=\omega BCA$, where $\omega=e^{2\pi i/3}$ is one of the roots of the unity, one obtains an algebraic structure, the so-called Z_3-graded algebras, as an extension of Grassmann algebras, Lie algebras and Clifford algebras. This ternary algebraic system can be applied for a description of quark fields, where the non-observability of isolated quarks is investigated as a phenomenon of algebraic confinement; see [22];

(ii) *Supersymmetric theories*: These theories are good places to look for exotic matter in the form of fermionic superpartners of bosonic particles that carry forces; see [20];

(iii) *Yang–Baxter equation*: This equation appears in many subjects such as statistical physics and quantum groups. There is a new way of solving the Yang–Baxter equation by using triple systems. More precisely, let $\{e_1, e_2, \ldots, e_N\}$ be a basis for a vector space V equipped with a bilinear non-degenerate form $\langle x \mid y\rangle$ and let $e^k=g^{km}e_m$, where g^{km} is the inverse metric tensor. Defining the θ-dependent triple product $[xyz]_\theta$ by $R^{ab}_{cd}(\theta)=\langle e^a \mid [e^b e_c e_d]_\theta\rangle$, we can rewrite the Yang–Baxter equation

$$R^{b'a'}_{a_1b_1}(\theta)R^{c'a_2}_{a'c_1}(\theta')R^{c_2b_2}_{b'c'}(\theta'')=R^{c'b'}_{b_1c_1}(\theta'')R^{c_2a'}_{a_1c'}(\theta')R^{b_2a_2}_{a'b'}(\theta)$$

with $\theta'=\theta+\theta''$ as the following form

$$[v\,[ue_jz]_{\theta'}\,[e^jxy]_\theta]_{\theta''}=[u\,[ve_jx]_{\theta'}\,[e^jzy]_{\theta''}]_\theta,$$

provided that the scattering matrix elements $R^{ab}_{cd}(\theta)$ fulfill $R^{ab}_{cd}(\theta)=R^{ba}_{dc}(\theta)$ (see [32]);

(iv) *Nambu mechanics*: It is a generalization of classical Hamiltonian mechanics introduced by Y. Nambu [31] who replaced a pair of variables in the Hamiltonian formulation by an n-tuple of variables in which $n\geq 3$, and the Poisson bracket by an n-ary operation, the so-called Nambu bracket. Recently, Takhtajan [39] introduced a canonical formalism for Nambu mechanics. He formulated its basic principles in an invariant geometrical way similar to that of Hamiltonian mechanics;

(v) *Lie theories*: The so-called ternary algebras may regard as building blocks of ordinary Lie (super) algebras, which have a significant role in theoretical physics; see [6]; and several other scientific discipline such as the non-standard statistics, the fractional quantum Hall effect and 4-dimensional Minkowskian space-time; cf. [1, 2, 21, 22, 41].

A ternary (associative) algebra $(\mathcal{A}, [\ ])$ is a linear space $\mathcal{A}$ over a scalar field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ equipped with a trilinear mapping, the so-called ternary product, $[\ ] : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that it is associative in the sense that $[[abc]\ d\ e] = [a\ [bcd]\ e] = [a\ b\ [cde]]$ for all $a, b, c, d, e \in \mathcal{A}$. If $\mathcal{A}$, as a linear space, is normed (Banach) and $\|[abc]\| \le \|a\|\ \|b\|\ \|c\|$ then it is called a normed (Banach) ternary algebra. This terminology differs from that of ternary algebra introduced by Brzozowski [9].

In particular, any subspace of a (normed) algebra, which is closed under triple product of its elements can be served as a (normed) ternary algebra via $[abc] = abc$. In general, a ternary algebra can not be derived from an associative binary algebra, see [22] for an interesting discussion. The study of ternary algebras may be regarded as the starting point of investigating other ternary structures on a given binary algebra such as the symmetric ternary product defined by $(abc) = abc + bca + cab$ or ω-skewsymmetric given by $\{abc\} = abc + \omega bca + \omega^2 cab$, where $\omega = e^{2\pi i/3}$.

In the case where the ternary algebra $(\mathcal{A}, [\ ])$ has an involution satisfying $[abc]^* = [c^* b^* a^*]$, one can define a new ternary product $[abc]_* = [ab^*c]$, which is linear in the outer variables, conjugate linear in the middle variable, and is associative of the second type, in the sense that $[xy[zts]_*]_* = [x[tzy]_*s]_* = [[xyz]_*ts]_*$.

Let $\mathcal{A}$ be a normed ternary algebra. By a *tri-module* we mean a linear space $\mathcal{M}$ equipped with the following trilinear mappings

$$
\begin{aligned}
&[\ ]_L : \mathcal{A} \times \mathcal{A} \times \mathcal{M} \to \mathcal{M}, \quad (a, b, m) \mapsto [abm]_L;\\
&[\ ]_C : \mathcal{A} \times \mathcal{M} \times \mathcal{A} \to \mathcal{M}, \quad (a, m, b) \mapsto [amb]_C;\\
&[\ ]_R : \mathcal{M} \times \mathcal{A} \times \mathcal{A} \to \mathcal{M}, \quad (m, a, b) \mapsto [mab]_R;
\end{aligned}
$$

satisfying the following compatible conditions:

$$
\begin{aligned}
&[a\ b\ [cdm]_L]_L = [[abc]\ d\ m]_L = [a\ [bcd]\ m]_L,\\
&[[mab]_R\ c\ d]_R = [m\ [abc]\ d]_R = [m\ a\ [bcd]]_R,\\
&[a\ [b\ [umc]_C\ d]_C\ v]_C = [[abu]\ m\ [cdv]]_C,\\
&[a\ [bcm]_L\ d]_C = [[abc]\ m\ d]_C = [a\ b\ [cmd]_C]_L,\\
&[a\ [mbc]_R\ d]_C = [[amb]_C\ c\ d]_R = [a\ m\ [bcd]]_C,\\
&[[abm]_L\ c\ d]_R = [a\ [bmc]_C\ d]_C = [a\ b\ [mcd]_R]_L,
\end{aligned}
$$

for all $a, b, c, d, u, v \in \mathcal{A}$ and $m \in \mathcal{A}$. This notion is a natural generalization of the binary case.

A linear mapping $\delta : \mathcal{A} \to \mathcal{M}$ is called a *ternary derivation* if

$$
\delta([abc]) = [\delta(a)\ b\ c]_R + [a\ \delta(b)\ c]_C + [a\ b\ \delta(c)]_L,
$$

for all $a, b, c \in \mathcal{A}$; cf. [7].

The conditions for tri-modules over ternary algebras of the second type can be easily restated. The notion of ternary derivation makes a link with Quantum Mechanics especially the version introduced by Nambu [31]. More precisely, let $(\mathcal{H}, \langle\, , \rangle)$ be a Hilbert space. Then $[abc] = \langle a, b\rangle c$ provides a ternary structure of second type and if δ is a ternary derivation from $\mathcal{H}$ into $\mathcal{H}$, then $\delta([abc]) = \langle a, b\rangle\delta(c) = [a\; b\; \delta(c)]$, which yield $\langle\delta(a), b\rangle = -\langle a, \delta(b)\rangle$. Hence $(\mathbf{i}\delta)^+ = \mathbf{i}\delta$. This gives us a one-to-one correspondence between hermitian operators and ternary derivations on $\mathcal{H}$.

3 Stability of Ternary Derivations

We start this section with a survey of some recent results on the stability of derivations of various ternary structures.

(I) A *C^*-ternary ring* is a Banach space $\mathcal{A}$ equipped with a ternary product $(x, y, z) \mapsto [x\; y\; z]$ of $\mathcal{A}^3$ into $\mathcal{A}$, which is linear in the outer variables, conjugate linear in the middle variable, and associative in the sense that $[x\; y\; [z\; t\; s]] = [x\; [t\; z\; y]\; s] = [[x\; y\; z]\; t\; s]$ and satisfies $\|[x\; y\; z]\| \le \|x\|\,\|y\|\,\|z\|$ and $\|[x\; x\; x]\| = \|x\|^3$; cf. [42].

A linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a *derivation* if $\delta([x\; y\; z]) = [\delta(x)\; y\; z] + [x\; \delta(y)\; z] + [x\; y\; \delta(z)]$ for all $x, y, z \in \mathcal{A}$. In [26], the following theorem was proved:

Theorem 3.1 *Suppose $f : \mathcal{A} \to \mathcal{A}$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : A^5 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y, u, v, w) := \frac{1}{2}\sum_{n=0}^{\infty} 2^{-n}\varphi(2^n x, 2^n y, 2^n u, 2^n v, 2^n w) < \infty,$$

and

$$\begin{aligned}&\|f(\mu x + \mu y + [u\; v\; w]) - \mu f(x) - \mu f(y) - [f(u)\; v\; w] - [u\; f(v)\; w] - [u\; v\; f(w)]\|\\&\quad \le \varphi(x, y, u, v, w),\end{aligned}$$

for all $\mu \in T^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x, y, u, v, w \in \mathcal{A}$. Then there exists a unique derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(x) - \delta(x)\| \le \widetilde{\varphi}(x, x, 0, 0, 0),$$

for all $x \in \mathcal{A}$.

Now we introduce an appropriate definition of almost derivation regarding to inequality (1.2).

Definition 3.2 The mapping $f : \mathcal{A} \to \mathcal{A}$ is called an *almost derivation* if $f(0) = 0$ and there exist numbers $\varepsilon > 0$ and $0 \le p < 1$ such that

$$\begin{aligned}&\|f(\mu x + \mu y + [u\; v\; w]) - \mu f(x) - \mu f(y) - [f(u)\; v\; w] - [u\; f(v)\; w] - [u\; v\; f(w)]\|\\&\quad \le \varepsilon(\|x\|^p + \|y\|^p + \|u\|^p + \|v\|^p + \|w\|^p),\end{aligned}$$

for all $x, y, u, v, w \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Theorem 3.3 [26] *Let $f : \mathcal{A} \to \mathcal{A}$ be an almost derivation. Then f is a derivation.*

(II) By a (complex) JB^*-triple we mean a complex Banach space $\mathcal{J}$ with a continuous triple product

$$\{\cdot,\cdot,\cdot\}:\mathcal{J}\times\mathcal{J}\times\mathcal{J}\to\mathcal{J},$$

which is linear in the outer variables and conjugate linear in the middle variable, and has the following properties:

(i) (commutativity) $\{x,y,z\}=\{z,y,x\}$;
(ii) (Jordan identity)

$$L(a,b)\{x,y,z\}=\{L(a,b)x,y,z\}-\{x,L(b,a)y,z\}+\{x,y,L(a,b)z\},$$

for all $a,b,x,y,z,\in\mathcal{J}$ in which $L(a,b)x:=\{a,b,x\}$;
(iii) For all $a\in\mathcal{J}$ the operator $L(a,a)$ is hermitian, i.e. $\|e^{itL(a,a)}\|=1$, and has positive spectrum in the Banach algebra $B(\mathcal{J})$;
(iv) $\|\{x,x,x\}\|=\|x\|^3$ for all $x\in\mathcal{J}$.

The class of JB^*-triples contains all C^*-algebras via $\{x,y,z\}=\frac{xy^*z+zy^*x}{2}$.

Let $\mathcal{J}$ be a complex JB^*-triple with norm $\|\cdot\|$ and $\theta:\mathcal{J}\to\mathcal{J}$ be a $\mathbb{C}$-linear mapping. A $\mathbb{C}$-linear mapping $D:\mathcal{J}\to\mathcal{J}$ is called a *θ-derivation* on $\mathcal{J}$ if

$$D(\{xyz\})=\{D(x)\theta(y)\theta(z)\}+\{\theta(x)D(y)\theta(z)\}+\{\theta(x)\theta(y)D(z)\},$$

for all $x,y,z\in\mathcal{J}$. The following theorem was established in [5]:

Theorem 3.4 *Let $f,h:\mathcal{J}\to\mathcal{J}$ be mappings with $f(0)=h(0)=0$ for which there exists a function $\varphi:\mathcal{J}^3\to[0,\infty)$ such that*

$$\widetilde{\varphi}(x,y,z):=\sum_{j=0}^{\infty}\frac{1}{2^j}\varphi(2^jx,2^jy,2^jz)<\infty,$$

$$\|f(\mu x+y)-\mu f(x)-f(y)\|\le\varphi(x,y,0),$$

$$\|h(\mu x+y)-\mu h(x)-h(y)\|\le\varphi(x,y,0),$$

$$\|f(\{xyz\})-\{f(x)h(y)h(z)\}-\{h(x)f(y)h(z)\}-\{h(x)h(y)f(z)\}\|\le\varphi(x,y,z),$$

for all $x,y,z\in\mathcal{J}$ and all $\mu\in S^1:=\{\lambda\in\mathbb{C}\mid|\lambda|=1\}$. Then there exist unique $\mathbb{C}$-linear mappings $D,\theta:\mathcal{J}\to\mathcal{J}$ such that

$$\|f(x)-D(x)\|\le\frac{1}{2}\widetilde{\varphi}(x,x,0),\qquad\|h(x)-\theta(x)\|\le\tfrac{1}{2}\widetilde{\varphi}(x,x,0)$$

for all $x\in\mathcal{J}$. Moreover, $D:\mathcal{J}\to\mathcal{J}$ is a θ-derivation on $\mathcal{J}$.

Now we are going to prove our main results concerning the generalized Hyers–Ulam–Rassias stability of ternary derivations associated with the generalized Jensen functional equation by using a fixed point method; cf. [10]. Throughout the rest, $\mathcal{A}$ and $\mathcal{M}$ denote a normed ternary algebra and a Banach ternary tri-module, respectively. We also assume that $s<r$. The cases where $t<r$ or $\min\{t,s\}>r$ can be formulated and proved by minor modifications.

We need the following known fixed point theorem.

Theorem 3.5 [25, 33] *Suppose $(\mathcal{X}, d)$ be a complete generalized metric space and let $J : \mathcal{X} \to \mathcal{X}$ be a strictly contractive mapping with a Lipschitz constant $L < 1$. Then, for each element $x \in \mathcal{X}$, either*

(A1) *$d(J^n x, J^{n+1}x) = \infty$ for all $n \geq 0$ or*
(A2) *there exists a natural number n_0 such that:*
  (A20) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$*
  (A21) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J*
  (A22) *y^* is the unique fixed point of J in the set $U = \{y \in \mathcal{X} : d(J^{n_0}x, y) < \infty\}$*
  (A23) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jx)$ for all $y \in U$.*

Theorem 3.6 *Let $f : \mathcal{A} \to \mathcal{M}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^5 \to [0, \infty)$ satisfying*

$$\lim_{n\to\infty} \frac{\varphi(\mu_i^n a, \mu_i^n b, \mu_i^n u, \mu_i^n v, \mu_i^n w)}{\mu_i^n} = 0,$$

for all $a, b, u, v, w \in \mathcal{A}$, where $\mu_i = \frac{r}{s} > 1$ if $i = 0$, and $\mu_i = \frac{s}{r} < 1$ if $i = 1$. Let the mapping $f : \mathcal{A} \to \mathcal{M}$ satisfy $f(0) = 0$ and

$$\left\| rf\left(\frac{\lambda sa + tb + [u\ v\ w]}{r}\right) - \lambda sf(a) - tf(b) - [f(u)\ v\ w]_R + [u\ f(v)\ w] + [u\ v\ f(w)]_L \right\| \leq \varphi(a, b, u, v, w), \tag{3.1}$$

for all $a, b, u, v, w \in \mathcal{A}$ and $\lambda \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Suppose that for $i = 0$ or $i = 1$ there exists $L_i < 1$ such that the mapping $\psi(a) := \frac{1}{s}\varphi(a, 0, 0, 0, 0)$ has the property $\psi(a) \leq \mu_i L_i \psi(\frac{a}{\mu_i})$ for all $a \in \mathcal{A}$.

Then there exists a unique ternary derivation $\delta_i : \mathcal{A} \to \mathcal{M}$ such that

$$\|f(a) - \delta_i(a)\| \leq \frac{L_i^{1-i}}{s(1-L_i)} \varphi(a, 0, 0, 0, 0) \quad (a \in \mathcal{A}). \tag{3.2}$$

Proof Consider the set $\mathcal{X} := \{g : \mathcal{A} \to \mathcal{M} : g(0) = 0\}$ and introduce a generalized metric on $\mathcal{X}$ by

$$d(g, h) = \inf\{c \in (0, \infty) : \|g(a) - h(a)\| \leq c\psi(a), \forall a \in \mathcal{A}\}.$$

It is easy to see that $(\mathcal{X}, d)$ is complete. For $i \in \{0, 1\}$, we define the mapping $J_i : \mathcal{X} \to \mathcal{X}$ by $(J_i g)(a) := \frac{1}{\mu_i} g(\mu_i a)$. For arbitrary elements $g, h \in \mathcal{X}$ we have

$$\begin{aligned} d(g, h) < c \quad &\Rightarrow \quad \|g(a) - h(a)\| \leq c\psi(a) \quad (a \in \mathcal{A}) \\ &\Rightarrow \quad \left\| \frac{1}{\mu_i} g(\mu_i a) - \frac{1}{\mu_i} h(\mu_i a) \right\| \leq \frac{1}{\mu_i} c\psi(\mu_i a) \quad (a \in \mathcal{A}) \\ &\Rightarrow \quad \left\| \frac{1}{\mu_i} g(\mu_i a) - \frac{1}{\mu_i} h(\mu_i a) \right\| \leq L_i c\psi(a) \quad (a \in \mathcal{A}) \\ &\Rightarrow \quad d(J_i g, J_i h) \leq L_i c. \end{aligned}$$

Therefore

$$d(J_i g, J_i h) \leq L_i d(g, h) \quad (g, h \in \mathcal{X}).$$

Hence J_i is a strictly contractive mapping on $\mathcal{X}$.

Set $b = u = v = w = 0$ and $\lambda = 1$ in (3.1) to obtain

$$\left\| \frac{r}{s} f\left(\frac{s}{r}a\right) - f(a) \right\| \le \frac{1}{s}\varphi(a,0,0,0,0) = \psi(a), \tag{3.3}$$

whence

$$\left\| \frac{r}{s} f\left(\frac{s}{r}a\right) - f(a) \right\| \le (L_1)^0 \psi(a),$$

for all $a \in \mathcal{A}$. By replacing a by $\frac{r}{s}a$ in (3.3) we have

$$\left\| f(a) - \frac{s}{r} f\left(\frac{r}{s}a\right) \right\| \le \frac{1}{r}\varphi\left(\frac{r}{s}a,0,0,0,0\right) = \frac{s}{r}\psi\left(\frac{r}{s}a\right),$$

whence

$$\left\| \frac{s}{r} f\left(\frac{r}{s}a\right) - f(a) \right\| \le (L_0)^1 \psi(a),$$

for all $a \in \mathcal{A}$.

Hence the following holds true:

$$\left\| \frac{1}{\mu_i} f(\mu_i a) - f(a) \right\| \le (L_i)^{1-i} \psi(a) \quad (i \in \{0,1\}, a \in \mathcal{A}). \tag{3.4}$$

Fix $i \in \{0,1\}$. It follows from (3.4) that $d(f, J_i f) \le (L_i)^{1-i} < \infty$. By the fixed point alternative there exists a natural number N_i such that the sequence $\{J_i^n f\}$ converges to a fixed point $\delta_i : \mathcal{A} \to \mathcal{M}$ of J_i and so $\delta_i(\mu_i a) = \mu_i \delta_i(a)$ $(a \in \mathcal{A})$. The mapping δ_i is the unique fixed point of J_i in the set $V_i = \{g \in \mathcal{X} : d(J_i^{N_i} f, g) < \infty\}$. Moreover,

$$d(g, \delta_i) \le \frac{d(g, J_i f)}{1 - L_i} \quad (g \in V_i).$$

Since $\lim_{n\to\infty} d(J_i^n f, \delta_i) = 0$ we easily conclude that

$$\lim_{n\to\infty} \frac{f(\mu_i^n a)}{\mu_i^n} = \delta_i(a) \quad (a \in \mathcal{A}).$$

Note that the sequence $\{\frac{f(\mu_i^n a)}{\mu_i^n}\}$ is the Hyers sequence when one use the direct method in establishing stability.

Obviously, $d(f, J_i^{N_i}) < \infty$ and so $f \in V_i$. Therefore

$$d(f, \delta_i) \le \frac{1}{1 - L_i} d(f, J_i f) \le \frac{L_i^{1-i}}{1 - L_i},$$

whence

$$\|f(a) - \delta_i(a)\| \le \frac{L_i^{1-i}}{s(1 - L_i)} \varphi(a,0,0,0,0),$$

for all $a \in \mathcal{A}$.

Fix $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{T}$. It follows from (3.1) that

$$\|f(\lambda a + b) - \lambda f(a) - f(b)\| \le \varphi(a, b, 0, 0, 0). \tag{3.5}$$

Let us replace a and b in (3.5) by $\mu_i^n a$ and $\mu_i^n b$, respectively, dividing the both sides by μ_i^n. Pass to the limit as $n \to \infty$ to get

$$\delta_i(\lambda a + b) = \lambda \delta_i(a) + \delta_i(b), \tag{3.6}$$

for all $\lambda \in \mathbb{T}$ and all $a, b \in \mathcal{A}$.

Next, let $\eta = \theta_1 + \mathbf{i}\theta_2 \in \mathbb{C}$, where $\theta_1, \theta_2 \in \mathbb{R}$. Let $\eta_1 = \theta_1 - [\theta_1]$, $\eta_2 = \theta_2 - [\theta_2]$. Then $0 \le \eta_j < 1$ $(1 \le j \le 2)$ and one can represent η_j as $\eta_j = \frac{\lambda_{j,1}+\lambda_{j,2}}{2}$ in which $\lambda_{j,k} \in \mathbb{T}$ $(1 \le j, k \le 2)$. Since δ_j satisfies (3.6) we infer that

$$\begin{aligned}
\delta_i(\eta a) &= \delta_i(\theta_1 a) + \mathbf{i}\delta_i(\theta_2 a)\\
&= [\theta_1]\delta_i(a) + \delta_i(\eta_1 a) + \mathbf{i}([\theta_2]\delta_i(a) + \delta_i(\eta_2 a))\\
&= \left([\theta_1]\delta_i(a) + \frac{1}{2}\delta_i(\lambda_{1,1}a + \lambda_{1,2}a)\right) + \mathbf{i}\left([\theta_2]\delta_i(a) + \frac{1}{2}\delta_i(\lambda_{2,1}a + \lambda_{2,2}a)\right)\\
&= \left([\theta_1]\delta_i(a) + \frac{1}{2}\lambda_{1,1}\delta_i(a) + \frac{1}{2}\lambda_{1,2}\delta_i(a)\right) + \mathbf{i}\left([\theta_2]\delta_i(a) + \frac{1}{2}\lambda_{2,1}\delta_i(a) + \frac{1}{2}\lambda_{2,2}\delta_i(a)\right)\\
&= \theta_1\delta_i(a) + \mathbf{i}\theta_2\delta_i(a)\\
&= \eta\delta_i(a),
\end{aligned}$$

for all $a \in \mathcal{A}$. So that δ_i is $\mathbb{C}$-linear (see also [4] for another proof of $\mathbb{C}$-linearity of δ_i).

Put $a = b = 0$, replace u, v, w by $\mu_i^n u, \mu_i^n v, \mu_i^n w$ in (3.1) and then divide the both sides of the obtained inequality by μ_i^{-3n} to get

$$\begin{aligned}
&\left\|\mu_i^{-n} r f\left(\frac{[\mu_i^n u\ v\ w]}{r}\right) - [\mu_i^{-n} f(\mu_i^n u)\ v\ w]_R + [u\ \mu_i^{-n} f(\mu_i^n v)\ w]_C + [u\ v\ \mu_i^{-n} f(\mu_i^n w)]_L\right\|\\
&\quad \le \mu_i^{-3n}\varphi(0, 0, \mu_i^n u, \mu_i^n v, \mu_i^n w).
\end{aligned}$$

Let n tend to infinity to get

$$r\delta_i\left(\frac{[uvw]}{r}\right) = [\delta_i(u)\ v\ w]_R + [u\ \delta_i(v)\ w]_C + [u\ v\ \delta_i(w)]_L,$$

for all $u, v, w \in \mathcal{A}$. Since δ_i is linear we conclude that δ_i is a ternary derivation. If σ_i is another linear mapping satisfying (3.2), then σ_i is trivially a fixed point of J_i in the set V_i, and so $\sigma_i = \delta_i$. □

As a consequences of our main theorem, we show the Hyers–Ulam–Rassias stability of ternary derivations.

Corollary 3.7 *Let $p_j \in [0, 1)$ $(1 \le j \le 5)$ and $\alpha, \beta > 0$. Suppose that the mapping $f : \mathcal{A} \to \mathcal{M}$ satisfies $f(0) = 0$ and*

$$\left\| rf\left(\frac{\lambda sa + tb + [u\ v\ w]}{r}\right) - \lambda sf(a) - tf(b) \right.$$
$$\left. - [f(u)\ v\ w]_R + [u\ f(v)\ w]_C + [u\ v\ f(w)]_L \right\|$$
$$\leq \alpha + \beta(\|a\|^{p_1} + \|b\|^{p_2} + \|u\|^{p_3} + \|v\|^{p_4} + \|w\|^{p_5}),$$

for all $\lambda \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ *and* $a, b, u, v, w \in \mathcal{A}$. *Then there exists a unique derivation* $\delta_0 : \mathcal{A} \to \mathcal{M}$ *such that*

$$\|f(a) - \delta_0(a)\| \leq \frac{\alpha + \beta\|a\|^{p_1}}{s^{p_1}(r^{1-p_1} - s^{1-p_1})},$$

for all $a \in \mathcal{A}$.

Proof Put $\varphi(a, b, u, v, w) = \alpha + \beta(\|a\|^{p_1} + \|b\|^{p_2} + \|u\|^{p_3} + \|v\|^{p_4} + \|w\|^{p_5})$ and $L_0 = (\frac{s}{r})^{1-p_1}$ in Theorem 3.6 (with $i = 0$). Then $\psi(a) = \frac{1}{s}(\alpha + \beta\|a\|^{p_1})$ and there exists a derivation δ_0 with the required property. □

Corollary 3.8 *Let* $p_j > 1$ $(1 \leq j \leq 5)$ *and* $\beta > 0$. *Suppose that the mapping* $f : \mathcal{A} \to \mathcal{M}$ *satisfies* $f(0) = 0$ *and*

$$\left\| rf\left(\frac{\lambda sa + tb + [u\ v\ w]}{r}\right) - \lambda sf(a) - tf(b) - [f(u)\ v\ w]_R \right.$$
$$\left. + [u\ f(v)\ w]_C + [u\ v\ f(w)]_L \right\|$$
$$\leq \beta(\|a\|^{p_1} + \|b\|^{p_2} + \|u\|^{p_3} + \|v\|^{p_4} + \|w\|^{p_5}),$$

for all $\lambda \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ *and* $a, b, u, v, w \in \mathcal{A}$. *Then there exists a unique derivation* $\delta_1 : \mathcal{A} \to \mathcal{M}$ *such that*

$$\|f(a) - \delta_1(a)\| \leq \frac{r^{p_1-1}\beta\|a\|^{p_1}}{s(r^{p_1-1} - s^{p_1-1})},$$

for all $a \in \mathcal{A}$.

Proof Put $\varphi(a, b, u, v, w) = \beta(\|a\|^{p_1} + \|b\|^{p_2} + \|u\|^{p_3} + \|v\|^{p_4} + \|w\|^{p_5})$ and $L_1 = (\frac{s}{r})^{p_1-1}$ in Theorem 3.6 (with $i = 1$). Then $\psi(a) = \frac{1}{s}\beta\|a\|^{p_1}$ and there exists a derivation δ_1 with the required property. □

4 Superstability of Ternary Derivations

Another nice version of stability is the so-called superstability. An equation $E(f) = 0$ is said to be superstable if the boundedness of $E(f)$ implies that either f is bounded or f is a solution of the equation. In this section, we aim to prove the superstability of ternary derivations associated to the generalized Jensen equation. First, we introduce an appropriate notion of ε-approximate derivation regarding to the so-called Hyers' inequality.

Definition 4.1 Given $\varepsilon > 0$, the mapping $f : \mathcal{A} \to \mathcal{A}$ is called an ε-approximate derivation if $f(0) = 0$ and

$$\left\| rf\left(\frac{\lambda sa + \lambda tb + [u\ v\ w]}{r}\right) - \lambda sf(a) - tf(b) - [f(u)\ v\ w] - [u\ f(v)\ w] - [u\ v\ f(w)] \right\| \le \varepsilon,$$

for all $a, b, u, v, w \in \mathcal{A}$ and $\lambda \in \mathbb{T}^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

The following theorem is one of our main results.

Theorem 4.2 *Let $f : \mathcal{A} \to \mathcal{A}$ be an almost derivation. Then f is a derivation.*

Proof Use Corollary 3.7 with $\alpha = \varepsilon$, $\beta = 0$ and $p_j = 0$ $(1 \le j \le 5)$ to get a ternary derivation δ defined by $\delta(a) := \lim_{n\to\infty} \frac{f(2^n a)}{2^n}$ such that

$$\|\delta(a) - f(a)\| \le \frac{\varepsilon}{r-s},$$

for all $a \in \mathcal{A}$. We have

$$\begin{aligned}
&\|2^n([u\ \ v\ \ f(2^m w)] - [u\ \ v\ \ 2^m f(w)])\| \\
&\quad \le \|f([2^n u\ \ v\ \ 2^m w]) - [f(2^n u)\ \ v\ \ 2^m w] - [2^n u\ \ f(v)\ \ 2^m w] - [2^n u\ \ v\ \ f(2^m w)]\| \\
&\qquad + \|f([2^n u\ \ v\ \ 2^m w]) - [f(2^n u)\ \ v\ \ 2^m w] - [2^n u\ \ f(v)\ \ 2^m w] - [2^n u\ \ v\ \ 2^m f(w)]\| \\
&\quad \le \varepsilon + \|f([2^n u\ \ v\ \ 2^m w]) - [f(2^n u)\ \ v\ \ 2^m w] - [2^n u\ \ f(v)\ \ 2^m w] - [2^n u\ \ v\ \ 2^m f(w)]\| \\
&\quad \le \varepsilon + \|f([2^n u\ \ v\ \ 2^m w]) - \delta([2^n u\ \ v\ \ 2^m w])\| \\
&\qquad + \|\delta([2^n u\ \ v\ \ 2^m w]) - [f(2^n u)\ \ v\ \ 2^m w] - [2^n u\ \ f(v)\ \ 2^m w] - [2^n u\ \ v\ \ 2^m f(w)]\| \\
&\quad \le \varepsilon + \frac{\varepsilon}{r-s} + 2^m \|\delta([2^n u\ \ v\ \ w]) - [f(2^n u)\ \ v\ \ w] - [2^n u\ \ f(v)\ \ w] - [2^n u\ \ v\ \ f(w)]\| \\
&\quad \le \varepsilon + \frac{\varepsilon}{r-s} + 2^m \|f([2^n u\ \ v\ \ w])\| - \|\delta([2^n u\ \ v\ \ w])\| \\
&\qquad + 2^m \|f([2^n u\ \ v\ \ w]) - [f(2^n u)\ \ v\ \ w] - [2^n u\ \ f(v)\ \ w] - [2^n u\ \ v\ \ f(w)]\| \\
&\quad \le \varepsilon + \frac{\varepsilon}{r-s} + \frac{2^m \varepsilon}{r-s} + 2^m \varepsilon,
\end{aligned}$$

for all nonnegative integers m, n and all $u, v, w \in \mathcal{A}$. Fix m, divide the both sides of the last inequality by 2^n and let n tend to infinity to obtain

$$\|[u\ \ v\ \ f(2^m w)] - [u\ \ v\ \ 2^m f(w)]\| \le 0,$$

for all m and all $u, v, w \in \mathcal{A}$. Therefore $\|[u\ \ v\ \ (\frac{f(2^m w)}{2^m} - f(w))]\| = 0$ for all m and all $u, v, w \in \mathcal{A}$. Letting m to infinity we get $\|[u\ \ v\ \ (\delta(w) - f(w))]\| = 0$ for all $u, v, w \in \mathcal{A}$. Putting $u = v = \delta(w) - f(w)$ we obtain

$$\|\delta(w) - f(w)\|^3 = \|[(\delta(w) - f(w))\ \ (\delta(w) - f(w))\ \ (\delta(w) - f(w))]\| = 0$$

and so $\delta(w) = f(w)$ for all $w \in \mathcal{A}$. □

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