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Lagrange Inversion: When and How

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Abstract The aim of the present paper is to show how the Lagrange Inversion Formula (LIF) can be applied in a straight-forward way i) to find the generating function of many combinatorial sequences, ii) to extract the coefficients of a formal power series, iii) to compute combinatorial sums, and iv) to perform the inversion of combinatorial identities. Particular forms of the LIF are studied, in order to simplify the computation steps. Some examples are taken from the literature, but their proof is different from the usual, and others are new.

Key words combinatorial sums · generating functions · Lagrange inversion formula · method of coefficients · Riordan arrays.

Mathematics Subject Classifications (2000) 05A10 · 05A15 · 05A19.

1 Introduction

The "method of coefficients" is an elegant technique for proving combinatorial identities, evaluating and inverting sums, essentially due to Egorychev [2] (see also [9]). It mainly consists in manipulating generating functions and their coefficients

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(see e.g., [6]), and we can summarize its most relevant aspects in the following three points which can overlap each other:

1. Given a sequence $(f_n)_{n\in N}$ of combinatorial interest, defined by a formula or by a recurrence relation, **find its generating function** $\mathcal{G}_t\left(f_n\right)_{n\in N} = \mathcal{G}\left(f_n\right) = \sum_{k=0}^{\infty} f_n t^n = f(t)$;
2. Given a formal power series $f(t) = \sum_{k=0}^{\infty} f_k t^k$, **extract the coefficient** of t^n : $[t^n]\ f(t) = f_n$ as an explicit expression;
3. Given a combinatorial identity such as a sum: $b_n = \sum_{k=0}^{n} d_{n,k} a_k$, find the transformation T connecting the two generating functions $b(t) = \mathcal{G}(b_n)$ and $a(t) = \mathcal{G}(a_n)$ and expressing the identity as $b(t) = T(a(t))$. This approach can solve two important problems:
   (a) If the a_k's are known, by extracting the coefficient $[t^n]b(t) = [t^n]T(a(t))$ we have a **closed form for the sum** of the right hand member;
   (b) If the a_k's (and hence $a(t)$) are not known, by **inverting the identity** we have $a(t) = T^{-1}(b(t))$ and hence the solution of the identity, that is, of the infinite system it represents (see, e.g., [12]).

The Lagrange Inversion Formula (LIF) assumes a central role in all these problems, and our aim is to show how its systematic use can produce very elegant and straightforward proofs. We take for granted the Lagrange formula, which we use below in the equivalent forms K6, K6′ and G6. Moreover, we point out (see Theorems 2.1, 2.3 and Corollary 2.2) that many applications of the LIF can be reduced to some particular cases which correspond to the generalized binomial and generalized exponential series in [7].

In general, the LIF is related to the computation of the compositional inverse of a formal power series and usually is given in this context; this approach often allows to solve the problem of coefficient extraction. Since the application of the LIF requires the solution of an equation, explicit forms of generating functions are only obtained when the equation can be actually solved; however, as we will see, in many cases also an implicit form of the generating function can be very useful, especially in finding the closed form of a sum and in inverting identities. Many variants of the LIF are known, and some of them are suited for being used in connection with the "method of coefficients"; Egorychev in [2] gives several different forms, which we have reduced and formalized in K6, K6′ and G6, but surely other approaches can be proposed. In fact, the literature about Lagrange inversion is extensive, and the reader is referred to [6, 8, 18] for proofs and applications, and to Stanley [18, p. 67] for more detailed and historical remarks. The original proof of Lagrange [1] was essentially algebraic, and the first combinatorial proof comes back to Raney [11] in 1960. We will use the LIF in its plain one variable version, but it has been generalized to the multivariate case, for example by Gessel [4], and to q-analogues, as Singer summarizes in [15]. Actually Gessel [3] extents the LIF and the q-LIF to non commutative power series.

In this expository paper, we show a number of examples, some of which are taken from the literature, but are proved in a different way. In particular, in Section 2 we introduce the method of coefficients and prove some theorems related to the Lagrange inversion formula. These theorems are used in Section 3 for finding some important generating functions and in Section 4 for performing coefficient extraction. In the same section we present a technique for transforming a formal power series in such a way that the LIF can become applicable; as far as we know, this approach is new, and allows us to apply the LIF in non-trivial situations. Finally, in Sections 5 and 6 we use Riordan arrays [14, 17] as an important example of generating function transformations; these transformations are used to close or invert combinatorial sums in connection with the formulas of Section 2.

2 The Lagrange Inversion Theorems

It is possible to specify some formal rules, which allow us to deduce, in a simple and "human" way [10], many generating functions and to extract the coefficients from a number of formal power series. For example, coefficient extraction can be performed by means of this set of rules:

$$\begin{array}{ll} K1.\ \textit{linearity} & [t^n]\left(\alpha f(t)+\beta g(t)\right)=\alpha[t^n]f(t)+\beta[t^n]g(t) \\ K2.\ \textit{shifting} & [t^n]tf(t)=[t^{n-1}]f(t) \\ K3.\ \textit{differentiation} & [t^n]f'(t)=(n+1)[t^{n+1}]f(t) \\ K4.\ \textit{convolution} & [t^n]f(t)g(t)=\sum_{k=0}^{n}\left([t^k]f(t)\right)\left([t^{n-k}]g(t)\right) \\ K5.\ \textit{composition} & [t^n]f(g(t))=\sum_{k=0}^{\infty}\left([t^k]f(t)\right)[t^n]g(t)^k \end{array}$$

Properly speaking $[t^n]$ is a functional $[t^n]:\mathcal{F}\to\mathbb{R}$, where $\mathcal{F}$ is the set of formal power series over $\mathbb{R}$ (the field of real numbers, but any field of 0-characteristic could be considered), defined by the rule: $[t^n]t^k=\delta_{n,k}$. This functional is a sort of inverse for the operator $\mathcal{G}$, acting on a sequence of real numbers and producing the formal power series which is the generating function of the sequence; formally, we have $\mathcal{G}\left([t^n]f(t)\right)=f(t)$ and $[t^n]\mathcal{G}\left(f_n\right)=f_n$, whenever the "principle of identity" is applicable: two formal power series $f(t)$ and $g(t)$ are equal if and only if $f_n=g_n$, for every $n\in\mathbb{N}$. In this sense, the rule for $\mathcal{G}$ corresponding to the rules for $[t^n]$ are:

$$\begin{array}{ll} G1.\ \textit{linearity} & \mathcal{G}\left(\alpha f_n+\beta g_n\right)=\alpha\mathcal{G}\left(f_n\right)+\beta\mathcal{G}\left(g_n\right) \\ G2.\ \textit{shifting} & \mathcal{G}\left(f_{n+1}\right)=\left(\mathcal{G}\left(f_n\right)-f_0\right)/t \\ G3.\ \textit{differentiation} & \mathcal{G}\left(nf_n\right)=tD_t\mathcal{G}\left(f_n\right) \\ G4.\ \textit{convolution} & \mathcal{G}\left(\sum_{k=0}^{n}f_kg_{n-k}\right)=\mathcal{G}\left(f_n\right)\mathcal{G}\left(g_n\right) \\ G5.\ \textit{composition} & \sum_{n=0}^{\infty}f_n\mathcal{G}\left(g_n\right)^n=\mathcal{G}\left(f_n\right)\circ\mathcal{G}\left(g_n\right) \end{array}$$

These rules are very powerful, despite their simple appearance and their obvious meaning (see also [7, Table 320]). However, they are not sufficient as soon as the concept of the compositional inverse of a formal power series is introduced. If $f(t)\in\mathcal{F}$, the compositional inverse $\bar{f}(t)\in\mathcal{F}$ is defined by the rule: $f(\bar{f}(t))=\bar{f}(f(t))=t$. It is possible to prove that $f(t)$ has a compositional inverse if and only if $f(0)=0$ and $f'(0)\neq 0$. In order to deal with this situation, the Lagrange Inversion Formula is appropriate.

Let us suppose that a formal power series $w=w(t)$ is implicitly defined by a relation $w=t\phi(w)$, where $\phi(t)$ is a formal power series such that $\phi(0)\neq 0$. The Lagrange Inversion Formula (LIF) states that:

$$[t^n]w(t)^k=\frac{k}{n}[t^{n-k}]\phi(t)^n. \tag{2.1}$$

If we set $f(t)=t/\phi(t)$, we immediately have $f(w(t))=w/\phi(w)=t$, showing that $w(t)$ is the compositional inverse of $f(t)$. Obviously, whenever $\phi(t)$ is "simpler" than $w(t)$, the coefficient extraction in the right hand member may be performed even if the coefficient extraction of the left hand member seems to be impossible. Formula (2.1) can be given the equivalent form:

$$K6.\ \textit{inversion}\ [t^n]F(w(t)) = \tfrac{1}{n}[t^{n-1}]F'(t)\phi(t)^n.$$

Here, $F(t)$ is any formal power series (actually, any formal Laurent series) and we have $w = t\phi(w)$ as before. For $F(t) = t^k$ we find again formula (2.1). The generating function counterpart is the following rule:

$$G6.\ \textit{diagonalization}\ \mathcal{G}\left([t^n]F(t)\phi(t)^n\right) = \left[\left.\frac{F(w)}{1 - t\phi'(w)}\right| w = t\phi(w)\right].$$

The notation $\left[f(w)|w = g(t)\right]$ is a linearization of $f(w)|_{w=g(t)}$ and denotes the substitution of $g(t)$ to every occurrence of w in $f(w)$ (that is, $f(g(t))$). In particular, $w = t\phi(w)$ is to be solved in $w = w(t)$ and w has to be substituted in the expression on the left of the | sign.

Formula K6 requires the evaluation of $F'(t)$; sometimes this derivative is rather complex and straight forward results can be obtained by using another version of the formula, which is proved by using the diagonalization rule G6. If we replace $F(t)$ with $F(t)(1 - \phi'(t)/\phi(t))$ in G6, we immediately get:

$$K6'.\ \textit{inversion}\ [t^n]F(w(t)) = [t^n]F(t)\phi(t)^{n-1}(\phi(t) - t\phi'(t)),$$

a formula involving $\phi'(t)$ instead of $F'(t)$.

The following power series will be used throughout the paper:

- the binomial series: $(1+t)^r = \sum_{k=0}^{\infty} \binom{r}{k} t^k,\ r \in \mathbb{R}$;
- the exponential series: $e^t = \exp(t) = \sum_{k=0}^{\infty} \frac{1}{k!} t^k$.

By rule K5 we have the more general formula (known as *Newton's rule*):

$$(1+\alpha t)^r = \sum_{n=0}^{\infty} \binom{r}{n} \alpha^n t^n \quad \text{or, equivalently,} \quad [t^n](1+\alpha t)^r = \binom{r}{n}\alpha^n.$$

In the sequel we also use the well-known property of binomial coefficients: $\binom{-r}{n} = \binom{r+n-1}{n}(-1)^n$.

Furthermore, some consequences of the diagonalization rule will play an important role in our developments; in fact, let us prove the following theorems (see [7, p. 200-204]).

Theorem 2.1 *Let us consider $a, q \in \mathbb{R}$; then we have the implicit generating functions:*

$$\mathcal{G}\left(\frac{a}{a+qk}\binom{a+qk}{k}\right) = \left[(1+w)^a \middle| w = t(1+w)^q\right]; \tag{2.2}$$

$$\mathcal{G}\left(\binom{a+qk}{k}\right) = \left[\left.\frac{(1+w)^{a+1}}{1-(q-1)w}\right| w = t(1+w)^q\right]. \tag{2.3}$$

Proof For formula (2.2) we find:

$$\mathcal{G}\left(\frac{a}{a+qk}\binom{a+qk}{k}\right) = \mathcal{G}\left(\frac{a+qk-qk}{a+qk}\binom{a+qk}{k}\right)=\mathcal{G}\left(\binom{a+qk}{k}-q\binom{a+qk-1}{k-1}\right)=$$
$$= \mathcal{G}\left([t^k](1+t)^{a+qk} - q[t^{k-1}](1+t)^{a+qk-1}\right) =$$
$$= \mathcal{G}\left([t^k](1-(q-1)t)(1+t)^{a+qk-1}\right)$$

This form is suitable for the LIF when we set $\phi(t) = (1+t)^q$; actually we have:

$$\mathcal{G}\left(\frac{a}{a+qk}\binom{a+qk}{k}\right) = \left[\frac{(1-(q-1)w)(1+w)^{a-1}}{1-tq(1+w)^{q-1}}\,\middle|\, w = t(1+w)^q\right] =$$
$$= \left[(1+w)^a \,\middle|\, w = t(1+w)^q\right]$$

since the $(1-(q-1)w)$ simplifies when we substitute $t = w/(1+w)^q$.

For formula (2.3) we find:

$$\mathcal{G}\left(\binom{a+qk}{k}\right) = \mathcal{G}\left([t^k](1+t)^{a+qk}\right) = \left[\frac{(1+w)^a}{1-tq(1+w)^{q-1}}\,\middle|\, w = t(1+w)^q\right] =$$
$$= \left[\frac{(1+w)^{a+1}}{1-(q-1)w}\,\middle|\, w = t(1+w)^q\right]. \qquad \square$$

When $q = 2$, the equation $w = t(1+w)^q$ can be solved explicitly, and we find a relation with the Catalan numbers. In fact, we have:

$$1 + w = \frac{1-\sqrt{1-4t}}{2t} = C(t)$$

which is the well-known Catalan number generating function. Therefore we can state the following:

Corollary 2.2 *Let us consider $a \in \mathbb{R}$; then we have the explicit generating functions:*

$$\mathcal{G}\left(\frac{a}{a+2k}\binom{a+2k}{k}\right) = C(t)^a; \tag{2.4}$$

$$\mathcal{G}\left(\binom{a+2k}{k}\right) = \frac{C(t)^a}{\sqrt{1-4t}}. \tag{2.5}$$

Analogous results can be obtained from the exponential function:

Theorem 2.3 *Let us consider $a, q \in \mathbb{R}$; then we have the implicit generating functions:*

$$\mathcal{G}\left(\frac{a(a+qk)^{k-1}}{k!}\right) = \left[e^{aw} \,\middle|\, w = te^{qw}\right]; \tag{2.6}$$

$$\mathcal{G}\left(\frac{(a+qk)^k}{k!}\right) = \left[\frac{e^{aw}}{1-qw}\,\middle|\, w = te^{qw}\right]. \tag{2.7}$$

Proof For the first formula we find:

$$\frac{a(a+qk)^{k-1}}{k!} = \frac{(a+qk)^k}{k!} - q\frac{(a+qk)^{k-1}}{(k-1)!} = [t^k]e^{(a+qk)t} - q[t^{k-1}]e^{(a+qk)t} =$$

$$= [t^k](1-qt)e^{at}e^{qkt} = [t^k]\left[\frac{(1-qw)e^{aw}}{1-qwe^{-qw}e^{qw}}\middle| w = te^{qw}\right] =$$

$$= [t^k]\left[e^{aw}\middle| w = te^{qw}\right].$$

For the second formula we have:

$$\frac{(a+qk)^k}{k!} = [t^k]e^{(a+qk)t} = [t^k]\left[\frac{e^{aw}}{1-qwe^{-qw}e^{qw}}\middle| w = te^{qw}\right] = \left[\frac{e^{aw}}{1-qw}\middle| w = te^{qw}\right].$$

□

3 Finding Generating Functions

Rules $G1$–$G5$ constitute a practical basis for finding the generating function of many sequences. For example, let us consider $(a^n)_{n\in N}$ and suppose we do not know anything about the geometric series. If we set $f_n = a^n$, we obviously have $f_{n+1} = af_n$, which is valid for every $n \in \mathbb{N}$ (principle of identity); therefore in terms of generating functions we find: $\mathcal{G}(f_{n+1}) = a\mathcal{G}(f_n)$ (linearity). Let us set $F(t) = \mathcal{G}(f_n)$ for simplicity sake; by the shifting rule we have:

$$\frac{F(t) - f_0}{t} = aF(t) \quad \text{or} \quad F(t) - 1 = atF(t) \quad \text{so that} \quad F(t) = \mathcal{G}\left(a^n\right) = \frac{1}{1-at}.$$

In a similar way, it is immediate to find the generating functions of a great number of combinatorial sequences, as Fibonacci, Catalan and Bell numbers.

As a simple example of applying equation (2.3) in the case $q = 1$, we wish to show that

$$\mathcal{G}\left(\binom{p+k}{m}\right) = \frac{t^{m-p}}{(1-t)^{m+1}} \qquad (m \geq p).$$

In fact by setting $p - m + k = h$ we have:

$$\mathcal{G}\left(\binom{p+k}{m}\right) = \sum_{k=0}^{\infty}\binom{p+k}{p+k-m}t^k = \sum_{h=p-m}^{\infty}\binom{m+h}{h}t^{m-p+h} =$$

$$= t^{m-p}\sum_{h=0}^{\infty}\binom{m+h}{h}t^h = t^{m-p}\left[(1+w)^{m+1}\middle| w = t(1+w)\right] = \frac{t^{m-p}}{(1-t)^{m+1}}.$$

A more complex example is given by the sequence $\left(\binom{2k}{k-r}\right)_{k\in N}$, where $r \in \mathbb{Z}$ is a parameter:

$$\mathcal{G}\left(\binom{2k}{k-r}\right) = \mathcal{G}\left([t^{k-r}](1+t)^{2k}\right) = \mathcal{G}\left([t^k]t^r(1+t)^{2k}\right)$$

Again, we have a sequence defined by diagonalization, and therefore we find

$$\mathcal{G}\left(\binom{2k}{k-r}\right) = \left[\frac{w^r}{1-2t(1+w)}\,\middle|\, w = t(1+w)^2\right].$$

As in the previous case, we substitute for t its expression $w/(1+w)^2$ and find an explicit value for $w = w(t)$. This requires a bit of attention; the second degree equation we obtain is $tw^2 - (1-2t)w + t = 0$, and therefore we can explicitly find an expression for $w = w(t)$:

$$w = \frac{1-2t \pm \sqrt{1-4t}}{2t}.$$

As we observed before, we are interested in the solution $w = w(t)$ such that $w(0) = 0$; this implies that we should choose the sign $-$ in the expression for $w = w(t)$. Now we have: $1 - 2t(1+w) = 1 - (1 - \sqrt{1-4t}) = \sqrt{1-4t}$ and therefore we find the generating function:

$$\mathcal{G}\left(\binom{2k}{k-r}\right) = \frac{1}{\sqrt{1-4t}}\left(\frac{1-2t-\sqrt{1-4t}}{2t}\right)^r. \tag{3.1}$$

For $r = 0$, we obtain the well-known generating function for central binomial coefficients. We can now observe that $(1-2t-\sqrt{1-4t})/(2t) = C(t) - 1 = tC(t)^2$, and therefore, formula (3.1) can be checked by setting $a = 2r$ in formula (2.5); in fact we have:

$$[t^k]\frac{t^r C(t)^{2r}}{\sqrt{1-4t}} = [t^{k-r}]\frac{C(t)^{2r}}{\sqrt{1-4t}} = \binom{2k}{k-r}.$$

In this application of the LIF, an unpleasant aspect is the fact that we have to solve an equation, and this equation can be rather complex. If it is an algebraic equation of first or second degree, as in the two previous examples, there is no problem and the substitution can be performed easily. However, the equation can have any degree, or can be not algebraic, and in that case the LIF can only give us an implicit expression. Notwithstanding this drawback, there can be situations, in which an implicit expression can be useful as an explicit one. Let us consider the following identity:

$$S_n^{[a,q]} = \sum_{k=0}^{n} \frac{a}{a+qk}\binom{a+qk}{k}\binom{qn-qk}{n-k} = \binom{a+qn}{n}.$$

The sum is clearly a convolution and by rule $G4$ we only need to determine the generating functions of the two factors. By Theorem 2.1 we have:

$$F_{a,q}(t) = \mathcal{G}\left(\frac{a}{a+qk}\binom{a+qk}{k}\right) = \left[(1+w)^a \,\middle|\, w = t(1+w)^q\right],$$

$$G_q(t) = \mathcal{G}\left(\binom{qk}{k}\right) = \left[\frac{1+w}{1-(q-1)w}\,\middle|\, w = t(1+w)^q\right].$$

At this point we can perform the convolution, take advantage from the fact that the two substitutions are the same and finally apply equation (2.3) from right to left:

$$S_n^{[a,q]} = [t^n]F_{a,q}(t)G_q(t) = [t^n]\left[\frac{(1+w)^{a+1}}{1-(q-1)w} \,\middle|\, w = t(1+w)^q \right] = \binom{a+qn}{n}.$$

4 Performing Coefficient Extraction

Let us consider identity (3.164) in Gould's collection of combinatorial identities [5]:

$$\begin{aligned}\textstyle\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k/2}{m} &= \frac{2^n}{(-4)^m}\left(\binom{2m-n-1}{m-1} - \binom{2m-n-1}{m}\right) = \\ &= \frac{n2^n}{m(-4)^m}\binom{2m-n-1}{m-n}.\end{aligned}$$

By applying Newton's rule, for the left hand member we have:

$$S = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k/2}{m} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}[t^m]\sqrt{1+t}^k = [t^m](1-\sqrt{1+t})^n.$$

A direct extraction of the coefficient is not feasible, but we can use formula (2.4) if we observe that $1-\sqrt{1+t} = -t/2C(-t/4)$. In fact we have:

$$\begin{aligned}[t^m](1-\sqrt{1+t})^n &= [t^m]\left(-\frac{t}{2}C(-t/4)\right)^n = \left(-\frac{1}{2}\right)^n [t^{m-n}]\,(C(-t/4))^n = \\ &= \left(-\frac{1}{2}\right)^n [t^{m-n}]\sum_{k=0}^{\infty}\frac{n}{n+2k}\binom{n+2k}{k}\left(-\frac{t}{4}\right)^k = \\ &= \left(-\frac{1}{2}\right)^n \frac{n}{2m-n}\binom{2m-n}{m-n}\left(-\frac{1}{4}\right)^{m-n} = \frac{n2^n}{m(-4)^m}\binom{2m-n-1}{m-n}.\end{aligned}$$

It can be instructive, however, to see how the LIF can be applied directly, without referring to the particular case of Theorem 2.1. If we set $w = w(t) = 1-\sqrt{1+t}$, we have $\sqrt{1+t} = 1-w$ or $1+t = 1-2w+w^2$, which is equivalent to $w = t/(w-2)$. This result suggests setting $\phi(w) = 1/(w-2)$, so that:

$$\begin{aligned}S = [t^m]w(t)^n &= \frac{n}{m}[w^{m-n}]\frac{1}{(w-2)^m} = \frac{n}{m2^m}[w^{m-n}]\frac{(-1)^m}{(1-w/2)^m} = \\ &= \frac{n(-1)^m}{m2^m}[w^{m-n}]\frac{1}{(1-w/2)^m} = \frac{n(-1)^m}{m2^m}\binom{-m}{m-n}\frac{(-1)^{m+n}}{2^{m-n}} = \\ &= \frac{n2^n(-1)^m}{m4^m}\binom{2m-n-1}{m-n}.\end{aligned}$$

The previous example is rather simple, because by setting $w = 1-\sqrt{1+t}$ we directly arrive to a manageable form for $\phi(t)$. This is not always the case, as one can easily imagine. Let us consider the following example: $[t^n]\left(1+2t+2\sqrt{t(1+t)}\right)^p$ which arises in the solution of some combinatorial identities (see Section 5). This coefficient can be computed by using equation (2.2) with $a = p$ and $q = 1/2$. In fact, if $w = t(1+w)^{1/2}$ then $w = t(t+\sqrt{4+t^2})/2$ and $1+w = 1+t^2/2+(t/2)\sqrt{4+t^2} = f(t^2/4)$, hence we have:

$$[t^n]f(t^2/4)^p = [t^n](1+w)^p = \sum_{k=0}^{\infty} \frac{p}{p+k/2}\binom{p+k/2}{k}t^k.$$

Finally,

$$[t^n]f(t)^p = [t^n]\sum_{k=0}^{\infty} \frac{p}{p+k/2}\binom{p+k/2}{k}2^k t^{k/2} = \frac{p}{p+n}\binom{p+n}{2n}4^n.$$

Again, this approach requires some clever considerations on the use of equation (2.2). A more complex, but also more direct method is to proceed as in the previous example. By the conditions for LIF applicability, we can exclude to set $w = 1+2t+2\sqrt{t(1+t)}$ since we would have $w(0) \neq 0$. Formally, we can try to apply the LIF if we are able to express $f(t) = 1+2t+2\sqrt{t(1+t)}$ as a *simple* function of w, for $t = w/\phi(w)$. More precisely, the idea is to determine two functions F and ϕ such that $w = t\phi(w)$, $\phi(0) \neq 0$, and $f(t) = F(w(t))$ so that:

$$[t^n]\left(1+2t+2\sqrt{t(1+t)}\right)^p = [t^n]\left[F(w)^p \,\middle|\, w = t\phi(w)\right] = \frac{1}{n}[w^{n-1}]\left(\frac{d}{dw}F(w)^p\right)\phi(w)^n \tag{4.1}$$

Let us formally set $t = w/\phi(w)$ in $f(t)$; we have:

$$F(w) = 1+2\tfrac{w}{\phi(w)}+2\sqrt{\tfrac{w}{\phi(w)}\left(1+\tfrac{w}{\phi(w)}\right)} \quad \text{or}$$
$$\phi(w)F(w)-\phi(w)-2w = 2\sqrt{w\phi(w)+w^2}$$

and simplifying we obtain:

$$\phi(w) = \frac{4wF(w)}{(1-F(w))^2}. \tag{4.2}$$

In order to be able to extract the coefficient (4.1) we would like a simple relation between $F(w)$ and $\phi(w)$, possibly $\phi(w) = F(w)$, $\phi(w) = 1-F(w)$ or something like that. Fortunately, in the right hand side of (4.2) we recognize the factors $F(w)$, $1/(1-F(w))$ and $1/(1-F(w))^2$ and if $\phi(w)$ were one of these factors, we would obtain an equation in the only unknown $F(w)$. In fact, $F(0) = 1$ and since we must have $\phi(w) \neq 0$, the only possibility among the three is to choose $\phi(w) = F(w)$; so we have:

$$1 = \frac{4w}{(1-F(w))^2} \quad \text{or} \quad F(w) = \phi(w) = 1 \pm 2\sqrt{w}.$$

Finally, by considering the plus sign, we obtain:[1]

$$\frac{1}{n}[w^{n-1}]\left(\frac{d}{dw}F(w)^p\right)\phi(w)^n = \frac{1}{n}[w^{n-1}]\left(\frac{d}{dw}(1+2\sqrt{w})^p\right)(1+2\sqrt{w})^n =$$
$$= \frac{p}{n}[w^{n-1}]\frac{(1+2\sqrt{w})^{n+p-1}}{\sqrt{w}}$$
$$= \frac{p}{n}[z^{2n-1}](1+2z)^{n+p-1} =$$
$$= \frac{p}{n}\binom{n+p-1}{2n-1}2^{2n-1} = \frac{p}{p+n}\binom{p+n}{2n}4^n.$$

In 1826 the first issue of the *Journal of reine und angewandte Mathematik*, better known as *Journal de Crelle*, was published. Five of the papers were written by Abel. In one of these papers, he presented the following generalization of the binomial formula, as given by Riordan [12, p. 18], which we prove by Theorem 2.3:

$$A_n = (x+y+an)^n = x\sum_{k=0}^{n}\binom{n}{k}(x+ka)^{k-1}(y+(n-k)a)^{n-k}.$$

This identity can be written:

$$\frac{(x+y+an)^n}{n!} = \sum_{k=0}^{n}\frac{x(x+ka)^{k-1}}{k!}\frac{(y+(n-k)a)^{n-k}}{(n-k)!}$$

and it is clearly the convolution of the generating functions:

$$\mathcal{G}\left(\frac{x(x+ak)^{k-1}}{k!}\right) = \left[e^{xw}\,\middle|\, w = te^{aw}\right], \qquad \mathcal{G}\left(\frac{(y+ak)^k}{k!}\right) = \left[\frac{e^{yw}}{1-aw}\,\middle|\, w = te^{aw}\right].$$

By performing the convolution we obtain:

$$A_n = [t^n]\left[e^{xw}\,\middle|\, w = te^{aw}\right]\left[\frac{e^{yw}}{1-aw}\,\middle|\, w = te^{aw}\right] = \left[\frac{e^{(x+y)w}}{1-aw}\,\middle|\, w = te^{aw}\right]$$

and by applying backwards formula (2.7) we finally find:

$$A_n = \frac{(x+y+an)^n}{n!}.$$

Alternatively, we wish to observe that we can directly apply rule $K6'$:

$$[t^n]\left[\frac{e^{(x+y)w}}{1-aw}\,\middle|\, w = te^{aw}\right] = \frac{e^{(x+y)t}}{1-at}\left(e^{at}\right)^{n-1}(e^{at} - ate^{at}) =$$
$$= [t^n]\frac{e^{(x+y)t}}{1-at}e^{atn}(1-at) = \frac{(x+y+an)^n}{n!}.$$

[1] In this case, as in many others, we consider formal power series in the *indeterminate* $\sqrt{t}$, or $t^{1/2}$, such as $f(\sqrt{t}) = \sum_{k=0}^{\infty} f_k(\sqrt{t})^k$. Obviously, the main property of $\sqrt{t}$ is that $(\sqrt{t})^2 = t$. Since no convergence problem is considered, this is a legitimate procedure. In general, other indeterminates can be considered such as $\sqrt[3]{t}$, $\sqrt[4]{t}$ and even $\ln(t)$.

5 Computing Combinatorial Sums

The rules of multiplication, differentiation, shifting and convolution are just examples of *transformations* of generating functions. As we shall see, combinatorial identities (among which an important role is reserved to sum inversions) are closely related to generating function transformations and to Lagrange inversion formula. In our opinion, the most important examples of transformations are induced by the concept of Riordan arrays, as introduced by Shapiro et al. [14] and used by Sprugnoli [17].

A (proper) Riordan array is a pair of formal power series $(d(t), h(t))$, $d(0) \neq 0$, $h(0) = 0, h'(0) \neq 0$, which defines an infinite, lower triangular array $d_{n,k} = [t^n]d(t)h(t)^k$. The simplest example is the Pascal triangle $P = (1/(1-t), t/(1-t))$ whose generic element is $d_{n,k} = \binom{n}{k}$. However, other important triangles, as Catalan, Motzkin and Schröder, are Riordan arrays and the Stirling numbers of both kinds are related to two Riordan arrays. As we shall see, a great variety of binomial coefficients constitute Riordan arrays and this will allow us to define important transformations and solve a large number of combinatorial identities and sum inversions. Obviously, Riordan arrays are not a panacea, and just to give two negative examples, we can quote Narayana and Eulerian triangles. Our reasoning starts from the following result: if $(d(t), h(t))$ is a Riordan array as above and $(f_k)_{k\in N}$ is a sequence of numbers with generating function $\mathcal{G}(f_k) = f(t)$, we have:

$$\sum_{k=0}^{n} d_{n,k} f_k = [t^n]d(t) f(h(t)) \tag{5.1}$$

In this sense, the Riordan array $(d(t), h(t))$ defines a transformation of the generating function $f(t)$. For example, by using the Pascal triangle we find:

$$\sum_{k=0}^{n} \binom{n}{k} f_k = [t^n]\frac{1}{1-t} f\left(\frac{t}{1-t}\right)$$

and this is called *Euler transformation* (see, Riordan [12]).

Let us consider a simple, but meaningful example; the relation

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^j = \left\{ {j \atop n} \right\} (-1)^n n!$$

is known with the pompous name of Govindarajulu and Suzuki identity in Gould's collection [5, (1.13)] and in Egorichev [2, p. 186]. Since k^j is a polynomial in k of degree j, Newton's rule for the differences of a polynomial immediately implies:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^j = 0, \quad \text{for } j < n \quad \text{and} \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^n = (-1)^n n!.$$

What happens when $j > n$? Flajolet and Sedgewick [13, p. 429] consider the generating function of Stirling numbers of the second kind $(e^t - 1)^j$ and by expanding the power they find the answer. Both these approaches require some sort of ingenuity or, if one prefers, they are *ad hoc* solutions. So we wonder if a more straightforward strategy exists to obtain the final result. The strategy should be constructive and pertain to combinatorial algebra. Actually, the theory of generating functions (and Riordan arrays) gives a simple and general solution: we apply the Euler transformation to the generating function (see [7, 7.46]):

$$\mathcal{G}\{(-1)^k k^j\} = \sum_{k=0}^{j} \left\{ {j \atop k} \right\} \frac{k!(-t)^k}{(1+t)^{k+1}};$$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k k^j = [t^n] \frac{1}{1-t} \sum_{k=0}^{j} \left\{ {j \atop k} \right\} \frac{k!(-t)^k}{(1-t)^k} (1-t)^{k+1} =$$

$$= [t^n] \sum_{k=0}^{j} \left\{ {j \atop k} \right\} k!(-1)^k t^k = \left\{ {j \atop n} \right\} (-1)^n n!,$$

since, as we know, $[t^n]t^k = \delta_{n,k}$ by definition.

In [17], Sprugnoli shows how the concept of Riordan arrays can be used to prove combinatorial identities involving binomial coefficients and Stirling numbers of both kinds. In particular, he proves the following two transformations:

$$\sum_{k=0}^{n} \binom{n+ak}{m+bk} f_k = [t^n] \frac{t^m}{(1-t)^{m+1}} f\left(\frac{t^{b-a}}{(1-t)^b}\right), \qquad b > a \tag{5.2}$$

$$\sum_{k=0}^{m} \binom{n+ak}{m+bk} f_k = [t^m](1+t)^n f\left(t^{-b}(1+t)^a\right), \qquad b < 0 \tag{5.3}$$

provided $m + bk \geq 0$ in both equations and $n + ak \geq 0$ in (5.2).

The previous examples on the Euler transformation is indicative, but let us give some other examples. We take them from Gould's collection [5], where more than 90% of identities in Chapters 1 and 3 can be proved by the two transformations above (see Sprugnoli [16]). Identity (3.26) is:

$$S = \sum_{k=0}^{n} \binom{2x}{2k} \binom{x-k}{n-k} = \frac{x}{x+n} \binom{x+n}{2n} 4^n.$$

The generating function of the first factor $\binom{2x}{2k}$ can be found by applying the bisection to $(1+t)^{2x}$; this generating function is then transformed according to the second factor and to rule (5.3) above. Thus we have:

$$S = \sum_{k=0}^{n} \binom{x-k}{n-k} \binom{2x}{2k} = [t^n](1+t)^x \left[\frac{(1+\sqrt{u})^{2x} + (1-\sqrt{u})^{2x}}{2} \,\middle|\, u = \frac{t}{1+t} \right] =$$

$$= [t^n] \frac{\left(1+2t+2\sqrt{t(1+t)}\right)^x + \left(1+2t-2\sqrt{t(1+t)}\right)^x}{2}.$$

We start with $[t^n]\left(1+2t+2\sqrt{t(1+t)}\right)^x$, calculated in the previous section. It is easily seen that $\left(1+2t-2\sqrt{t(1+t)}\right)^x$ gives an analogous result, so that we obtain the desired formula using linearity.

The identity (3.144) of Gould's collection is ascribed to Jensen (1902), and is an example of a non-closed formula:

$$S = \sum_{k=0}^{n} \binom{y-kz}{n-k} \binom{x+kz}{k} = \sum_{k=0}^{n} \binom{x+y-k}{n-k} z^k.$$

In this case, we can prove the identity by showing that the two members are just the same coefficient of the same generating function. We apply formula (5.3) to the generating function of $S_k^{[x,z]} = \binom{x+kz}{k}$ (see formula (2.3)):

$$\sum_{k=0}^{n} \binom{y-kz}{n-k} \binom{x+kz}{k} = [t^n](1+t)^y \left[\left[\frac{(1+w)^{x+1}}{1-(z-1)w} \,\middle|\, w = u(1+w)^z \right] \,\middle|\, u = \frac{t}{(1+t)^z} \right]$$

where we have a substitution in a substitution, which however simplifies to $w = t$. We therefore conclude:

$$S = [t^n] \frac{(1+t)^{x+y+1}}{1-(z-1)t}.$$

On the other hand, for the right hand member we apply rule (5.3) again and obtain:

$$\sum_{k=0}^{n} \binom{x+y-k}{n-k} z^k = [t^n](1+t)^{x+y} \left[\frac{1}{1-zu} \,\middle|\, u = \frac{t}{1+t} \right] = [t^n] \frac{(1+t)^{x+y+1}}{1-(z-1)t}$$

and this proves that the two sums are equal, as desired.

Identity (3.110) in Gould's collection is rather unusual in its conclusion:

$$S = \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-k}{n} \frac{k}{2^k(2n-k)} = (-4)^n \binom{-1/4}{n}.$$

We begin by rewriting the left hand member in order to eliminate $k/(2n-k)$:

$$S = \sum_{k=0}^{n} \left(2\binom{2n-k-1}{n-1} - \binom{2n-k}{n} \right) \binom{2k}{k} \frac{1}{2^k}.$$

The generating function of $\binom{2k}{k}/2^k$ is obviously $1/\sqrt{1-2t}$ and therefore we can apply the transformation (5.2) above:

$$S = \left(2[t^{2n-1}] \frac{t^{n-1}}{(1-t)^n} - [t^{2n}] \frac{t^n}{(1-t)^{n+1}} \right) \left[\frac{1}{\sqrt{1-2u}} \,\middle|\, u = t \right].$$

The substitution is rather easy, and we have:

$$S = [t^n] \frac{1-2t}{(1-t)^{n+1}} \cdot \frac{1}{\sqrt{1-2t}} = [t^n] \frac{\sqrt{1-2t}}{(1-t)^{n+1}}.$$

Here we apply the diagonalization with $\phi(t) = 1/(1-t)$, i.e., $1 - t\phi'(w) = (1-2w)/(1-w)$:

$$S = [t^n]\left[\frac{\sqrt{1-2w}}{1-w}\frac{1-w}{1-2w}\,\bigg|\, w = \frac{t}{1-w}\right] = [t^n]\left[\frac{1}{\sqrt{1-2w}}\,\bigg|\, w = \frac{1-\sqrt{1-4t}}{2}\right] =$$

$$= [t^n]\frac{1}{\sqrt[4]{1-4t}} = [t^n](1-4t)^{-1/4} = \binom{-1/4}{n}(-4)^n.$$

We conclude by showing how Riordan arrays and generating functions allow us to solve problems related to Abel identities. Let us consider identity (1.118) in [5], but the proof of the others can follow the same lines:[2]

$$S = \sum_{k=0}^{n}\binom{n}{k}\frac{(yk)^{n-k}}{k}(x-yk)^k = \frac{x^n}{n}.$$

Actually, the sum can be rewritten in the following form:

$$S = n!\sum_{k=0}^{n}\frac{(yk)^{n-k}}{k(n-k)!}\cdot\frac{(x-yk)^k}{k!}.$$

We observe that the first factor is related to the Riordan array $(1+yt, te^{yt})$:

$$\frac{(yk)^{n-k}}{k(n-k)!} = y\cdot\frac{(yk)^{n-k-1}}{(n-k)!} = \frac{1}{n}\left(\frac{(yk)^{n-k}}{(n-k)!} + y\cdot\frac{(yk)^{n-k-1}}{(n-k-1)!}\right) = \frac{1}{n}[t^{n-k}](1+yt)e^{ykt}.$$

The generating function of the second term corresponds to formula (2.7) with $a = x$ and $q = -y$:

$$\mathcal{G}\left(\frac{(x-yk)^k}{k!}\right) = \left[\frac{e^{xw}}{1+yw}\,\bigg|\, w = te^{-yw}\right].$$

Therefore, Abel identity reduces to a Riordan array transformation:

$$S = \frac{n!}{n}(1+yt)[t^n]\left[\left[\frac{e^{xw}}{1+yw}\,\bigg|\, w = ue^{-yw}\right]\bigg|\, u = te^{yt}\right] = \frac{n!}{n}[t^n](1+yt)\cdot\frac{e^{xt}}{1+yt} = \frac{x^n}{n};$$

in fact, we observe that $w = te^{yt}e^{-yw}$ implies $w = t$.

6 Inverting Combinatorial Identities

The use of Riordan arrays in proving identities related to sum inversion can be illustrated by a simple application involving the Pascal triangle. Let us suppose we have an identity:

$$b_n = \sum_{k=0}^{n}\binom{n}{k}a_k \tag{6.1}$$

[2] As observed by a referee, the "y" in the formula is redundant; however, this is the form given in [5].

and wish to invert it, that is to find an expression relating a_n to the b_k's; in other words, if the sequence $(b_k)_{k\in N}$ is given and the sequence $(a_k)_{k\in N}$ is unknown, we wish to solve the infinite system (6.1). By Euler transformation we know:

$$b_n = [t^n]\frac{1}{1-t}\,a\left(\frac{t}{1-t}\right) \quad \text{or} \quad b(t) = \frac{1}{1-t}\,a\left(\frac{t}{1-t}\right).$$

If we set $y = t/(1-t)$, we can substitute $t = y/(1+y)$ in:

$$(1-t)b(t) = a\left(\frac{t}{1-t}\right) \quad \text{obtaining} \quad a(y) = \frac{1}{1+y}\,b\left(\frac{y}{1+y}\right).$$

Here we recognize the Riordan array $(1/(1+y), y/(1+y))$ with elements $\binom{n}{k}(-1)^{n-k}$, and therefore we obtain the inverse relation:

$$a_n = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}b_k.$$

This is a classical example, but we can go on to more sophisticated cases. Suppose we have:

$$a_n = \sum_k\left(\binom{rn+p}{k} - (qr-1)\binom{rn+p}{k-1}\right)b_{n-qk}$$

and we wish to find the inverse sum, relating the b_n's to the a_k's. Since $b_{n-qk} = [t^{n-qk}]b(t) = [t^n]b(t)t^{qk}$, we can imagine that b_{n-qk} be the generic element of the Riordan array $(b(t), t^q)$ and use it as a transformation. Since the generating functions for the two terms in the right hand member are obvious, we find:

$$\begin{aligned} a_n &= [t^n]b(t)(1+t^q)^{rn+p} - (qr-1)[t^n]b(t)t^q(1+t^q)^{rn+p} = \\ &= [t^n]b(t)(1-(qr-1)t^q)(1+t^q)^{rn+p}. \end{aligned}$$

We wish to find a relation between generating functions, and because of the exponent n in the right hand member, we should apply diagonalization:

$$a(t) = \left[\frac{b(w)(1-qrw^q+w^q)(1+w^q)^{p+1}}{1-qrw^q+w^q}\;\middle|\; w = t(1+w^q)^r\right];$$

here we computed: $1-t\phi'(w) = (1-qrw^q+w^q)/(1+w^q)$. The expression simplifies:

$$a(t) = \left[b(w)(1+w^q)^{p+1}\;\middle|\; w = t(1+w^q)^r\right]$$

and we can invert, since we know that $t = w/(1+w^q)^r$:

$$b(w) = \frac{1}{(1+w^q)^{p+1}}\,a\left(\frac{w}{(1+w^q)^r}\right).$$

The indeterminate t disappears and we recognize the transformation related to the Riordan array: $((1+w^q)^{-p-1}, w(1+w^q)^{-r})$ whose generic element is:

$$\begin{aligned} d_{n,k} &= [w^{n-k}]\frac{1}{(1+w^q)^{p+1+rk}} = \binom{-p-1-rk}{(n-k)/q} = \\ &= \binom{(pq+n+(qr-1)k)/q}{(n-k)/q}(-1)^{(n-k)/q}. \end{aligned}$$

Therefore, we conclude by setting $n-k=qh$, since a binomial coefficient is zero when its "denominator" is not an integer:

$$b_n = \sum_h \binom{p+rn-(rq-1)h}{h}(-1)^h a_{n-qh}.$$

When $q=r=1$ we find the inverse relations:

$$a_n = \sum_k \binom{p+n}{k} b_{n-k} \quad \Leftrightarrow \quad b_n = \sum_k \binom{p+n}{k}(-1)^k a_{n-k},$$

a variant of (6.1). When $q=1$ and $r=2$ we have:

$$a_n = \sum_k \left(\binom{2n+p}{k} - \binom{2n+p}{k-1}\right) b_{n-k} \quad \Leftrightarrow \quad b_n = \sum_k \binom{2n+p-k}{k}(-1)^k a_{n-k}.$$

When $q=2$ and $r=1$ we obtain the inverse relations:

$$a_n = \sum_k \left(\binom{n+p}{k} - \binom{n+p}{k-1}\right) b_{n-2k} \quad \Leftrightarrow \quad b_n = \sum_k \binom{n+p-k}{k}(-1)^k a_{n-2k}.$$

When $q=r=2$ we find inversion (2.5.6) in Riordan's book [12].

We conclude with an inversion of Abel's type, which is better proved by using exponential generating functions: by $\widehat{a}(t)$ we denote the formal power series $\sum_k a_k t^k/k!$, the ordinary generating function of the sequence $(a_k/k!)_{k\in N}$. Let us suppose we are given the sum:

$$a_n = \sum_k \binom{n}{k}(x+k)^{n-k} b_k \qquad \text{that is} \qquad \frac{a_n}{n!} = \sum_k \frac{(x+k)^{n-k}}{(n-k)!} \cdot \frac{b_k}{k!}$$

where the first factor in the right hand member is the generic element of the Riordan array $(e^{xt},\ te^t)$. In fact we have:

$$d_{n,k} = [t^{n-k}]e^{xt}e^{kt} = [t^{n-k}]e^{(x+k)t} = \frac{(x+k)^{n-k}}{(n-k)!}.$$

Therefore, we find $\widehat{a}(t) = e^{xt}\widehat{b}(te^t)$ or $\widehat{b}(te^t) = e^{-xt}\widehat{a}(t)$. By setting $y = te^t$ or $t = ye^{-t}$, we have $\widehat{b}(y) = \left[e^{-xt}\widehat{a}(t) \;\middle|\; t = ye^{-t}\right]$. By using Theorem $K6'$ we have:

$$\begin{aligned}\frac{b_n}{n!} &= [t^n]e^{-xt}\widehat{a}(t)e^{-(n-1)t}\left(e^{-t}+te^{-t}\right) = [t^n](1+t)e^{-(x+n)t}\widehat{a}(t) = \\ &= \sum_k \frac{(-1)^{n-k}(x+n)^{n-k}}{(n-k)!} \cdot \frac{a_k}{k!} + \sum_k \frac{(-1)^{n-k-1}(x+n)^{n-k-1}}{(n-k-1)!} \cdot \frac{a_k}{k!} = \\ &= \sum_k (-1)^{n-k}\frac{(x+n)^{n-k-1}}{(n-k)!} \cdot (x+k) \cdot \frac{a_k}{k!}.\end{aligned}$$

Finally, by multiplying everything by $n!$:

$$b_n = \sum_k \binom{n}{k} (-1)^{n-k} (x+n)^{n-k-1} (x+k) a_k.$$

This is inversion (3.1.3) in Riordan's book [12].

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