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RESEARCH ARTICLE

Asset pricing and hedging in financial markets with fixed and proportional transaction costs

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Abstract
We establish the asset pricing and hedging principle in a financial market model, which is a specific case of the von Neumann-Gale dynamical system, with both fixed and proportional transaction costs and trading constraints. The main results are hedging criteria stated in terms of consistent valuation systems, generalizing the notion of an equivalent martingale measure.

Keywords Von Neumann–Gale dynamical systems · Asset pricing · Hedging · Consistent valuation systems · Transaction costs · Portfolio constraints

JEL Classification G10 · G12 · G13 · C61 · C65 · C67

1 Introduction

The classical result in Mathematical Finance states that in a complete frictionless market with no transaction costs and no portfolio constraints the "fair" (no-arbitrage) price of a derivative security equals the expectation, with respect to the unique martingale measure, of the security's discounted payoff (see, for instance, Pliska (1997); Björk (1998), or Föllmer and Schied (2002)). This result has been extended to financial markets with transaction costs and trading constraints by numerous researchers, including Bensaid et al. (1992); Cvitanić and Karatzas (1996); Soner et al. (1995); Jouini and Kallal (1995a, b, 1999); Jouini (2000, 2001); Föllmer and Kramkov (1997); Carassus et al. (2001); Napp (2001); Jouini and Napp (2001); Kabanov and Stricker (2001); Kabanov (1999, 2001); Stettner (2002); Schachermayer (2004); Roux (2011), and others.

Dempster et al. (2006) established the pricing principle which unifies previous results by leveraging the parallelism between paths of economic dynamics in the von

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Neumann-Gale model and hedging strategies in a dynamic model of a financial market. Von Neumann-Gale dynamical systems, originally introduced by Von Neumann (1937); Gale (1956), and Rockafellar (1967) for deterministic economic growth modelling, have been extended to stochastic settings by Dynkin, Radner, and others in the 1970 s Dynkin (1971, 1972); Dynkin and Yushkevich (1979); Radner (1971, 1972). Recent advancements (see, e.g., Evstigneev and Schenk-Hoppé 2006, 2008; Bahsoun et al. 2008; Zhitlukhin 2019) have surmounted earlier challenges, and shown that these stochastic systems provide a natural framework for modelling financial markets with frictions. Dempster et al. (2006) addressed no-arbitrage pricing and hedging with proportional transaction costs, and later was extended to a broader context by Evstigneev and Zhitlukhin (2013). Babaei et al. (2020a, b, 2021) explored von Neumann-Gale dynamics in the capital growth theory under proportional transaction costs, further extending the results in this field.

The primary contribution of this paper lies in the extension of the hedging and pricing principle in multiple directions: incorporating fixed and proportional transaction costs, allowing short selling under specified constraints, including margin requirements, and accounting for assets paying dividends with potentially different rates for long and short positions. The main results of the paper present general hedging criteria expressed in terms of *consistent discount factors* and *consistent valuation systems*, extending the notion of an equivalent martingale measure.

Let us briefly mention other results related to the topic of the present paper. The equivalence between the absence of arbitrage and the existence of risk-neutral measures under fixed transaction costs has been studied by Jouini et al. (2001). Lépinette and Tran (2016, 2017) investigated the separation of risk measures in the same context. Brown and Zastawniak (2020) extended this analysis to situations where both fixed and proportional transaction costs apply concurrently. They demonstrated that the absence of arbitrage in a model consisting of one risky asset and one risk-free asset with both fixed and proportional transaction costs is equivalent to the existence of a family of absolutely continuous single-step probability measures. This family, together with an adapted process within the bid-ask intervals satisfying the martingale property for each measure, ensures the absence of arbitrage.

The paper's structure is as follows. Section 2 describes the model. Section 3 states the hedging problem and provides some basic results about it in the model at hand. Section 4 establishes the hedging criteria in terms of consistent discount factors. Section 5 defines and discusses the notion of a consistent valuation system and its relation to the hedging problem. The Appendix assembles general mathematical facts used in this work.

2 The model

We consider a model of a financial market with fixed and proportional transaction costs, as well as portfolio constraints, which is based on von Neumann-Gale dynamical systems.

Let $(\Omega, \mathcal{F}, P)$ be a finite probability space and $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_T = \mathcal{F}$ a sequence of algebras representing a filtration on this space. Sets in the algebra $\mathcal{F}_t$

are interpreted as events observable prior to date t. Without loss of generality the probability of each $\omega \in \Omega$ is positive. Equalities and inequalities for random variables will be understood to hold for all ω.

We consider a market where m assets are traded at dates $t = 0, 1, \ldots, T$. The positions of a portfolio $a \in \mathbb{R}^m$ will be expressed in terms of their value.

For each $t = 0, 1, \ldots, T$ and $i = 1, \ldots, m$ the following $\mathcal{F}_t$-measurable random variables are given: market prices $s_{t,i} > 0$, transaction cost rates for selling and buying assets $0 \le \lambda^+_{t,i} < 1$, $\lambda^-_{t,i} \ge 0$, fixed transaction costs $C_t \ge 0$, dividend yield for long and short positions $0 \le D^+_{t,i} \le D^-_{t,i}$. Let $R_{t,i} = s_{t,i}/s_{t-1,i}$ denote the return of assets.

Portfolio constraints in the model are specified by the cones[1]

$$X_t(\omega) = \left\{ a \in \mathbb{R}^m : \sum_{i=1}^m \left(1 - \lambda^+_{t,i}(\omega)\right) a^i_+ \ge \mu_t(\omega) \sum_{i=1}^m \left(1 + \lambda^-_{t,i}(\omega)\right) a^i_- \right\}, \quad (1)$$

where $\mu_t(\omega) > 1$ are $\mathcal{F}_t$-measurable random variables which can be interpreted as margin requirement coefficients: a trader must be able to liquidate the long positions of her portfolio to cover the short positions with excess determined by μ_t.

Trading in the model at hand goes on as follows. At each date t, $t = 1, \ldots, T-1$, a trader pays C_t as fixed transaction costs. We will assume that $C_0 = C_T = 0$. Then she receives the dividend on her portfolio $a(\omega)$ that she purchased at the previous date. The amount of dividend is specified by the function $d_t(\omega, a)$ defined by the formula

$$d_t(a) = \sum_{i=1}^m \left(D^+_{t,i} a^i_+ - D^-_{t,i} a^i_- \right).$$

Here $D^{\pm}_{t,i}$ specify the amount of dividend received or returned[2] for each dollar invested in asset i. The amount of dividend received or returned for 1 physical unit of asset i will be $D^{\pm}_{t,i} s_{t-1,i}$.

After that, the trader rearranges her portfolio $a(\omega)$ with added dividend to a portfolio $b(\omega)$ subject to the self-financing constraint. The possibility of rearrangement is specified by the inequality

$$\sum_{i=1}^m \left(1 - \lambda^+_{t,i}\right) \left(R_{t,i} a^i - b^i\right)_+ + d_t(a) \ge \sum_{i=1}^m \left(1 + \lambda^-_{t,i}\right) \left(R_{t,i} a^i - b^i\right)_- + C_t. \quad (2)$$

The left-hand side of (2) is the amount of money the trader receives for selling the assets plus the dividends, the right-hand side is the amount of money she pays for buying the assets, including fixed transaction costs. This inequality means that the trader does not use external funds to rearrange her portfolio, so it can be regarded as a self-financing condition (with a possibility of free disposal).

[1] A set X in a linear space is called a cone if it contains with any its elements x, y any non-negative linear combination $\lambda x + \mu y$ ($\lambda, \mu \ge 0$) of these elements. The cone X is called pointed if the inclusions $x \in X$ and $-x \in X$ imply $x = 0$.

[2] We assume that dividend on short positions must be returned.

Let

$$\psi_t(\omega, a, b) = \sum_{i=1}^{m} \left(1 - \lambda_{t,i}^{+}\right) \left(R_{t,i} a^i - b^i\right)_+ \\ - \sum_{i=1}^{m} \left(1 + \lambda_{t,i}^{-}\right) \left(R_{t,i} a^i - b^i\right)_- + d_t(a). \quad (3)$$

Then inequality (2) can be stated as $\psi_t(\omega, a, b) \geq C_t(\omega)$. The above description of the model corresponds to the requirement that pairs of portfolios (a, b) belong to the sets

$$Z_t(\omega) := \{(a, b) \in X_{t-1}(\omega) \times X_t(\omega) : \psi_t(\omega, a, b) \geq C_t(\omega)\}. \quad (4)$$

A natural instance of this model is when asset 1 represents cash deposited with a bank account, and the other assets represent holdings in shares of stock. Then it is natural to put $s_{t,1} = 1$, $\lambda_{t,1}^{\pm} = 0$ (value is expressed in terms of cash, and there are no transaction costs for cash). The random variables $D_{t,1}^{\pm}$ may be $\mathcal{F}_{t-1}$-measurable and represent *risk-free* interest rates for lending and borrowing money. The random variables $D_{t,i}^{\pm}$, $i \geq 2$, may be $\mathcal{F}_t$-measurable and represent dividend yield rates on stock. The possibility to have different dividend rates for long and short positions may be used e.g. when some assets pay dividends in a currency different from asset 1 and there is a bid–ask spread in the exchange rates.

Observe that $Z_t(\omega)$ is a convex set but not a cone: it is convex, since the function $\psi_t(a, b)$ is concave as follows from the representation

$$\psi_t(a, b) = \sum_{i=1}^{m} \left[\left(1 - \lambda_{t,i}^{+}\right) \left(R_{t,i} a^i - b^i\right) + D_{t,i}^{+} a^i\right] \\ - \sum_{i=1}^{m} \left[\left(\lambda_{t,i}^{-} + \lambda_{t,i}^{+}\right) \left(R_{t,i} a^i - b^i\right)_- + \left(D_{t,i}^{-} - D_{t,i}^{+}\right) a_-^i\right],$$

where the first sum is a linear function of a, b and the second sum is a convex function of a, b. However, it does not contain with any pair (a, b) all pairs $\lambda(a, b)$, where $\lambda \geq 0$.

The sets $X_t(\omega)$ and $Z_t(\omega)$ described above generate a stochastic dynamical system over the time interval $t = 0, 1, \ldots, T$. Let $\mathcal{L}_t^m$ $(t = 0, 1, ...)$ be a linear space of $\mathcal{F}_t$-measurable vector functions $x(\omega)$ with values in $\mathbb{R}^m$. We say that a vector function $x(\omega)$ is a *random state* of the system and write $x \in \mathcal{X}_t$ if $x \in \mathcal{L}_t^m$ and $x(\omega) \in X_t(\omega)$.

A sequence of random states $x_0 \in \mathcal{X}_0, x_1 \in \mathcal{X}_1, ...$ is called a *feasible trading strategy* if

$$(x_{t-1}(\omega), x_t(\omega)) \in Z_t(\omega),\ t = 1, \ldots, T - 1.$$

Note that a feasible trading strategy in the model is nothing but a *path* of the dynamical system under consideration. Thus, this model is a version of a von Neumann-Gale model in which the sets X_t are cones, but the sets Z_t are just convex. This extends previous literature on applications of von Neumann-Gale models to mathematical finance which dealt with conic models (see, e.g., Dempster et al. 2006; Evstigneev and Zhitlukhin 2013).

3 The hedging problem

Let us define the following cones describing possibilities of constructing initial portfolios and liquidating terminal ones:

$$V_0(\omega) = \{(a, b) \in R_+ \times X_0 : a \geq -\psi_0(\omega, 0, b)\}, \tag{5}$$

and

$$V_T(\omega) = \{(a, b) \in X_{T-1}(\omega) \times R : \psi_T(\omega, a, 0) \geq b\}, \tag{6}$$

where ψ_0 and ψ_T are defined by (3) (in the formula for ψ_0 one can formally put $R_{0,i} = 1$ and $d_0(a) \equiv 0$, since we only consider $\psi_0(\omega, 0, \cdot)$).

Thus, we state that to create a portfolio b at time 0, it is essential to have at least an amount of money $-\psi_0(\omega, 0, -b)$. When liquidating a portfolio a created at time $T-1$ to obtain funds at time T, one must ensure that the resulting amount of money $\psi_T(\omega, a, 0)$ exceeds the value of b. This rationale underlies the definitions of the cones V_0 and V_T in equations (5) and (6).

A sequence $(v_0, x_0, x_1, \ldots, x_{T-1}, v_T)$ is called a *hedging strategy* if

(a) $(x_0, x_1, \ldots, x_{T-1})$ is a feasible trading strategy,
(b) $v_0 \in \mathcal{L}_0^1$ and $(v_0(\omega), x_0(\omega)) \in V_0(\omega)$,
(c) $v_T \in \mathcal{L}_T^1$ and $(x_{T-1}(\omega), v_T(\omega)) \in V_T(\omega)$.

We shall say that an initial endowment $v_0 \in \mathcal{L}_0^1$ *allows the hedging* of a contingent claim $v_T \in \mathcal{L}_T^1$ if there exists a hedging strategy of the form $(v_0, x_0, x_1, \ldots, x_{T-1}, v_T)$.

Denote by $\mathcal{H}$ the set of pairs $(v_0, v_T) \in \mathcal{L}_0^1 \times \mathcal{L}_T^1$ such that v_0 allows the hedging of v_T. The main question we are interested in is how to characterize the set $\mathcal{H}$. This question is of fundamental importance for asset pricing: the smallest element of the set $\{v_0 : (v_0, v_T) \in \mathcal{H}\}$ represents the *hedging price* of the contingent claim v_T.

Let us introduce two basic assumptions that will be assumed to hold throughout the paper. Define $\Lambda_{t,i}^+(\omega) = 1 - \lambda_{t,i}^+$ and $\Lambda_{t,i}^-(\omega) = 1 + \lambda_{t,i}^-$. Then we require the following to hold.

(**B1**) For each t, there exist constants $\underline{R}_t, \overline{R}_t, \underline{\Lambda}_t, \overline{\Lambda}_t, \overline{D}_t$, and $\overline{C}_t$ such that $0 < \underline{R}_t \leq R_{t,i}(\omega) \leq \overline{R}_t$, $0 < \underline{\Lambda}_t \leq \Lambda_{t,i}^+(\omega)$, $\Lambda_{t,i}^-(\omega) \leq \overline{\Lambda}_t$, $D_{t,i}^-(\omega) \leq \overline{D}_t$, and $C_t(\omega) \leq \overline{C}_t$ for all i, ω.

(**B2**) For each t, we have $\mu_t > \nu_t$ where

$$\nu_t := \max\{(\overline{\Lambda}_{t+1}\overline{R}_{t+1} + \overline{D}_{t+1})/(\underline{\Lambda}_{t+1}\underline{R}_{t+1} + \underline{D}_{t+1}); \overline{\Lambda}_t/\underline{\Lambda}_t\}$$

and $\underline{D}_t \geq 0$ is a constant such that $\underline{D}_t \leq D_{t,i}^+(\omega)$ for all ω, i.

These assumptions are not restrictive, and in fact, (**B1**) is always satisfied since the probability space is finite.

The following propositions prove some technical properties of the model that will be used in what follows. For convenience of notation, denote

$$Z_0(\omega) := V_0(\omega), \quad Z_T(\omega) := V_T(\omega).$$

The first proposition shows that the model has a property of "strict monotonicity".

Proposition 1 *Suppose $(a, b) \in Z_t(\omega)$ for some $t = 0, \dots, T$. Then for any $a' \geq a$ such that $a' \neq a$, there exists b' such that $b' \geq b$, $b' \neq b$ and $(a', b') \in Z_t(\omega)$.*

Proof We first prove it for $t = 0$. Let $(a, b) \in Z_0(\omega) = V_0(\omega)$, and $a' > a$. Let $r = (\frac{a'-a}{\overline{\Lambda}_1}, 0, \dots, 0)$, and $b' = b + r$. As $X_t(\omega) \supseteq R^m_+$ and is a cone, we have $b' \in X_0(\omega)$. Then

$$-\psi_0(\omega, 0, b') = -\psi_0(\omega, 0, b + r) \leq -\psi_0(\omega, 0, b) - \psi_0(\omega, 0, r) \leq a',$$

where the first inequality holds because $\psi_0(\omega, 0, x)$ is concave and positively homogeneous (of degree one) in x, and the second inequality holds by using condition (**B1**) and the fact that $-\psi_0(\omega, 0, r) = \left(1 + \lambda^+_{0,1}\right) \frac{a'-a}{\overline{\Lambda}_1} \leq a' - a$. This proves that $(a', b') \in Z_0(\omega)$

Let us prove it for $t = T$. Let $(a, b) \in Z_T(\omega) = V_T(\omega)$, and $a' \geq a$, $a' \neq a$. We can write $a' = a + r$, where $r = \left(r^1, \dots, r^m\right)$, $r^i \geq 0$ for all i, and $r^j > 0$ for some j. Then

$$\psi_T\left(\omega, a', 0\right) \geq \psi_T(\omega, a, 0) + \psi_T(\omega, r, 0) \geq b + r_j\left(\underline{D}_T + \underline{R}_T \underline{\Lambda}_T\right) > b.$$

Let $b' = b + r_j\left(\underline{D}_T + \underline{R}_T \underline{\Lambda}_T)\right)$. Then $\left(a', b'\right) \in Z_T(\omega)$.

We now consider $t = 1, \dots, T-1$. Let $(a, b) \in Z_t(\omega)$, and $a' \geq a$, $a' \neq a$. Then we have $\psi_t(\omega, a, b) \geq C_t$, where ψ_t is defined by (3), and $a' = a + r$, where $r = (r^1, \dots, r^m)$, $r^i \geq 0$ for all i, and $r^j > 0$ for some j. Let $b' = b + r'$, where $r' = (0, \dots, \underline{R}_t r^j, \dots, 0)$. Note that $b' \in X_t(\omega)$. We will prove that $(a', b') \in Z_t(\omega)$. Indeed,

$$\psi_t(\omega, a', b') \geq \psi_t(\omega, a, b) + \psi_t(\omega, r, r') > C_t(\omega),$$

where the first inequality holds because $\psi_t(\omega, x, y)$ is concave and positively homogeneous (of degree one) in (x, y), and the second inequality holds in view of condition (**B1**) and the relation

$$\psi_t(\omega, r, r') = \sum_{i \neq j}^{m} (1 - \lambda^+_{t,i})(R_{t,i} r^i) + (1 - \lambda^+_{t,j})(R_{t,j} r^j - \underline{R}_t r^j) + d_t(r) > 0.$$

□

Let $|\cdot|$ denote the norm of a vector in a finite-dimensional space defined as the sum of the absolute values of its coordinates. For a finite-dimensional vector a, we will denote by $\mathbb{B}(a, r)$ the ball $\{b : |b-a| \le r\}$. In the next proposition, we prove that the sets Z_t have non-empty interior.

Proposition 2 *For each $t = 1, 2, \ldots, T-1$, there exists an $\mathcal{F}_t$-measurable vector function $\mathring{z}_t = (\mathring{x}_t, \mathring{y}_t)$ such that for all $\omega \in \Omega$, we have*

$$\mathbb{B}(\mathring{z}_t, \varepsilon_t) \subseteq Z_t(\omega), \tag{7}$$

where $\varepsilon_t > 0$ is some constant.

We will need the following auxiliary result to prove Proposition 2.

Lemma 1 *(a) For each t there exists a constant $\tau_t > 0$ such that if $a \in X_t(\omega)$ then $|a_+| - \nu_t |a_-| \ge \tau_t |a|$.*
(b) For each t there exist positive constants κ_t^1 and κ_t^2 such that if $a \in X_{t-1}(\omega)$, $b \in X_t(\omega)$ and $|b| \le \kappa_t^1 |a| - \kappa_t^2$, then $(a, b) \in Z_t(\omega)$.

Proof (a) Observe that $X_t(\omega) \subseteq \tilde{X}_t$, where $\tilde{X}_t = \{a \in \mathbb{R}^m : \mu_t |a_-| \le |a_+|\}$, and since $\mu_t > 1$ we have $\tilde{X}_t \cap (-\tilde{X}_t) = \{0\}$. The continuous function $h_t(a) = |a_+| - \nu_t |a_-|$ is strictly positive on the compact set $\hat{X}_t = \tilde{X}_t \cap \{a : |a| = 1\}$. Indeed, since $h_t(a) \ge (\mu_t - \nu_t)|a_-|$ on $\tilde{X}_t$, then the equality $h_t(a) = 0$ would imply $|a_-| = 0$, and hence $|a_+| = h_t(a) = 0$, so that $|a| = 0$. Then $h_t(a)$ attains a strictly positive minimum on $\hat{X}_t$, which can be taken as τ_t.

(b) Let $b \in X_t(\omega)$. It is straightforward to check that for any numbers x, y we have $(x-y)_+ \ge x_+ - y_+$ and $(x-y)_- \le x_- + y_+$. Using this, we obtain for any $a \in X_{t-1}(\omega)$

$$\begin{aligned}
\psi_t(a, b) &\ge \sum_{i=1}^m ((\Lambda_{t,i}^+ R_{t,i} + D_{t,i}^+) a_+^i - (\Lambda_{t,i}^- R_{t,i} + D_{t,i}^-) a_-^i) - \sum_{i=1}^m (\Lambda_{t,i}^+ + \Lambda_{t,i}^-) b_+^i \\
&\ge (\underline{\Lambda}_t \underline{R}_t + \underline{D}_t)|a_+| - (\overline{\Lambda}_t \overline{R}_t + \overline{D}_t)|a_-| - 2\overline{\Lambda}_t |b_+| \\
&\ge (\underline{\Lambda}_t \underline{R}_t + \underline{D}_t)(|a_+| - \nu_{t-1}|a_-|) - 2\overline{\Lambda}_t |b| \\
&\ge \tau_{t-1}(\underline{\Lambda}_t \underline{R}_t + \underline{D}_t)|a| - 2\overline{\Lambda}_t |b|,
\end{aligned} \tag{8}$$

where the third inequality follows from (**B2**). Then statement (b) can be fulfilled with the constant $\kappa_t^1 = \tau_{t-1}(\underline{\Lambda}_t \underline{R}_t + \underline{D}_t)/(2\overline{\Lambda}_t)$ and $\kappa_t^2 = \overline{C}_t / 2\overline{\Lambda}_t$, since in that case $\psi_t(a, b) \ge C_t$, implying $(a, b) \in Z_t$. □

Proof of Proposition 2 Let $\mathring{x}_t = \frac{2\kappa_t^2}{\kappa_t^1 m}(1, \ldots, 1) \in \mathbb{R}^m$. Put $\mathring{z}_t = (\mathring{x}_t, \mathring{y}_t)$ with $\mathring{y}_t = (\kappa_t^1/4)\mathring{x}_t$. Note that $|\mathring{x}_t| = 2\kappa_t^2/\kappa_t^1$, and $|\mathring{y}_t| = \kappa_t^2/2 < \kappa_t^1 |\mathring{x}_t| - \kappa_t^2 = \kappa_t^2$, thus statement (b) of Lemma 1 implies $\mathring{z}_t \in Z_t$. Observe that there exists $\delta_t > 0$ such that $\mathbb{B}(\mathring{z}_t, \delta_t) \subset \mathbb{R}_+^{2m}$ and therefore $\mathbb{B}(\mathring{z}_t, \delta_t) \subset X_{t-1} \times X_t$. Since $|\mathring{y}_t| < \kappa_t^1 |\mathring{x}_t| - \kappa_t^2$, then one can find $0 < \varepsilon_t \le \delta_t$ such that $|b| \le \kappa_t^1 |a| - \kappa_t^2$ for any $(a, b) \in \mathbb{B}(\mathring{z}_t, \varepsilon_t)$. Indeed, we have

$$|b| \le |b - \mathring{y}_t| + |\mathring{y}_t| \le \varepsilon_t + \kappa_t^2/2 = \varepsilon_t + 2\kappa_t^2 - 3\kappa_t^2/2$$
$$\le \varepsilon_t + \kappa_t^1|a| + \kappa_t^1\varepsilon_t - 3\kappa_t^2/2 \le \kappa_t^1|a| - \kappa_t^2.$$

The third inequality holds because $|a| \ge |\mathring{x}_t| - \varepsilon_t$, and the last inequality holds as long as $\varepsilon_t \le \kappa_t^2/2(1+\kappa_t^1)$. Hence, $\mathring{z}_t$ and ε_t satisfy conditions of proposition. □

Proposition 3 *For any $b \in X_t(\omega)$, we have $\psi_t(\omega, 0, -b) \ge 0$, and $\psi_t(\omega, 0, b) \le 0$. If $b \ne 0$, then $\psi_t(\omega, 0, -b) > 0$, and $\psi_t(\omega, 0, b) < 0$.*

Proof This follows from conditions (**B1**), (**B2**) and Lemma 1. Indeed,

$$\psi_t(\omega, 0, -b) = \sum_{i=1}^{m}(1 - \lambda_{t,i}^{+}(\omega))b_+^i - \sum_{i=1}^{m}(1 + \lambda_{t,i}^{-}(\omega))b_-^i$$
$$\ge \underline{\Lambda}_t|b_+| - \overline{\Lambda}_t|b_-| \ge \underline{\Lambda}_t(|b_+| - \nu_t|b_-|) \ge \underline{\Lambda}_t\tau_t|b|.$$

In this chain, the first inequality follows from (**B1**), the second follows from (**B2**), and the last one follows from Lemma 1.

The second claim of the proposition follows from the chain of relations

$$-\psi_t(\omega, 0, b) = \sum_{i=1}^{m}(1 + \lambda_{t,i}^{-}(\omega))b_+^i - \sum_{i=1}^{m}(1 - \lambda_{t,i}^{+}(\omega))b_-^i$$
$$\ge |b_+| - |b_-| \ge |b_+| - \nu_t|b_-| \ge \tau_t|b|.$$

□

4 Consistent discount factors and solution of the hedging problem

Let us call *consistent discount factors with a transaction costs term* (hereinafter, simply *consistent discount factors*) any triple (q_0, q_T, r), where $q_0 \in \mathcal{L}_0^1$, $q_T \in \mathcal{L}_T^1$ are strictly positive functions and $r \ge 0$ is a constant such that

$$Eq_0v_0 \ge Eq_Tv_T + r \text{ for any } (v_0, v_T) \in \mathcal{H}.$$

The next theorem provides the general solution to the hedging problem in terms of consistent discount factors.

Theorem 1 *(a) Consistent discount factors exist.*

(b) We have $(v_0, v_T) \in \mathcal{H}$ if and only if $Eq_0v_0 \ge Eq_Tv_T + r$ for all consistent discount factors (q_0, q_T, r).

To prove the theorem, we will need several auxiliary definitions and preparatory results.

In what follows, let us fix the market prices, proportional transaction cost rates and dividend yields, but allow the transaction costs $C_t \ge 0$ to vary. Introduce the set $\mathcal{U} \subseteq$

$\mathcal{L}_0^1 \times \mathcal{L}_1^1 \times \cdots \times \mathcal{L}_{T-1}^1 \times \mathcal{L}_T^1$ which consists of sequences $u = (v_0, C_1, \ldots, C_{T-1}, v_T)$ such that in the market with transaction costs C_t the initial endowment v_0 allows the hedging of the contingent claim v_T. Observe that $\mathcal{U}$ is a cone.

Proposition 4 *The cone $\mathcal{U}$ is closed.*

To prove this result, we first establish a lemma.

Lemma 2 *There exist a constant $\gamma > 0$ not depending on the transaction costs C_t such that for any hedging strategy $(v_0, x_0, \ldots, x_{T-1}, v_T)$ it holds that $|x_t| \le \gamma v_0$ for all $t = 0, \ldots, T-1$.*

Proof First observe that there exists a constant $\delta > 0$ such that if $b \in X_t(\omega)$, then $|b| \le \delta \sum_{i=1}^m b^i$. Indeed, since the underlying probability space is finite, we can find a constant $\underline{\mu} > 1$ such that $\underline{\mu} \le \mu_t$ for all $t = 0, \ldots, T-1$. If $b \in X_t(\omega)$, then, as follows from the definition of X_t, we have $|b_+| \ge \mu_t |b_-| \ge \underline{\mu}|b_-|$. Consequently, $\sum_{i=1}^m b^i = |b_+| - |b_-| \ge (1 - \frac{1}{\underline{\mu}})|b_+|$ and $|b| = |b_+| + |b_-| \le (1 + \frac{1}{\underline{\mu}})|b_+|$. Hence, we can take $\delta = (\underline{\mu} + 1)/(\underline{\mu} - 1)$.

Next observe that the inequality $\psi_t(a, b) \ge c$ implies

$$\sum_{i=1}^m (R_{t,i} a^i - b^i + D_{t,i}^- a^i) \ge c.$$

Let $\overline{R} = \max_t \overline{R}_t$ and $\overline{D} = \max_t \overline{D}_t$ be constants such that $R_{t,i} \le \overline{R}$ and $D_{t,i}^- \le \overline{D}$ for all $t = 1, \ldots, T-1$ and $i = 1, \ldots, m$. Consequently, if $\psi_t(a, b) \ge c$ and $b \in X_t(\omega)$, then

$$|b| \le \delta \sum_{i=1}^m b^i \le \delta((\bar{R} + \bar{D})|a| - c).$$

Suppose $(v_0, x_0, \ldots, x_{T-1}, v_T)$ is a hedging strategy. For $t = 0$, from the condition $\psi(0, x_0) \ge -v_0$, we find that $|x_0| \le \delta v_0$. For $t = 1, \ldots, T-1$, the condition $\psi_t(x_{t-1}, x_t) \ge C_t$ implies $|x_t| \le \delta(\bar{R} + \bar{D})|x_{t-1}|$, since it is assumed that $C_t \ge 0$. Now the claim of the proposition follows by induction. □

Proof of Proposition 4 Consider $u^n = (v_0^n, C_1^n, \ldots, C_{T-1}^n, v_T^n) \in \mathcal{U}$ such that $u^n \to u = (v_0, C_1, \ldots, C_{T-1}, v_t)$. We have to show that $u \in \mathcal{U}$.

Let $h^n = (v_0^n, x_0^n, \ldots, x_{T-1}^n, v_T^n)$ be the corresponding hedging strategies. From Lemma 2, it follows that x_t^n are bounded sequences for each $t = 0, \ldots, T-1$. The sequences v_0^n and v_T^n are also bounded, which follows from the convergence $u^n \to u$.

Consequently, it is possible to choose a convergent subsequence $h^{n_k} \to h = (v_0, x_0, \ldots, x_{T-1}, v_T)$. In view of the continuity of the functions $\psi_t(\omega, a, b)$, we have $v_0 \ge -\psi_0(0, x_0)$, $\psi_t(x_{t-1}, x_t) \ge C_t$ for all $t = 1, \ldots, T-1$, and $\psi_T(x_{T-1}, 0) \ge v_T$. Thus h is hedging strategy in the model with transaction costs C_t. Consequently, $u \in \mathcal{U}$. □

Next introduce the closed cone

$$\mathcal{K} = \{(\tilde{v}_0, C_1, \ldots, C_{T-1}, v_T) : \\ \tilde{v}_0 \in \mathcal{L}_0^1,\ \tilde{v}_0 \le 0,\ C_t \in \mathcal{L}_t^1,\ C_t \ge 0 \text{ for all } t,\ v_T \in \mathcal{L}_T^1,\ v_T \ge 0\}.$$

The following result can be regarded as the no-arbitrage hypothesis for the model at hand.

Proposition 5 *We have* $\mathcal{U} \cap \mathcal{K} = \{0\}$.

Proof If $u = (v_0, x_0, \ldots, x_{T-1}, v_T) \in \mathcal{U} \cap \mathcal{K}$, then necessarily $v_0 = 0$ as follows from the definition of a hedging strategy. Proposition 2 implies that in this case $x_0 = \cdots = x_{T-1} = 0$. Consequently, the inequality $\psi_t(x_{t-1}, x_t) \ge C_t$ implies that $C_t = 0$ for all t, and $v_T \le 0$. Thus $u = 0$. □

Proposition 6 *Let* $u = (v_0, C_1, \ldots, C_{T-1}, v_T) \in \mathcal{U} - \mathcal{K}$ *be such that* $C_t \ge 0$ *for all* $t = 1, \ldots, T$. *Then* $u \in \mathcal{U}$.

Proof Let $u = u' - k$, where $u' = (v_0', C_1', \ldots, C_{T-1}', v_T') \in \mathcal{U}$ and $k \in \mathcal{K}$. Denote by $h = (v_0', x_0, \ldots, x_{T-1}, v_T')$ the corresponding hedging strategy which allows the hedging of the contingent claim v_T' from the initial endowment v_0' in the market with transaction costs C_t'.

We have $v_0' \ge -\psi_0(0, x_0)$, hence $v_0 \ge -\psi_0(0, x_0)$, since $v_0 \ge v_0'$. From the condition $\psi_t(x_{t-1}, x_t) \ge C_t'$, we find that $\psi_t(x_{t-1}, x_t) \ge C_t$ for all $t = 1, \ldots, T-1$. Finally, as $\psi_T(x_{T-1}, 0) \ge v_T'$, we have $\psi_T(x_{T-1}, 0) \ge v_T$ because $v_T' \ge v_T$.

As a result, we see that h is a strategy which allows to hedge v_T from the initial endowment v_0 in the market with transaction costs C_t. Consequently, $u \in \mathcal{U}$. □

Proof of Theorem 1 We apply Proposition 7 (see Appendix) to the cones $\mathcal{U}$ and $\mathcal{K}$. The conditions of applicability of this proposition are met in view of the above results.

Both cones $\mathcal{U}$ and $\mathcal{K}$ are contained in the linear space $\mathcal{L}_0^1 \times \mathcal{L}_1^1 \times \cdots \times \mathcal{L}_T^1$. Since the probability space is finite, this space is finite dimensional.

Any linear functional $l(u)$ on this space can be represented as

$$l(u) = -Eq_0 v_0 + \sum_{t=1}^{T-1} Eq_t C_t + Eq_T v_T, \tag{9}$$

where $q_t \in \mathcal{L}_t^1$.

By virtue of Proposition 7, there exists a linear functional l of the form (9) such that $l(u) > 0$ for any $u \in \mathcal{K} \setminus \{0\}$, and $l(u) \le 0$ for any $u \in \mathcal{U}$. The first property implies that all $q_t > 0$. Then the triple (q_0, q_T, r) with $r = \sum_{t=1}^{T-1} Eq_t C_t$ is a consistent discount factor in the model with given transaction costs C_t. This proves the first claim of the theorem.

Let us prove the second claim. If $(v_0, v_T) \in \mathcal{H}$, then $Eq_0 v_0 \ge Eq_T v_T + r$ for any consistent discount factors as follows from their definition. Conversely, suppose $Eq_0 v_0 \ge Eq_T v_T + r$ for any consistent discount factors. Then, in particular, this inequality holds for any (q_0, q_T, v) for which there exists a functional $l(u)$ defined through factors $(q_0, q_1, \ldots, q_T)$ by formula (9) such that $r = \sum_{t=1}^{T-1} Eq_t C_t$ and which is strictly positive on $\mathcal{K} \setminus \{0\}$ and non-positive on $\mathcal{U}$. Let $u = (v_0, C_1, \ldots, C_{T-1}, v_T)$ with given transaction costs C_t. By Proposition 8, we have $u \in \mathcal{U} - \mathcal{K}$. Then $u \in \mathcal{U}$ by Proposition 6. Consequently, $(v_0, v_T) \in \mathcal{H}$. This proves the second claim. □

5 Consistent valuation systems

Now we provide a solution of the hedging problem in terms of *consistent valuation systems*, a notion which extends the notion of consistent discount factors.

A *consistent price system with transaction costs terms* (which we will call simply a *consistent price system*) is pair consisting of a sequence of non-negative vector functions $p_0(\omega), \ldots, p_T(\omega)$ such that $p_t \in \mathcal{L}_t^m$, and a sequence of non-negative constants $r_1, \ldots, r_{T-1}$ which satisfy the relations

$$\bar{p}_1(\omega)a \leq p_0(\omega)a \text{ for all } a \in X_0(\omega), \tag{10}$$

$$\bar{p}_{t+1}(\omega)b + r_t \leq p_t(\omega)a \text{ for all } (a, b) \in Z_t(\omega),\ t = 1, ..., T-1, \tag{11}$$

where $\bar{p}_{t+1}(\omega) := E_t p_{t+1}(\omega)$ and $E_t(\cdot) = E(\cdot|\mathcal{F}_t)$.

Note that by virtue of conditions (10) and (11), for any trading strategy $x_0, x_1, \ldots, x_{T-1}$, the sequence

$$p_0x_0,\ p_1x_0,\ p_2x_1 + r_1,\ p_3x_2 + r_1 + r_2,\ \ldots\ p_Tx_{T-1} + (r_1 + \cdots + r_{T-1})$$

is a non-negative supermartingale.

The idea of the notion of a consistent price system goes back to the notion of competitive prices supporting competitive paths maximizing profits over each time period $t-1,\ t$ in the theory of economic dynamics – see, e.g. Malinvaud (1953); Radner (1967); Gale (1967); Peleg (1974); Dasgupta and Mitra (1999), and Clark (2008). Consistent price systems generalize the concept of an equivalent martingale measure in classical no-arbitrage criteria pertaining to frictionless markets, see Dempster et al. (2006).

Let $q_0 \in \mathcal{L}_0^1$ and $q_T \in \mathcal{L}_T^1$ be strictly positive scalar functions, and $((p_0, \ldots, p_T),\ (r_1, \ldots, r_{T-1}))$ be a consistent price system. Then the pair of sequences $((q_0, p_0, \ldots, p_T, q_T),\ (r_1, \ldots, r_{T-1}))$ is called a *consistent valuation system* if

$$p_0(\omega)b \leq q_0(\omega)a \text{ for all } (a, b) \in V_0(\omega), \tag{12}$$

$$p_T(\omega)a \geq q_T(\omega)b \text{ for all } (a, b) \in V_T(\omega). \tag{13}$$

According to (12) and (13), the prices and discount factors under consideration are such that the profit one can get in the course of portfolio creation or liquidation cannot be strictly positive.

Theorem 2 *The following claims hold true.*

(a) If $((q_0, p_0, \ldots, p_T, q_T),\ (r_1, \ldots, r_{T-1}))$ *is a consistent valuation system, then* (q_0, q_T, r)*, where* $r = r_1 + \ldots + r_{T-1}$*, are consistent discount factors.*
(b) If (q_0, q_T, r) *are consistent discount factors, then there exists a consistent valuation system* $((q_0, p_0, \ldots, p_T, q_T),\ (r_1, \ldots, r_{T-1}))$ *with* $r = r_1 + \ldots + r_{T-1}$.

From Theorems 1 and 2, we immediately get the following corollary.

Corollary 1 *Consistent valuation systems exist and we have* $(v_0, v_T) \in \mathcal{H}$ *if and only if* $Eq_0 v_0 \geq Eq_T v_T + \sum_{t=1}^{T-1} r_t$ *for all consistent valuation systems* $((q_0, p_0, \ldots, p_T, q_T),\ (r_1, \ldots, r_{T-1}))$.

In the course of proof of Theorem 2, the following notion will be needed. We shall call a sequence of vector functions

$$\xi := \{v_0, x_0, y_0, x_1, y_1, \ldots, x_{T-1}, y_{T-1}, x_T, v_T\}, \tag{14}$$

where

$$v_0 \in \mathcal{L}_0^1,\ (x_t, y_t) \in \mathcal{L}_t^m \times \mathcal{L}_t^m\ [t = 0, \ldots, T-1],\ x_T \in \mathcal{L}_T^m,\ v_T \in \mathcal{L}_T^1, \tag{15}$$

a *generalized hedging strategy* if

$$(v_0, x_0) \in V_0(\omega), \quad y_0 \in X_0(\omega), \tag{16}$$
$$(x_t, y_t) \in Z_t(\omega)\ [t = 1, \ldots, T-1], \tag{17}$$
$$(x_T, v_T) \in V_T(\omega), \tag{18}$$
$$x_0 \geq y_0,\ y_0 \geq x_1,\ y_1 \geq x_2, \ldots,\ y_{T-1} \geq x_T. \tag{19}$$

Note that if the inequalities in (19) hold as equalities, we obtain an ordinary hedging strategy.

Lemma 3 *For any generalized hedging strategy, there is a hedging strategy* $(v_0', y_0', \ldots, y_{T-1}', v_T')$ *such that* $v_0' = v_0,\ y_t' \geq y_t\ [t = 0, \ldots, T-1]$ *and* $v_T' \geq v_T$.

Proof This is easily proved by induction using Proposition 1. □

Lemma 4 *There exists a generalized hedging strategy*

$$\hat{\xi} = \{\hat{v}_0, \hat{x}_0, \hat{y}_0, \hat{x}_1, \hat{y}_1, \ldots, \hat{x}_{T-1}, \hat{v}_{T-1}, \hat{x}_T, \hat{v}_T\}$$

such that

$$\hat{x}_0 > \hat{y}_0,\ \hat{y}_0 > \hat{x}_1,\ \hat{y}_1 > \hat{x}_2, \ldots,\ \hat{y}_{T-1} > \hat{x}_T.$$

Furthermore, there exists a hedging strategy $(v_0', y_0', \ldots, y_{T-1}', v_T')$ *such that* $v_0' > 0,\ y_t' > 0\ [t = 0, \ldots, T-1]$ *and* $v_T' > 0$.

Proof In the proof of Proposition 2, we have constructed constant vectors $\mathring{x}_t$ and $\mathring{y}_t$ for each $t = 1, 2, \ldots, T-1$. These vectors can be expressed as $\mathring{x}_t = \alpha_t(1, \ldots, 1)$ and $\mathring{y}_t = \beta_t(1, \ldots, 1)$, where α_t and β_t are positive constants. Importantly, for all $\omega \in \Omega$, it holds that $(\mathring{x}_t, \mathring{y}_t) \in Z_t(\omega)$.

Now, let us proceed with a backward induction argument. We begin by setting $(\hat{x}_{T-1}, \hat{y}_{T-1}) = (\mathring{x}_{T-1}, \mathring{y}_{T-1})$, $\hat{x}_T = \frac{1}{2}\mathring{y}_{T-1}$, and $\hat{v}_T = \psi_T(\omega, \hat{x}_T, 0)$. Note that $(\hat{x}_T, \hat{v}_T) \in V_T(\omega)$, $\hat{y}_{T-1} > \hat{x}_T$, $(\hat{x}_{T-1}, \hat{y}_{T-1}) \in Z_{T-1}(\omega)$, and according to Proposition 3, $\hat{v}_T > 0$.

Now, if $\mathring{y}_{T-2} > \mathring{x}_{T-1}$ $(\beta_{T-2} > \alpha_{T-1})$, we can simply set $(\hat{x}_{T-2}, \hat{y}_{T-2}) = (\mathring{x}_{T-2}, \mathring{y}_{T-2})$. However, if $\mathring{y}_{T-2} \leq \mathring{x}_{T-1}$, we can find a $\lambda > 1$ such that $\lambda\mathring{y}_{T-2} > \mathring{x}_{T-1}$. In this case, we set $(\hat{x}_{T-2}, \hat{y}_{T-2}) = \lambda(\mathring{x}_{T-2}, \mathring{y}_{T-2})$. Since $\lambda > 1$ and $(\mathring{x}_{T-2}, \mathring{y}_{T-2}) \in Z_{T-2}(\omega)$, it follows that $(\hat{x}_{T-2}, \hat{y}_{T-2}) \in Z_{T-2}(\omega)$.

By induction, let us assume that we have already constructed vectors $\hat{x}_1$ and $\hat{y}_1$ such that $(\hat{x}_1, \hat{y}_1) \in Z_1(\omega)$, and $\hat{y}_1 > \hat{x}_2$. Now, let $\hat{y}_0 = 2\hat{x}_1$, $\hat{x}_0 = 4\hat{x}_1$, and $v_0 = -\phi_0(\omega, -\hat{x}_0)$. Notably, according to Proposition 3, $\hat{v}_0 > 0$. With this, we have successfully constructed

$$\hat{\xi} = \{\hat{v}_0, \hat{x}_0, \hat{y}_0, \hat{x}_1, \hat{y}_1, \ldots, \hat{x}_{T-1}, \hat{v}_{T-1}, \hat{x}_T, \hat{v}_T\}$$

that satisfies all the required conditions. The second part of the proof is immediate from Lemma 3. □

Lemma 4 is an analogue of Slater's constraint qualification. It says that there exists a generalized hedging strategy (which can be regarded as a hedging strategy *with consumption*) such that an investor following it can sell, with a view to consumption, strictly positive amounts of assets of each type at every date. In the literature, such models are called *regular* (see, e.g., Dempster et al. 2006).

Now we are ready to give a proof of Theorem 2.

Proof of Theorem 2 Claim (a) clearly follows from the definitions of consistent valuation systems and consistent discount factors.

Let us prove claim (b). We will mainly follow the lines of the proof of Theorem 7.1 in Dempster et al. (2006). Consider consistent discount factors (q_0, q_T, r). By virtue of their definition, we have $Eq_T v_T - Eq_0 v_0 + r \leq 0$ for all $(v_0, v_T) \in \mathcal{H}$.

Denote by Ξ the set of generalized hedging strategies. For $\xi \in \Xi$, define

$$F(\xi) = Eq_T v_T - Eq_0 v_0 + r.$$

In view of Lemma 3, we have $F(\xi) \leq 0$ for any $\xi \in \Xi$. Thus, the maximum of F over Ξ is equal to zero (the maximum is attained since one can increase v_0 without changing the other components of ξ). By applying the Kuhn–Tucker theorem (see Proposition 9) to this maximization problem, we relax constraints (19) in the definition of a generalized hedging strategy. The Kuhn–Tucker theorem can be applied in view of the regularity of the model.

As a result, there exist non-negative functions $p_t \in \mathcal{L}_t^m$ such that

$$Eq_T v_T - Eq_0 v_0 + r + Ep_0(x_0 - y_0) + \sum_{t=1}^{T} Ep_t(y_{t-1} - x_t) \leq 0$$

for all sequences $(v_0, x_0, y_0, x_1, y_1, \ldots, x_{T-1}, y_{T-1}, x_T, v_T)$ satisfying conditions (15)–(18). Rearranging the terms, we obtain

$$\begin{aligned} &E(p_0 x_0 - Eq_0 v_0) + (Ep_1 y_0 - Ep_0 y_0) \\ &+ \sum_{t=1}^{T-1} (Ep_{t+1} y_t - Ep_t x_t) + (Eq_T v_T - Ep_T x_T) \leq -r. \end{aligned} \tag{20}$$

Let $-r_0, -r_0', -r_1, \ldots, -r_{T-1}$, and $-r_T$ denote the maximal values of the corresponding terms in the left-hand side of this inequality, which are taken over, respectively, $(x_0, y_0) \in V_0$, $y_0 \in X_0$, $(x_t, y_t) \in Z_t$, $t = 1, \ldots, T-1$, and $(x_T, v_T) \in V_T$. Since V_0, X_0 and V_T are cones, we have $r_0 = r_0' = r_T = 0$. Therefore, $\sum_{t=1}^{T-1} r_t \geq r$. Also observe that if $(x_t, y_t) \in Z_t$, then $(\alpha x_t, \alpha y_t) \in Z_t$ for any $\alpha \geq 1$, which implies that $r_1, \ldots, r_{T-1}$ should be non-negative.

Consequently, we have inequalities

$$Ep_0 x_0 - Eq_0 v_0 \leq 0 \text{ for all } (v_0, x_0) \in V_0(\omega), \tag{21}$$
$$E\bar{p}_1 y_0 - Ep_0 y_0 \leq 0 \text{ for all } y_0 \in X_0(\omega), \tag{22}$$
$$E\bar{p}_{t+1} y_t - Ep_t x_t \leq -r_t \text{ for all } (x_t, y_t) \in Z_t(\omega),\ t = 1, \ldots, T-1, \tag{23}$$
$$Eq_T v_T - Ep_T x_T \leq 0 \text{ for all } (x_T, v_T) \in V_T(\omega). \tag{24}$$

Then (21), (22), (23) and (24) imply (12), (10), (11) and (13), respectively. As a result, $((q_0, p_0, \ldots, p_T, q_T),\ (r_1, \ldots, r_{T-1}))$ is a consistent valuation system.

Finally, observe that if we decrease some of the values r_t (but keeping them non-negative), we will still have a consistent valuation system. Thus, we can choose $r_1, \ldots, r_{T-1}$ such that $r_1 + \cdots + r_{T-1} = r$. □

6 Appendix

In this appendix we collect well-known auxiliary results that were used in the proofs.

Proposition 7 *Let H and K be two closed cones in $\mathbb{R}^n$, and the cone K being pointed. Then the following assertions are equivalent.*

(a) $H \cap K = \{0\}$.
(b) There exists $l \in \mathbb{R}^n$ such that $lh \leq 0$ for all $h \in H$, and $lk > 0$ for all $k \in K \setminus \{0\}$.

Proposition 8 *Again, consider two closed cones $H, K \subseteq \mathbb{R}^n$, with K being pointed. Suppose that $H \cap K = \{0\}$. Then, for any $u \in \mathbb{R}^n$, the following assertions are equivalent.*

(a) $u \in H - K$.
(b) $lu \leq 0$ *for all* $l \in \mathbb{R}^n$ *such that* $lh \leq 0$ *for all* $h \in H$ *and* $lk > 0$ *for all* $k \in K \setminus \{0\}$.

Let $g_1(x), \ldots, g_m(x)$ and $F(x)$ be concave functions defined on a convex set X in $\mathbb{R}^n$. Put $g(x) = (g_1(x), \ldots, g_m(x))$ and consider the following optimization problem:
($\mathcal{P}$) Maximize $F(x)$ on X subject to $g(x) \geq 0$.
Assume that the following assumption (*Slater's condition*) holds.
(S) There exists an element $y \in X$ such that $g(y) > 0$
(with all inequalities being componentwise).

Proposition 9 *(Kuhn-Tucker theorem). Let* $\bar{x}$ *be a point in* X *such that* $g(\bar{x}) \geq 0$. *Then the following* assertions are equivalent.

(a) $\bar{x}$ *is a solution to problem* ($\mathcal{P}$).
(b) There exists a vector $p \in \mathbb{R}^m$ *with non-negative coordinates such that* $pg(\bar{x}) = 0$ *and*

$$f(x) + pg(x) \leq f(\bar{x}) \text{ for all } x \in X. \tag{25}$$

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