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Characters of Fundamental Representations of Quantum Affine Algebras

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Abstract We give closed formulae for the q-characters of the fundamental representations of the quantum loop algebra of a classical Lie algebra, in terms of a family of partitions satisfying some simple properties. We also give the multiplicities of the eigenvalues of the imaginary subalgebra in terms of these partitions.

Key words q-characters · quantum affine algebra · braid group action

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1. Introduction

In this paper we study the q-characters of the fundamental finite-dimensional representations of the quantum loop algebra $\mathbf{U}_q$ associated to a classical simple Lie algebra. The notion of q-characters defined in [7] is analogous to the usual notion of a character of a finite-dimensional representation of a simple Lie algebra. These characters and their generalizations have been studied extensively [6, 9, 11] using combinatorial and geometric methods. A more representation theoretic approach was developed in [4]. In particular, that paper approached the problem of studying whether the q-characters admitted a Weyl group invariance which was analogous to the invariance of characters of finite-dimensional representations of simple Lie algebras. In the quantum case, it is reasonable to expect that the Weyl group be replaced by the braid group, [2, 3, 7] but it is easy to see that this is false even for sl_2. However, it was shown in [4] that in a suitably modified way, the q-characters of the fundamental representations of the quantum loop algebra of a classical Lie algebra do admit an invariance under the braid group action. It was also shown that the q-character of such representations could then be calculated in a certain inductive way.

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In this paper, we use that inductive method to give closed formulas for the q-characters of all the fundamental representations of the quantum loop algebras of a classical simple Lie algebra. To describe the results a bit further, recall that the quantum loop algebra admits a commutative subalgebra $\mathbf{U}_q(0)$ corresponding to the imaginary root vectors. Any finite-dimensional representation V of the quantum loop algebra, breaks up as a direct sum of generalized eigenspaces for the action of $\mathbf{U}_q(0)$. These are called the ℓ-weight spaces and the eigenvalues corresponding to the non-zero eigenspaces are called the ℓ-weights of the representation. The ℓ-weights lie in a free abelian multiplicative group $\mathcal{P}_q$. Let $\mathbf{Z}[\mathcal{P}_q]$ be the integral group ring over $\mathcal{P}_q$ and for $\varpi \in \mathcal{P}_q$, let V_ϖ be the corresponding eigenspace of V. The element of $\mathrm{Z}[\mathcal{P}_q]$ defined by,

$$\mathrm{ch}_\ell(V) = \sum_{\varpi\in\mathcal{P}_q} \dim(V_\varpi)\; e(\varpi),$$

is called the q-character of V. If P is the usual weight lattice of the simple Lie algebra, then it was shown in [4] that there exists a canonical group homomorphism $\mathrm{wt} : \mathcal{P}_q \to \mathcal{P}$.

Assume now that V is a fundamental representation of the quantum loop algebra. Roughly speaking, this means that V corresponds to a canonical generator of $\mathcal{P}_q$. It was shown in [4] that the problem of determining the ℓ-weights of V is reduced to determining V_ϖ where $\mathrm{wt}(\varpi)$ is in the dominant chamber P^+ of P. Assume from now on that $\mathrm{wt}(\varpi) \in P^+$. We give explicit formulas for ϖ with $\mathrm{V}_\varpi \neq 0$. In the case of B_n, C_n, we see as a consequence that $\dim V_\varpi = 1$ (this was proved by different methods in [9]). In the case of D_n it can happen that $\dim(V_\varpi) > 1$ and we compute this dimension in Section 5. The idea is to show that every ℓ-weight ϖ comes from a partition $\mathbf{j}$ with certain properties and we find that $\dim V_\varpi = 2^{M_\mathbf{j}}$ where $M_\mathbf{j}$ is defined in a canonical way in Section 5.

2. Preliminaries

2.1. Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra of rank n and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Set $I = \{1, 2, \cdots, n\}$ and let $\{\alpha_i : i \in I\}$ (resp. $\{\omega_i : i \in I\}$) be the set of simple roots (resp. fundamental weights) of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let also $\check{\alpha}_i$ denote the simple co-roots. As usual, Q, (resp. P) denotes the root (resp. weight) lattice of $\mathfrak{g}$, $Q^+ = \sum_{i=1}^n \mathbf{N}\alpha_i$, and $P^+ = \sum_{i=1}^n \mathbf{N}\omega_i$. Let W be the Weyl group of $\mathfrak{g}$ generated by simple reflections $\{s_i : i \in I\}$. For $w \in W$, let $\ell(w)$ denote the length of a reduced expression for w. Given $\lambda = \sum_{i\in I} \lambda_{i\omega_i} \in P^+$ let $W(\lambda)$ be the subgroup of W generated by $\{s_i : i \in I,\ \lambda_i = 0\}$ and let W_λ be the set of left coset representatives of $W/W(\lambda)$ of minimal length. The braid group $\mathcal{B}$ associated to $\mathfrak{g}$ is generated by elements T_i, $i \in I$ and relations

$$\begin{aligned} T_iT_j &= T_jT_i, \quad \text{if} \quad a_{ij} = 0,\\ T_iT_jT_i &= T_jT_iT_j, \quad \text{if} \quad a_{ij}a_{ji} = 1,\\ (T_iT_j)^2 &= (T_jT_i)^2, \quad \text{if} \quad a_{ij}a_{ji} = 2,\\ (T_iT_j)^3 &= (T_jT_i)^3, \quad \text{if} \quad a_{ij}a_{ji} = 3, \end{aligned}$$

where $i, j \in \{1, 2, \cdots, n\}$ and $A = (a_{ij})\ 1 \le i, j \le n$ is the Cartan matrix of $\mathfrak{g}$. For $i \in I$, fix integers $d_i \in \mathbf{N}$ minimal such that $d_ia_{ij} = d_ja_{ji}$ for all $j \in I$. Given $w \in W$ and a reduced expression $w = s_{i_1} \cdots s_{i_k}$ let $T_w = T_{i_1} \cdots T_{i_k}$ be the corresponding element of $\mathcal{B}$. It is well-known that T_w is independent of the choice of the reduced expression.

2.2. Let $q \in \mathbf{C}^\times$ and assume that q is not a root of unity. For $r, m \in \mathbf{N}$, $m \geq r$, define complex numbers,

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}, \qquad [m]_q! = [m]_q[m-1]_q \ldots [2]_q[1]_q, \qquad \begin{bmatrix} m \\ r \end{bmatrix}_q = \frac{[m]_q!}{[r]_q![m-r]_q!}.$$

Set $q_i = q^{d_i}$ and $[m]_i = [m]_{q_i}$.

Let $\mathcal{P}_q$ be the (multiplicative) subgroup of $\mathbf{C}(u)^n$ generated by the elements, $\omega_{i,a}$, $i \in I$, a $\in \mathbf{C}^\times$, where $\omega_{i,a}$ is the n-tuple of elements in $\mathbf{C}(u)$ whose i^{th} entry is $1 - au$ and all other entries 1. The elements $\omega_{i,a}$ are called ℓ-fundamental weights. It is obvious that $\mathcal{P}_q$ is generated freely as an abelian group by the fundamental ℓ-weights. $\mathcal{P}_q$ is called the ℓ-weight lattice. Given any element $\varpi \in \mathcal{P}_q$ and $1 \leq j \leq n$, let ϖ_j be the j^{th} entry of ϖ.

Let $\mathcal{P}_q^+$ be the monoid generated by 1 and the elements $\omega_{i,a}$, $i \in I$, $a \in \mathbf{C}^\times$, clearly $\mathcal{P}_q^+$ consists of n-tuples of polynomials with constant term one and an element of $\mathcal{P}_q^+$ is called an ℓ-dominant weight. Let $\mathrm{wt} : \mathcal{P}_q \to P$ be the group homomorphism defined by extending, $\mathrm{wt}(\omega_{i,a}) = \omega_i$.

The group $\mathcal{B}$ acts on on $\mathcal{P}_q$ as follows [2, 3, 7]: For $i \in I$ and $\varpi = (\varpi_1, \cdots, \varpi_n) \in \mathcal{P}_q$, we have

$$\begin{aligned}
&(T_i\varpi)_j = \varpi_j, \quad \text{if } a_{ji} = 0,\\
&(T_i\varpi)_j = \varpi_j(u)\varpi_i(q_iu), \quad \text{if } a_{ji} = -1,\\
&(T_i\varpi)_j = \varpi_j(u)\varpi_i(q^3u)\varpi_i(qu), \quad \text{if } a_{ji} = -2,\\
&(T_i\varpi)_j = \varpi_j(u)\varpi_i(q^5u)\varpi_i(q^3u)\varpi_i(qu), \quad \text{if } a_{ji} = -3,\\
&(T_i\varpi)_i = \frac{1}{\varpi_i(q_i^2u)}.
\end{aligned}$$

For $i \in I$, set

$$\alpha_{i,a} = (T_i(\omega_{i,a}))^{-1}\omega_{i,a},$$

and let $\mathcal{Q}_\mathrm{q}$ be the subgroup of $\mathcal{P}_q$ generated by the $\alpha_{i,a}$. Let $\mathcal{Q}_q^+$ the monoid generated by 1 and $\alpha_{i,a}$, $i \in I$, $a \in \mathbf{C}^\times$, and $\mathcal{Q}_q^- = (\mathcal{Q}_q^+)^{-1}$.

2.3. The quantum loop algebra $\mathbf{U}_q$ of $\mathfrak{g}$ is the algebra with generators $x_{i,r}^\pm$ ($i \in I$, $r \in \mathbf{Z}$), $K_i^{\pm 1}$ ($i \in I$), $h_{i,r}$ ($i \in I$, $r \in \mathbf{Z}\backslash\{0\}$) and the following defining relations:

$$\begin{aligned}
K_iK_i^{-1} &= K_i^{-1}K_i = 1, \quad K_iK_j = K_jK_i,\\
K_ih_{j,r} &= h_{j,r}K_i,\\
K_ix_{j,r}^\pm K_i^{-1} &= q_i^{\pm a_{ij}}x_{j,r}^\pm,\\
[h_{i,r}, h_{j,s}] &= 0, \quad [h_{i,r}, x_{j,s}^\pm] = \pm\tfrac{1}{r}[ra_{ij}]_{q^i}x_{j,r+s}^\pm,\\
x_{i,r+1}^\pm x_{j,s}^\pm - q_i^{\pm a_{ij}}x_{j,s}^\pm x_{i,r+1}^\pm &= q_i^{\pm a_{ij}}x_{i,r}^\pm x_{j,s+1}^\pm - x_{j,s+1}^\pm x_{i,r}^\pm,\\
[x_{i,r}^+, x_{j,s}^-] &= \delta_{i,j}\tfrac{\psi_{i,r+s}^+ - \psi_{i,r+s}^-}{q_i - q_i^{-1}},
\end{aligned}$$

$$\sum_{\pi \in \Sigma_m}\sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_i x_{i,r_{\pi(1)}}^\pm \ldots x_{i,r_{\pi(k)}}^\pm x_{j,s}^\pm x_{i,r_{\pi(k+1)}}^\pm \ldots x_{i,r_{\pi(m)}}^\pm = 0, \quad \text{if } i \neq j,$$

for all sequences of integers $r_1, \ldots, r_m$, where $m = 1 - a_{ij}$, Σ_{m} is the symmetric group on m letters, and the $\psi^{\pm}_{i,r}$ are determined by equating powers of u in the formal power series

$$\sum_{r=0}^{\infty} \psi^{\pm}_{i,\pm r} u^{\pm r} = K_i^{\pm 1} \exp\left(\pm (q_i - q_i^{-1}) \sum_{s=1}^{\infty} h_{i,\pm s} u^{\pm s} \right).$$

Let $\mathbf{U}_q(\mathfrak{g})$ be the subalgebra of $\mathbf{U}_q$ generated by the elements $x^{\pm}_{i,0}$, $K_i^{\pm 1}$ for $1 \leq i \leq n$.

2.4. For $i \in I$, set

$$h^{\pm}_i(u) = \sum_{k=1}^{\infty} \frac{q^{\pm k} h_{i,\pm k}}{[k]_i} u^k,$$

and define elements $P_{i,\pm k}$, $i \in I$, $k \in \mathbf{Z}$, $k \geq 0$, by the generating series,

$$P^{\pm}_i(u) = \sum_{k=0}^{\infty} P_{i,\pm k} u^k = \exp(-h^{\pm}_i(u)). \tag{1.1}$$

Let $\mathbf{U}^{\pm}_q(0)$ be the subalgebra of $\mathbf{U}_q$ generated by the elements $h_{i,\pm k}$ $i \in I$, $k \in \mathbf{Z}$, $k > 0$, or equivalently, the subalgebra generated by the elements $P_{i,\pm k}$, $i \in I$, $k \in \mathbf{Z}$, $k > 0$, and let $\mathbf{U}_q(0)$ be the subalgebra generated by $\mathbf{U}^{\pm}_q(0)$. An element $\varpi = (\varpi_1, \cdots, \varpi_{\mathrm{n}}) \in \mathcal{P}_q$ can be regarded as an element of $\mathrm{Hom}(\mathbf{U}_q(0), \mathbf{C})$ by extending the assignment,

$$\varpi(P^{\pm}_i(u)) = \varpi^{\pm}_i(u),$$

where $\varpi^{+}_i(u) = \varpi_i$, $\varpi^{-}_i = u^{\deg \varpi_{\mathrm{i}}} \varpi_i(u^{-1}) / (u^{\deg \varpi_{\mathrm{i}}} \varpi_i(u^{-1}))|_{u=0}$.

2.5. Given a $\mathbf{U}_q$-module V and $\mu = \sum_i \mu_i \omega_i \in P$, set

$$V_\mu = \{ v \in V : K_i.v = q_i^{\mu_i} v, \quad \forall\, i \in I \}.$$

We say that V is a module of type 1 if

$$V = \bigoplus_{\mu \in P} V_\mu.$$

Set

$$\mathrm{wt}(V) = \{ \mu \in P : V_\mu \neq 0 \},$$

and given $v \in V_\mu$ set $\mathrm{wt}(v) = \mu$. An element $\varpi \in \mathcal{P}_q$ is an ℓ-weight of V if there exists a non-zero element $v \in V$ such that

$$(P_{i,\pm r} - (\varpi^{\pm}_i)_r)^N v = 0, \quad N \equiv N(i, r, v) \in \mathbf{Z}^+,$$

for all $i \in I$ and $r \in \mathbf{Z}^+$ and v is called an ℓ-weight vector in V with ℓ-weight ϖ. Let V_ϖ be the subspace of V spanned by ℓ-weight vectors with ℓ-weight ϖ. If V is a finite-dimensional $\mathbf{U}_q$-module, then,

$$V = \bigoplus_{\varpi \in \mathcal{P}_q} V_\varpi, \qquad V_\mu = \bigoplus_{\varpi \in \mathcal{P}_q} V_\varpi \cap V_\mu.$$

Denote by $\mathrm{wt}_\ell(V)$ the set of ℓ-weights of V and define $\mathrm{wt}_\ell(v)$ in the obvious way.

2.6. Let $\mathcal{C}_q$ be the category of finite-dimensional $\mathbf{U}_q$-modules of type 1. A module $V \in \mathcal{C}_q$ is ℓ-highest weight with ℓ-highest weight $\varpi \in \mathcal{P}_q$ if there exists a non-zero vector $0 \neq v \in V$ such that $V = \mathbf{U}_q v$ and,

$$x_{i,r}^+ v = 0, \quad P_i^\pm(u)v = (\varpi)_i^\pm v, \quad K_i^{\pm 1} v = q^{\pm \mathrm{wt}\varpi(\check{\alpha}_i)} v, \quad (x_{i,r}^-)^{\mathrm{wt}\varpi(\check{\alpha}_i)+1} v = 0, \tag{1.2}$$

for all $i \in I$, $r \in \mathbf{Z}$. The element v is called the ℓ-highest weight vector.

Any ℓ-highest weight module has a unique irreducible quotient which is also a highest weight module with the same highest weight. There exists a bijective correspondence between elements of $\mathcal{P}_q^+$ and isomorphism classes of irreducible finite-dimensional modules, [5]. Given $\omega \in \mathcal{P}_q^+$, let $V(\omega) \in \mathcal{C}_q$ be an element in the corresponding isomorphism class, and let v_ω be the ℓ-highest weight vector. Then, $V(\omega)_{\mathrm{wt}\,\omega} = \mathbf{C} v_\omega$.

2.7. From now on we suppose $\mathfrak{g}$ is of classical type, i.e., $\mathfrak{g}$ is of type A_n, B_n, C_n or D_n. The following result was proved in [4].

THEOREM. *Let $i \in I$, $a \in \mathbf{C}^\times$ and $V = V(\omega_{i,a})$. Assume that $\varpi \in \mathrm{wt}_\ell(V)$ is such that* $\mathrm{wt}(\varpi) = \lambda \in P^+$.

(i) For all $w \in W_\lambda$ we have

$$\dim(V_\varpi) = \dim(V_{T_w \varpi}),$$

and

$$T_w(\mathrm{wt}_\ell(V_\lambda)) = \mathrm{wt}_\ell(V_{w\lambda}).$$

(ii) Suppose that $\varpi \neq \omega_{i,a}$. There exists $\varpi' \in \mathrm{wt}_\ell(V)$, $\mu = \mathrm{wt}(\varpi') \in P^+$, $w \in W_\mu$, $j \in I$ with $\ell(s_j w) = \ell(w) + 1$, and $c \in \mathbf{C}^\times$ such that

$$(T_w(\varpi'))_j = (1 - cu)(1 - c'u) \qquad \text{and} \qquad \varpi = T_w(\varpi')(\alpha_{j,c})^{-1}$$

for some $c' \neq cq_j^2$, and

$$\dim(V_\varpi) \geq 2 \qquad \text{if} \qquad c = c'.$$

Further, for all $v \in V_{T_w(\varpi')}$ and $s \in \mathbf{Z}$,

$$x_{j,s}^- v \in V_\varpi + V_{T_w(\varpi')(\alpha_{j,c'})^{-1}}. \tag{1.3}$$

COROLLARY. *We have*

$$\mathrm{ch}_\ell(V) = \sum_{\lambda \in P^+} \sum_{w \in W_\lambda} \sum_{\varpi \in \mathrm{wt}_\ell(V_\lambda)} (\dim V_\varpi) e(T_w(\varpi)).$$

From now on, we will let ϖ also denote the element $e(\varpi)$ of $\mathbf{Z}[\mathcal{P}_q]$. Notice that since the group $\mathcal{P}_q$ is multiplicative, this should cause no confusion.

2.8. It follows from the corollary that if $V(\omega_{i,a})$ is a minuscule representation of $\mathfrak{g}$, i.e., $V(\omega_{i,a})_\lambda = 0$ for all $\lambda \in P^+$ with $\lambda < \omega_i$, then

$$\mathrm{ch}_\ell(V(\omega_{i,a})) = \sum_{w \in W_{\omega_i}} T_w(\omega_{i,a}).$$

It follows from [5] that this is the case for all fundamental representations of A_n, the spin nodes for the orthogonal algebras, and the natural representations of C_n and D_n. In the rest of the paper we consider the remaining cases.

2.9. We conclude this section with a stronger version of Theorem 1.7*(ii)*. *Let* $\mathbf{U}_{q_j}(\widehat{\mathfrak{g}}_j)$ be the subalgebra of $\mathbf{U}_q$ generated by the elements $x^{\pm}_{j,m}$, $h_{j,s}$, $K_j^{\pm 1}$, $m, s \in \mathbf{Z}$, $s \neq 0$. It is known that $\mathbf{U}_{q_j}(\widehat{\mathfrak{g}}_j)$ is isomorphic to $\mathbf{U}_{q_j}(\widehat{\mathfrak{sl}}_2)$.

PROPOSITION. *Let* $i \in I$, $a \in \mathbf{C}^\times$, $V = V(\omega_{i,a})$. *Let* $\varpi' \in \mathrm{wt}_\ell(V)$ *satisfy the following:* $\mu = \mathrm{wt}(\varpi') \in P^+$,

$$(T_w(\varpi'))_j = (1 - cu)(1 - c'u),$$

for some $c' \neq cq_j^2$ *and* $w \in W_\mu$, $j \in I$ *satisfying* $(w(\mu) - \alpha_j) \in P^+$. *Set,* $\varpi = T_w(\varpi') \times (\alpha_{j,c})^{-1}$. *Then*

$$\dim(V_\varpi) \geq \dim(V_{\varpi'}).$$

Moreover, if $c = c'$, *then*

$$\dim(V_\varpi) \geq 2\dim(V_{\varpi'}).$$

Proof. First note that if $\mu = \mathrm{wt}(\varpi') \in P^+$ then $\mu = \omega_r$ for some $r \leq i$, (see [4, Section 1] for instance). Further, since $w \in W$ and $j \in I$ are such that $\ell(s_j w) = \ell(w) + 1$ it follows that $w\omega_r + \alpha_j \notin \mathrm{wt}(V)$. Since $\mathrm{wt}(T_w(\varpi')) = w\omega_r$, it follows that

$$x^+_{j,m} V_{T_w(\varpi')} = 0, \quad \forall\, m \in \mathrm{Z}.$$

Choose a basis $\{v_1, \cdots, v_p\}$ of $V_{T_w}(\varpi')$ such that

$$\mathbf{U}_q(0) v_m \in \sum_{s \leq m} \mathbf{C}(q) v_s.$$

Let $U_m = \mathbf{U}_{q_j}(\widehat{\mathfrak{g}}_j) v_m$. Then U_m / U_{m-1} is an ℓ-highest weight module for $\mathbf{U}_{q_j}(\widehat{\mathfrak{g}}_j) v_m$ with highest weight $(1 - cu)(1 - c'u)$ and hence by [4, 7] there exists a unique (up to scalar multiple) non-zero element in the span of $\{x^-_{j,r} v_m : r \in \mathbf{Z}\}$ which is an eigenvector (modulo U_{m-1}) for the $P_{j,r}$ with eigenvalue ϖ_j if $c \neq c'$ and two linearly independent elements if $c = c'$. Equation (1.3) of Theorem 1.7*(ii)* now implies that such vectors are in V_ϖ and hence the proposition is proved. □

Remark. It will actually follow from Theorem 2.3 and its proof that equality holds in Proposition 1.9.

3. Closed Formulae for q-Characters

In this section we state the main theorem which gives closed formulas for the ℓ-weights ϖ with $\mathrm{wt}(\varpi) \in P^+$ of the fundamental representations of quantum affine algebras.

3.1. Assume that the Dynkin diagram of $\mathfrak{g}$ is labeled as in [1]. Throughout this section we shall assume that we have fixed an integer i such that

$$\begin{array}{ll} 1 < i \le n & \text{if } \mathfrak{g} = C_n, \\ 1 \le i < n & \text{if } \mathfrak{g} = B_n, \\ 1 < i \le n-2 & \text{if } \mathfrak{g} = D_n. \end{array}$$

Define a subset I_i of I by,

$$I_i = \{r : 0 \le r \le i\}, \qquad \text{if } \mathfrak{g} = B_n, \tag{2.1}$$

$$= \{r : 0 \le r \le i, r \equiv i \bmod 2\}, \qquad \text{if } \mathfrak{g} = C_n, D_n. \tag{2.2}$$

From now on, given $r \in I_i$, we shall denote by M the greatest integer less than or equal to $(i-r)/2$.

3.2. For $r \in I_i$, let $\mathbf{J}_{k,r}$ be the set of partitions $r < j_1 < j_2 < \cdots < j_k \le n$ of length k and satisfying

$$\begin{array}{ll} j_s \le n - i + r + 2s - 1, \quad 1 \le s \le k, & \text{if } \mathfrak{g} = C_n, \\ j_k < n, & \text{if } \mathfrak{g} = D_n. \end{array}$$

Set

$$\mathbf{J}_r = \mathbf{J}_{M,r} \qquad \text{if } \mathfrak{g} = C_n, \tag{2.3}$$

$$= \cup_{0 \le k \le M} \mathbf{J}_{k,r} \qquad \text{if } \mathfrak{g} = B_n, D_n, \tag{2.4}$$

where $\mathbf{J}_{0,r}$ consists of the empty partition.

3.3. Given $j \in I$, $r \in I_i$ and an integer $2d_1 s \in \mathbf{N}$, define elements $\pi_r(j,s) \in \mathcal{P}_q$ by

$$\pi_r(j,s) = \omega^{-1}_{j,q_1^{i+j-2s-2r}} \omega_{j,q_1^{h-i-j+2s}},$$

where h is the dual Coxeter number of $\mathfrak{g}$ if $\mathfrak{g}$ is of type B_n or D_n and is twice the dual Coxeter number if $\mathfrak{g}$ is of type C_n. Given $\mathbf{j} \in \mathbf{J}_r$, set

$$\begin{aligned} \pi_r(\mathbf{j}) &= \omega_{r,q_1^{r-i}} \prod_{s=1}^{k} \pi_r(j_s - 1, s-1) \pi_r^{-1}(j_s, s), \qquad \text{if either } \mathfrak{g} = C_n \text{ or } j_k < \overline{n}, \\ &= \omega_{r,q_1^{r-i}} \left(\prod_{s=1}^{k-1} \pi_r(j_s - 1, s-1) \pi_r^{-1}(j_s, s) \right) \pi_r(n-1, k-1) \\ &\quad \times \pi_r^{-1}(n, k - 1/4), \qquad \text{if } \mathfrak{g} = B_n \text{ and } j_k = n, \\ &= \omega_{r,q^{r-i}} \left(\prod_{s=1}^{k} \pi_r(j_s - 1, s-1) \pi_r^{-1}(j_s, s) \right) \pi_r^{-1}(n, k + 1/2), \\ &\qquad \text{if } \mathfrak{g} = D_n \text{ and } j_k = n-1. \end{aligned}$$

where $\overline{n} = n$ (resp. $\overline{n} = n-1$) if $\mathfrak{g} = B_n$ (resp. if $\mathfrak{g} = D_n$). We understand that if $\mathbf{j}$ is the empty partition, then $\pi_r(\mathbf{j}) = \omega_{r,q_1^{r-i}}$ and, if $r = 0$, that $\omega_{0,a} = 1$ and $\omega_0 = 0$.

If $\mathfrak{g}$ is of type B_n, define $\pi_r(\mathbf{j}, *) \in \mathcal{P}_q$ by

$$\begin{aligned}\pi_r(\mathbf{j}, *) &= \omega_{n,q^{2(n-i+2k)-1}}\omega^{-1}_{n,q^{2(n+i-2k-2r)-1}}, &&\text{if } k \neq M \text{ and } j_k \neq n,\\ &= 1, &&\text{otherwise.}\end{aligned}$$

If $\mathfrak{g}$ is of type D_n, define elements $\pi_r(\mathbf{j}, \pm) \in \mathcal{P}_q$ by

$$\begin{aligned}\pi_r(\mathbf{j}, +) &= \omega_{n,q^{n-i+2k-1}}\omega^{-1}_{n,q^{n+i-2r-2k-1}},\\ \pi_r(\mathbf{j}, -) &= \omega_{n-1,q^{n-i+2k-1}}\omega^{-1}_{n-1,q^{n+i-2r-2k-1}}.\end{aligned}$$

THEOREM. *Let* $V = V(\omega_{i,1})$.

(i) *If* $\mathfrak{g}$ *is of type* C_n*, the assignment* $\mathbf{J}_r \to \mathcal{P}_q$ *defined by* $\mathbf{j} \mapsto \pi_r(\mathbf{j})$ *is injective and the image is* $\mathrm{wt}_\ell(V_{\omega_r})$*. In particular,*

$$\mathrm{ch}_\ell(V) = \sum_{r\in I_i}\sum_{w\in W_{\omega_r}}\sum_{\mathbf{j}\in\mathbf{J}_r} T_w(\pi_r(\mathbf{j})).$$

(ii) *If* $\mathfrak{g}$ *is of type* B_n*, the assignment* $\mathbf{J}_r \to \mathcal{P}_q$ *defined by* $\mathbf{j} \mapsto \pi_r(\mathbf{j})\pi_r(\mathbf{j}, *)$ *is injective and the image is* $\mathrm{wt}_\ell(V_{\omega_r})$*. In particular,*

$$\mathrm{ch}_\ell(V) = \sum_{r\in I_i}\sum_{w\in W_{\omega_r}}\sum_{\mathbf{j}\in\mathbf{J}_r} T_w(\pi_r(\mathbf{j})\pi_r(\mathbf{j}, *)).$$

(iii) *If* $\mathfrak{g}$ *is of type* D_n*, then*

$$\mathrm{wt}_\ell(V_{\omega_r}) = \{\pi_r(\mathbf{j})\pi_r(\mathbf{j}, \pm) : \mathbf{j} \in \mathbf{J}_{k,r}, 0 \le k < M\} \cup \{\pi_r(\mathbf{j}) : \mathbf{j} \in \mathbf{J}_{M,r}\}.$$

Moreover

$$\mathrm{ch}_\ell(V) = \sum_{r\in I_i}\sum_{w\in W_{\omega_r}}\left(\sum_{\mathbf{j}\in\mathbf{J_r}\setminus J_{M,r}}(T_w(\pi_r(\mathbf{j})\pi_r(\mathbf{j}, +)) + T_w(\pi_r(\mathbf{j})\pi_r(\mathbf{j}, -))) + \sum_{\mathbf{j}\in\mathbf{J}_{M,r}} T_w(\pi_r(\mathbf{j}))\right).$$

We prove the theorem in the next three sections using Theorem 1.7 in an inductive way.

4. The Case of C_n

4.1. Observe that the set $\mathbf{J}_r$ depends on n, i, r and it will be necessary for the proofs to write $\mathbf{J}_r$ as $\mathbf{J}_r(i)$. Notice moreover that

$$(j_1, \cdots, j_M) \in \mathbf{J}_r(i) \Leftrightarrow (j_2, \cdots, j_M) \in \mathbf{J}_{j_1}(i - r + j_1 - 2), \tag{3.1}$$

and also that

$$(j_1, \cdots, j_{M-1}) \in \mathbf{J}_{r+2}(i) \Leftrightarrow (j_1 - 2, j_2 - 2, \cdots j_{M-1} - 2, j_M) \in \mathbf{J}_r(i) \quad \forall \;\; j_{M-1} - 2 < j_M < n. \tag{3.2}$$

LEMMA. *We have*

$$|\mathbf{J}_r(i)| = \binom{n-r}{M} - \binom{n-r}{M-1} = \dim V(\omega_i)_{\omega_r}.$$

Proof. It suffices to prove the first equality, the second being well-known (see [8] for instance). If $M = 0, 1$ the result clearly holds for all $n \in \mathbf{N}$ and $1 \leq i \leq n$. Assume now that we know the result for all $M' < M$, $n \in \mathbf{N}$, and $1 \leq i \leq n$. By (3.1) we get,

$$|\mathbf{J}_r(i)| = \sum_{j=r+1}^{n-2M+1} |\mathbf{J}_{j_1}(i - r + j_1 - 2)| = \sum_{j=r+1}^{n-2M+1} \left(\binom{n-j}{M-1} - \binom{n-j}{M-2} \right),$$

where in the last equality we used the induction hypothesis. The identity

$$\binom{m}{l} = \binom{m-1}{l} + \binom{m-1}{l-1}, \quad 1 \leq l \leq m-1,$$

now gives the result. □

4.2.

LEMMA. *The map* $\mathbf{j} \mapsto \pi_r(\mathbf{j})$ *from* $\mathbf{J}_r(i) \to \mathcal{P}_q$ *is injective.*

Proof. We proceed by induction on M. Suppose that $\pi_r(\mathbf{j}) = \pi_r(\mathbf{j}')$ for some $\mathbf{j}, \mathbf{j}' \in \mathbf{J}_r(i)$. Writing $\mathbf{j} = (j_1, \cdots, j_M)$ and $\mathbf{j}' = (j'_1, \cdots, j'_M)$ and comparing the $(j_1 - 1)^{th}$ entries in $\pi_r(\mathbf{j})$ and $\pi_r(\mathbf{j}')$ we find that $j_1 = j'_1$. This proves that induction starts at $M = 1$. The inductive step follows by (3.1). □

4.3. For $r > 1$ and $r - 1 \leq j < n$ define elements $w_{r,j} \in W$ and $T_{r,j} \in \mathcal{B}(\mathfrak{g})$ by

$$\begin{aligned} w_{r,j} &= s_{j-1}s_{j-2}\cdots s_{r-1}s_{j+1}s_{j+2}\cdots s_{n-1}s_n\cdots s_r, \\ T_{r,j} &= T_{w_{r,j}}. \end{aligned}$$

It is not hard to check (see [10] for instance) that $w_{r,j} \in W_{\omega_r}$.

The next proposition is a straightforward if a somewhat tedious computation.

PROPOSITION.

(i) *For all* $r \in I$, *and* $r - 1 \leq j < n$, *we have*

$$w_{r,j}\omega_r = \omega_{r-2} + \alpha_j.$$

$$\text{(ii)} \quad T_{r,j}(\omega_{l,a}) = \begin{cases} \omega_{l-2,aq^2}\omega^{-1}_{j-1,aq^{j-l+3}}\omega_{j,aq^{j-l+2}}\omega_{j,aq^{2n-j-l+2}}\omega^{-1}_{j+1,aq^{2n-j-l+3}}, & \text{if } r \leq l \leq j, \\ \omega_{l,aq^2}\omega_{j,aq^{l-j}}\omega_{j,aq^{2n-j-l+2}}\omega^{-1}_{j+1,aq^{l-j+1}}\omega^{-1}_{j+1,aq^{2n-j-l+3}}, & \text{if } l > j. \end{cases} \tag{3.3}$$

4.4. Part (i) of Theorem 2.3 now follows from Lemma 3.2 and the next proposition.

PROPOSITION. *We have*

$$\mathrm{wt}_\ell(V_{\omega_r}) = \{\pi_r(\mathbf{j}) : \mathbf{j} \in \mathbf{J}_r\}, \tag{3.4}$$

and

$$\dim V_{\pi_r(\mathbf{j})} = 1 \qquad \text{for all} \qquad \mathbf{j} \in \mathbf{J}_r. \tag{3.5}$$

Proof. Recall from [5] that

$$V \cong V(\omega_i)$$

as $U_q(\mathfrak{g})$-modules. It suffices to prove that,

$$\{\pi_r(\mathbf{j}) : \mathbf{j} \in \mathbf{J}_r\} \subset \mathrm{wt}_\ell(V_{\omega_r}), \tag{3.6}$$

since Lemma 3.1 and Lemma 3.2 then imply both (3.4) and (3.5).

To prove (3.6) we proceed by induction on M with induction beginning at $M = 0$. Assume that we know the result for $M - 1$. To prove the inductive step it follows from (3.2) that if $\mathbf{j} = (\mathbf{j}_1, \cdots, \mathbf{j}_M) \in \mathbf{J}_r$ then $\mathbf{j}' = (\mathbf{j}_1 + 2, \cdots, \mathbf{j}_{M-1} + 2) \in \mathbf{J}_{r+2}$. The induction hypothesis implies that $\pi_{r+2}(\mathbf{j}') \in \mathrm{wt}_\ell(V)$ and hence by Theorem 1.7 we see that $T_{r+2,\, j}(\pi_{r+2}(\mathbf{j}')) \in \mathrm{wt}_\ell(V)$ for all $r < j < n$.

Using Proposition 3.3(*ii*) we find that

$$T_{r+2,\, j}(\pi_{r+2}(\mathbf{j}')) = \pi_r(\mathbf{j})\boldsymbol{\alpha}_{j,q^{2n+i-j-2r-2M}},$$

and also that the j^{th}-coordinate of $T_{r+2,\, j}(\pi_{r+2}(\mathbf{j}'))$ is

$$(1 - q^{i+j-2r-2M}u)(1 - q^{2n+i-j-2r-2M}u).$$

Hence by Theorem 1.7(*ii*) we see that

$$\pi_r(\mathbf{j}) = T_{r+2,\, j}(\pi_{r+2}(\mathbf{j}'))\boldsymbol{\alpha}^{-1}_{j,q^{2n+i-j-2r-2M}} \in \mathrm{wt}_\ell(V_{\omega_r}).$$ □

5. The Case of B_n

5.1.

LEMMA. *We have*

$$|\mathbf{J}_r| = \sum_{k=0}^{M} \binom{n-r}{k} = \dim V_{\omega_r}.$$

Proof. The first equality is clear. For the second, recall that it was proved in [5] that $V \cong \bigoplus_{l=0}^{[i/2]} V(\omega_{i-2l})$ as $U_q(\mathfrak{g})$-modules. Since

$$\dim V(\omega_j)_{\omega_r} = \binom{n-r}{[\frac{j-r}{2}]}, \quad j < n,$$

it follows that

$$\dim V_{\omega_r} = \sum_{l=0}^{M} \binom{n-r}{[\frac{i-2l-r}{2}]} = \sum_{k=0}^{M} \binom{n-r}{k}. \qquad \square$$

5.2.

LEMMA. *The map* $\mathbf{j} \mapsto \pi_r(\mathbf{j})\pi_r(\mathbf{j},*)$ *from* $\mathbf{J}_r$ *to* $\mathcal{P}_q$ *is injective.*

Proof. Suppose that $0 \leq k, k' \leq M$, $\mathbf{j}, \mathbf{j}' \in \mathbf{J}_r$, $\mathbf{j} = (j_1, \cdots, j_k)$, $\mathbf{j}' = (j'_1, \cdots, j'_{k'})$ are such that

$$\pi_r(\mathbf{j})\pi_r(\mathbf{j},*) = \pi_r(\mathbf{j}')\pi_r(\mathbf{j}',*). \tag{4.1}$$

We first show that $k = k'$. For this, notice that for any $\mathbf{j}'' = (j''_1, \cdots, j''_{k''}) \in \mathbf{J}_r$ we have

$$(\pi_r(\mathbf{j}'')\pi_r(\mathbf{j}'',*))_n = 1 \Longleftrightarrow k'' = M \quad \text{and} \quad j''_{k''} < n.$$

Hence, $(\pi_r(\mathbf{j})\pi_r(\mathbf{j},*))_n = 1$ implies $k = k' =$ M. Otherwise, the equation

$$(\pi_r(\mathbf{j})\pi_r(\mathbf{j},*))_n = (\pi_r(\mathbf{j}')\pi_r(\mathbf{j}',*))_n$$

gives that either $j_k, j'_{k'} < n$ or $j_k = j'_{k'} = n$. In the first case we get $\pi_r(\mathbf{j},*)_n = \pi_r(\mathbf{j}',*)_n$ and in the second case we get $\pi_r(\mathbf{j})_n = \pi_r(\mathbf{j}')_n$. In any case it follows that $k = k'$.

Now, suppose $\mathbf{j} \neq \mathbf{j}'$ and let $1 \leq s_0 \leq n$ be minimal such that $j_{s_0} \neq j'_{s_0}$. Assume without loss of generality that $j'_{s_0} > j_{s_0}$. This means that $j'_{s_0-1} = j_{s_0-1} < j'_{s_0} - 1$ and hence

$$\pi_r(\mathbf{j}')_{j'_{s_0}-1} = (1 - q_1^{2n-i-j'_{s_0}+2s_0-2}u)(1 - q_1^{i+j'_{s_0}-2s_0-2r+1}u)^{-1} \neq 1. \tag{4.2}$$

We claim now that there exists $s_1 \geq s_0$ such that $j_{s_1} = j'_{s_0} - 1$. Assuming the claim, we get a contradiction to the fact that $\mathbf{j} \neq \mathbf{j}'$ as follows. Since

$$\pi_r(\mathbf{j}')_{j'_{s_0}-1} = (1 - q_1^{i+j'_{s_0}-2s_1-2r-1}u)(1 - q_1^{2n-i-j'_{s_0}+2s_1}u)^{-1} = \pi_r(\mathbf{j})_{j_{s_1}},$$

comparing with (4.2) gives

$$2n - i - j'_{s_0} + 2s_0 - 2 = i + j'_{s_0} - 2s_1 - 2r - 1,$$

which is obviously impossible. To prove the claim, set

$$s_1 = \max\{1 \leq s \leq k : j_s < j'_{s_0}\}.$$

The claim follows if we prove that $j_{s_1+1} > j'_{s_0}$. The maximality of s_1 implies that $j_{s_1+1} \geq j'_{s_0}$ and hence it suffices to prove that $j_{s_1+1} \neq j'_{s_0}$. If $j_{s_1+1} = j'_{s_0}$ then the same argument that gave (4.2) gives,

$$\pi_r(\mathbf{j})_{j_{s_1+1}-1} = (1 - q_1^{2n-i-j_{s_1+1}+2(s_1+1)-2}u)(1 - q_1^{i+j'_{s_1+1}-2(s_1+1)-2r+1}u)^{-1} \neq 1,$$

which implies that $s_1 + 1 = s_0$ contradicting $s_1 \geq s_0$. $\square$

5.3. For $r>0$ and $r\le j<n$ define elements $w_{r,\,j}\in W_{\omega_r}$ by,

$$\begin{aligned} w_{r+1,\,j} &= s_{j-1}s_{j-2}\cdots s_r s_{j+1}s_{j+2}\cdots s_{n-1}s_n\cdots s_{r+1},\\ w_{r,n} &= s_{n-1}\cdots s_r,\\ T_{r,\,j} &= T_{w_{r,\,j}}. \end{aligned}$$

The proof of the next proposition is along the same lines as the proof of Proposition 3.3 and we omit the details.

PROPOSITION.

(i) *For all $r\in I\backslash\{n\}$ and $r-1\le j<n$ we have*

$$w_{r,\,j}\omega_r=\omega_{r-2}+\alpha_j \quad \textit{and} \quad w_{r,n}\omega_r=\omega_{r-1}+\alpha_n.$$

(ii) *For $r-1\le j<n-1$ and $r\le l<n$ we have:*

$$T_{r,\,j}(\omega_{l,1})=\begin{cases} \omega_{l-2,q_1^2}\omega^{-1}_{j-1,q_1^{j-l+3}}\omega_{j,q_1^{j-l+2}}\omega_{j,q_1^{2n-j-l-1}}\omega^{-1}_{j+1,q_1^{2n-j-l}}, & \textit{if } l\le j,\\ \omega_{l,q_1^2}\omega_{j,q_1^{l-j}}\omega_{j,q_1^{2n-j-l-1}}\omega^{-1}_{j+1,q_1^{l-j+1}}\omega^{-1}_{j+1,q_1^{2n-j-l}}, & \textit{if } l>j. \end{cases} \tag{4.3}$$

Further,

$$T_{r,\,j}(\omega_{n,q^{-1}})=\omega_{j,q_1^{n-j-1}}\omega^{-1}_{j+1,q_1^{n-j}}\omega_{n,q^3},$$

$$\begin{aligned} T_{r,n-1}(\omega_{l,1}) &= \omega_{l-2,q_1^2}\omega^{-1}_{n-2,q_1^{n-l+2}}\omega_{n-1,q_1^{n-l+1}}\omega_{n-1,q_1^{n-l}}\omega^{-1}_{n,q^{2(n-l)+1}}\omega^{-1}_{n,q^{2(n-l)+3}},\\ T_{r,n-1}(\omega_{n,q^{-1}}) &= \omega_{n-1,1}\omega^{-1}_{n,q},\\ T_{r,n}(\omega_{l,1}) &= \omega_{l-1,q_1}\omega^{-1}_{n-1,q_1^{n-l+1}}\omega_{n,q^{2(n-l)-1}}\omega_{n,q^{2(n-l)+1}}. \end{aligned} \tag{4.4}$$

5.4. To prove Theorem 2.3(*ii*) we proceed by induction on M. Induction clearly begins when $M=0$. The inductive step is immediate from the following proposition, Lemma 4.1, and Lemma 4.2.

PROPOSITION. *Assume that $M>0$ and let $\mathbf{j}=(j_1,\cdots,j_k)\in\mathbf{J}_r$.*

(i) *If $k<M$ and $j_k<n$ we have:*

$$\pi_r(\mathbf{j})\pi_r(\mathbf{j},*)=T_{r+1,n}(\pi_{r+1}(\mathbf{j}')\pi_{r+1}(\mathbf{j}',*))\alpha^{-1}_{n,a}\in \mathrm{wt}_\ell(V_{\omega_r}),$$

where $\mathbf{j}'=(j_1+1,\cdots,j_k+1)\in\mathbf{J}_{r+1}$ and $a=q^{2(n+i-2r-2k)-3}$.

(ii) *If $j_k=n$, we have*

$$\pi_r(\mathbf{j})\pi_r(\mathbf{j},*)=T_{r+1,n}(\pi_{r+1}(\mathbf{j}')\pi_{r+1}(\mathbf{j}',*))\boldsymbol{\alpha}^{-1}_{n,a}\in \mathrm{wt}_\ell(V_{\omega_r}),$$

where $\mathbf{j}'=(j_1+1,\cdots,j_{k-1}+1)\in\mathbf{J}_{r+1}$ and $a=q^{2(n-i+2k)-5}$.

(iii) *If $k=M$ and $j_k<n$, then*

$$\pi_r(\mathbf{j})\pi_r(\mathbf{j},*)=T_{r+2,\,j}(\pi_{r+2}(\mathbf{j}')\pi_{r+2}(\mathbf{j}',*))\alpha^{-1}_{j,a}\in \mathrm{wt}_\ell(V_{\omega_r}),$$

where $\mathbf{j}'=(j_1+2,\cdots,j_{k-1}+2)\in\mathbf{J}_{r+2}$, $a=q_1^{2n-j-r-3}$, and $j=j_k$.

Proof. Observe first that it is clear that the elements $\mathbf{j}'$ defined in the proposition are in $\mathbf{J}_{r+1}$ in the first two cases and in $\mathbf{J}_{r+2}$ in the third case. The fact that $\pi_r(\mathbf{j})\pi_r(\mathbf{j},*)$ and $\pi_{r+1}(\mathbf{j}')\pi_{r+1}(\mathbf{j}',*)$ (resp. $\pi'_{r+2}(\mathbf{j}')\pi_{r+2}(\mathbf{j}',*)$) are related as in the proposition is again a tedious but simple checking using the formulas in Proposition 4.3(*ii*). The main point is to notice that this implies $\pi_r(\mathbf{j})\pi_r(\mathbf{j},*) \in \mathrm{wt}_\ell(V_{\omega_r})$. For that, one observes that the calculation gives, respectively:

$$(i)\ (T_{r+1,n}(\pi_{r+1}(\mathbf{j}')\pi_{r+1}(\mathbf{j}',*)))_n$$
$$= (1-q^{2(n+i-2r-2k)-3}u)(1-q^{2(n-i+2k)-1}u), \text{ if } j_k = n-1 \text{ or } k < M-1/2,$$
$$= (1-q^{2(n+i-2r-2k)-3}u)(1-q^{2(n+i-2r-2k)-5}u) \text{ if } j_k < n-1 \text{ and } k = M-1/2.$$

$$(ii)\quad (T_{r+1,n}(\pi_{r+1}(\mathbf{j}')\pi_{\mathrm{r}+1}(\mathbf{j}',*)))_n = (1-q^{2(n-i+2k)-5}u)(1-q^{2(n+i-2r-2k)+1}u)\ .$$
$$(iii)\quad (T_{r+2,j}(\pi_{r+2}(\mathbf{j}')\pi_{r+2}(\mathbf{j}',*)))_j = (1-q_1^{j-r}u)(1-q_1^{2n-j-r-3}u)\ .$$

The result then follows from Theorem 1.7. □

6. The Case of D_n

6.1.

LEMMA. *We have:*

$$|\mathbf{J}_{M,r}| + 2\sum_{k=0}^{M-1}|\mathbf{J}_{k,r}| = \sum_{l=0}^{M}\binom{n-r}{l} = \dim V_{\omega_r}.$$

Proof. It follows from the definition of $\mathbf{J}_{k,r}$ that

$$|\mathbf{J}_{k,r}| = \binom{n-r-1}{k},$$

and hence to prove the the first equality we must show that

$$\sum_{l=0}^{M}\binom{n-r}{l} = \binom{n-r-1}{M} + 2\sum_{k=0}^{M-1}\binom{n-r-1}{k}.$$

Using the binomial identity

$$\binom{n-r}{l} = \binom{n-r-1}{l} + \binom{n-r-1}{l-1},$$

we find that

$$\sum_{l=0}^{M}\binom{n-r}{l} = 1 + \sum_{l=1}^{M}\left(\binom{n-r-1}{l} + \binom{n-r-1}{l-1}\right)$$
$$= 1 + \binom{n-r-1}{M} + \sum_{l=1}^{M-1}\binom{n-r-1}{l} + \sum_{l=0}^{M-1}\binom{n-r-1}{l}$$
$$= \binom{n-r-1}{M} + 2\sum_{l=0}^{M-1}\binom{n-r-1}{l}.$$

For the second equality, recall that it was proved in [5] that as $U_q(\mathfrak{g})$-modules

$$V \cong \bigoplus_{l=0}^{[i/2]} V(\omega_{i-2l}).$$

The result now follows since

$$\dim V(\omega_j)_{\omega_r} = \binom{n-r}{(j-r)/2}, \quad 1 \le j \le n-2.$$

6.2. Given $r > 1$ and $r-1 \le j \le n$, define elements $w_{r,j} \in W_{\omega_r}$ by,

$$\begin{aligned} w_{r,j} &= s_{j-1}s_{j-2}\cdots s_{r-1}s_{j+1}\cdots s_{n-2}s_n s_{n-1}\cdots s_r, \quad j \le n-2, \\ &= s_{n-2}\cdots s_{r-1}s_{j'}s_{n-2}\cdots s_r, \quad j,j' \in \{n-1,n\}, \quad j' \ne j, \\ T_{r,j} &= T_{w_{r,j}}. \end{aligned}$$

PROPOSITION. *For all* $1 < r \le n-2$ *and* $r-1 \le j \le n$ *we have:*

(i) $w_{r,j}\omega_r = \omega_{r-2} + \alpha_j$.

(ii)

$$T_{r,j}(\omega_{l,a}) = \begin{cases} \omega_{l,aq^2}\omega_{j,aq^{l-j}}\omega_{j,aq^{2n-l-2-j}}\omega^{-1}_{j+1,aq^{l-j+1}}\omega^{-1}_{j+1,aq^{2n-l-j-1}}, & \text{if } j < l, \\ \omega_{l-2,aq^2}\omega^{-1}_{j-1,aq^{j-l+3}}\omega_{j,aq^{j-l+2}}\omega_{j,aq^{2n-l-j-2}}\omega^{-1}_{j+1,aq^{2n-l-j-1}}, & \text{if } l \le j \le n-2, \\ \omega_{l-2,aq^2}\omega^{-1}_{n-2,aq^{n-l+2}}\omega_{j,aq^{n-l-1}}\omega_{j,aq^{n-l+1}}, & \text{if } j = n-1, n, \end{cases} \tag{5.1}$$

if $1 \le l \le n-2$ *and*

$$T_{r,j}(\omega_{l,a}) = \begin{cases} \omega_{j,aq^{n-1-j}}\omega^{-1}_{j+1,aq^{n-j}}\omega_{l',aq^2}, & \text{if } j < n-2, \\ \omega_{n-2,aq}\omega^{-1}_{l,aq^2}, & \text{if } j = n-2, \\ \omega_{l,a}, & \text{if } j = l, \\ \omega_{n-3,aq^2}\omega^{-1}_{n-2,aq^3}\omega_{l',aq^2}, & \text{if } j = l', \end{cases} \tag{5.2}$$

if $l = n-1, n$, *where* $l' \in \{n-1,n\}\backslash\{l\}$.

6.3. The next proposition is proved in a similar manner to the corresponding one for B_n and C_n. We omit the details this time.

PROPOSITION. *For* $M \ge 0$, *we have*

$$\{\pi_r(\mathbf{j}) : \mathbf{j} \in \mathbf{J}_{M,r}\} \cup \{\pi_r(\mathbf{j})\pi_r(\mathbf{j},\pm) : \mathbf{j} \in \mathbf{J}_{k,r}, 0 \le k < M\} \subset \mathrm{wt}_\ell(V_{\omega_r}).$$

6.4. To complete the proof of Theorem 2.3(*iii*), we must prove that in fact

$$\{\pi_r(\mathbf{j}) : \mathbf{j} \in \mathbf{J}_{M,r}\} \cup \{\pi_r(\mathbf{j})\pi_r(\mathbf{j},\pm) : \mathbf{j} \in \mathbf{J}_{k,r}, 0 \le k < M\} = \mathrm{wt}_\ell(V_{\omega_r}).$$

For D_n this is more difficult, since it is no longer true that the maps $\mathbf{j} \mapsto \pi_r(\mathbf{j})\pi_r(\mathbf{j},\pm)$ are injective.

The next lemma is a simple checking.

LEMMA. *Let* $\mathbf{j} \in \mathbf{J}_{k,r}$ *and* $\mathbf{j}' \in \mathbf{J}_{k',r}$.

(i) *We have* $\pi_r(\mathbf{j}, \pm) = \pi_r(\mathbf{j}', \pm)$ *iff* $k = k'$. *Moreover, if* $k = M$, *then* $\pi_{\rm r}(\mathbf{j}, \pm) = 1$.
(ii) *If* $\pi_r(\mathbf{j})\pi_r(\mathbf{j}, +) = \pi_r(\mathbf{j}')\pi_r(\mathbf{j}', \pm)$, *then* $k = k'$. *Moreover, if* $k = k' < M$, *then* $\pi_r(\mathbf{j})\pi_r(\mathbf{j}, +) \neq \pi_r(\mathbf{j}')\pi_r(\mathbf{j}', -)$.

6.5. Define an equivalence relation $\sim$ on $\mathbf{J}_{k,r}$ by

$$\mathbf{j} \sim \mathbf{j}' \iff \pi_r(\mathbf{j}) = \pi_r(\mathbf{j}').$$

Let $\bar{\mathbf{j}}$ be the equivalence class of $\mathbf{j}$.

PROPOSITION. *Let* $\mathbf{j} \in \mathbf{J}_{k,r}$. *Then*

$$\dim V_{\pi_r(\mathbf{j})\pi_r(\mathbf{j},\pm)} \geq |\bar{\mathbf{j}}|.$$

Assuming this proposition the proof of Theorem 2.3(*iii*) is completed as follows. For each $0 \leq k \leq M$, fix a set $\mathbf{S}_{k,r} \subset \mathbf{J}_{k,r}$ of representatives of the distinct equivalence classes with respect to $\sim$. Proposition 5.5 and Lemma 5.4 imply that

$$\sum_{\mathbf{j}\in\mathbf{S}_{M,r}} \dim(V_{\pi_r(\mathbf{j})}) \; + \; \sum_{k=0}^{M-1}\sum_{\mathbf{j}\in\mathbf{S}_{k,r}} \left(\dim(V_{\pi_r(\mathbf{j})\pi_r(\mathbf{j},+)}) + \dim(V_{\pi_r(\mathbf{j})\pi_r(\mathbf{j},-)})\right)$$
$$\geq \sum_{\mathbf{j}\in\mathbf{S}_{M,r}} |\bar{\mathbf{j}}| + 2\sum_{k=0}^{M-1}\sum_{\mathbf{j}\in\mathbf{S}_{k,r}} |\bar{\mathbf{j}}| = |\mathbf{J}_{M,r}| + 2\sum_{k=0}^{M-1} |\mathbf{J}_{k,r}|.$$

Since $V_{\pi_r(\mathbf{j})\pi_r(\mathbf{j},\pm)} \subset V_{\omega_r}$ it follows from Lemma 5.1 that

$$V_{\omega_r} = \left(\bigoplus_{\mathbf{j}\in\mathbf{S}_{M,r}} (V_{\pi_r(\mathbf{j})})\right) \oplus \left(\bigoplus_{k=0}^{M-1}\bigoplus_{\mathbf{j}\in\mathbf{S}_{k,r}} (V_{\pi_r(\mathbf{j})\pi_r(\mathbf{j},+)} \oplus V_{\pi_r(\mathbf{j})\pi_r(\mathbf{j},-)})\right),$$

which proves Theorem 2.3(*iii*).

6.6. It remains to prove Proposition 5.5. This requires some combinatorial definitions and results which we now establish and which allow us to actually compute $|\bar{\mathbf{j}}|$ in terms of $\mathbf{j} \in \mathbf{J}_{k,r}$. Thus we shall see that

$$|\bar{\mathbf{j}}| = 2^{M_{\mathbf{j}}}, \tag{5.3}$$

where $M_{\mathbf{j}} = |\{s : 1 \leq s \leq k, j_s \in \{n - i + r + 2s - 2, n - i + r + 2s - 1\}\}|$.

6.7. Recall that a strictly increasing partition $\mathbf{n}$ of positive integers of length k is an increasing sequence $0 < j_1 < j_2 < \cdots < j_k$ of natural numbers. Let

$$\mathrm{supp}(\mathbf{n}) = \{j_s : 1 \leq s \leq k\},$$

and let $\iota_{\mathbf{n}} : \mathrm{supp}(\mathbf{n}) \to \mathbf{N}$ be defined by $\iota_{\mathbf{n}}(j) = s$, if $j = j_s$ for $1 \leq s \leq k$. Clearly any finite subset of $\mathbf{N}$ defines a strictly increasing partition.

Given a finite subset $S \subset \mathbf{N}$ and a partition $\mathbf{n}$ of length k, let $\mathbf{n}_S$ be the partition corresponding to the set,

$$(\mathrm{supp}(\mathbf{n}) \setminus S) \cup (S \setminus \mathrm{supp}(\mathbf{n})).$$

Clearly,

$$\mathbf{n}_S \neq \mathbf{n}_{S'} \quad if \quad S \neq S',$$

and

$$(\mathbf{n}_S)_S = \mathbf{n}, \quad (\mathbf{n}_S)_{S'} = (\mathbf{n}_{S'})_S = (\mathbf{n})_{S\cup S'}, \quad \text{if } S \cap S' = \emptyset. \tag{5.4}$$

We shall adopt the convention that if $k, k' \in \mathbf{Z}$, then $[k, k'] = \{\min\{k, k'\}, \min\{k, k'\} + 1, \cdots, \max\{\mathrm{k}, \mathrm{k}'\}\}$. Define also $(k, k']$ and $[k', k)$ in the obvious way.

Associated with a partition $\mathbf{n}$, define functions $\sigma_{\mathbf{n}}^{\pm}, \tau_{\mathbf{n}}^{\pm} : \mathrm{supp}(\mathbf{n}) \to \mathbf{N}$ by,

$$\sigma_{\mathbf{n}}^{+}(j) = \max\{j' \in \mathrm{supp}(\mathbf{j}) : j' \geq j \text{ and } j'' - j < 2(\iota_{\mathbf{n}}(j'') - \iota_{\mathbf{n}}(j)) \; \forall \, j'' \in \mathrm{supp}(\mathbf{j}) \cap (j, j']\},$$
$$\sigma_{\mathbf{n}}^{-}(j) = \min\{j' \in \mathrm{supp}(\mathbf{j}) : j' \leq j \text{ and } j - j'' < 2(\iota_{\mathbf{n}}(j) - \iota_{\mathbf{n}}(j'')) \; \forall \, j'' \in \mathrm{supp}(\mathbf{j}) \cap [j', j)\},$$

and

$$\tau_{\mathbf{n}}^{\pm}(j) = j + 2(\iota_{\mathbf{n}}(\sigma_{\mathbf{n}}^{\pm}(j)) - \iota_{\mathbf{n}}(j)) \pm 1.$$

The following lemma is easy.

LEMMA. *Let* $j \in \mathrm{supp}(\mathbf{n})$. *Then,*

$$|\mathrm{supp}(\mathbf{n}_{[j, \tau_{\mathbf{n}}^{\pm}(j)]})| = |\mathrm{supp}(\mathbf{n})|.$$

From now on we set

$$\mathbf{n}^{\pm}(j) = \mathbf{n}_{[j, \tau_{\mathbf{n}}^{\pm}(j)]}, \quad j \in \mathrm{supp}(\mathbf{n}). \tag{5.5}$$

Moreover, if $S \subset \mathrm{supp}(\mathbf{n})$ is such that

$$[j, \tau_{\mathbf{n}}^{\pm}(j)] \cap [j', \tau_{\mathbf{n}}^{\pm}(j')] = \emptyset, \quad \forall j \neq j', j, j' \in S, \tag{5.6}$$

then set

$$\mathbf{n}^{\pm}(S) = (\mathbf{n}^{\pm}(j))^{\pm}(S \backslash \{j\}), \quad j \in S \subset \mathrm{supp}(\mathbf{n}). \tag{5.7}$$

Notice that $\mathbf{n}^{\pm}(S)$ is well-defined by (5.4).

6.8. For a partition $\mathbf{n}$ and an integer $m > 0$, define

$$\mathrm{supp}_m(\mathbf{n}) = \{j \in \mathrm{supp}(\mathbf{n}) : 2\iota_{\mathbf{n}}(j) = j - m\}.$$

The following lemma is easy.

LEMMA. *Let* $j \in \mathrm{supp}_m(\mathbf{n})$ *for some* $m > 1$. *Then* $\tau_{\mathbf{n}}^{\pm}(j) \in \mathrm{supp}_{m\pm1}(\mathbf{n}^{\pm}(j))$, *and*

$$\mathrm{supp}_{m\pm1}(\mathbf{n}^{\pm}(j)) = \mathrm{supp}_{m\pm1}(\mathbf{n}) \sqcup \{\tau_{\mathbf{n}}^{\pm}(j)\},$$
$$\mathrm{supp}_{m}(\mathbf{n}^{\pm}(j)) = \mathrm{supp}_{m}(\mathbf{n}) \backslash \{j\}.$$

In particular $\tau_{\mathbf{n}^{\pm}(j)}^{\mp}(\tau_{\mathbf{n}}^{\pm}(j)) = j$.

Remark. Notice that $\mathbf{n}^{\pm}(S)$ is well defined for all $S \subset \mathrm{supp}_m(\mathbf{j})$. The Lemma then implies that $\mathrm{supp}_m(\mathbf{n}^{\pm}(\mathrm{supp}_m(\mathbf{n})) = \emptyset$.

6.9. From now on we set

$$N = n - i + r - 2, \quad \text{supp}^+(\mathbf{j}) = \text{supp}_N(\mathbf{j}), \quad \text{supp}^-(\mathbf{j}) = \text{supp}_{N+1}(\mathbf{j}). \tag{5.8}$$

Let $\mathbf{j} \in \mathbf{J}_{k,r}$. The next proposition describes $\bar{\mathbf{j}}$ for $\mathbf{j} \in \mathbf{J}_{k,r}$.

PROPOSITION. *Let* $\mathbf{j}, \mathbf{j}' \in \mathbf{J}_{k,r}$. *Then,*

$$\mathbf{j} \sim \mathbf{j}' \Longleftrightarrow \mathbf{j}' = (\mathbf{j}^-(S^-))^+(S^+), \tag{5.9}$$

for some $S^- \subset \text{supp}^-(\mathbf{j})$ *and* $S^+ \subset \text{supp}^+(\mathbf{j}^-(S^-))$. *In particular,*

$$|\bar{\mathbf{j}}| = 2^{|\text{supp}^+(\mathbf{j})| + |\text{supp}^-(\mathbf{j})|}. \tag{5.10}$$

COROLLARY. *Let* $\mathbf{j} \in \mathbf{J}_{k,r}$ *be such that* $\text{supp}^-(\mathbf{j}) = \emptyset$. *Then* $|\bar{\mathbf{j}}| = 2^{|\text{supp}^+(\mathbf{j})|}$ *and*

$$|\mathbf{J}_{k,r}| = \sum_{\substack{\mathbf{j} \in \mathbf{J}_{k,r}, \\ \text{supp}^- \mathbf{j} = \emptyset}} \mathbf{2}^{|\text{supp}^+(\mathbf{j})|}.$$

Notice that (5.10) is immediate from (5.9) and Lemma 5.8.
For the proof, it is useful to notice that,

$$\pi_r(j, s) = 1 \Leftrightarrow 2s = j - (N+1). \tag{5.11}$$

6.10. We now see that Proposition 5.9 can be deduced from the following.

PROPOSITION. *Let* $\mathbf{j} \in \mathbf{J}_{k,r}$. *For any* $S \subset \text{supp}^{\pm}(\mathbf{j})$ *we have* $\mathbf{j}^{\pm}(S) \sim \mathbf{j}$.

If $\mathbf{j}', \mathbf{j} \in \mathbf{J}_{k,r}$ are related as in the right hand side of (5.9), we have by Proposition 5.10 that,

$$\mathbf{j} \sim \mathbf{j}^-(S^-) \sim (\mathbf{j}^-(S^-))^+(S^+) = \mathbf{j}'.$$

For the converse, let $\mathbf{j} \sim \mathbf{j}'$ and assume that

$$\mathbf{j}^-(\text{supp}^-(\mathbf{j})) = \mathbf{j}'^-(\text{supp}^-(\mathbf{j}'). \tag{5.12}$$

Set

$$S^- = \text{supp}^-(\mathbf{j}) \qquad \text{and} \qquad S^+ = \tau_{\mathbf{j}}^-(\text{supp}^-(\mathbf{j}')).$$

It follows from Lemma 5.8 that

$$S^+ \subset \text{supp}^+(\mathbf{j}'^-(\text{supp}^-(\mathbf{j}'))).$$

Then, using (5.4), Lemma 5.8 and (5.12) we see that

$$\mathbf{j}' = (\mathbf{j}'^-(\text{supp}^-(\mathbf{j}')))^+(S^+) = (\mathbf{j}^-(S^-))^+(S^+).$$

It remains to show that (5.12) is always satisfied if $\mathbf{j} \sim \mathbf{j}'$. Observe that Proposition 5.10 gives

$$\mathbf{j}^-(\text{supp}^-(\mathbf{j})) \sim \mathbf{j}'^-(\text{supp}^-(\mathbf{j}')),$$

and that by Remark 5.8

$$\text{supp}^-(\mathbf{j}^-(\text{supp}^-(\mathbf{j}))) = \text{supp}^-(\mathbf{j}'^-(\text{supp}^-(\mathbf{j}'))) = \emptyset.$$

In other words, to prove (5.12) it suffices to prove,

$$\mathbf{j}_1 \sim \mathbf{j}_2, \text{ and } \operatorname{supp}^-(\mathbf{j}_1) = \operatorname{supp}^-(\mathbf{j}_2) = \emptyset \Rightarrow \mathbf{j}_1 = \mathbf{j}_2. \tag{5.13}$$

Indeed, if $\mathbf{j}_1 \neq \mathbf{j}_2$ set

$$j = \max\{ j' : j' \in \operatorname{supp}(\mathbf{j}_1) \cup \operatorname{supp}(\mathbf{j}_2) \text{and } j' \notin \operatorname{supp}(\mathbf{j}_1) \cap \operatorname{supp}(\mathbf{j}_2)\}.$$

Assume $j \in \operatorname{supp}(\mathbf{j}_1)$. Then, either $j+1 \in \operatorname{supp}(\mathbf{j}_1) \cap \operatorname{supp}(\mathbf{j}_2)$ or $j+1 \notin \operatorname{supp}(\mathbf{j}_1) \cup \operatorname{supp}(\mathbf{j}_2)$. In any case it follows that

$$\pi_r(\mathbf{j}_1)_j = \pi_r(\mathbf{j}_2)_j = 1.$$

If $j+1 \notin \operatorname{supp}(\mathbf{j}_1) \cup \operatorname{supp}(\mathbf{j}_2)$ then $\pi_r(\mathbf{j}_1)_j = \pi_r(j, \iota_{\mathbf{j}_1}(j))$ and (5.11) gives that $j \in \operatorname{supp}^-(\mathbf{j}_1)$. Otherwise we have $\pi_r(\mathbf{j}_2)_j = \pi_r(j, \iota_{\mathbf{j}_1}(j+1)-1)$ and (5.11) shows that $j \in \operatorname{supp}^-(\mathbf{j}_1)$ in this case as well, contradicting $\operatorname{supp}^-(\mathbf{j}_1) = \emptyset$. This completes the proof of Proposition 5.9.

6.11. Proof of Proposition 5.10. It clearly suffices to prove the result when $S = \{j\}$ since the general case follows by transitivity. We can assume that $j \in \operatorname{supp}^+(\mathbf{j})$ since then using Lemma 5.8 the case $j \subset \operatorname{supp}^-(\mathbf{j})$ follows. Since,

$$\operatorname{supp}(\mathbf{j}) \cap [r, j-1] = \operatorname{supp}(\mathbf{j}^+(S)) \cap [r, j-1] \quad \text{and}$$

$$\operatorname{supp}(\mathbf{j}) \cap [\tau_{\mathbf{j}}{}^+(j)+1, n] = \operatorname{supp}(\mathbf{j}^+(S)) \cap [\tau_{\mathbf{j}}{}^+(j)+1, n],$$

it follows immediately that

$$\pi_r(\mathbf{j})_l = \pi_r(\mathbf{j}^+(S))_l \ \forall\, l \in [r, j-1) \cup (\tau_{\mathbf{j}}^+(j), n].$$

If $l \in [j-1, \tau_{\mathbf{j}}^+(j)]$ we proceed by induction on l. To see that induction starts at $l = j-1$, observe that

$$j \notin \operatorname{supp}(\mathbf{j}^+(S)), \ \ 2(\iota_{\mathbf{j}}(j)-1) = (j-1)-(N+1).$$

The conclusion follows from (5.11). For the inductive step, assume that

$$\pi_r(\mathbf{j})_{l'} = \pi_r(\mathbf{j}^+(j))_{l'}, \ \forall\, j-1 \le l' < l \le \tau_{\mathbf{j}}^+(j).$$

Assume first that $l \in \operatorname{supp}(\mathbf{j})$, in particular $l < \tau_{\mathbf{j}}^+(j)$. If $(l+1) \in \operatorname{supp}(\mathbf{j})$ then,

$$\pi_r(\mathbf{j})_l = \pi_r(\mathbf{j}^+(S))_l = 1$$

and we are done. If $(l+1) \notin \operatorname{supp}(\mathbf{j})$, then $(l+1) \in \operatorname{supp}(\mathbf{j}^+(j))$ and we have from the definition of π_r that,

$$\pi_r(\mathbf{j})_l = \pi_r(l, \iota_{\mathbf{j}}(l))_l \qquad \text{and } \ \pi_r(\mathbf{j}^+(j))_l = \pi_r(l, \iota_{\mathbf{j}^+(j)}(l+1)-1)_l,$$

which gives,

$$\pi_r(\mathbf{j})_l = (1 - q^{i-2r+l-2\iota_{\mathbf{j}}(l)}u)(1 - q^{2n-i-l+2\iota_{\mathbf{j}}(l)-2}u)^{-1},$$

and

$$\pi_r(\mathbf{j}^+(j))_l = (1 - q^{2n-i-l+2\iota_{\mathbf{j}^+(j)}(l+1)-4}u)(1 - q^{i+l-2r-2\iota_{\mathbf{j}^+(j)}(l+1)+2}u)^{-1}.$$

Let

$$l_0 = \min\{m \ge j : m \le l'' \le l \Rightarrow l'' \in \operatorname{supp}(\mathbf{j})\}.$$

Then, either $l_0 = j$, in which case we have

$$\iota_{\mathbf{j}}(l) = \iota_{\mathbf{j}}(j) + l - j = (i - r - n - j + 2l + 2)/2, \qquad \iota_{\mathbf{j}^+(j)}(l+1) = \iota_{\mathbf{j}}(j),$$

which implies

$$\pi_r(\mathbf{j})_l = \pi_r(\mathbf{j}^+(j))_l,$$

and we are done, or $l_0 > j$. In that case $(l_0 - 1) \in \operatorname{supp}(\mathbf{j}^+(j))$, $\iota_{\mathbf{j}}(l) = \iota_{\mathbf{j}}(l_0) + l - l_0$, and

$$\pi_r(\mathbf{j})_{l_0-1} = \pi_r(l_0 - 1, \iota_{\mathbf{j}}(l_0) - 1)_{l_0-1}, \qquad \text{and}$$
$$\pi_r(\mathbf{j}^+(j))_{l_0-1} = \pi_r(l_0 - 1, \iota_{\mathbf{j}^+(j)}(l_0 - 1))_{l_0-1}.$$

Since $l_0 - 1 < l$, the induction hypothesis gives $\pi_r(\mathbf{j})_{l_0-1} = \pi_r(\mathbf{j}^+(j))_{l_0-1}$, which implies that

$$\iota_{\mathbf{j}^+(j)}(l_0 - 1) = i + l_0 - n - r - \iota_{\mathbf{j}}(l_0) + 1$$

and, therefore,

$$\iota_{\mathbf{j}^+(j)}(l+1) = i + l_0 - n - r - \iota_{\mathbf{j}}(l_0) + 2.$$

The conclusion follows.

Finally, suppose that $l \notin \operatorname{supp}(\mathbf{j})$. If $l = \tau_{\mathbf{j}}^+(j)$, then (5.11) and Lemma 5.7 imply

$$\pi_r(\mathbf{j})_l = \pi_r(\mathbf{j}^+(j))_l = 1,$$

and we are done. If $l < \tau_{\mathbf{j}}^+(j)$ and $l + 1 \in \operatorname{supp}(\mathbf{j}^+(j))$ we again have

$$\pi_r(\mathbf{j})_l = \pi_r(\mathbf{j}^+(j))_l = 1$$

and we are done. Otherwise we have $(l+1) \in \operatorname{supp}(\mathbf{j})$, and

$$\pi_r(\mathbf{j})_l = \pi_r(l, \iota_{\mathbf{j}}(l+1) - 1)_l, \qquad \pi_r(\mathbf{j}^+(j))_l = \pi_r(l, \iota_{\mathbf{j}^+(j)}(l))_l,$$

which gives,

$$\pi_r(\mathbf{j})_l = (1 - q^{2n-i-l+2\iota_{\mathbf{j}}(l+1)-4}u)(1 - q^{i-2r+l-2\iota_{\mathbf{j}}(l+1)+2}u)^{-1}$$

and

$$\pi_r(\mathbf{j}^+(j))_l = (1 - q^{i+l-2r-2\iota_{\mathbf{j}^+(j)}(l)}u)(1 - q^{2n-i-l+2\iota_{\mathbf{j}^+(j)}(l)-2}u).$$

Let

$$l_0 = \max\{l' : j \le l' < l, l' \in \operatorname{supp}(\mathbf{j})\}.$$

Then, $l_0 + 1 \in \operatorname{supp}(\mathbf{j}^+(j))$ and

$$\iota_{\mathbf{j}^+(j)}(l) = \iota_{\mathbf{j}^+(j)}(l_0 + 1) + l - (l_0 + 1), \qquad \iota_{\mathbf{j}}(l_0) = \iota_{\mathbf{j}}(l+1) - 1.$$

In particular,

$$\pi_r(\mathbf{j})_{l_0} = \pi_r(l_0, \iota_{\mathbf{j}}(l_0))_{l_0} \qquad \text{and} \qquad \pi_r(\mathbf{j}^+(j))_{l_0} = \pi_r(l_0 + 1, \iota_{\mathbf{j}^+(j)}(l_0 + 1) - 1)_{l_0}.$$

Since $l_0 < l$, the induction hypothesis gives

$$\pi_r(\mathbf{j})_{l_0} = \pi_r(\mathbf{j}^+(j))_{l_0}$$

which implies that

$$\iota_{\mathbf{j}^+(j)}(l_0+1) = i - r - n + l_0 + \iota_{\mathbf{j}}(l_0) + 2.$$

It follows that $\pi_r(\mathbf{j}^+(j))_l = \pi_r(\mathbf{j})_l$ and the proof of the Proposition is complete. □

6.12 Proof of Proposition 5.5. For this proof it is necessary to indicate that $\text{supp}^{\pm}(\mathbf{j})$ depends on r and so from now on, if $\mathbf{j} \in \mathbf{J}_{k,r}$ we denote the set $\text{supp}^{\pm}(\mathbf{j})$ by $\text{supp}_r^{\pm}(\mathbf{j})$. We also assume that the representatives of the equivalence classes of $\mathbf{J}_{k,r}$ are chosen so that $\text{supp}_r^-(\mathbf{j}) = \emptyset$. We will show that

$$\dim V_{\pi_r(\mathbf{j})\pi_s(\mathbf{j},\pm)} \geq |\bar{\mathbf{j}}| = 2^{|\text{supp}_r^+(\mathbf{j})|} \quad \forall\, \mathbf{j} \in \mathbf{J}_r \ \text{ satisfying } \ \text{supp}_r^-(\mathbf{j}) = \emptyset.$$

We proceed by induction on $M = (i-r)/2$ noting that induction starts at $M = 0$. For the inductive step, assume that we know the result for all $M' < M$, i.e., we know that for all $s \in I_i$, $s > r$ we have

$$\dim V_{\pi_s(\mathbf{j}')\pi_s(\mathbf{j}',\pm)} \geq 2^{|\text{supp}_s^+(\mathbf{j}')|} \ \forall\, \mathbf{j}' \in \mathbf{J}_s \ \text{ satisfying } \ \text{supp}_s^-(\mathbf{j}') = \emptyset.$$

Given $j \in \text{supp}_r^+(\mathbf{j})$, let $\mathbf{j}' \in \mathbf{J}_{k-1,r+2}$ be the partition whose support is given by,

$$\text{supp}(\mathbf{j}') = \{j' + 2 : r < j' < \sigma_{\mathbf{j}}^+(j),\ j' \in \text{supp}(\mathbf{j})\} \cup \{j' \in \text{supp}(\mathbf{j}) : \sigma_{\mathbf{j}}^+(j) < j' < n\}.$$

Observe that

$$\text{supp}_{r+2}^-(\mathbf{j}') = \emptyset,$$

and

$$\text{supp}_{r+2}^+(\mathbf{j}') = \{j' + 2 : r < j' < \sigma_{\mathbf{j}}^+(j),\ j' \in \text{supp}_r^+(\mathbf{j})\} \cup \{j' \in \text{supp}_r^+(\mathbf{j}) : \sigma_{\mathbf{j}}^+(j) < j' < n\}.$$

This gives,

$$\sigma_{\mathbf{j}}^+(j) > j \Rightarrow |\text{supp}_{r+2}^+(\mathbf{j}')| = |\text{supp}_r^+(\mathbf{j})|,$$
$$\sigma_{\mathbf{j}}^+(j) = j \Rightarrow |\text{supp}_{r+2}^+(\mathbf{j}')| = |\text{supp}_r^+(\mathbf{j})| - 1.$$

The inductive step is proved if we show that,

$$\sigma_{\mathbf{j}}^+(j) > j \Rightarrow \dim V_{\pi_r(\mathbf{j})\pi_r(\mathbf{j},\pm)} \geq \dim V_{\pi_{r+2}(\mathbf{j}')\pi_{r+2}(\mathbf{j}',\pm)},$$
$$\sigma_{\mathbf{j}}^+(j) = j \Rightarrow \dim V_{\pi_r(\mathbf{j})\pi_r(\mathbf{j},\pm)} \geq 2\dim V_{\pi_{r+2}(\mathbf{j}')\pi_{r+2}(\mathbf{j}',\pm)}.$$

Set $l = \sigma_{\mathbf{j}}^+(j)$. Using Lemma 5.2 we find that,

$$T_{r+2,l}(\pi_{r+2}(\mathbf{j}')\pi_{r+2}(\mathbf{j}',\pm)) = \pi_r(\mathbf{j})\pi_r(\mathbf{j},\pm)\alpha_{l,q^{2n-i-l+2\iota_{\mathbf{j}}(l)-4}},$$

and that

$$(T_{r+2,l}(\pi_{r+2}(\mathbf{j}')\pi_{r+2}(\mathbf{j}',\pm)))_l = (1 - q^{2n-i-l+2\iota_{\mathbf{j}}(l)-4}u)(1 - q^{i+l-2r-2\iota_{\mathbf{j}}(l)}u).$$

It now follows from Proposition 1.9 that $\dim V_{\pi_r(\mathbf{j})\pi_r(\mathbf{j},\pm)} \geq \dim V_{\pi_{r+2}(\mathbf{j}')\pi_{r+2}(\mathbf{j}',\pm)}$. Further,

$$\sigma^+_{\mathbf{j}}(j) = j \Rightarrow (T_{r+2,l}(\boldsymbol{\pi}_{r+2}(\mathbf{j}')\boldsymbol{\pi}_{r+2}(\mathbf{j}', \pm)))_l = (1 - q^{n-r-2}u)^2,$$

and we are done by using Proposition 1.9 once more. □

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