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RESEARCH ARTICLE

The no-arbitrage pricing of non-traded assets

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Abstract
This paper shows how to uniquely price non-traded assets using no-arbitrage in an otherwise frictionless market setting. The approach requires the assumption that the hedging error, properly defined, is non-priced or idiosyncratic risk. This methodology can be applied to private loans, illiquid publicly traded debt, insurance contacts, private equity, real estate, and real options.

Keywords Arbitrage pricing · Non-traded assets · Idiosyncratic risk · Private debt · Private equity · Insurance contracts · Real estate · Real options

JEL Classification G12 · G13 · G22

1 Introduction

The pricing of non-traded assets is an important area within finance. By non-traded we mean an asset whose payoffs cannot be synthetically constructed via an admissible and dynamic trading strategy using the traded assets in a standard continuous time, continuous trading, frictionless, and competitive market model.[1] Non-traded assets include private loans, illiquid publicly traded debt, insurance contacts, private equity, real estate, and real options. The purpose of the paper is to provide a no-arbitrage methodology for pricing these non-traded assets in an otherwise frictionless and competive market setting. This approach uses the notion of non-priced hedging risk in conjunction with a complete market for a collection of related traded assets.

Non-traded assets are a special case of pricing derivatives in an incomplete market. See (Bingham and Kiesel 2000), Chapter 7, for an excellent summary of this literature. As is well-known in this literature, in an incomplete market, there is no unique price for a derivative whose payoff is non-linear in the traded assets. This is because the set of

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[1] All of these terms will be defined in Sect. 2 below.

equivalent martingale measures contains a continuum of elements. Various methods have been employed to select a unique element from this set. One approach is to choose a price implied by a given objective function; this includes variance and risk minimizing hedging and indifference pricing. A second approach is to choose a unique martingale measure assuming certain risks are non-priced, which uniquely determines the martingale measure. For example (Merton 1976) assumes jump risk is non-priced, Hull and White (1987) assume volatility risk is non-priced, and Jarrow et al. (2005) assume the same for default risk. This paper revisits, formalizes, and generalizes this later approach to general semimartingale price processes.

This method for pricing non-traded assets can be used to value private loans, illiquid publicly traded debt, insurance contacts, private equity, real estate, and real options. It is an important approach in practice because:

(i) it avoids the necessity of assuming a particular preference or objective function to determine a unique price, and

(ii) it can be argued that with the abundance of assets traded in current markets, sufficient securities have already been issued to hedge most systematic risks. This implies that the only remaining non-traded risks are non-priced or idiosyncratic, hence, the above methodology applies.

An outline of this paper is as follows. Section 2 sets up the model structure. Section 3 provides the key result under the non-priced hedging error assumption, and Sect. 4 illustrates the approach with various examples including private debt, private equity, and real options. Section 5 concludes.

2 The set-up

We consider a continuous time, continuous trading model on a finite horizon $[0, T]$. The randomness is represented by $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, which is a filtered complete probability space where the filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ satisfies the usual hypothesis with $\mathcal{F}_0$ the trivial σ algebra, $\mathcal{F}_T = \mathcal{F}$, and where $\mathbb{P}$ is the statistical probability measure.

2.1 The original market

The market is assumed to be *competitive* and *frictionless*. Competitive means that traders act as price takers, believing their trades have no quantity impact on the market price. Frictionless means that there are no transaction costs and no trading constraints.

Traded in the economy are n risky assets and a money market account (mma) that is locally riskless. Without loss of generality, we assume the mma's value is unity for all times. The (normalized) market prices of the risky assets are given by a non-negative semimartingale $S_t := (S_1(t), \dots, S_n(t))$ for $0 \le t \le T$ that is adapted to $\mathbb{F}$. Without loss of generality, we assume that no cash flows are paid to the risky assets. Let $\mathbb{F}^s = (\mathcal{F}_t^s)_{0 \le t \le T}$ be the filtration generated by S. We assume that $\mathbb{F}^s \subset \mathbb{F}$ and $\mathbb{F}^s \neq \mathbb{F}$. The economic interpretation is that $\mathbb{F}$ reflects additional randomness present in the market and not reflected in the stock price processes.

In the subsequent analysis, the different filtrations will be important. Consequently, when defining the various concepts, we will make the underlying filtration explicit. *Trading strategies* are defined to be holdings in the risky assets that are $\mathbb{F}$ - predictable (i.e. depending on only current and past information) and holdings in the mma that are $\mathbb{F}$ - optional. To exclude doubling strategies from discussion, we only consider trading strategies that are *admissible* (the value of the trading strategy is bounded below). A trading strategy is *self-financing* if it requires no cash inflows or outflows except at times 0 and T. Specifically, denote by $\mathscr{O}$ the $\mathbb{F}$ - optional σ-algebra and $\mathscr{L}(S, \mathbb{F})$ the set of $\mathbb{F}$- predictable processes for which the stochastic integral with respect to S exists.

An *admissible self financing trading strategy* (s.f.t.s) with initial wealth x and wealth process X is the $n+1$ - tuple of stochastic processes $(\alpha_0, \alpha := (\alpha_1, \dots, \alpha_n)) \in (\mathscr{O}, \mathscr{L}(S, \mathbb{F}))$ such that there exists some constant c where

$$X_t = \alpha_0(t) + \alpha_t \cdot S_t = x + \int_0^t \alpha_u \cdot dS_u \geq c, \forall t \in [0, T].$$

Here, $x \cdot y$ with $x, y \in \mathbb{R}^n$ denotes the inner product.

The first equality represents the trading strategy's wealth at time t, which equals the number of units of the mma plus the number of shares of the risky assets times their market prices. The second equality is the self-financing condition, which states that the trading strategy's time t wealth equals the initial wealth plus the accumulated capital gains from the trading strategy over $[0, T]$. The third inequality is the admissibility condition, which represents a uniform lower bound on the wealth process. This lower bound represents a borrowing constraint. We denote by $\mathcal{A}(x, \mathbb{F})$ the set of admissible s.f.t.s. $(\alpha_0, \alpha) \in (\mathscr{O}, \mathscr{L}(S, \mathbb{F}))$ given an initial wealth x.

A *simple arbitrage opportunity* is an admissible s.f.t.s. $(\alpha_0, \alpha) \in \mathcal{A}(x, \mathbb{F})$ with initial wealth $x = 0$ and wealth process X such that

$$\mathbb{P}(X_T \geq 0) = 1, \qquad \text{and} \qquad \mathbb{P}(X_T > 0) > 0.$$

A *Free Lunch with Vanishing Risk* (FLVR) is an admissible s.f.t.s. that is an extension of a simple arbitrage opportunity that includes (the limits of) approximate simple arbitrage opportunities. We say the market satisfies *No Free Lunch with Vanishing Risk* (NFLVR) if there exists no FLVR.

An equivalent local martingale measure $\mathbb{Q}$ is any probability measure on $(\Omega, \mathcal{F})$ such that for $A \in \mathcal{F}$, $\mathbb{Q}(A) = 0$ iff $\mathbb{P}(A) = 0$ (in symbols $\mathbb{Q} \sim \mathbb{P}$) and S is a $\mathbb{Q}$ local martingale with respect to $\mathbb{F}$.

Define $\mathcal{M}_l(\mathbb{F})$ to be the set of *equivalent local martingale measures* (ELMM) with respect to $\mathbb{F}$. The first fundamental theorem of asset pricing states that $\mathcal{M}_l(\mathbb{F}) \neq \emptyset$ if and only if the market satisfies NFLVR.

An admissible s.f.t.s. with wealth process X is said to be a *dominating* for asset i if there exists an admissible s.f.t.s $(\alpha_0, \alpha) \in \mathscr{A}(x, \mathbb{F})$ such that $x < S_i(0)$ and

$$x + \int_0^T \alpha_u \cdot dS_u = S_i(T) \quad \text{a.s.}$$

The market is said to satisfy No Dominance (ND) if for all assets $i = 0, 1, ..., n$ there exist no such dominating s.f.t.s.

Define $\mathcal{M}(\mathbb{F})$ to be the set of *equivalent martingale measures (EMM) under which* S is a $\mathbb{Q}$ martingale.

The third fundamental theorem states that $\mathcal{M}(\mathbb{F}) \neq \emptyset$ if and only if the market satisfies NFLVR and ND.

A market is defined to be *complete* with respect to some $\mathbb{Q} \in \mathcal{M}_l(\mathbb{F})$ if for any non-negative payoff $C_T \in L^1_+(\Omega, \mathcal{F}_T, \mathbb{Q})$ at time T, there exists a $x \geq 0$ and $(\alpha_0, \alpha) \in \mathscr{A}(x, \mathbb{F})$ such that

$$x + \int_0^T \alpha_u \cdot dS_u = C_T,$$

and the wealth process

$$C_t = \alpha_0(t) + \alpha_t \cdot S_t = x + \int_0^t \alpha_u \cdot dS_u$$

is a $\mathbb{Q}$ martingale with respect to $\mathbb{F}$ and $L^1_+(\Omega, \mathcal{F}_T, \mathbb{Q})$ is the set of $\mathcal{F}_T$ measurable, non-negative, real-valued random variables $C_T(\omega)$ with $\mathbb{E}^{\mathbb{Q}}[C_T] < \infty$. The payoff $C_T \in L^1_+(\Omega, \mathcal{F}_T, \mathbb{Q})$ can be interpreted as the cash flow to a traded or a non-traded asset.

By the second fundamental theorem of asset pricing, given there exists a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$, the market is complete with respect to $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ if and only if the EMM is unique. In a complete market, $\mathbb{E}^{\mathbb{Q}}[\cdot]$ gives the unique present value operator to determine the arbitrage-free price of any $C_T \in L^1_+(\Omega, \mathcal{F}_T, \mathbb{Q})$ at time t, which is $\mathbb{E}^{\mathbb{Q}}[C_T|\mathcal{F}_t]$. For more discussion on all of these topics, see (Jarrow 2021b), Chapter 2.

For the following analysis, we invoke the following assumption for the original market.

Assumption 1 (*NFLVR, ND, and Incomplete Original Market*)
(i) $\mathcal{M}(\mathbb{F}) \neq \emptyset$.
(ii) Fix a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$,[2] the original market is incomplete with respect to $\mathbb{Q}$.

In an incomplete market satisfying NFLVR and ND, there exists payoffs $C \in L^1_+(\Omega, \mathcal{F}_T, \mathbb{Q})$ that cannot be replicated using the mma and the n risky assets. And, there are an infinite number of martingale measures $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$. Hence, there is no unique arbitrage-free price for any such payoff. For subsequent use, we define the original market as the collection

$$\left(S, \mathbb{F}, L^1_+(\Omega, \mathcal{F}_T, \mathbb{Q})\right)$$

for a given $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$.

[2] This determines the set of integrable random variables at time T.

2.2 The restricted market

This section introduces the restricted market consisting of just the traded assets and its derivatives. Given the original market satisfies NFLVR and ND, $\mathcal{M}(\mathbb{F}) \neq \emptyset$. Fix a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ and define $\mathbb{Q}^s := \mathbb{Q} \mid_{\mathcal{F}_T^s}$ on $(\Omega, \mathcal{F}_T^s)$.

For the given $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$, the restricted market is defined as the collection

$$\left(S, \mathbb{F}^s, L_+^1(\Omega, \mathcal{F}_T^s, \mathbb{Q}^s)\right).$$

That is, given $\mathbb{Q}^s$, the restricted market is the set of traded risky assets S, the filtration generated by the risky assets $\mathbb{F}^s$, and the set of payoffs $\tilde{C} \in L_+^1(\Omega, \mathcal{F}_T^s, \mathbb{Q}^s)$ that are $\mathcal{F}_T^s$ measurable.

Lemma 1 (The Restricted Market satisfies NFLVR and ND) *Fix a* $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ *and define* $\mathbb{Q}^s := \mathbb{Q} \mid_{\mathcal{F}_T^s}$. *Then,*

S is a $\mathbb{Q}^s$ martingale with respect to $\mathbb{F}^s$, i.e. $\mathbb{Q}^s \in \mathcal{M}(\mathbb{F}^s)$.

Proof Given a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$, $\mathbb{E}^{\mathbb{Q}}[S_T \mid \mathcal{F}_t] = S_t$.

Taking conditional expectations with respect to $\mathcal{F}_t^s$ of both sides yields

$$\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}[S_T \mid \mathcal{F}_t] \,\middle|\, \mathcal{F}_t^s\right] = \mathbb{E}^{\mathbb{Q}}\left[S_t \,\middle|\, \mathcal{F}_t^s\right].$$

But $\mathbb{F}^s \subset \mathbb{F}$ implies $\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}[S_T \mid \mathcal{F}_t] \,\middle|\, \mathcal{F}_t^s\right] = \mathbb{E}^{\mathbb{Q}}\left[S_T \,\middle|\, \mathcal{F}_t^s\right]$ and $\mathbb{E}^{\mathbb{Q}}\left[S_t \,\middle|\, \mathcal{F}_t^s\right] = S_t$. Substitution gives

$$\mathbb{E}^{\mathbb{Q}}\left[S_T \,\middle|\, \mathcal{F}_t^s\right] = S_t.$$

Finally, since S is $\mathbb{F}^s$ adapted, $\mathbb{E}^{\mathbb{Q}}\left[S_T \,\middle|\, \mathcal{F}_t^s\right] = \mathbb{E}^{\mathbb{Q}^s}\left[S_T \,\middle|\, \mathcal{F}_t^s\right]$, which completes the proof. □

By the first and third fundamental theorems of asset pricing, this lemma implies that the restricted market satisfies both NFLVR and ND.

In this restricted market, the set of admissible s.f.t.s. only depend on the information in the filtration $\mathbb{F}^s$, denoted $(\alpha_0, \alpha) \in \mathscr{A}(x, \mathbb{F}^s)$. We add the following assumption on the restricted market.

Assumption 2 (*Complete Restricted Market*)

Fix a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$.

The restricted market is complete with respect to $\mathbb{Q}^s = \mathbb{Q} \mid_{\mathcal{F}_T^s}$.

This assumption states that for any non-negative $\mathcal{F}_T^s$ measurable payoff $\tilde{C}_T \in L_+^1(\Omega, \mathcal{F}_T^s, \mathbb{Q}^s)$ at time T, there exists a $x \geq 0$ and $(\alpha_0, \alpha) \in \mathscr{A}(x, \mathbb{F}^s)$ such that

$$x + \int_0^T \alpha_u \cdot dS_u = \tilde{C}_T,$$

and the wealth process

$$\tilde{C}_t = \alpha_0(t) + \alpha_t \cdot S_t = x + \int_0^t \alpha_u \cdot dS_u$$

is a $\mathbb{Q}^s$ martingale with respect to $\mathbb{F}^s$. It is also a $\mathbb{Q}$ martingale with respect to $\mathbb{F}^s$ because $\mathbb{Q}^s = \mathbb{Q} \mid_{\mathcal{F}_T^s}$ and $\tilde{C}_t$ is $\mathbb{F}^s$ adapted. This implies that

$$\tilde{C}_t = \mathbb{E}^{\mathbb{Q}^s}[\tilde{C}_T | \mathcal{F}_t^s] = \mathbb{E}^{\mathbb{Q}}[\tilde{C}_T | \mathcal{F}_t^s],$$

and the initial investment satisfies

$$x = \mathbb{E}^{\mathbb{Q}^s}[\tilde{C}_T] = \mathbb{E}^{\mathbb{Q}}[\tilde{C}_T]. \tag{1}$$

Last, by the second fundamental theorem of asset pricing, the EMM $\mathbb{Q}^s$ on $(\Omega, \mathcal{F}_T^s)$ is unique. Hence, all $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ generate the same the probability measure $\mathbb{Q}^s$.

3 The Theorem

This section solves the following problem.

Problem. Given a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ and the cash flow from a non-traded asset $C \in L_+^1(\Omega, \mathcal{F}_T, \mathbb{Q}) \cap L_+^1(\Omega, \mathcal{F}_T, \mathbb{P})$ that cannot be replicated using the mma and the n risky assets, what is its unique arbitrage-free price?

Because there are an infinite number of EMM $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$, and any one of them gives a possible price, we need to introduce an additional assumption to solve the problem. We will state this assumption later, after motivating it during the derivation of the problem's solution.

Choose an arbitrary $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$. We will later show that the price of the non-traded asset is independent of the EMM selected. Fix a non-traded asset's payoff $C_T \in L_+^1(\Omega, \mathcal{F}_T, \mathbb{Q}) \cap L_+^1(\Omega, \mathcal{F}_T, \mathbb{P})$ in the original market that cannot be replicated using the mma and the n risky assets.

Next, consider the related payoff $\tilde{C}_T := \mathbb{E}^{\mathbb{P}}[C_T | \mathcal{F}_T^s]$. This payoff is in the restricted market $\left(S, \mathbb{F}^s, L_+^1(\Omega, \mathcal{F}_T^s, \mathbb{Q}^s)\right)$ because it is $\mathcal{F}_T^s$ measurable. Because the restricted market is complete, there exists a $x \geq 0$ and $(\alpha_0, \alpha) \in \mathscr{A}(x, \mathbb{F}^s)$ such that

$$\begin{aligned} x + \int_0^T \alpha_u \cdot dS_u &= \tilde{C}_T, \\ x &= \mathbb{E}^{\mathbb{Q}^s}[\tilde{C}_T], \end{aligned}$$

and the wealth process

$$\tilde{C}_t = \alpha_0(t) + \alpha_t \cdot S_t = \mathbb{E}^{\mathbb{Q}^s}[\tilde{C}_T] + \int_0^t \alpha_u \cdot dS_u$$

is a $\mathbb{Q}^s$ martingale with respect to $\mathbb{F}^s$. That is, the payoff $\tilde{C}_T := \mathbb{E}^{\mathbb{P}}[C_T|\mathcal{F}_T^s]$ can be replicated using the mma and traded risky assets, and it's wealth process is $\tilde{C}_t$. And, $\mathbb{E}^{\mathbb{Q}^s}[\tilde{C}_T]$ is the unique risk neutral value of the cash flow $\tilde{C}_T$ in the restricted market.

Next, we use this s.f.t.s. $(\alpha_0, \alpha) \in \mathscr{A}(x, \mathbb{F}^s)$, but in the original market, to construct a partial hedge for the non-traded asset's payoff. The hedging error ε_T is defined by the following expression:

$$\varepsilon_T = C_T - \tilde{C}_T.$$

Here, the payoff $\tilde{C}_T$ represents the "traded" part of the payoff C_T and ε_T represents the "non-traded" part. The following lemma follows easily from the definition.

Lemma 2 (Expected Hedging Error with respect to $\mathbb{F}^s$) $\mathbb{E}^{\mathbb{P}}\left(\varepsilon_T \left|\mathcal{F}_t^s\right.\right) = 0$ *for all* $t \in [0, T]$, *which implies* $\mathbb{E}^{\mathbb{P}}\left(\varepsilon_T\right) = 0$.

Proof Taking expectations

$$\mathbb{E}^{\mathbb{P}}\left(C_T \left|\mathcal{F}_t^s\right.\right) = \mathbb{E}^{\mathbb{P}}\left(\tilde{C}_T \left|\mathcal{F}_t^s\right.\right) + \mathbb{E}^{\mathbb{P}}\left(\varepsilon_T \left|\mathcal{F}_t^s\right.\right)$$

But, $\mathbb{E}^{\mathbb{P}}\left(\tilde{C}_T \left|\mathcal{F}_t^s\right.\right) = \mathbb{E}^{\mathbb{P}}\left(\mathbb{E}^{\mathbb{P}}\left(C_T \left|\mathcal{F}_T^s\right.\right) \left|\mathcal{F}_t^s\right.\right) = \mathbb{E}^{\mathbb{P}}\left(C_T \left|\mathcal{F}_t^s\right.\right)$

which implies $\mathbb{E}^{\mathbb{P}}\left(\varepsilon_T \left|\mathcal{F}_t^s\right.\right) = 0$. And, $\mathbb{E}^{\mathbb{P}}\left(\varepsilon_T\right) = 0$. □

This lemma states that under the statistical probability $\mathbb{P}$, the hedging error has zero conditional expectation with respect to the filtration $\mathbb{F}^s$ generated by the traded risky assets.

Using the given $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$, the time 0 arbitrage-free value of the non-traded risky asset's payoff is:

$$\mathbb{E}^{\mathbb{Q}}\left(C_T\right) = \mathbb{E}^{\mathbb{Q}^s}(\tilde{C}_T) + \mathbb{E}^{\mathbb{Q}}\left(\varepsilon_T\right) \tag{2}$$

where the equality uses expression (1).

The first term $\mathbb{E}^{\mathbb{Q}^s}(\tilde{C}_T)$ is the price of the "traded" part of the non-traded asset's payoff, determined by replication in the restricted market. The second term, $\mathbb{E}^{\mathbb{Q}}\left(\varepsilon_T\right)$, represents the arbitrage-free price of the hedging error. The first term does not depend on the particular $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ selected, but the second term does. To remove this dependence, we add the following assumption.

Assumption 3 (*Non-priced Hedging Error Risk*)
For all $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$, $\quad \mathbb{E}^{\mathbb{Q}}\left(\varepsilon_T\right) = \mathbb{E}^{\mathbb{P}}\left(\varepsilon_T\right)$.

The assumption that hedging error risk is non-priced has be used in different contexts. In Merton (1976) jump risk is assumed to be non-priced, Hull and White (1987) volatility risk is assumed to be non-priced, and in Jarrow et al. (2005) default risk is assumed to be non-priced. This assumption is implied if the hedging error is diversifiable in a large portfolio (see the "Appendix" for a justification of this assumption in a discrete time model).

Under this assumption, we get the solution to the problem posed at the beginning of the section.

Theorem 3 (Arbitrage-Free Price of the Non-traded Asset)

$$\mathbb{E}^{\mathbb{Q}}(C_T) = \mathbb{E}^{\mathbb{Q}^s}(\mathbb{E}^{\mathbb{P}}\left(C_T \left| \mathcal{F}_T^s\right.\right)). \tag{3}$$

Proof By expression (2), $\mathbb{E}^{\mathbb{Q}}(C_T) = \mathbb{E}^{\mathbb{Q}^s}(\tilde{C}_T) + \mathbb{E}^{\mathbb{Q}}(\varepsilon_T)$.

Assumption 3 and Lemma 1 give $\mathbb{E}^{\mathbb{P}}(\varepsilon_T) = 0 = \mathbb{E}^{\mathbb{Q}}(\varepsilon_T)$. Substitution completes the proof. □

This price is uniquely determined by the arbitrage-free traded assets' prices in the original market. It is independent of the original EMM $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ selected because the hedging error is non-priced and $\mathbb{Q}^s$ is uniquely determined in the restricted market which is complete.

A special case of this theorem, when the non-traded asset's cash flows are independent of market prices S under $\mathbb{P}$, is worth noting. Formally, the special case is when C_T is independent of $\mathbb{F}^s$ under $\mathbb{P}$ as given in the following corollary.

Corollary 4 (Market Prices Independent of the Non-traded Asset's Cash Flows under $\mathbb{P}$)

If C_T is independent of $\mathbb{F}^s$ under $\mathbb{P}$, then

$$\mathbb{E}^{\mathbb{Q}}(C_T) = \mathbb{E}^{\mathbb{P}}(C_T). \tag{4}$$

Proof Under the hypothesis, $\mathbb{E}^{\mathbb{P}}\left(C_T \left| \mathcal{F}_T^s\right.\right) = \mathbb{E}^{\mathbb{P}}(C_T)$ is a constant. Then, using Theorem 3, yields the result. □

As noted in the corollary, when the non-traded asset's cash flows are independent of market prices, then the arbitrage-free price of the non-traded asset is equal to the expected cash flow under the statistical probability $\mathbb{P}$. This special case is useful in the determination of arbitrage-free insurance premiums (see Sect. 4.2 below).

In summary, we state the solution to the arbitrage-free pricing of a non-traded asset.

Solution. To price the cash flow to a non-traded asset $C_T \in L_+^1(\Omega, \mathcal{F}_T, \mathbb{Q}) \cap L_+^1(\Omega, \mathcal{F}_T, \mathbb{P})$ that cannot be replicated using the mma and the n risky assets:

(1) first, consider the part of the asset's payoff $\tilde{C}_T := \mathbb{E}^{\mathbb{P}}[C_T | \mathcal{F}_T^s]$ that can be hedged using the traded assets. This component can be uniquely priced by

$$\mathbb{E}^{\mathbb{Q}^s}(\tilde{C}_T)$$

where $\mathbb{Q}^s := \mathbb{Q}\,|_{\mathcal{F}^s}$.

(2) Second, consider the remaining hedging error $\varepsilon_T = C_T - \tilde{C}_T$. Assuming this hedging error represents non-priced or idiosyncratic risk, then

$$\mathbb{E}^{\mathbb{Q}}(C_T) = \mathbb{E}^{\mathbb{Q}^s}(\tilde{C}_T) = \mathbb{E}^{\mathbb{Q}^s}\left(\mathbb{E}^{\mathbb{P}}\left(C_T \left| \mathcal{F}_T^s\right.\right)\right)$$

is the unique arbitrage-free price in the original market for the non-traded asset's payoff C_T.

We will illustrate this solution in Sect. 4 below for various non-traded assets.

4 Examples

This section presents some examples to illustrate the use of this methodology for pricing various non-traded assets. The simplest examples are selected for clarity, and it will become clear that the examples are easily generalized to make them more realistic and suitable for practice.

4.1 Private debt

This first example is to price non-traded private debt. Let a privately owned company have outstanding debt of various types, one of which is a zero-coupon bond promising to pay 1 dollar at time T. This debt is private and not traded.

For simplicity, we assume that the spot rate of interest is zero, so the mma's value is unity for all times.

Trading is equity for a similar company with price process

$$S_t = S_0 e^{\mu t - \frac{1}{2}\sigma^2 t + \sigma W_t}$$

where S_0, μ, σ are strictly positive constants and W_t is a standard Brownian motion with $W_0 = 0$ under $\mathbb{P}$.

Let $Z_T(\omega) \in \{0, 1\}$ be a $\mathcal{F}_T$ measurable, binomial random variable with probability $\lambda(S_T)$ of $\{Z_T = 1\}$ under $\mathbb{P}$ where $\lambda(\cdot) : \mathbb{R} \to [0, \infty)$ is Borel measurable. Here, $\lambda(S_T)$ is the probability of the private debt defaulting at time T, which is assumed to depend on the value of the similar company's stock price at time T.

$\mathbb{F}$ is the filtration generated by W_t for all $t \in [0, T)$ and (W_T, Z_T) at time T.

The zero-coupon bond issued by the private company has time T payoff

$$C_T(\omega) = \begin{cases} \delta \ if \ Z_T(\omega) = 1 \\ 1 \ if \ Z_T(\omega) = 0 \end{cases} \tag{5}$$

where $\delta \in (0, 1)$ is the recovery rate.

We assume the original market consisting of the similar company's stock and the mma satisfies NFLVR and ND, so that there exists a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ under which S is a $\mathbb{Q}$ - martingale with respect to $\mathbb{F}$.[3]

The restricted market is complete, hence there exists a unique $\mathbb{Q}^s \in \mathcal{M}(\mathbb{F}^s)$ where $\mathbb{Q}^s = \mathbb{Q} \mid_{\mathcal{F}_T^s}$ such that S_t is a $\mathbb{Q}^s$ martingale with respect to $\mathbb{F}^s$, and

$$S_t = S_0 e^{-\frac{1}{2}\sigma^2 t + \sigma \tilde{W}_t} \tag{6}$$

where $\tilde{W}_t = (\frac{\mu}{\sigma})t + W_t$ is a Brownian motion under $\mathbb{Q}^s$.

Assumptions 1–2 are satisfied by construction. Assuming assumption 3 holds, i.e. $\varepsilon_T = C_T - \mathbb{E}^{\mathbb{P}}\left(C_T \,\middle|\, \mathcal{F}_T^s\right)$ is idiosyncratic risk, we have that the time 0 arbitrage-free

[3] This precludes the information in $\mathbb{F}$ generating a NFLVR with trading in the stock S and mma.

price of non-traded private debt is:

$$\mathbb{E}^{\mathbb{Q}}(C_T) = \mathbb{E}^{\mathbb{Q}^s}\left(\mathbb{E}^{\mathbb{P}}\left(C_T \left| \mathcal{F}_T^s \right.\right)\right)$$
$$= \mathbb{E}^{\mathbb{Q}^s}\left(\delta\lambda(S_T) + (1 - \lambda(S_T))\right).$$

The only randomness underlying this expectation is that due to the traded similar equity's price, S_T. Given a functional form of λ, this is easily computed using a normal distribution based on expression (6).

This example also applies to publicly traded debt that, although issued, is very illiquid. As illustrated, it should be noted that this example is a simple case of the models contained in the credit risk literature for the pricing of credit derivatives. The only difference is that typically, in this literature (see Jarrow 2009), it is assumed that the debt is traded and the original market is complete so that the EMM is uniquely determined. Here, in contrast, we do not assume that the expanded market is complete and we allow the risky debt and credit derivatives to be non-traded.

4.2 Insurance contracts

This section discusses the valuation of insurance contracts, the purpose of which is to determine the arbitrage-free insurance premium (see Jarrow 2021a for a related discussion of arbitrage-free insurance premiums). We consider two cases, each increasing in complexity. For both cases, for simplicity, we assume that the spot rate of interest is zero, so the mma's value is unity for all times.

Consider a term insurance contract on an event over the time period $[0, T]$, where for the purposes of discussion, T is a short time horizon, e.g. 1 year. The contract is repriced and repurchased again by the insured, if desired, at time T. The prime example is a yearly term life insurance policy.

We assume that the insurance premium of p dollars is paid at time 0 to insure the event over $[0, T]$. If the event occurs over $[0, T]$, K dollars is paid at time T. The payoff K could be a random variable.

Assume that it costs the insurance company c dollars to issue the insurance contract, and that this cost is incurred at time 0 as well.

4.2.1 Independent event risk (life insurance)

We first consider a term life insurance contract. Here, the contract's payoff K is a constant, determined at the time the insurance contract is issued.

Let the insured event be denoted by the indicator variable $Z_T(\omega) \in \{0, 1\}$, which is a $\mathcal{F}_T$ measurable, binomial random variable with probability λ of $\{Z_T = 1\}$ under $\mathbb{P}$. The probability λ is the actuarial probability that the insured dies over $[0, T]$.

Let S be the market prices of the traded risky assets, and $\mathbb{F}$ the filtration generated by S_t for all $t \in [0, T)$ and (S_T, Z_T) at time T.

We assume that Z_T is independent of market prices S under $\mathbb{P}$. This is a reasonable assumption for the death of an individual.

Then, the cash flow to the insurance company from issuing such a policy at time T is

$$C_T(\omega) = \begin{cases} p - c - K & if \ Z_T(\omega) = 1 \\ p - c & if \ Z_T(\omega) = 0 \end{cases} \tag{7}$$

This is the insurance premium received less costs incurred $(p - c)$, less the payoff if the insured dies (K). Recall that interest rates are assumed to be zero.

We assume the original market consisting of the risky assets and the mma satisfies NFLVR and ND, so that there exists a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ under which S is a $\mathbb{Q}$ - martingale with respect to $\mathbb{F}$.

Finally, we also assume that the restricted market is complete, hence there exists a unique $\mathbb{Q}^s \in \mathcal{M}(\mathbb{F}^s)$ where $\mathbb{Q}^s = \mathbb{Q} \mid_{\mathcal{F}_T^s}$ such that S_t is a $\mathbb{Q}^s$ martingale with respect to $\mathbb{F}^s$.

Assumptions 1–2 are satisfied by construction. Assuming assumption 3 holds, i.e. $\varepsilon_T = C_T - \mathbb{E}^{\mathbb{P}}\left(C_T \,\middle|\, \mathcal{F}_T^s\right)$ is idiosyncratic risk, then the arbitrage-free value of the life insurance contract, using Corollary 4, is

$$\mathbb{E}^{\mathbb{Q}}(C_T) = \mathbb{E}^{\mathbb{P}}\left(C_T\right) = p - c - \lambda K. \tag{8}$$

Hence, the arbitrage-free insurance premium is that p such that $\mathbb{E}^{\mathbb{Q}}(C_T) = 0$, i.e.

$$p = \lambda K + c.$$

This is the actuarial value of the insurance contract's payoff (λK) plus costs (c).

4.2.2 Dependent event risk (car insurance)

Next, we consider a term car insurance contract. Here, the car insurance contract's payoff is the random variable $K(\omega)$ having a uniform distribution over $[0, k]$ with mean $\frac{k}{2}$ under $\mathbb{P}$, where k corresponds to the value of the car at time 0 (another non-traded asset). K represents the damage to the car in the event of an auto accident.

Trading is oil with a price process

$$S_t = S_0 e^{\mu t - \frac{1}{2}\sigma^2 t + \sigma W_t}$$

where S_0, μ, σ are strictly positive constants and W_t is a standard Brownian motion with $W_0 = 0$ under $\mathbb{P}$.

Let the insured event be denoted by the indicator variable $Z_T(\omega) \in \{0, 1\}$, which is a $\mathcal{F}_T$ measurable, binomial random variable with probability $\lambda(S_T)$ of $\{Z_T = 1\}$ under $\mathbb{P}$ where $\lambda(\cdot) : \mathbb{R} \to [0, \infty)$ is Borel measurable. Here, $\lambda(S_T)$ is the probability of an car accident, which is assumed to be a decreasing function of oil prices. The reason is that as oil prices decrease, cars are driven more frequently, and the probability of an accident increases.

We assume that Z_T, K (the loss to the car in the event of an accident), and market prices S are independent under $\mathbb{P}$. This is a reasonable assumption because the car accident event itself and the damages resulting are independent of the price of oil, due to random events surrounding the accident while driving of a car. Note that the probability of the event, however, depends on the price of oil.

Then, the cash flow to the insurance company from issuing such a policy at time T is

$$C_T(\omega) = \begin{cases} p - c - K(\omega) \; if \;\; Z_T(\omega) = 1 \\ \quad p - c \qquad if \;\; Z_T(\omega) = 0 \end{cases} \tag{9}$$

This is the insurance premium received less costs incurred $(p - c)$, less the losses if the accident occurs $(K(\omega))$. Recall that interest rates are assumed to be zero.

$\mathbb{F}$ is the filtration generated by W_t for all $t \in [0, T)$ and (W_T, K, Z_T) at time T.

We assume the original market satisfies NFLVR and ND, so that there exists a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ under which S is a $\mathbb{Q}$ - martingale with respect to $\mathbb{F}$.

The restricted market is complete, hence there exists a unique $\mathbb{Q}^s \in \mathcal{M}(\mathbb{F}^s)$ where $\mathbb{Q}^s = \mathbb{Q} \,|_{\mathcal{F}_T^s}$ such that S_t is a $\mathbb{Q}^s$ martingale with respect to $\mathbb{F}^s$, and

$$S_t = S_0 e^{-\frac{1}{2}\sigma^2 t + \sigma \tilde{W}_t} \tag{10}$$

where $\tilde{W}_t = (\frac{\mu}{\sigma})t + W_t$ is a Brownian motion under $\mathbb{Q}^s$.

Assumptions 1 - 2 are satisfied by construction. Assuming assumption 3 holds, i.e. $\varepsilon_T = C_T - \mathbb{E}^{\mathbb{P}}\left(C_T \,\middle|\, \mathcal{F}_T^s\right)$ is idiosyncratic risk, then the arbitrage-free value of the car insurance contract is

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}(C_T) &= \mathbb{E}^{\mathbb{Q}^s}\left(\mathbb{E}^{\mathbb{P}}\left(C_T \,\middle|\, \mathcal{F}_T^s\right)\right) \\ &= p - c - \frac{k}{2}\mathbb{E}^{\mathbb{P}}\left(\lambda(S_T)\right) \end{aligned}$$

where by the independence assumption $\mathbb{E}^{\mathbb{P}}\left(C_T \,\middle|\, \mathcal{F}_T^s\right) = \mathbb{E}^{\mathbb{P}}(K)\lambda(S_T)$ and by the uniform distribution assumption $\mathbb{E}^{\mathbb{P}}(K) = \frac{k}{2}$. Hence, the arbitrage-free insurance premium is that p such that $\mathbb{E}^{\mathbb{Q}}(C_T) = 0$, i.e.

$$p = \frac{k}{2}\mathbb{E}^{\mathbb{Q}^s}\left(\lambda(S_T)\right) + c.$$

This is *not* the actuarial value of the insurance contract's payoff $(\frac{k}{2}\lambda)$ plus costs (c). The reason is that the probability of a car accident has a systematic risk associated with it. This is easily computed given the oil price process (10).

4.3 Private equity

The next example is to price non-traded private equity. Let a privately owned company have outstanding equity. For simplicity, we assume that the spot rate of interest is zero, so the mma's value is unity for all times.

Trading is equity for a similar company with price process

$$S_t = S_0 e^{\mu t - \frac{1}{2}\sigma^2 t + \sigma W_t}$$

where S_0, μ, σ are strictly positive constants and W_t is a standard Brownian motion with $W_0 = 0$ under $\mathbb{P}$.

Let $Z_T(\omega)$ be a $\mathcal{F}_T$ measurable, normally distributed (0, 1) random variable under $\mathbb{P}$.

The cash flow to the private equity at time T is

$$C_T = S_T e^{\alpha(S_T) - \frac{1}{2}\beta(S_T)^2 + \beta(S_T) Z_T}$$

where $\alpha(\cdot) : \mathbb{R} \to [0, \infty)$ and $\beta(\cdot) : \mathbb{R} \to [0, \infty)$ are Borel measurable. As stated, the time T cash flow to the private equity is the cash flow to the similar company's equity S_T modified by the random variable $e^{\alpha(S_T) - \frac{1}{2}\beta(S_T)^2 + \beta(S_T) Z_T}$ that depends on S_T.

$\mathbb{F}$ is the filtration generated by W_t for all $t \in [0, T)$ and (W_T, Z_T) at time T.

We assume the original market satisfies NFLVR and ND, so that there exists a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ under which S is a $\mathbb{Q}$ - martingale with respect to $\mathbb{F}$.

The restricted market is complete, hence there exists a unique $\mathbb{Q}^s \in \mathcal{M}(\mathbb{F}^s)$ where $\mathbb{Q}^s = \mathbb{Q}\,|_{\mathcal{F}_T^s}$ such that S_t is a $\mathbb{Q}^s$ martingale with respect to $\mathbb{F}^s$, and

$$S_t = S_0 e^{-\frac{1}{2}\sigma^2 t + \sigma \tilde{W}_t} \tag{11}$$

where $\tilde{W}_t = (\frac{\mu}{\sigma})t + W_t$ is a Brownian motion under $\mathbb{Q}^s$.

Assumptions 1–2 are satisfied by construction. Assuming assumption 3 holds, i.e. $\varepsilon_T = C_T - \mathbb{E}^{\mathbb{P}}\left(C_T \,\middle|\mathcal{F}_T^s\right)$ is idiosyncratic risk, we have that the time 0 arbitrage-free price of non-traded private equity is:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}(C_T) &= \mathbb{E}^{\mathbb{Q}^s}\left(\mathbb{E}^{\mathbb{P}}\left(C_T \,\middle|\mathcal{F}_T^s\right)\right)\\ &= \mathbb{E}^{\mathbb{Q}^s}\left(S_T e^{\alpha(S_T)}\right).\end{aligned}$$

The only randomness underlying this expectation is that due to the traded similar equity's price, S_T. Given functional form for (α), this is easily computed using a normal distribution function based on expression (11).

4.4 Real estate

This section values a privately owned home, which trades in a very illiquid market. Again, for simplicity, we assume that the spot rate of interest is zero, so the mma's value is unity for all times.

Trading is a REIT (real estate investment trust) or a real estate based ETF (electronic traded fund), a traded house index, with price process

$$S_t = S_0 e^{\mu t - \frac{1}{2}\sigma^2 t + \sigma W_t} \tag{12}$$

where S_0, μ, σ are strictly positive constants and W_t is a standard Brownian motion with $W_0 = 0$ under $\mathbb{P}$.

Let $Z_T(\omega)$ be a $\mathcal{F}_T$ measurable, normally distributed (0, 1) random variable under $\mathbb{P}$.

The cash flow to selling the house at time T is

$$C_T = S_T e^{\alpha(S_T) - \frac{1}{2}\eta^2 + \eta Z_T}$$

where η is a strictly positive constant and $\alpha(\cdot) : \mathbb{R} \to [0, \infty)$ is Borel measurable.

$\mathbb{F}$ is the filtration generated by W_t for all $t \in [0, T)$ and (W_T, Z_T) at time T.

We assume the original market satisfies NFLVR and ND, so that there exists a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ under which S is a $\mathbb{Q}$ - martingale with respect to $\mathbb{F}$.

The restricted market is complete, hence there exists a unique $\mathbb{Q}^s \in \mathcal{M}(\mathbb{F}^s)$ where $\mathbb{Q}^s = \mathbb{Q}\,|_{\mathcal{F}^s_T}$ such that S_t is a $\mathbb{Q}^s$ martingale with respect to $\mathbb{F}^s$, and

$$S_t = S_0 e^{-\frac{1}{2}\sigma^2 t + \sigma \tilde{W}_t} \tag{13}$$

where $\tilde{W}_t = (\frac{\mu}{\sigma})t + W_t$ is a Brownian motion under $\mathbb{Q}^s$.

Assumptions 1 - 2 are satisfied by construction. Assuming assumption 3 holds, i.e. $\varepsilon_T = C_T - \mathbb{E}^{\mathbb{P}}\left(C_T \,\middle|\, \mathcal{F}^s_T\right)$ is idiosyncratic risk, we have that the time 0 arbitrage-free price of the house is:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}(C_T) &= \mathbb{E}^{\mathbb{Q}^s}\left(\mathbb{E}^{\mathbb{P}}\left(C_T \,\middle|\, \mathcal{F}^s_T\right)\right)\\ &= \mathbb{E}^{\mathbb{Q}^s}\left(S_T e^{\alpha(S_T)}\right).\end{aligned}$$

Given expression (13), this expectation is easily computed.

4.5 Real options

The last example illustrates how to price non-traded real options. For a review of the real option's literature, see (Lambrecht 2017). For simplicity, we assume that the spot rate of interest is zero, so the mma's value is unity for all times. Consider an oil company that is deciding whether or not to extract oil from a well at time T. The market price of oil, a traded commodity, is given by

$$S_t = S_0 e^{\mu t - \frac{1}{2}\sigma^2 t + \sigma W_t}$$

where S_0, μ, σ are strictly positive constants and W_t is a standard Brownian motion with $W_0 = 0$ under $\mathbb{P}$.

Due to the oil extraction methods, after the taking into account impurities which affect the price of oil received before refinement, the cash flow received from the extracted oil at time T is

$$S_T^* = S_T e^{-\frac{1}{2}\eta^2 + \eta Z_T}$$

where $Z_T(\omega)$ is a $\mathcal{F}_T$ measurable, normally distributed (0, 1) random variable under $\mathbb{P}$ and $\eta > 0$ is a constant.

$\mathbb{F}$ is the filtration generated by W_t for all $t \in [0, T)$ and (W_T, Z_T) at time T.

The (real) option to extract oil at time T has payoff

$$C_T = max\left[S_T^* - K, 0\right] = max\left[S_T e^{-\frac{1}{2}\eta^2 + \eta Z_T} - K, 0\right]$$

where $K > 0$ is the cost of the extraction.

We assume the original market satisfies NFLVR and ND, so that there exists a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ under which S is a $\mathbb{Q}$ - martingale with respect to $\mathbb{F}$.

The restricted market is complete, hence there exists a unique $\mathbb{Q}^s \in \mathcal{M}(\mathbb{F}^s)$ where $\mathbb{Q}^s = \mathbb{Q}\,|_{\mathcal{F}_T^s}$ such that S_t is a $\mathbb{Q}^s$ martingale with respect to $\mathbb{F}^s$, and

$$S_t = S_0 e^{-\frac{1}{2}\sigma^2 t + \sigma \tilde{W}_t} \tag{14}$$

where $\tilde{W}_t = (\frac{\mu}{\sigma})t + W_t$ is a Brownian motion under $\mathbb{Q}^s$. Assumptions 1 – 2 are satisfied by construction.

Assuming assumption 3 holds, i.e. $\varepsilon_T = C_T - \mathbb{E}^{\mathbb{P}}\left(C_T \,\middle|\, \mathcal{F}_T^s\right)$ is idiosyncratic risk, we have that the time 0 arbitrage-free price of option to extract is:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}(C_T) &= \mathbb{E}^{\mathbb{Q}^s}\left(\mathbb{E}^{\mathbb{P}}\left(C_T \,\middle|\, \mathcal{F}_T^s\right)\right) \\ &= \mathbb{E}^{\mathbb{Q}^s}\left(S_T N(d_1) - K N(d_2)\right)\end{aligned}$$

where $N(\cdot)$ is the standard (0, 1) normal distribution function,

$$d_1 := \frac{log(S_T/K) + \frac{1}{2}\eta^2}{\eta}, \qquad \text{and} \qquad d_2 := d_1 - \eta.$$

The only randomness remaining underlying this expectation is that due to the traded oil price, S_T. Given the functional form of $N(\cdot)$, this is easily computed using a normal distribution function based on expression (14).

5 Conclusion

Using the arbitrage-free pricing methodology in a related complete market, this paper shows how to price non-traded assets in an otherwise frictionless market. This is an important application of the arbitrage-free pricing methodology because it applies to a wide range of assets in the economy, including private debt, illiquid publicly traded debt, insurance contracts, private equity, real estate, and real options. The methodology can be applied without assuming a particular preference or objective function. Its application only requires that the hedging error, properly defined, is non-priced. This non-priced hedging error condition is a very reasonable approximation in current markets given the plethora of traded securities.

Appendix: Diversifiable risk

The idea of diversifiable risk being non-priced is due to Merton (1976) and Ross (1976). This section provides a set of sufficient conditions for the satisfaction of assumption 3 in a discrete time model.

First, partition the time interval $[0, T]$ into discrete time intervals of unit length $t = 0, 1, ..., T$. All of the notation in the previous section applies. Consider the admissible s.f.t.s. $(\alpha_0, \alpha) \in \mathscr{A}(x, \mathbb{F}^s)$ that generates $\tilde{C}_T$ with initial investment $\tilde{C}_0$. We assume that this trading strategy (α_0, α) is constant over the subintervals. Then, the value process can be written as

$$\tilde{C}_0 = \mathbb{E}^{\mathbb{Q}}(\tilde{C}_T) = \alpha_0(0) + \alpha_0 \cdot S_0$$

with

$$\tilde{C}_{t+1} - \tilde{C}_t = \alpha_t \cdot (S_{t+1} - S_t)$$

so that

$$\tilde{C}_T = \tilde{C}_0^i + \sum_{t=0}^{T} \alpha_t \cdot (S_{t+1} - S_t).$$

Define $C_t = \tilde{C}_t + \varepsilon_t$. Then,

$$C_{t+1} - C_t = \alpha_t \cdot (S_{t+1} - S_t) + (\varepsilon_{t+1} - \varepsilon_t)$$

So, that

$$C_T = C_0 + \sum_{t=0}^{T} \alpha_t \cdot (S_{t+1} - S_t) + (\varepsilon_T - \varepsilon_0)$$

or,

$$C_T = \tilde{C}_0 + \sum_{t=0}^{T} \alpha_t \cdot (S_{t+1} - S_t) + \varepsilon_T = \tilde{C}_T + \varepsilon_T.$$

To introduce the notion of diversifiable risk, we suppose that there exists an infinite collection of such hedging errors $(\varepsilon_t^i)_{i=1,...,\infty}$ associated with the payoff $\tilde{C}_T$ for $i = 1, ..., \infty$. These could be due to unique frictions faced by individuals or trading venues or assets (e.g. different houses). We add the following assumption.

Assumption 4 (*Diversifiable Risk*)
(i) For every $t = 0, 1, ..., T$, $\varepsilon_{t+1}^i - \varepsilon_t^i$ given $\mathcal{F}_t$ are uniformly bounded and (cross sectionally) independent and identically distributed under $\mathbb{P}$ for $i = 1, ..., \infty$.
(ii) $\mathbb{E}^{\mathbb{P}}(\varepsilon_T^i) = 0$.

Condition (i) of assumption 4 captures the cross sectional independence of the hedging errors at any intermediate time t. Condition (ii) states that the unconditional expected hedging error is zero, under the statistical probability $\mathbb{P}$.

Define the conditional expected hedging error $\varepsilon_t^i := \mathbb{E}^{\mathbb{P}}\left(\varepsilon_T^i \,|\mathcal{F}_t\right)$ with respect to $\mathbb{F}$. To prove the required result we consider the hedging error's return over the time interval $[t, t+1]$ for $t \geq 1$, i.e.

$$\frac{\varepsilon_{t+1}^i - \varepsilon_t^i}{\varepsilon_t^i}.$$

Note that $\frac{\mathbb{E}^{\mathbb{P}}(\varepsilon_{t+1}^i|\mathcal{F}_t)-\varepsilon_t^i}{\varepsilon_t^i} := \mu_t^P$ because ε_t^i are identically distributed, and $\mu_t^P = 0$ because ε_t^i is a $\mathbb{P}$ martingale with respect to $\mathbb{F}$.

At $t = 0$, consider the return on the portfolio consisting of the hedging error plus a dollar in the mma, i.e.

$$\frac{\varepsilon_1^i - \varepsilon_0^i}{\varepsilon_0^i + 1}.$$

To apply an existing theorem in the literature, first a collection of assets representing the hedging errors needs to be constructed. For each i, this can be obtained by holding the cash flows C_t^i and shorting $\tilde{C}_t$ for $t \geq 1$. For $t = 0$ add a unit investment to this trading strategy in the mma. Shorting $\tilde{C}_t$ is feasible because this wealth process can be obtained using just the traded assets.

Then, applying (Jarrow 2021a), Theorem 5, the hedging error's return under the martingale measure $\mathbb{Q}$ is the same as that under the statistical probability $\mathbb{P}$, i.e.

$$\frac{\mathbb{E}^{\mathbb{Q}}\left(\varepsilon_{t+1}^i \,|\mathcal{F}_t\right) - \varepsilon_t^i}{\varepsilon_t^i} = \mu_t^P$$

for $t \geq 1$ and

$$\frac{\mathbb{E}^{\mathbb{Q}}\left(\varepsilon_1^i\right) - \varepsilon_0^i}{\varepsilon_0^i + 1} = \mu_0^P.$$

The idea underlying the theorem's proof is that the return to an equally weighted portfolio of the payoffs ε_{t+1}^i converges to the constant μ_t^P by the law of large numbers, which in the limit has no risk. Hence, the risk-adjustment for the individual hedging error returns under $\mathbb{Q}$ is the same as that under $\mathbb{P}$.

To complete the proof, note that because $\mu_t^P = 0$ for all t, this implies

$$\mathbb{E}^{\mathbb{Q}}\left(\varepsilon_{t+1}^i \,|\mathcal{F}_t\right) - \varepsilon_t^i = 0$$

for all t. Therefore, $\mathbb{E}^{\mathbb{Q}}(\varepsilon_T^i) = \varepsilon_0^i$. But, $\varepsilon_0^i = \mathbb{E}^{\mathbb{P}}(\varepsilon_T^i) = 0$, by assumption 4 (ii). Hence,

$$\mathbb{E}^{\mathbb{Q}}\left(C_T^i\right) = \tilde{C}_0 = \mathbb{E}^{\mathbb{Q}^s}(\tilde{C}_T),$$

which completes the argument.

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