Annals of Finance (2023) 19:141–168
https://doi.org/10.1007/s10436-023-00426-1

RESEARCH ARTICLE

No-arbitrage conditions and pricing from discrete-time to continuous-time strategies

Dorsaf Cherif[1,3] · Emmanuel Lépinette[2,3]

Received: 15 July 2021 / Accepted: 24 March 2023 / Published online: 24 April 2023

Abstract
In this paper, a general framework is developed for continuous-time financial market models defined from simple strategies through conditional topologies that avoid stochastic calculus and do not necessitate semimartingale models. We then compare the usual no-arbitrage conditions of the literature, e.g. the usual no-arbitrage conditions NFL, NFLVR and NUPBR and the recent AIP condition. With appropriate pseudo-distance topologies, we show that they hold in continuous time if and only if they hold in discrete time. Moreover, the super-hedging prices in continuous time coincide with the discrete-time super-hedging prices, even without any no-arbitrage condition.

Keywords No-arbitrage condition · Super-hedging price · AIP condition · NFL condition · Discrete-time financial model · Continuous-time financial market model

JEL Classification G1 · G11 · G12 · G13

1 Introduction

Absence of arbitrage opportunities is an usual condition imposed on financial market models to deduce a characterization of super-hedging prices. In continuous-time,

Dorsaf Cherif and Emmanuel Lépinette contributed equally to this work.

✉ Emmanuel Lépinette
emmanuel.lepinette@ceremade.dauphine.fr

Dorsaf Cherif
dorsaf-cherif@hotmail.fr

[1] Latao, Faculty of Sciences of Tunis, El Manar University, 94-Rommana, 1068 Tunis, Tunisia

[2] Ceremade, CNRS, Paris Dauphine University, PSL, Place du Maréchal De Lattre De Tassigny, Paris, Cedex 16, France

[3] Gosaef, Faculty of Sciences of Tunis, El Manar University, 94-Rommana, 1068 Tunis, Tunisia

Delbaen and Schachermayer (1994) have introduced the famous no-arbitrage condition NFLVR as equivalent to the existence of a local martingale measure, see also the well known NFL condition by Kreps (1981) at the origin of the arbitrage theory in continuous time. More recently, the weaker NUPBR no-arbitrage condition (Karatzas and Kardaras 2007) has been introduced as the minimal one necessary to solve utility maximization problems.

However, models where the price processes are not semi-martingales are also considered in the literature, e.g. fractional Brownian motion, see Pakkanen (2010) and Lo (1991) for empirical studies. Moreover, in the papers Rogers (1997) and Sottinen (2001), it is shown that arbitrage opportunities exist in fractional Brownian motion models. Also Guasoni considers (Guasoni 2006) non-semimartingale models with transaction costs. In the paper (Carassus and Lépinette 2021), the no-arbitrage condition AIP ensures the finiteness of the super-hedging prices in non-semimartingale frictionless models and a dynamic programming principle allows to compute them in discrete time.

Absence of arbitrage opportunities in non-semimartingale models has also been considered by restricting the class of admissible trading strategies as initiated by Cheredito (2003), Bender et al. (2008), Bayraktar and Sayit (2010), Sayit (2013) among others. Precisely, only simple strategies with a minimal deterministic time between two trades are allowed. It is then possible to show that fractional Brownian motions, and more general processes, are arbitrage free with respect to this so-called Cheridito's class of simple strategies, see Jarrow et al. (2009). In other words, this specific restricted class of simple strategies is adapted to the non-semimartingale price processes of consideration in such a way that a no-arbitrage condition holds.

Our approach is different: We fix an a priori given class of strategies that are interpreted as simple discrete-time strategies (discrete-time or simple strategies in short) and the continuous-time strategies are defined as convergent sequences of simple strategies. Here, convergence should be understood with respect to a topology induced by a (conditional) pseudo-distance we introduce in such a way that, by definition, a terminal continuous-time portfolio value is attainable from a terminal discrete-time portfolio process, up to an arbitrarily small error. Precisely, if $\overline{v}_T$ is a terminal continuous-time portfolio value, then for every $\varepsilon > 0$, there exists a terminal discrete-time portfolio value v_T such that $v_T \geq \overline{v}_T - \varepsilon$.

We aim to show that the usual no-arbitrage conditions NFL, NFLVR and NUPBR in discrete-time are respectively equivalent to their analogous conditions in continuous time, with an appropriate choice of a pseudo-distance topology which is financially meaning. The same holds for the weaker AIP condition which means that non negative payoffs admit non negative prices, or equivalently, the infimum super-hedging price of a non negative price cannot be $-\infty$, see Carassus and Lépinette (2021). Moreover, we then show that the infimum super-hedging prices in discrete time and in continuous time coincide, without supposing any no-arbitrage condition. Of course, such prices may be numerically estimated only if AIP holds, which is the weaker no-arbitrage condition of consideration.

In the following, we first present the general framework that generates the continuous-time portfolios from the discrete-time ones without any semi-martingale setting. Then, we successively compare in discrete time and in continuous time the

NFL, NFLVR, AIP and NUPBR no-arbitrage conditions. Finally, we compare the super-hedging prices in discrete time and in continuous time. The last section exposes the theory we have developed on pseudo-distance topologies. In the appendix, some auxiliary results are collected.

2 Model

Let $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbf{P})$ be a complete stochastic basis which is right-continuous. We consider a financial market model defined by d risky assets described by a continuous-time right-continuous price process $S_t = (S_t^1, ..., S_t^d) \in \mathbf{R}_+^d$, $t \in [0, T]$, adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Moreover, we suppose that there exists a bond whose price is $S^0 = 1$, without loss of generality. The quantities invested in a portfolio are described, as usual, by a real-valued adapted process θ^0 that describes the quantity invested in the bond and an adapted process $\theta = (\theta^1, ..., \theta^d) \in \mathbf{R}^d$, called strategy, that describes the quantities invested in the risky assets. Without transaction costs, the liquation value of the strategy θ is given by the portfolio process $V = V^\theta = \theta S$ where the product needs to be understood as the Euclidean inner product on $\mathbf{R}^d$. Recall that, in discrete-time $t = 0, 1, \cdots, T$, $V = V^\theta$ is said self-financing if $\theta_t S_{t+1} = \theta_{t+1} S_{t+1}$, i.e. $\Delta V_{t+1} := V_{t+1} - V_t = \theta_t \Delta S_{t+1}$. Then, the terminal value of a self-financing portfolio process starting from the zero initial capital is of the form $V_{t,T} = \sum_{u=t}^T \theta_{u-1} \Delta S_u$.

In the following, $T > 0$ is the horizon time and we consider for any time $t \leq T$, a set $\mathcal{V}_{t,T}$ of T- terminal discrete-time portfolios, starting from the zero initial capital at time t. An element of $\mathcal{V}_{t,T}$ may be seen as a portfolio value generated by a simple strategy, as in Cheredito (2003) or generated by specific discrete-time strategies more generally.

A first typical example is when the trades are only executed at arbitrary deterministic times:

$$\mathcal{V}_{t,T}^{\text{det}} = \left\{ \sum_{i=1}^n \theta_{t_{i-1}} \Delta S_{t_i},\ t = t_0 < \cdots < t_n = T,\ \theta_{t_i} \in L^0(\mathbb{R}^d, \mathcal{F}_{t_i}),\ n \geq 1 \right\}. \tag{1}$$

A second example is when the portfolios are revised at some stopping times, e.g. when some market conditions are satisfied. Let us denote by $\mathcal{T}_{t,T}$ the set of all $[t, T]$-valued stopping times. We denote by $\hat{\mathcal{T}}_{t,T}^n$, $n \geq 1$, the set of all increasing sequences of stopping times $(\tau_i)_{i=0}^n$ such that $t = \tau_0 < \cdots < \tau_n = T$. We then consider the set:

$$\mathcal{V}_{t,T}^{\text{rand}} = \left\{ \sum_{i=1}^n \theta_{\tau_{i-1}} \Delta S_{\tau_i},\ (\tau_i)_{i=0}^n \in \hat{\mathcal{T}}_{t,T}^n,\ \theta_{\tau_i} \in L^0(\mathbf{R}^d, \mathcal{F}_{\tau_i}),\ n \geq 1 \right\}. \tag{2}$$

Remark 1 In the common cases, the discrete-time portfolio processes $V_{t,T} \in \mathcal{V}_{t,T}$ are explicitly characterized by a priori given "simple" strategies $\theta^{t,T} \in \mathcal{S}_{t,T}$, i.e. $V_{t,T} = \mathcal{I}(\theta^{t,T})$ for some operator $\mathcal{I}$. In that case, we also denote by $V_{t,u}$ the u-time value of $V_{t,T}$, i.e. $V_{t,u} = \mathcal{I}(\theta^{t,T,u})$, $u \in [t, T]$, where $\theta^{t,T,u}$ is the restriction of $\theta^{t,T}$ to the interval $[t, u]$ so that $\theta_v^{t,T,u} = 0$ if $v > u$. This is the case in the two examples above and we write $\mathcal{V}_{t,T} = \mathcal{I}(\mathcal{S}_{t,T})$. In continuous-time, this is usual to require the strategies to be admissible. In the example given by (2), we have

$$\mathcal{I}_u(\theta) := \mathcal{I}(\theta^{t,T,u}) = \sum_{i=1}^{n} \theta_{\tau_{i-1}} \left(S_{\tau_i \wedge u} - S_{\tau_{i-1} \wedge u} \right), \quad u \in [t, T]. \tag{3}$$

We say that θ is admissible if there exists $m \in \mathbf{R}$ such that $\mathcal{I}_u(\theta) \geq m$ a.s. for all $u \in [t, T]$. In that case, the corresponding set of terminal portfolio processes is denoted by ${}^a\mathcal{V}_{t,T}$ instead of $\mathcal{V}_{t,T}$. ◊

In the following, we consider $L^0(\mathbf{R}^d, \mathcal{F}_T)$, $d \geq 1$, the set of all equivalence classes of random variables defined on $(\Omega, \mathcal{F}_T, \mathbf{P})$ with values in $\mathbf{R}^d$. The following definitions allow to define continuous-time portfolio processes (resp. strategies) from discrete-time portfolio processes (resp. simple strategies).

Definition 1 Let $t \leq T$ and let $\mathcal{O}_t$ be a topology on $L^0(\mathbf{R}, \mathcal{F}_T)$. We say that a sequence $(V^n_{t,T})_{n\geq 1}$ of $\mathcal{V}_{t,T}$ is $\mathcal{O}_t$-integrable if $(V^n_{t,T})_{n\geq 1}$ is convergent with respect to $\mathcal{O}_t$.

The definition above is designed for an arbitrary topology $\mathcal{O}_t$. It will be used for the particular topology $\mathcal{O}_t$ as defined in Sect. 4.2 below:

Definition 2 Let $t \leq T$ and let $\mathcal{O}_t$ be a topology on $L^0(\mathbf{R}, \mathcal{F}_T)$. We denote by $\mathcal{V}^c_{t,T} = \mathcal{V}^c_{t,T}(\mathcal{O}_t)$ the family of all limits for the topology $\mathcal{O}_t$ of $\mathcal{O}_t$-integrable sequences $(V^n_{t,T})_{n\geq 1}$ of $\mathcal{V}_{t,T}$. An element of $\mathcal{V}^c_{t,T}$ is called a terminal continuous-time portfolio.

Definition 3 Let $t \leq T$ and let $\mathcal{O}_t$ be a topology on $L^0(\mathbf{R}, \mathcal{F}_T)$. Suppose that $\mathcal{V}_{t,T} = \mathcal{I}(\mathcal{S}_{t,T})$ for some operator $\mathcal{I}$ and simple strategies $\mathcal{S}_{t,T}$. We say that a sequence $(\theta^n)_{n\geq 1}$ of $\mathcal{S}_{t,T}$ is $\mathcal{O}_t$-integrable if $(V^n_{t,u} = \mathcal{I}_u(\theta^n))_{n\geq 1}$ is $\mathcal{O}_t$-integrable for all $u \leq T$.

Definition 4 Let $t \leq T$ and let $\mathcal{O}_t$ be a topology on $L^0(\mathbf{R}, \mathcal{F}_T)$. Suppose that $\mathcal{V}_{t,T} = \mathcal{I}(\mathcal{S}_{t,T})$ for some operator $\mathcal{I}$ and simple strategies $\mathcal{S}_{t,T}$. A continuous-time strategy θ on $[t, T]$ is an $\mathcal{O}_t$-integrable sequence $\theta = (\theta^n)_{n\geq 1}$ of simple strategies $\theta^n \in \mathcal{S}_{t,T}$. In that case, for any $u \in [t, T]$, we define $V^c_{t,T}(u) = \mathcal{I}_u(\theta)$ as a limit in $\mathcal{O}_t$ of the convergent sequence $(\mathcal{I}_u(\theta^n))_{n\geq 1}$. We then have $V^c_{t,T} = V^c_{t,T}(T) \in \mathcal{V}^c_{t,T} = \mathcal{V}^c_{t,T}(\mathcal{O}_t)$ by definition.

The aim of the paper is to understand whether a no-arbitrage condition imposed on the set of all discrete-time portfolio processes (or simple strategies) at any time t also holds on the set of all continuous-time portfolio processes (resp. strategies). Clearly, that should depend on the topologies $(\mathcal{O}_t)_{t\in[0,T]}$. Also, it is interesting to compare the super-hedging prices obtained by the discrete-time portfolio processes from the continuous-time ones.

In the following, we shall consider at any time $t \leq T$ a topology $\mathcal{O}_t$ that satisfies the Fatou property defined as follows:

Definition 5 A topology $\mathcal{O}_t$ on $L^0(\mathbf{R}, \mathcal{F}_T)$ satisfies the Fatou property if for any sequence $(X^n)_{n\geq 1}$ of $L^0(\mathbf{R}, \mathcal{F}_T)$ that converges to X in $\mathcal{O}_t$, we have $X \leq \liminf_n X_{k_n}$ for some subsequence $(k_n)_{n\geq 1}$.

Note that the Fatou property holds as soon as $X = \liminf_n X_{k_n}$ for some subsequence $(k_n)_{n\geq 1}$. This is the case for the usual topologies, in particular the topologies defined with respect to the convergence in probability or the L^p norms

$\|X\|_p = (E|X|^p)^{1/p}$, $p \in [1, \infty]$. We shall see that this is also the case for the topology of Sect. 4.2. This is a non Hausdorff topology which satisfies the following properties:

Definition 6 A topology $\mathcal{O}_t$ on $L^0(\mathbf{R}, \mathcal{F}_T)$ is said $\mathcal{F}_t$-positively homogeneous if for any sequence $(X^n)_{n\geq 1}$ of $L^0(\mathbf{R}, \mathcal{F}_T)$ that converges to X in $\mathcal{O}_t$, and for all $\alpha_t \in L^0(\mathbf{R}^+, \mathcal{F}_t)$, $(\alpha_t X^n)_{n\geq 1}$ converges to $\alpha_t X$ in $\mathcal{O}_t$.

Definition 7 A topology $\mathcal{O}_t$ on $L^0(\mathbf{R}, \mathcal{F}_T)$ is said $\mathcal{F}_t$-lower bond preserving if, for any $X \in L^0(\mathbf{R}, \mathcal{F}_T)$ such that $X \geq m_t$ for some $m_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ and for any sequence $(X^n)_{n\geq 1}$ of $L^0(\mathbf{R}, \mathcal{F}_T)$ that converges to X in $\mathcal{O}_t$, there exists a subsequence $(X^{k_n})_{n\geq 1}$ such that $X^{k_n} \geq \mu_t$ for some $\mu_t \in L^0(\mathbf{R}, \mathcal{F}_t)$.

3 The NFL and the NFLVR conditions

Let us define $\mathcal{A}_{t,T} := \mathcal{V}_{t,T} - L^0(\mathbf{R}_+, \mathcal{F}_T)$ (resp. $\mathcal{A}^c_{t,T} := \mathcal{V}^c_{t,T} - L^0(\mathbf{R}_+, \mathcal{F}_T)$) the set of all attainable claims from discrete-time (resp. continuous-time) portfolio processes. We denote by $L^\infty(\mathbf{R}, \mathcal{F}_T)$ the set of all equivalence classes of bounded random variables X such that $\|X\|_\infty < \infty$. Consider the corresponding sets $\mathcal{A}^\infty_{t,T} := \mathcal{A}_{t,T} \cap L^\infty(\mathbf{R}, \mathcal{F}_T)$ and $\mathcal{A}^{c,\infty}_{t,T} := \mathcal{A}^c_{t,T} \cap L^\infty(\mathbf{R}, \mathcal{F}_T)$ of bounded attainable claims. Then, we denote by $\overline{\mathcal{A}}^{w,\infty}_{t,T}$ and $\overline{\mathcal{A}}^{c,w,\infty}_{t,T}$ the weak closures of $\mathcal{A}^\infty_{t,T}$ and $\mathcal{A}^{c,\infty}_{t,T}$ respectively with respect to the topology $\sigma(L^\infty, L^1)$.

3.1 The NFL condition

The NFL condition is very well known in mathematical finance. It means that it is not possible to asymptotically get (in limit) a strictly positive profit when starting from a zero initial capital and following a bounded self-financing portfolio process. Here, asymptotically means that we complete the set of bounded self-financing portfolio processes by their limits in L^∞ w.r.t. $\sigma(L^\infty, L^1)$.

Definition 8 Let $(\mathcal{O}_t)_{t\leq T}$ be a collection of topologies on $L^0(\mathbf{R}, \mathcal{F}_T)$ and $\mathcal{V}^c_{t,T} = \mathcal{V}^c_{t,T}(\mathcal{O}_t)$, $t \leq T$. The No Free Lunch condition (NFL, Kreps 1981) is defined at time t by $\overline{\mathcal{A}}^{w,\infty}_{t,T} \cap L^\infty(\mathbf{R}^+, \mathcal{F}_t) = \{0\}$ (resp. $\overline{\mathcal{A}}^{c,w,\infty}_{t,T} \cap L^\infty(\mathbf{R}^+, \mathcal{F}_t) = \{0\}$) for the model defined by the discrete-time (resp. continuous-time) portfolio processes. We say that the NFL condition holds if it holds at any time $t \leq T$.

In the following, if $\mathcal{O}$ and $\mathcal{O}'$ are two topologies, we say that $\mathcal{O} \subseteq \mathcal{O}'$ if any open set of $\mathcal{O}$ is an open set of $\mathcal{O}'$. We consider a collection $(\mathcal{O}_t)_{t\leq T}$ of topologies on $L^0(\mathbf{R}, \mathcal{F}_T)$ so that $\mathcal{V}^c_{t,T} = \mathcal{V}^c_{t,T}(\mathcal{O}_t)$, $t \leq T$.

Lemma 1 *Suppose that $\mathcal{O}_0 \subseteq \mathcal{O}_t$ and $\mathcal{V}_{t,T} \subseteq \mathcal{V}_{0,T}$ for all $t \in [0, T]$. Then, the NFL condition holds for the continuous-time (resp. discrete-time) portfolio processes if and only if NFL holds at time $t = 0$.*

Proof By the assumptions, we deduce that $\mathcal{V}^c_{t,T} \subseteq \mathcal{V}^c_{0,T}$ for all $t \in [0, T]$. We deduce that $\overline{\mathcal{A}}^{c,w,\infty}_{t,T} \subseteq \overline{\mathcal{A}}^{c,w,\infty}_{0,T}$ for all $t \in [0, T]$. The conclusion follows. □

Proposition 2 *Suppose that the topology $\mathcal{O}_t$, $t \leq T$, satisfies the Fatou property, is $\mathcal{F}_t$-positively homogeneous and is $\mathcal{F}_t$-lower bond preserving. Assume that $\mathcal{V}_{t,T}$ is a $\mathcal{F}_t$ positive cone, i.e. $\mathcal{V}_{t,T}$ is convex and $\alpha_t \mathcal{V}_{t,T} \subseteq \mathcal{V}_{t,T}$ for all $\alpha_t \in L^0(\mathbf{R}^+, \mathcal{F}_t)$. Then, with $\mathcal{V}^c_{t,T} = \mathcal{V}^c_{t,T}(\mathcal{O}_t)$, the following statement are equivalent:*

1. *NFL holds at time t for the model defined by the discrete-time portfolio processes.*
2. *There exists $Q_t \sim P$ such that $E_{Q_t}(V) \leq 0$ for all $V \in \mathcal{V}_{t,T}$ such that V is bounded from below by a constant.*
3. *NFL holds at time t for the model defined by the continuous-time portfolio processes.*
4. *There exists $Q_t \sim P$ such that $E_{Q_t}(V) \leq 0$ for all $V \in \mathcal{V}^c_{t,T}$ such that V is bounded from below by a constant.*

Proof By the assumptions, $\overline{\mathcal{A}}^{w,\infty}_{t,T}$ and $\overline{\mathcal{A}}^{c,w,\infty}_{t,T}$ are positive cones. Therefore the equivalences between 1.) and 2.) and between 3.) and 4.) are immediate consequences of the Kreps-Yan theorem, see Kabanov and Safarian (2009, Theorem 2.1.4). Indeed, if $E_{Q_t}(V) \leq 0$ for all $V \in \mathcal{A}^c_{t,T} \cap L^\infty(\mathbf{R}, \mathcal{F}_T)$ (resp. $\mathcal{A}_{t,T} \cap L^\infty(\mathbf{R}, \mathcal{F}_T)$), it suffices to apply the Fatou lemma to the sequence $V^m = V 1_{\{V \leq m\}} \in L^\infty(\mathbf{R}, \mathcal{F}_T)$, as $m \to \infty$, if V is bounded from below, to deduce 4.) (resp. 2.)). It is clear that 4.) implies 2.) since $\mathcal{V}_{t,T} \subseteq \mathcal{V}^c_{t,T}$. It remains to show that 2.) implies 4.). We first observe that 2.) implies that $E_{Q_t}(V|\mathcal{F}_t) \leq 0$ for all bounded from below $V \in \mathcal{V}_{t,T}$, since $\mathcal{V}_{t,T}$ is a $\mathcal{F}_t$ positive cone. We then deduce by rescaling that the inequality $E_{Q_t}(V|\mathcal{F}_t) \leq 0$ also holds if V is bounded from below by an $\mathcal{F}_t$-measurable random variable. Then, consider $V \in \mathcal{V}^c_{t,T}$ such that $V \geq m$ a.s. for some $m \in \mathbf{R}$. By definition, $V = \lim_n V^n$ in $\mathcal{O}_t$, for some convergent sequence of elements $V^n \in \mathcal{V}_{t,T}$. As $\mathcal{O}_t$ satisfies the Fatou property, we may suppose w.l.o.g. that $V \leq \liminf_n V^n$. Moreover, as $\mathcal{O}_t$ is $\mathcal{F}_t$- lower bond preserving, we may also suppose that $V^n \geq \mu_t$ a.s., for all $n \geq 1$, where $\mu_t \in L^0(\mathbf{R}, \mathcal{F}_t)$. Then, $E_{Q_t}(V|\mathcal{F}_t) \leq \lim_n E_{Q_t}(V^n|\mathcal{F}_t)$ by the Fatou lemma. As $E_{Q_t}(V^n|\mathcal{F}_t) \leq 0$ by the remark above, the conclusion follows. □

Remark 2 The equivalent probability measure $Q_t \sim P$ in Statemement 2.) is generally interpreted as a risk-neutral probability measure, see Dalang et al. (1990).

Definition 9 The price process is said locally bounded if there exists a sequence of increasing stopping times $(T^n)_{n\geq 1}$ and a real-valued sequence $(M^n)_{n\geq 1}$ such that $\lim_{n\to\infty} T^n = +\infty$ and the stopped processes S^{T^n} are bounded by M^n.

Note that, if the jumps $\Delta S_t = S_t - S_{t-}$ are uniformly bounded by a constant $M \geq 0$, it suffices to consider $T^n = \inf\{t \geq T_{n-1} : S_t \geq n\}$ so that $S^{T^n} \leq M + n$.

Corollary 3 *Suppose that $\mathcal{O}_0 \subseteq \mathcal{O}_t$ for all $t \leq T$. Suppose that the topology $\mathcal{O}_0$ satisfies the Fatou property, is $\mathcal{F}_0$-positively homogeneous and is $\mathcal{F}_0$- lower bond preserving. Assume that $\mathcal{V}_{t,T}$ is given by (2) for all $t \leq T$ and S is a locally bounded process. Then, if NFL holds for the discrete-time (resp. continuous-time) portfolios, there exists a local martingale measure for S. Moreover, if $\mathcal{V}_{t,T} =^a \mathcal{V}_{t,T}$, the existence of a local martingale measure for S implies NFL for both discrete-time and continuous-time portfolios.*

Proof Note that $\mathcal{V}_{t,T} \subseteq \mathcal{V}_{0,T}$ by (2). Therefore, as $\mathcal{O}_0 \subseteq \mathcal{O}_t$, it suffices to consider the NFL condition at time $t = 0$ by Lemma 1. By Proposition 2, NFL in discrete time and in continuous time are equivalent. In the following, we use the notations of Definition 9. If NFL holds, the local martingale measure $Q = Q_0$ for S is given by Proposition 2. Indeed, for each $n \geq 1$, and $t_1 \leq t_2$ such that $t_2 \leq T$, $V = \pm \left(S_{t_2 \wedge T^n} - S_{t_1 \wedge T^n}\right) 1_{F_{t_1}} \in \mathcal{V}_{0,T}$ for all $F_{t_1} \in \mathcal{F}_{t_1}$ and V is bounded from below by $-M^n$. So, we deduce that $E_Q(\left(S_{t_2 \wedge T^n} - S_{t_1 \wedge T^n}\right) 1_{F_{t_1}}) = 0$ and finally $E_Q(S_{t_2}^{T^n} | \mathcal{F}_{t_1}) = S_{t_1}^{T^n}$. This implies that S is a local martingale under Q. At last, if $\mathcal{V}_{0,T} =^a \mathcal{V}_{0,T}$, consider an admissible simple strategy θ such that $\mathcal{I}_u(\theta) \geq m$ for all $u \in [0, T]$, see (3). Suppose that there exists a local martingale measure Q for S. So, there exists an increasing sequence $(T^n)_{n \geq 1}$ of stopping times such that $\lim_n T^n = \infty$ and the stopped process S^{T^n} is a martingale, for all $n \geq 1$. It is easily seen that $E_Q[\mathcal{I}_{T \wedge T^n}(\theta)] = 0$. Indeed, it suffices to successively apply to tower property knowing that the generalized conditional expectation $E_Q(\theta_{\tau_{i-1}} \left(S_{\tau_i \wedge T^n} - S_{\tau_{i-1} \wedge T^n}\right) | \mathcal{F}_{\tau_{i-1}}) = 0$. Moreover, $\mathcal{I}_{T \wedge T^n}(\theta) \geq m$ by the admissibility property. Therefore, $E_Q[\mathcal{I}_T(\theta)] \leq \liminf_n E_Q[\mathcal{I}_{T \wedge T^n}(\theta)] \leq 0$, by the Fatou lemma. The conclusion follows by Proposition 2. □

3.2 The NFLVR condition

The NFLVR condition is also well known in mathematical finance. The financial interpretation is the same as the NFL one, i.e. it is an asymptotic no-arbitrage condition, but the topology $\sigma(L^\infty, L^1)$ is replaced by the strong topology defined by the L^∞ norm.

Let $\mathcal{A}_{t,T}^\infty := \mathcal{A}_{t,T} \cap L^\infty(\mathbf{R}, \mathcal{F}_T)$ and $\mathcal{A}_{t,T}^{c,\infty} := \mathcal{A}_{t,T}^c \cap L^\infty(\mathbf{R}, \mathcal{F}_T)$ be the sets of bounded attainable claims. Then, we denote by $\overline{\mathcal{A}}_{t,T}^\infty$ and $\overline{\mathcal{A}}_{t,T}^{c,\infty}$ the norm closures of $\mathcal{A}_{t,T}^\infty$ and $\mathcal{A}_{t,T}^{c,\infty}$ respectively with respect to the topology induced by the norm $\| \cdot \|_\infty$.

Definition 10 The condition NFLVR holds at time $t \leq T$ for the discrete-time portfolios (resp. continuous-time portfolios) if $\overline{\mathcal{A}}_{t,T}^\infty \cap L^\infty(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$ (resp. $\overline{\mathcal{A}}_{t,T}^{c,\infty} \cap L^\infty(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$). We say that NFLVR holds if NFLVR holds at any time $t \leq T$.

We easily observe that NFL implies NFLVR. Actually, under some conditions on the price process, NFL and NFLVR are equivalent Delbaen and Schachermayer (1994, Corollary 1.2) to the existence of a local martingale measure, as we shall see. Note that it is not trivial whether the NFLVR condition for discrete-time portfolios is equivalent to the NFLVR condition for continuous-time portfolios. This is not true in general, see Delbaen and Schachermayer (1994, Example 6.5.). But we have the following:

Proposition 4 *Suppose that $\mathcal{O}_0 \subseteq \mathcal{O}_t$ for all $t \leq T$. Suppose that the topology $\mathcal{O}_0$ satisfies the Fatou property, is positively homogeneous and is $\mathcal{F}_0$-lower bond preserving. Assume that $\mathcal{V}_{t,T} =^a \mathcal{V}_{t,T}$ is given by (2) for all $t \leq T$ and S is a continuous process. Then, the conditions NFL and NFLVR for discrete-time portfolios and the conditions NFL and NFLVR for continuous-time portfolios are equivalent to the existence of a local martingale measure for S.*

Proof Recall that the NFL condition for discrete-time portfolios implies the NFLVR condition for discrete-time portfolios. By Delbaen and Schachermayer (1994, Theorem 7.6), there exists a local martingale measure for S. By Corollary 3, we deduce that NFL holds both for discrete-time and continuous-time portfolio processes. The conclusion follows. □

The result above implies that the price process S needs to be a semi-martingale for the NFL condition to hold. The same holds if the NFLVR condition holds even for discrete-time portfolio processes, see Delbaen and Schachermayer (1994, Theorem 7.2) for locally bounded processes S. The next no-arbitrage condition AIP we consider does not necessitate the price process to be a semimartingale.

4 The AIP condition

The AIP condition has been initially introduced in Carassus and Lépinette (2021) for discrete-time models. The financial interpretation is that the hedging prices of non negative European claims are non negative or, equivalently, the hedging prices of non negative hedgeable European claims are finite. The advantage of this condition is that it is sufficient, at least in discrete-time, to deduce the super-hedging prices without supposing that the price process is a semimartingale.

Our goal is to study the AIP condition for continuous-time processes and relate it to the same condition for discrete-time processes. To do so, we shall use the notion of conditional essential infimum and supremum, see Kabanov and Safarian (2009, Sect. 5.3.1). We recall that, if $\mathcal{H}$ is a sub σ-algebra, the $\mathcal{H}$-measurable essential supremum $\mathrm{ess\,sup}_{\mathcal{H}}(\Gamma)$ of a collection Γ of real-valued random variables is the smallest $\mathcal{H}$-measurable random variable that dominates Γ a.s. and we define $\mathrm{ess\,inf}_{\mathcal{H}}(\Gamma) = -\mathrm{ess\,sup}_{\mathcal{H}}(-\Gamma)$. If the elements of Γ are $\mathcal{H}$-measurable, we use the notation $\mathrm{ess\,sup}(\Gamma) := \mathrm{ess\,sup}_{\mathcal{H}}(\Gamma)$. If $\Gamma = \{\gamma\}$ is a singleton, we write $\mathrm{ess\,sup}_{\mathcal{H}}\,\Gamma = \mathrm{ess\,sup}_{\mathcal{H}}\,\gamma$.

Theorem 5 *Let Γ be a family of $\mathcal{F}_T$-measurable random variables in $L^0(\mathbf{R}, \mathcal{F}_T)$ and let $\mathcal{H}$ be a sub σ-algebra of $\mathcal{F}_T$. There exists a unique $\mathcal{H}$-measurable random variable denoted by* $\mathrm{ess\,sup}_{\mathcal{H}}\,\Gamma$ *such that:*

(1) $\mathrm{ess\,sup}_{\mathcal{H}}\,\Gamma \geq \gamma$ *a.s. for all* $\gamma \in \Gamma$.
(2) If $\gamma_{\mathcal{H}}$ is $\mathcal{H}$-measurable and satisfies $\gamma_{\mathcal{H}} \geq \gamma$ a.s. for all $\gamma \in \Gamma$, then $\gamma_{\mathcal{H}} \geq \mathrm{ess\,sup}_{\mathcal{H}}\,\Gamma$ *a.s.*.

Definition 11 A contingent claim $h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$ is said to be super-hedgeable in discrete time (resp. continuous time) at time t if there exists $p_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ (called a super-hedging price) and a discrete-time (resp. continuous-time) portfolio process $V_{t,T}$ such that $p_t + V_{t,T} \geq h_T$.

Recall that the set of all super-hedgeable claims in discrete time (resp. continuous time) from the zero initial endowment at time t is given by the set $\mathcal{A}_{t,T} = \mathcal{V}_{t,T} - L^0(\mathbf{R}_+, \mathcal{F}_T)$ (resp. $\mathcal{A}^c_{t,T}$). We denote by $\mathcal{P}_{t,T}(h_T)$ (resp. $\mathcal{P}^c_{t,T}(h_T)$) the set of super-hedging prices in discrete time (resp. in continuous time) for the claim

$h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$. The infimum super-hedging price in discrete time (resp. in continuous time) is $\pi_{t,T}(h_T) = \text{ess inf}(\mathcal{P}_{t,T}(h_T))$ (resp. $\pi^c_{t,T}(h_T) = \text{ess inf}(\mathcal{P}^c_{t,T}(h_T))$). We adopt the notation $\mathcal{P}_{t,T}(0) = \mathcal{P}_{t,T}$ (resp. $\mathcal{P}^c_{t,T}(0) = \mathcal{P}^c_{t,T}$), etc..when $h_T = 0$. We observe that $\mathcal{P}_{t,T} = \mathcal{A}_{t,T} \cap L^0(\mathbf{R}, \mathcal{F}_t)$ and $\mathcal{P}^c_{t,T} = \mathcal{A}^c_{t,T} \cap L^0(\mathbf{R}, \mathcal{F}_t)$. Moreover,

$$\mathcal{P}_{t,T} = \{\text{ess sup}_{\mathcal{F}_t}(-v_{t,T}) : v_{t,T} \in \mathcal{V}_{t,T}\} + L^0(\mathbf{R}_+, \mathcal{F}_t), \tag{4}$$

$$\mathcal{P}^c_{t,T} = \{\text{ess sup}_{\mathcal{F}_t}(-v_{t,T}) : v_{t,T} \in \mathcal{V}^c_{t,T}\} + L^0(\mathbf{R}_+, \mathcal{F}_t). \tag{5}$$

Indeed, p_t is a price in discrete time for 0 if there exists $v_{t,T} \in \mathcal{V}_{t,T}$ such that $p_t + v_{t,T} \geq 0$ i.e $p_t \geq -v_{t,T}$, which is equivalent to $p_t \geq \text{ess sup}_{\mathcal{F}_t}(-v_{t,T})$. We have a similar characterization for $\mathcal{P}^c_{t,T}$.

Definition 12 An instantaneous profit in discrete time (resp. in continuous time) at time $t < T$ is a strategy that super-replicates in discrete time (resp. in continuous time) the zero contingent claim starting from a negative price $p_{t,T} \in \mathcal{P}_{t,T} \cap L^0(\mathbf{R}_-, \mathcal{F}_t)$ (resp. $p_{t,T} \in \mathcal{P}^c_{t,T} \cap L^0(\mathbf{R}_-, \mathcal{F}_t)$) such that $p_{t,T} \neq 0$. In the absence of such an instantaneous profit, we say that the Absence of Instantaneous Profit (AIP) holds at time t, i.e.

$$\mathcal{P}_{t,T} \cap L^0(\mathbf{R}_-, \mathcal{F}_t) = \mathcal{A}_{t,T} \cap L^0(\mathbf{R}_+, \mathcal{F}_t) = \{0\}. \tag{6}$$

Respectively, $\mathcal{P}^c_{t,T} \cap L^0(\mathbf{R}_-, \mathcal{F}_t) = \mathcal{A}^c_{t,T} \cap L^0(\mathbf{R}_+, \mathcal{F}_t) = \{0\}$ in continuous time. We say that AIP holds if AIP holds at any $t \leq T$.

Remark 3 The NFLVR condition implies AIP. ◊

Remark 4 AIP in discrete time at time $t \leq T$ is equivalent to $\pi_{t,T}(0) = 0$ or equivalently $\mathcal{P}_{t,T} = L^0(\mathbf{R}_+, \mathcal{F}_t)$. Indeed $\pi_{t,T}(0) \leq 0$ as $0 \in \mathcal{P}_{t,T}$. Moreover, if AIP holds then $\mathcal{P}_{t,T} \subset L^0(\mathbf{R}_+, \mathcal{F}_t)$. To see it, consider $p_{t,T} \in \mathcal{P}_{t,T}$. Then $1_{\{p_{t,T} \leq 0\}} p_{t,T} \in \mathcal{P}_{t,T}$ hence $1_{\{p_{t,T} \leq 0\}} p_{t,T} = 0$ by AIP and $p_{t,T} \geq 0$. Conversely, any $p_t \geq 0$ is a price for the zero claim since $0 \in \mathcal{P}_{t,T}$. The same holds in continuous time. ◊

The following lemma provides another financial interpretation of the AIP condition. Precisely, when starting from the zero initial endowment, it is not possible to obtain a terminal wealth which, estimated at time t, is strictly positive on a non null $\mathcal{F}_t$-measurable set. In particular, under AIP, there is a possibility to face a loss when starting from zero.

Lemma 6 *The AIP condition holds in discrete time (resp. in continuous time) if and only if, for any $t \leq T$ and for all $v_{t,T} \in \mathcal{V}_{t,T}$ (resp. $\mathcal{V}^c_{t,T}$), we have* $\text{ess inf}_{F_t}(v_{t,T}) \leq 0$.

Proof This is a direct consequence of (4). □

4.1 The AIP condition for discrete-time portfolio processes

The following two propostions are direct consequences deduced from Carassus and Lépinette (2021).

Proposition 7 *Suppose that* $d = 1$ *and the discrete-time portfolio processes are given by (1). The AIP condition holds in discrete time if and only if, for all* $t_1 < t_2 < T$, $S_{t_1} \in \left[\text{ess inf}_{\mathcal{F}_{t_1}}(S_{t_2}), \text{ess sup}_{\mathcal{F}_{t_1}}(S_{t_2})\right]$.

In the following, if $\mathcal{H}$ is a sub σ-algebra, we denote by $\text{supp}_{\mathcal{H}}(X)$ the $\mathcal{H}$-measurable conditional support of any random variable X, i.e. the smallest $\mathcal{H}$-measurable random set $\text{supp}_{\mathcal{H}}(X)$ such that $X \in \text{supp}_{\mathcal{H}}(X)$ a.s., see El Mansour and Lépinette (2020). The convex envelop of any $A \subseteq \mathbf{R}^d$ is denoted by $\text{conv}(A)$.

Proposition 8 *Suppose that* $d \geq 1$ *and the discrete-time portfolio processes are given by (1). Then, AIP holds in discrete time if and only if* $S_{t_1} \in \text{conv}(\text{supp}_{\mathcal{F}_{t_1}}(S_{t_2}))$ *for any* $t_1 \leq t_2 \leq T$.

Similarly, we may show the following:

Proposition 9 *Suppose that the discrete-time portfolio processes are given by* (2). *Then, AIP holds in discrete time if and only if* $S_{\tau_1} \in \text{conv}(\text{supp}_{\mathcal{F}_{\tau_1}}(S_{\tau_2}))$ *for every stopping times* $\tau_1, \tau_2 \in \mathcal{T}_{0,T}$ *such that* $\tau_1 \leq \tau_2$.

Proof Suppose that AIP holds and consider two stopping times $\tau_1 \leq \tau_2$ in $[0, T]$. Then, AIP holds for the two time steps smaller model defined by $(S_{\tau_i})_{i=1,2}$ and $(\mathcal{F}_{\tau_i})_{i=1,2}$. By Carassus and Lépinette (2021), we deduce that the minimal price of the zero claim for $(S_{\tau_i})_{i=1,2}$ is given by

$$0 = \pi_{\tau_1,\tau_2}(S_{\tau_1}, S_{\tau_2}) = -\delta_{\text{conv}(\text{supp}_{\mathcal{F}_{\tau_1}}(S_{\tau_2}))}(S_{\tau_1}),$$

where, for any $I \subseteq \mathbf{R}^d$, $\delta_I = (+\infty)1_I$ with the convention $(+\infty) \times (0) = 0$. Therefore, $S_{\tau_1} \in \text{conv}(\text{supp}_{\mathcal{F}_{\tau_1}}(S_{\tau_2})$.

Reciprocally, suppose that, for any $\tau_1 \leq \tau_2 \leq T$, $S_{\tau_1} \in \text{conv}(\text{supp}_{\mathcal{F}_{\tau_1}}(S_{\tau_2})$. Then, $0 = \pi_{\tau_1,\tau_2}(S_{\tau_1}, S_{\tau_2})$ for any $\tau_1 \leq \tau_2 \leq T$. Consider $p_t \in \mathcal{P}_{t,T}$ such that $p_t + \sum_{i=1}^n \theta_{\tau_{i-1}} \Delta S_{\tau_i} \geq 0$ for some strategies $\theta_{\tau_i} \in L^0(\mathbb{R}^d, \mathcal{F}_{\tau_i})$ and stopping times $t = \tau_0 < \tau_1 < \cdots < \tau_n = T$. Then, $p_t + \sum_{i=0}^{n-2} \theta_{\tau_i} \Delta S_{\tau_{i+1}}$ is a price for the zero claim in the two time steps model $(S_{\tau_i})_{i=n-1,n}$. As $0 = \pi_{\tau_{n-1},\tau_n}(S_{\tau_{n-1}}, S_{\tau_n})$, we get that $p_t + \sum_{i=0}^{n-2} \theta_{\tau_i} \Delta S_{\tau_{i+1}} \geq 0$. By induction, we finally deduce that $p_t \geq 0$, i.e. AIP holds. □

We know reformulate the proposition above when $d = 1$ in term of sub-maxingales, see Barron et al. (2003).

Definition 13 We say that a continuous-time process $M = (M_t)_{t \leq T}$ adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is a sub-maxingale (resp. super-maxingale) if, for any $u, t \in [0, T]$ such that $u \leq t$, we have $\text{ess sup}_{\mathcal{F}_u} M_t \geq M_u$ (resp. we have $\text{ess sup}_{\mathcal{F}_u} M_t \leq M_u$). Moreover, M is said a maxingale if it is both a super-maxingale and a sub-maxingale.

Note that the notion of maxingale is an adaptation of the martingale concept to the conditional supremum operator. Observe that, for a super-maxingale M, $\text{ess sup}_{\mathcal{F}_u} M_t \leq M_u$ implies that $M_u \geq M_t$ and we deduce that the super-maxingales coincide with the non increasing processes.

Definition 14 We say that a continuous-time process $M = (M_t)_{t \le T}$ adapted to the filtration $(\mathcal{F}_t)_{t \in [0,T]}$ is a strong sub-maxingale if, for any $\tau \in \mathcal{T}_{0,T}$, the stopped process M^τ is a sub-maxingale.

An open issue is whether a sub-maxingale may be a strong sub-maxingale. When the operator is the conditional expectation, the Doob's stopping Theorem (Jacod and Shiryaev 2003) states that this is the case, at least when M is bounded from above by a martingale, see Jacod and Shiryaev (2003, Theorem 1.39). By Lemma 37, we have:

Proposition 10 *Let $M = (M_t)_{t \le T}$ be a right-continuous continuous-time process adapted to the filtration $(\mathcal{F}_t)_{t \in [0,T]}$. Then, M is a strong sub-maxingale if and only if for all stopping times $\tau, S \in \mathcal{T}_{0,T}$, $\operatorname{ess\,sup}_{\mathcal{F}_S}(M_\tau) \ge M_{S \wedge \tau}$.*

Proof Suppose that M is a strong sub-maxingale. Let $S, \tau \in \mathcal{T}_{0,T}$. As S^τ is a sub-maxingale, we apply Lemma 37 with the stopping time S and the deterministic stopping time T. We get that $\operatorname{ess\,sup}_{\mathcal{F}_S}(M_{\tau \wedge T}) \ge M_{\tau \wedge S \wedge T}$, i.e. $\operatorname{ess\,sup}_{\mathcal{F}_S}(M_\tau) \ge M_{\tau \wedge S}$. The reverse implication is immediate. □

Proposition 11 *Suppose that $d = 1$ and the discrete-time portfolio processes are given by (2). The following statements are equivalent:*

1. *AIP condition holds in discrete-time.*
2. *We have $S_{\tau_1} \in \left[\operatorname{ess\,inf}_{\mathcal{F}_{\tau_1}}(S_{\tau_2}), \operatorname{ess\,sup}_{\mathcal{F}_{\tau_1}}(S_{\tau_2})\right]$, for all $\tau_1, \tau_2 \in \mathcal{T}_{0,T}$ such that $\tau_1 \le \tau_2$.*
3. *S and $-S$ are strong sub-maxingales.*

Proof Suppose that AIP holds. Condition AIP for the discrete-time portfolios of (2) implies the statement 2.) by Proposition 9. In particular, S and $-S$ are sub-maxingales and, for any $t \in [0, T]$ and $\tau \in \mathcal{T}_{0,T}$ such that $\tau \ge t$ a.s., we have by 2.) the inequality

$$\operatorname{ess\,sup}_{\mathcal{F}_t} S_\tau \ge S_t. \tag{7}$$

For fixed $\tau \in \mathcal{T}_{0,T}$, we deduce that S^τ is a sub-maxingale. To see it, consider $t_1 < t_2 \le T$. On the set $A = \{\tau \wedge t_2 < t_1\} \in \mathcal{F}_{t_1}$, we have

$$1_A \operatorname{ess\,sup}_{\mathcal{F}_{t_1}} S^\tau_{t_2} = 1_A \operatorname{ess\,sup}_{\mathcal{F}_{t_1}} S_{t_2 \wedge \tau \wedge t_1} = 1_A S_{\tau \wedge t_1}.$$

On $B = \Omega \backslash A$, as $(t_2 \wedge \tau) \vee t_1 \ge t_1$, we deduce from (7) that

$$1_B \operatorname{ess\,sup}_{\mathcal{F}_{t_1}} S^\tau_{t_2} = 1_B \operatorname{ess\,sup}_{\mathcal{F}_{t_1}} S_{(t_2 \wedge \tau) \vee t_1} \ge 1_B S_{t_1} = 1_B S_{t_1 \wedge \tau}.$$

Therefore, we conclude that $\operatorname{ess\,sup}_{\mathcal{F}_{t_1}} S^\tau_{t_2} \ge S^\tau_{t_1}$ and, finally, S is a strong sub-maxingale. By the same reasoning, $-S$ is also a strong sub-maxingale. Therefore, 1.) implies 2.), which implies 3.). Moreover, 3.) implies 2.) by Proposition 10. At last, if 2.) holds, we conclude that 1.) holds by Proposition 9. □

4.2 The AIP condition for continuous-time portfolio processes

In this section, we consider topologies $(\mathcal{O}_t)_{t\in[0,T]}$ such that $\mathcal{V}^c_{t,T} = \mathcal{V}^c_{t,T}(\mathcal{O}_t)$ for all $t \le T$, and such that the AIP condition in continuous time and in discrete time are equivalent, as stated in our main Theorem 13. Precisely, we consider for any time $t \le T$, the topology on $L^0(\mathbf{R}, \mathcal{F}_T)$ induced by the pseudo-distance:

$$\hat{d}_t^+(X, Y) = E(\text{ess sup}_{\mathcal{F}_t}(X - Y)^+ \wedge 1), \quad X, Y \in L^0(\mathbf{R}, \mathcal{F}_T). \tag{8}$$

We send the readers to Sect. 7 for the definition and the main properties of a pseudo-distance topology.

We notice that a sequence of discrete-time portfolios $(V^n_{t,T})_{n\ge1}$ of $\mathcal{V}_{t,T}$ is convergent in $\mathcal{O}_t$ if and only if $\inf_{n\ge1} V^n_{t,T} > -\infty$ a.s., see Proposition 25. So, $\mathcal{V}^c_{t,T} = \mathcal{V}^c_{t,T}(\mathcal{O}_t)$ is an a priori large class of so-called continuous-time portfolios. In particular, if $(V^n_{t,T})_{n\ge1}$ is a sequence of usual stochastic integrals that converge to some stochastic integral $\mathcal{I}_{t,T}(\theta)$, then the convergence holds in probability hence so does in $\mathcal{O}_t$ by Proposition 25. Any limit $V^c_{t,T} \in \mathcal{V}^c_{t,T}$ satisfies $V^c_{t,T} \le \mathcal{I}_{t,T}(\theta)$ by Proposition 29 but $\mathcal{I}_{t,T}$ does not necessarily belong to $V^c_{t,T}$. This means that $\mathcal{I}_{t,T}$ cannot necessarily be super-hedged asymptotically by simple strategies.

Let us give a financial interpretation of the convergence in $\mathcal{O}_t$. By Proposition 32, $V^n_{t,T}$ converges to $V^c_{t,T} \in \mathcal{V}^c_{t,T}$ if $V^c_{t,T} \le V^n_{t,T} + \alpha^n_t$ for all $n \ge 1$, where $\alpha^n_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ converges to 0 in probability. Therefore, it is possible to reach (actually super-replicates) the continuous-time portfolio value $V^c_{t,T}$ from discrete-time portfolios up to an arbitrary small error. This is why we believe that this topology is well adapted to finance. By Propositions 29, 23 and 32, we obtain that $\mathcal{O}_t$ satisfies the Fatou property, is $\mathcal{F}_t$-positively homogeneous and is $\mathcal{F}_t$-low bound preserving. This implies that the NFL and the NFLVR conditions in discrete-time and continuous-time are equivalent as stated in Sect. 3 for these pseudo-distance topologies. We also have:

Lemma 12 *Suppose that, for any $t \le T$, $\mathcal{O}_t$ is the pseudo-distance topology defined by* (8) *and $\mathcal{V}^c_{t,T} = \mathcal{V}^c_{t,T}(\mathcal{O}_t)$. Then, the NFLVR condition in continuous-time is equivalent to the NA condition $\mathcal{A}^c_{t,T} \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$ in continuous-time, for all $t \le T$.*

Proof Notice that by Proposition 32, $\mathcal{A}^{c,\infty}_{t,T}$ is closed in L^∞ hence we have $\overline{\mathcal{A}}^{c,\infty}_{t,T} = \mathcal{A}^{c,\infty}_{t,T}$ and NFLVR reads as $\mathcal{A}^c_{t,T} \cap L^\infty(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$, which is equivalent to the NA condition as $\mathcal{A}^c_{t,T} - L^0(\mathbf{R}_+, \mathcal{F}_T) \subseteq \mathcal{A}^c_{t,T}$. □

The main result of this section is the following:

Theorem 13 *Suppose that, for any $t \le T$, $\mathcal{O}_t$ is the pseudo-distance topology defined by* (8) *and $\mathcal{V}^c_{t,T} = \mathcal{V}^c_{t,T}(\mathcal{O}_t)$. Then, AIP holds in continuous time if and only if AIP holds in discrete time.*

Proof It suffices to prove that AIP holds in continuous time if it holds in discrete time. By Lemma 6, we have $\text{ess inf}_{F_t}(v_{t,T}) \le 0$ for all $v_{t,T} \in \mathcal{V}_{t,T}$. We have to show the same for $v^c_{t,T} \in \mathcal{V}^c_{t,T}$. By Proposition 32, $V^c_{t,T} \le V^n_{t,T} + \alpha^n_t$ for all $n \ge 1$, where $\alpha_t^n \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ converges to 0 in probability and $V_{t,T}^n \in \mathcal{V}_{t,T}$. As α_t^n is $\mathcal{F}_t$-measurable, we deduce that

$$\text{ess inf}_{F_t} V_{t,T}^c \leq \text{ess inf}_{F_t} V_{t,T}^n + \alpha_t^n \leq \alpha_t^n.$$

As $n \to +\infty$, we deduce that ess $\inf_{F_t} V_{t,T}^c \leq 0$ hence AIP holds in continuous time by Lemma 6. □

5 The NUPBR no-arbitrage condition

The No Unbounded Profit with Bounded Risk no-arbitrage condition NUPBR has been introduced in Karatzas and Kardaras (2007). In our setting, this condition may be adapted if we only consider admissible portfolios. This is why, we suppose that the portfolio processes are generated by an operator $\mathcal{I}$ as in Remark 1. We define for $m \in (0, \infty)$, ${}^a\mathcal{V}_{t,T}(m)$ (resp. ${}^a\mathcal{V}_{t,T}^c(m)$ in continuous time) the set of all admissible portfolio values $V_{t,T} = \mathcal{I}(\theta) \in^a \mathcal{V}_{t,T}$ such that $V_{t,u} = \mathcal{I}_u(\theta) \geq -m$ for all $u \in [t, T]$.

Let us define, for every $t \in [0, T]$, the space $\text{SP}(\mathbf{R}, (\mathcal{F}_u)_{u\in[t,T]})$ of all $(\mathcal{F}_u)_{u\in[t,T]}$-adapted real-valued stochastic processes on $[t, T]$. We consider the family of topologies $(\mathcal{O}_t)_{t\in[0,T]}$ such that, for every $t \in [0, T]$, $\mathcal{O}_t$ is the topology on $\text{SP}(\mathbf{R}, (\mathcal{F}_u)_{u\in[t,T]})$ which is induced by the pseudo-distance:

$$\begin{aligned} &\hat{d}_t^+(X, Y) = E(\text{ess sup}_{u\in[t,T]} \text{ess sup}_{\mathcal{F}_t} (X_u - Y_u)^+ \wedge 1), \\ &X, Y \in \text{SP}(\mathbf{R}, (\mathcal{F}_u)_{u\in[t,T]}). \end{aligned} \tag{9}$$

By the same reasoning as in the proof of Proposition 32, a sequence $(X^n)_{n\geq 1} \in \text{SP}(\mathbf{R}, (\mathcal{F}_u)_{u\in[t,T]})$ converges to $X \in \text{SP}(\mathbf{R}, (\mathcal{F}_t)_{u\in[u,T]})$ in $\mathcal{O}_t$ if and only if there exists a sequence $(\alpha_t^n)_{n\geq 1}$ such that α_t^n tends to 0 in probability as $n \to \infty$ and $X_u \leq X_u^n + \alpha_t^n$ for all $u \in [t, T]$. Moreover, adapting the Proposition 25, we may show that a sequence $(X^n)_{n\geq 1} \in \text{SP}(\mathbf{R}, (\mathcal{F}_u)_{u\in[t,T]})$ is convergent in $\mathcal{O}_t$ if and only if $\inf_n X_u^n > -\infty$ a.s. for all $u \in [t, T]$.

With $(\mathcal{O}_t)_{t\in[0,T]}$ given by (9), we define $\mathcal{V}_{t,T}^c$ as the terminal values $V_{t,T}^c(T)$ of limit processes $V_{t,T}^c$ such that $V_{t,T}^c = \lim_n V_{t,T}^n$ where $V_{t,T}^n = (V_{t,T}^n(u))_{u\in[t,T]}$ are the discrete time processes associated to $\mathcal{V}_{t,T}$, see Remark 1.

Definition 15 We say that NUPBR holds in discrete time (resp. in continuous time) at time $t \leq T$ if, for any $m > 0$, ${}^a\mathcal{V}_{t,T}(m)$ (resp. ${}^a\mathcal{V}_{t,T}^c(m)$) is bounded in probability. We say that NUPBR holds if it holds at any time.

Recall that a sequence $(X^n)_{n\geq 0}$ of random variables is bounded in probability if, for all $\epsilon > 0$, there exists $n_0 \geq 1$ and $M > 0$ such that, for all $n \geq n_0$, $P(|X^n| > M) \leq \epsilon$. More generally, a set $C \subseteq L^0(\mathbf{R}, \mathcal{F}_T)$ is bounded in probability if any sequence $(X^n)_{n\geq 0}$ of C is bounded in probability.

In the setting of semimartingales, it is shown in Karatzas and Kardaras (2007) that NUPBR + NA, i.e. $\mathcal{V}_{0,T} \cap L^0(\mathbf{R}^+, \mathcal{F}_T) = \{0\}$, is equivalent to NFLVR. In particular, NUPBR alone does not necessarily implies NA. This is due to the fact that a portfolio $V_{t,T} \in \mathcal{V}_{t,T}$ such that $V_{t,T} \geq 0$ is not necessary admissible. Otherwise, if $V_{t,T}$ is admissible, then by Karatzas and Kardaras (2007, Theorem 3.12), we get that $V_{t,T}(u) \geq 0$ for all $u \in [t, T]$ by the super-martingale property. Then, necessary $V_{t,T} = 0$, i.e. NA would hold since, otherwise, the sequence $V^n_{t,T} = nV_{t,T}$, $n \geq 1$, is unbounded in probability. In conclusion, NUPBR holds at time t in continuous time (resp. in discrete time) implies NA (and so AIP) at time t only for the restricted sets ${}^a\mathcal{V}^c_{t,T}$ and ${}^a\mathcal{V}_{t,T}$ respectively.

Our main result of this section is the following. Before, we recall a definition:

Definition 16 We say that a subset Γ of $L^0(\mathbf{R}, \mathcal{F}_T)$ is infinitely $\mathcal{F}_t$-decomposable (resp. $\mathcal{F}_t$-decomposable) if for any partition of Ω (resp. finite partition) by elements $(F^n_t)^\infty_{n=1}$ of $\mathcal{F}_t$ and any sequence $(X^n)_{n\geq 1}$ of Γ, we have $\sum_{n=1}^\infty X^n 1_{F^n_t} \in \Gamma$.

Theorem 14 *Suppose that, for $t \leq T$, $\mathcal{O}_t$ is the pseudo-distance topology defined by (9) and $\mathcal{V}^c_{t,T} = \mathcal{V}^c_{t,T}(\mathcal{O}_t)$. Suppose that $\mathcal{V}_{t,T}$ is infinitely $\mathcal{F}_t$-decomposable. Then, NUPBR holds in discrete time if and only if it holds in continuous time.*

Proof It suffices to show that NUPBR holds in continuous time if it holds in discrete time. To do so, suppose that ${}^a\mathcal{V}^c_{t,T}(m)$ is not bounded in probability for some $m > 0$. Then, there exists a sequence $(V^{c,n}_{t,T})_{n\geq 1} \in^a \mathcal{V}^c_{t,T}(m)$ and $\epsilon \in (0, 1)$ such that $P(V^{c,n}_{t,T} > n) > \epsilon$ for all $n \geq 1$. By Proposition 32, for all $n \geq 1$, there exists a sequence $(V^{n,m}_{t,T})_{m\geq 1} \in^a \mathcal{V}_{t,T}$ and a sequence $(\alpha^{n,m}_t)_{m\geq 1} \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ such that $\alpha^{n,m}_t$ converges to 0 in probability as $m \to \infty$ and $V^{c,n}_{t,T}(u) \leq V^{n,m}_{t,T}(u) + \alpha^{n,m}_t$, for all $m \geq 1$ and $u \in [0, T]$. We may assume w.l.o.g. that $\alpha^{n,m}_t$ converges to 0 a.s. as $m \to \infty$. Then, there exists an integer-valued $\mathcal{F}_t$-measurable random variable m^n_t such that $\alpha^{n,m^n_t}_t \in L^0([0, 1], \mathcal{F}_t)$. As $\mathcal{V}_{t,T}$ is infinitely $\mathcal{F}_t$-decomposable, we deduce that $V^{n,m^n_t}_{t,T} \in \mathcal{V}_{t,T}$. Note that $V^{c,n}_{t,T}(u) \leq V^{n,m^n_t}_{t,T}(u) + 1$ hence $V^{n,m^n_t}_{t,T} \in^a \mathcal{V}_{t,T}(m + 1)$ for all $n \geq 1$. Moreover, $\epsilon < P(V^{c,n}_{t,T} > n) \leq P(V^{n,m^n_t}_{t,T} > n - 1)$, for all $n \geq 1$. This implies that the sequence $(V^{n,m^n_t}_{t,T})_{n\geq 1}$ is not bounded in probability, contrarily to the assumption NUPBR for $\mathcal{V}_{t,T}$. This contradiction allows one to conclude that NUPBR holds in continuous time. □

6 Super-hedging prices

6.1 Super-hedging prices without no-arbitrage condition

Recall that the super-hedging prices (resp. the infimum super-hedging price) of a payoff $h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$ are defined after Definition 11. Our main result is the following:

Theorem 15 *Suppose that, for any $t \leq T$, $\mathcal{O}_t$ is the pseudo-distance topology defined by (8) and $\mathcal{V}^c_{t,T} = \mathcal{V}^c_{t,T}(\mathcal{O}_t)$. Then, the infimum super-hedging prices of a payoff $h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$, in discrete time and in continuous time respectively, coincide i.e.*

$$\pi_{t,T}(h_T) = \operatorname{ess\,inf}(\mathcal{P}_{t,T}(h_T)) = \pi^c_{t,T}(h_T) = \operatorname{ess\,inf}(\mathcal{P}^c_{t,T}(h_T)).$$

Proof As $\mathcal{P}_{t,T}(h_T) \subseteq \mathcal{P}^c_{t,T}(h_T)$, we have $\pi^c_{t,T}(h_T) \le \pi_{t,T}(h_T)$. Consider a price $p_t \in \mathcal{P}^c_{t,T}(h_T)$ such that $p_t + V^c_{t,T} \ge h_T$ for some $V^c_{t,T} \in \mathcal{V}^v_{t,T}$. By Proposition 32, we have $V^c_{t,T} \le V^n_{t,T} + \alpha^n_t$ for all $n \ge 1$, where $\alpha^n_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ converges to 0 in probability and $V^n_{t,T} \in \mathcal{V}_{t,T}$. We deduce that $p_t + \alpha^n_t \in \mathcal{P}_{t,T}(h_T)$ hence $p_t + \alpha^n_t \ge \pi_{t,T}(h_T)$. As $n \to \infty$, we deduce that $p_t \ge \pi_{t,T}(h_T)$ hence $\pi^c_{t,T}(h_T) \ge \pi_{t,T}(h_T)$. The conclusion follows. □

Remark 5 1. Note that, at any time, $\mathcal{P}^c_{t,T}(h_T)$ may be empty. In that case, we also have $\mathcal{P}^c_{t,T}(h_T) = \emptyset$ and $\pi_{t,T}(h_T) = \pi^c_{t,T}(h_T) = \infty$. Reciprocally, if we have $\mathcal{P}^c_{t,T}(h_T) = \emptyset$, then $\pi_{t,T}(h_T) = \infty$ and we deduce that $\pi^c_{t,T}(h_T) = \infty$ by Theorem 15.

2. If $\mathcal{V}_{t,T}$ is a positive cone, then $\mathcal{P}_{t,T}$ and $\mathcal{P}^c_{t,T}$ are positive cones if $\mathcal{O}_t$ is $\mathcal{F}_t$-positively homogeneous. Therefore, $\pi_{t,T} = \pi_{t,T}(0) < 0$ implies that $\pi_{t,T} = \pi^c_{t,T}(h_T) = -\infty$. Let us consider a payoff $h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$ such that $h_T \le \alpha S_T + \beta$ for some $\alpha, \beta \in \mathbf{R}$. Then, for all price $p_{t,T} \in \mathcal{P}_{t,T}$, we deduce that $p_{t,T} + \alpha S_t + \beta \in \mathcal{P}_{t,T}(h_T)$. Therefore, $\pi_{t,T} = \pi^c_{t,T} = -\infty$ implies that $\pi_{t,T}(h_T) = \pi^c_{t,T}(h_T) = -\infty$. This is why the condition AIP is financially meaning as it avoids this unrealistic situation where the prices of a positive payoff h_T may be as negatively large as possible so that it is not possible to compute the infimum price.
3. If $\mathcal{V}_{t,T} = \cup_{n\ge n}\mathcal{V}_{t,T}(n)$ where $\mathcal{V}_{t,T}(n)$ is an increasing sequence of discrete-time models, then observe that $\pi_{t,T}(h_T) = \inf_n \pi^n_{t,T}(h_T)$ where $\pi^n_{t,T}(h_T)$ are the infimum prices associated to the models $\mathcal{V}_{t,T}(n)$, $n \ge 1$. Moreover, if $\mathcal{V}_{t,T}(n)$ is a model only composed of a finite number of dates, then $\pi^n_{t,T}(h_T)$ may be computed as in Carassus and Lépinette (2021). This is the case in practice, if the trades only may be executed at deterministic dates, e.g. every second.

♢

6.2 Infinitely $\mathcal{F}_t$-decomposable extension of the discrete-time prices

In the following, we show that the discrete-time portfolio processes may be extended without changing the infimum prices and we get a precise form of the set of super-hedging prices. We denote by $\mathrm{Part}_t(\Omega)$ the set of all $\mathcal{F}_t$-measurable partitions of Ω and we consider

$$\mathcal{V}^{\mathrm{id}}_{t,T} = \left\{ \sum_{n=1}^{\infty} X^n 1_{F^n_t} \,:\, X^n \in \mathcal{V}_{t,T},\ (F^n_t)_{n=1}^{\infty} \in \mathrm{Part}_t(\Omega) \right\}. \tag{10}$$

Note that $\mathcal{V}^{\mathrm{id}}_{t,T}$ is infinitely $\mathcal{F}_t$-decomposable. We say that $\mathcal{V}^{\mathrm{id}}_{t,T}$ is the discrete-time infinitely $\mathcal{F}_t$-decomposable extension of $\mathcal{V}_{t,T}$. We then denote by $\mathcal{P}^{\mathrm{id}}_{t,T}(h_T)$ the set of all prices obtained from $\mathcal{V}^{\mathrm{id}}_{t,T}$ and $\pi^{\mathrm{id}}_{t,T}(h_T) := \operatorname{ess\,inf}_{\mathcal{F}_t} \mathcal{P}^{\mathrm{id}}_{t,T}(h_T)$. We denote by $\mathcal{V}^{\mathrm{id},c}_{t,T}$ the continuous-time processes deduced from $\mathcal{V}^{\mathrm{id}}_{t,T}$.

Lemma 16 *The AIP condition holds for $\mathcal{V}_{t,T}$ if and only AIP holds for its infinitely $\mathcal{F}_t$-decomposable extension.*

Proof It suffices to show that AIP holds for its $\mathcal{F}_t$-decomposable extension as soon as it holds for $\mathcal{V}_{t,T}$. By Lemma 6, let us show that $\text{ess inf}_{\mathcal{F}_t}(V_{t,T}) \le 0$ for all $V_{t,T} \in \mathcal{V}^{\text{id}}_{t,T}$. Suppose that $\mathcal{V}^{\text{id}}_{t,T} = \sum_{n=1}^{\infty} X^n 1_{F_t^n}$ where $X^n \in \mathcal{V}_{t,T}$ and $(F_t^n)_{n=1}^{\infty} \in \text{Part}_t(\Omega)$. Then,

$$1_{F_t^m}\,\text{ess inf}_{\mathcal{F}_t}(V_{t,T}) = 1_{F_t^m}\,\text{ess inf}_{\mathcal{F}_t}(V_{t,T}1_{F_t^m}) = 1_{F_t^m}\,\text{ess inf}_{\mathcal{F}_t}(X^n) \le 0.$$

The conclusion follows. □

Lemma 17 *Suppose that $\mathcal{V}_{t,T}$ is $\mathcal{F}_t$-decomposable, $t \le T$, and consider a payoff $h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$. Then, we have $\pi^{\text{id}}_{t,T}(h_T) = \pi_{t,T}(h_T)$ and*

$$\mathcal{P}_{t,T}(h_T) \subseteq \mathcal{P}^{\text{id}}_{t,T}(h_T) \subseteq \overline{\mathcal{P}}_{t,T}(h_T),$$

where $\overline{\mathcal{P}}_{t,T}(h_T)$ is the closure of $\mathcal{P}_{t,T}(h_T)$ in L^0.

Proof As $\mathcal{V}_{t,T} \subseteq \mathcal{V}^{\text{id}}_{t,T}$, we have $\mathcal{P}_{t,T}(h_T) \subseteq \mathcal{P}^{\text{id}}_{t,T}((h_T)$ and $\pi^{\text{id}}_{t,T}(h_T) \le \pi_{t,T}(h_T)$. Moreover, if $p_t \in \mathcal{P}^{\text{id}}_{t,T}((h_T)$, then we have $p_t + \sum_{i=1}^{\infty} V^i_{t,T} 1_{F_t^i} \ge h_T$ for some $V^i_{t,T} \in \mathcal{V}_{t,T}$, $i \ge 1$ and a partition $(F_t^i)_{i \ge 1}$ of Ω by elements of $\mathcal{F}_t$. Consider $p_t^0 \in \mathcal{P}_{t,T}(h_T)$ and define $p_t^n = p_t 1_{\cup_{i=1}^n F_t^i} + p_t^0 1_{\Omega \setminus \cup_{i=1}^n F_t^i}$, $n \ge 1$. As $\mathcal{V}_{t,T}$ is $\mathcal{F}_t$-decomposable, $p_t^n \in \mathcal{P}_{t,T}(h_T)$. Moreover, $p_t = \lim_{n \to \infty} p_t^n$. We then deduce that $\mathcal{P}^{\text{id}}_{t,T}((h_T) \subseteq \overline{\mathcal{P}}_{t,T}(h_T)$ hence $\pi^{\text{id}}_{t,T}(h_T) \ge \pi_{t,T}(h_T)$. The conclusion follows. □

Proposition 18 *Consider a payoff $h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$. Then, there exists $\Lambda_t \in \mathcal{F}_t$ such that $\mathcal{P}^{\text{id}}_{t,T}(h_T) = L^0(J_{t,T}(h_T), \mathcal{F}_t)$ and*

$$J_{t,T}(h_T) = [\pi^{\text{id}}_{t,T}(h_T), \infty)1_{\Lambda_t} + (\pi^{\text{id}}_{t,T}(h_T), \infty)1_{\Omega \setminus \Lambda_t}.$$

Proof It suffices to argue on the set of all ω such that $\pi^{\text{id}}_{t,T}(h_T) < \infty$. Therefore, we suppose w.l.o.g. that there exists $p_t^0 \in \mathcal{P}^{\text{id}}_{t,T}(h_T)$. Let us consider

$$\Gamma_t = \left\{ \Lambda_t \in \mathcal{F}_t :\ \pi^{\text{id}}_{t,T}(h_T)1_{\Lambda_t} + p_t^0 1_{\Omega \setminus \Lambda_t} \in \mathcal{P}^{\text{id}}_{t,T}(h_T) \right\}.$$

Note that $\emptyset \in \Gamma_t$. As $\mathcal{V}^{\text{id}}_{t,T}$ is infinitely $\mathcal{F}_t$-decomposable, $\mathcal{P}^{\text{id}}_{t,T}(h_T)$ is infinitely $\mathcal{F}_t$-decomposable by Lemma 39. We deduce that $\Lambda_t^1 \cup \Lambda_t^2 \in \Gamma_t$ if $\Lambda_t^1, \Lambda_t^2 \in \Gamma_t$. Then, the family $\{1_{\Lambda_t} : \Lambda_t \in \Gamma_t\}$ is directed upward. We deduce that $\text{ess sup}_{\Lambda_t \in \Gamma_t} 1_{\Lambda_t} = 1_{\Lambda_t^\infty}$ where Λ_t^∞ is an increasing union of elements of Γ_t. As $\mathcal{P}^{\text{id}}_{t,T}(h_T)$ is infinitely $\mathcal{F}_t$-decomposable, we get that $\Lambda_t^\infty \in \Gamma_t$. We may also show that Λ_t^∞ is independent of p_t^0. We then define $J_{t,T}(h_T)$ as above with $\Lambda_t = \Lambda_t^\infty$. We claim that $\mathcal{P}^{\text{id}}_{t,T}(h_T) = L^0(J_{t,T}(h_T), \mathcal{F}_t)$. To see it, consider a price $p_t^0 \in \mathcal{P}^{\text{id}}_{t,T}(h_T)$ and suppose that $p_t^0 = \pi^{\text{id}}_{t,T}(h_T)$ on a non null set of $\Omega \setminus \Lambda_t$. Then, we get a contradiction with the maximality of Λ_t. So, we obtain that $\mathcal{P}^{\text{id}}_{t,T}(h_T) \subseteq L^0(J_{t,T}(h_T), \mathcal{F}_t)$. Reciprocally, consider $p_t \in L^0(J_{t,T}(h_T), \mathcal{F}_t)$. Then, $p_t^0 = p_t + 1_{\Lambda_t} > \pi^{\text{id}}_{t,T}(h_T)$ a.s. hence

$p_t^0 \in \mathcal{P}^{\rm id}_{t,T}(h_T)$ by Lemma 40. Moreover, $p_t \geq \pi^{\rm id}_{t,T}(h_T)1_{\Lambda_t} + p_t^0 1_{\Omega\setminus\Lambda_t}$ by construction. Since $\pi^{\rm id}_{t,T}(h_T)1_{\Lambda_t} + p_t^0 1_{\Omega\setminus\Lambda_t} \in \mathcal{P}^{\rm id}_{t,T}(h_T)$ by definition of Λ_t, we deduce that $p_t \in \mathcal{P}^{\rm id}_{t,T}(h_T)$. Therefore, $\mathcal{P}_{t,T}(h_T) = L^0(J_{t,T}(h_T), \mathcal{F}_t)$. □

Corollary 19 *Suppose that $\mathcal{V}_{t,T}$ is $\mathcal{F}_t$-decomposable, $t \leq T$, and consider a payoff $h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$. Then, the closure in L^0 of $\mathcal{P}_{t,T}(h_T)$, $\mathcal{P}^{\rm id}_{t,T}(h_T)$ and $\mathcal{P}^{\rm id,c}_{t,T}(h_T)$ coincide with $L^0([\pi_{t,T}, \infty), \mathcal{F}_t)$.*

The natural question is whether $\mathcal{P}^{\rm id}_{t,T}(h_T) = \mathcal{P}_{t,T}(h_T)$. Actually, this is not the case in general, as shown in the following example:

Example 1 We consider the framework of our paper between time $t = 1$ and $t = 2$. Suppose that $\Omega = \{\omega_i : i = 1, 2, 3, 4\}$, $\mathcal{F}_1 = \{A, A^c, \emptyset, \Omega\}$ where $A = \{\omega_1, \omega_2\}$, $A^c = \Omega \setminus A$, and $\mathcal{F}_2$ is the family of all subsets of Ω. We consider any probability measure P on $\mathcal{F}_2$ such that $P(\{\omega_i\}) > 0$ for all $i = 1, 2, 3, 4$. We assume that $\mathcal{V}_{t,T} = \{V^1, V^2\}$ where $V^1(\omega_i) = i - 1$ for $i = 1, 2, 3, 4$ and $(V^2(\omega_i))_{i=1}^4 = \{-1, 2, 3, 4\}$. At last, we suppose that the payoff is $h(\omega_i) = i$ for $i = 1, 2, 3, 4$. Then, the minimal prices at time $t = 1$ associated to V^1, V^2 are respectively $p_1(V^1) = 1$ and $p_1(V^2) = 21_A$. Therefore, $\mathcal{P}_{1,2}(h) = L^0([1, \infty), \mathcal{F}_1) \cup L^0([21_A, \infty), \mathcal{F}_1)$. Then, $\pi_{1,2}(h) = 1_A \notin \mathcal{P}_{1,2}(h)$. On the other hand, we may see that $\mathcal{V}^{\rm id}_{t,T} = \{V^1, V^2, V^3, V^4\}$ where $V^3 = V^1 1_A + V^2 1_{A^c}$ and $V^4 = V^2 1_A + V^1 1_{A^c}$. We then show that $p_1(V^3) = 1_A$ and $p_1(V^4) = 1 + 1_A$. It follows that $\pi^{\rm id}_{1,2}(h) = \pi_{1,2}(h) = 1_A \in \mathcal{P}^{\rm id}_{1,2}(h)$ and $\mathcal{P}^{\rm id}_{t,T}(h) = L^0([1_A, \infty), \mathcal{F}_1)$. We conclude that $\mathcal{P}^{\rm id}_{t,T}(h) \neq \mathcal{P}_{1,2}(h)$. ♢

7 Topology defined by a semi-distance

Definition 17 Let E be a vector space. A semi-distance is a mapping d defined on $E \times E$ with values in $\mathbf{R}_+$ such that the triangular inequality holds:

$$d(X, Y) \leq d(X, Z) + d(Z, Y), \quad X, Y, Z \in E.$$

Example 2 At time $t \leq T$, we define on $L^0(\mathbf{R}, \mathcal{F}_T) \times L^0(\mathbf{R}, \mathcal{F}_T)$ the pseudo-distance:

$$\hat{d}_t^+(X, Y) = E(\text{ess sup}_{\mathcal{F}_t}((X - Y)^+) \wedge 1), \quad X, Y \in L^0(\mathbf{R}, \mathcal{F}_T).$$

Observe that only the triangle inequality is satisfied by d_t^+. In general $d_t^+(X, Y) \neq d_t^+(Y, X)$. For example, if $X + 1 \leq Y$ a.s., then $d_t^+(X, Y) = 0$ but $d_t^+(Y, X) = 1$. In particular, $d_t^+(X, Y) = 0$ does not necessarily imply that $X = Y$ a.s. ♢

Example 3 Another pseudo-distance is given by

$$d^+(X, Y) = E((X - Y)^+ \wedge 1).$$

Notice that $d^+ \leq \hat{d}_t^+$. ♢

A pseudo-distance d allows us to define a topologie on $L^0(\mathbf{R}, \mathcal{F}_T)$. To do so, let us define, for every $X_0 \in L^0(\mathbf{R}, \mathcal{F}_T)$, the set

$$\mathcal{B}_\varepsilon(X_0) = \left\{X \in L^0(\mathbf{R}, \mathcal{F}_T) : d(X_0, X) \leq \varepsilon\right\}$$

that we call ball of radius $\varepsilon \in \mathbf{R}^+$, centered at $X_0 \in L^0(\mathbf{R}, \mathcal{F}_T)$. A set $V \subseteq L^0(\mathbf{R}, \mathcal{F}_T)$ is said a neighborhood of $X \in L^0(\mathbf{R}, \mathcal{F}_T)$ if there is $\varepsilon \in (0, \infty)$ such that $\mathcal{B}_\varepsilon(X) \subset V$. A set $O \subset L^0(\mathbf{R}, \mathcal{F}_T)$ is said open if it is a neighborhood of all $X \in O$. We denote by $\mathcal{T}_d$ the collection of all open sets.

Lemma 20 *The family $\mathcal{T}_d$ of open sets defined from the pseudo-distance d is a topology.*

Proof It is clear that $L^0(\mathbf{R}, \mathcal{F}_T)$ is a neighborhood of all its elements, i.e. $L^0(\mathbf{R}, \mathcal{F}_T) \in \mathcal{T}_d$, and $\emptyset \in \mathcal{T}_d$ by convention. Let $(O_i)_{i \in I}$ be a family of open sets. Let $x \in \bigcup_{i \in I} O_i$, so that $x \in O_i$ for some $i \in I$. As O_i is open, O_i is a neighborhood of x and, consequently, $\bigcup_{i \in I} O_i$ is a neighborhood of x.

Let $(O_i)_{i \in I}$ be a finite family of open sets. Let $x \in \bigcap_{i \in I} O_i$, so that $x \in O_i$ for every $i \in I$. So, for every $i \in I$, there exist $\varepsilon_i \in (0, \infty)$ such that $\mathcal{B}_{\varepsilon_i}(x) \subset O_i$. Let $\varepsilon = \inf_{i \in I}(\varepsilon_i) \in (0, \infty)$. We have $\mathcal{B}_\varepsilon(x) \subset O_i$ for every $i \in I$. We conclude that $\bigcap_{i \in I} O_i$ is open. □

In the following, we denote by $\widehat{\mathcal{T}_t}$ the topology associated to the pseudo-distance $\hat{d}_t^+$ given in Example 2. Similarly, we denote by $\widehat{\mathcal{B}_\varepsilon}(x)$ the associated balls. We also denote by $\mathcal{T}$ the topology defined by d^+ as in Example 3 while the associated balls are just denoted by $\mathcal{B}_\varepsilon(x)$.

Remark 6 We observe several basic properties which are of interest:

(1) The topology defined by the pseudo-distance is not separated in general. Take for example $X, Y \in L^0(\mathbf{R}, \mathcal{F}_T)$ such that $Y > X$ a.s. For every $\varepsilon \in \mathbf{R}^+$, $X - Y < 0 \leq \varepsilon$ hence $(X - Y)^+ = 0 \leq \varepsilon$. So,

$$\hat{d}_t^+(X - Y) = E(\text{ess sup}_{\mathcal{F}_t}(X - Y)^+ \wedge 1) \leq \varepsilon \wedge 1$$

and we conclude that $Y \in \widehat{\mathcal{B}_\varepsilon}(X)$.

(2) A sequence $(X_n)_{n \in \mathbb{N}}$ of elements in $L^0(\mathbf{R}, \mathcal{F}_T)$ converges to $X \in L^0(\mathbf{R}, \mathcal{F}_T)$ with respect to $\mathcal{T}_d$ if, for all $\varepsilon \in \mathbf{R}^+$, there exist $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$, $X_n \in \mathcal{B}_\varepsilon(X)$.

(3) If A is a subset of E, then X belongs to the closure of A with respect to $\mathcal{T}_d$ if and only if $X = \lim_n(X_n)$, i.e. $d(X, X^n) \to 0$, where $(X_n)_{n \in \mathbb{N}}$ is a sequence of elements of A. Indeed, this is a direct consequence of the construction of the balls from d.

(4) If $(X_n)_{n \in \mathbb{N}}$ converges to X with respect to $\widehat{\mathcal{T}_t}$ then $(X_n)_{n \in \mathbb{N}}$ converges to X with respect to $\mathcal{T}$, see Examples 2 and 3.

(5) If $(X_n)_{n \in \mathbb{N}}$ converges to X with respect to $\widehat{\mathcal{T}_t}$ and $(\tilde{X}_n)_{n \in \mathbb{N}}$ is another sequence such that $\tilde{X}_n \geq X_n$ a.s., for all $n \in \mathbb{N}$, then $(\tilde{X}_n)_{n \in \mathbb{N}}$ converges to X with respect to $\widehat{\mathcal{T}_t}$. ♢

Remark 7 We recall that $d(X,Y)=E(|X-Y|\wedge 1)$ is the distance generating the convergence in probability. So, a sequence $(X_n)_{n\in\mathbb{N}}$ of elements in $L^0(\mathbf{R},\mathcal{F}_T)$ converges to $X\in L^0(\mathbf{R},\mathcal{F}_T)$ with respect to $\widehat{\mathcal{T}}_t$, see Example 2, if and only ess $\sup_{\mathcal{F}_t}(X-X_n)^+$ converges to 0 in probability. Consequently there exists a subsequence $(X_{n_k})_k$ of $(X_n)_n$ such that ess $\sup_{\mathcal{F}_t}(X-X_{n_k})^+$ converges to 0 almost surely, i.e. for every $\varepsilon\in\mathbf{R}^+$ there exists k_0 such that, for all $k>k_0$, we have ess $\sup_{\mathcal{F}_t}(X-X_{n_k})^+\le\varepsilon$, which implies that $X\le\varepsilon+X_{n_k}$. ◊

Lemma 21 *If F is a closed set for $\mathcal{T}$ (resp. for $\widehat{\mathcal{T}}_t$), then F is a lower set, i.e. $F-L^0(\mathbf{R}_+,\mathcal{F}_T)\subseteq F$.*

Proof Indeed, consider $Z\le\gamma$ where $\gamma\in F$. Then, $(Z-\gamma)^+=0$ hence the constant sequence $(\gamma_n=\gamma)_{n\ge1}$ converges to Z and, finally, $Z\in F$. Note that, if F is closed for $\mathcal{T}$, it is closed for $\widehat{\mathcal{T}}_t$. □

Lemma 22 *Let d be a pseudo-distance on $E\times E$. Consider two sequences $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ of elements in E which converge to $X,Y\in L^0(\mathbf{R},\mathcal{F}_T)$ respectively with respect to $\mathcal{T}_d$. If $d(a+b,a+c)\le d(b,c)$ for all $a,b,c\in E$, then $(X_n+Y_n)_{n\in\mathbb{N}}$ converges to $X+Y$.*

Proof It suffices to observe that

$$\begin{aligned} d(X+Y,X_n+Y_n) &\le d(X+Y,X_n+Y)+d(X_n+Y,X_n+Y_n)\\ &\le d(X,X_n)+d(Y,Y_n). \end{aligned}$$

□

Proposition 23 *Consider the pseudo-distance $\hat{d}_t^+$ from Example 2. Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ be two sequences of elements in $L^0(\mathbf{R},\mathcal{F}_T)$ which converge respectively to $X,Y\in L^0(\mathbf{R},\mathcal{F}_T)$ with respect to $\widehat{\mathcal{T}}_t$. The following convergences hold with respect to $\widehat{\mathcal{T}}_t$:*

(1) The sequence $(\alpha_tX_n)_{n\in\mathbb{N}}$ converges to α_tX, for all $\alpha_t\in L^0(\mathbf{R}_+,\mathcal{F}_t)$.
(2) The sequence $(\alpha X_n)_{n\in\mathbb{N}}$ converges to αX, for all $\alpha\in L^\infty(\mathbf{R}_+,\mathcal{F}_T)$.
(3) The sequence (ess $\sup_{\mathcal{F}_t}(X_n))_{n\ge1}$ *converges to* ess $\sup_{\mathcal{F}_t}(X)$.

Moreover, the two first statements remain true if we replace $\widehat{\mathcal{T}}_t$ by $\mathcal{T}$.

Proof Recall that ess $\sup_{\mathcal{F}_t}(\alpha_tX-\alpha_tX_n)^+=\alpha_t$ ess $\sup_{\mathcal{F}_t}(X-X_n)^+$ if α_t belongs to $L^0(\mathbf{R}_+,\mathcal{F}_t)$. Then, for all $\gamma>0$,

$$\begin{aligned} d_t^+(\alpha_tX,\alpha_tX_n) &= E(\alpha_t\,\text{ess}\sup_{\mathcal{F}_t}(X-X_n)^+\wedge 1.1_{\text{ess}\sup_{\mathcal{F}_t}(X-X_n)^+<\gamma})\\ &\quad +E(\alpha_t\,\text{ess}\sup_{\mathcal{F}_t}(X-X_n)^+\wedge 1.1_{\text{ess}\sup_{\mathcal{F}_t}(X-X_n)^+\ge\gamma})\\ &\le E(\alpha_t\gamma\wedge 1)+P(\text{ess}\sup_{\mathcal{F}_t}(X-X_n)^+\ge\gamma). \end{aligned}$$

By the dominated convergence theorem, we may fix γ small enough such that $E(\alpha_t\gamma\wedge 1)\le\epsilon/2$, where $\epsilon>0$ is arbitrarily chosen. Moreover, by assumption, $P(\text{ess}\sup_{\mathcal{F}_t}(X-X_n)^+\ge\gamma)\le\epsilon/2$, if n is large enough. We get that $d_t^+(\alpha_tX,\alpha_tX_n)\le\epsilon$, if n is large enough, i.e. $\alpha_tX_n\to\alpha_tX$.

The second statement is a consequence of the first one as we may observe that, for all $\alpha \in L^\infty(\mathbf{R}_+, \mathcal{F}_T)$,

$$d_t^+(\alpha X, \alpha X_n) \le d^+(\|\alpha\|_\infty X, \|\alpha\|_\infty X_n).$$

At last, notice that the following inequality holds

$$\text{ess sup}_{\mathcal{F}_t}(X) = \text{ess sup}_{\mathcal{F}_t}(X + X_n - X_n) \le \text{ess sup}_{\mathcal{F}_t}(X - X_n) + \text{ess sup}_{\mathcal{F}_t}(X_n).$$

Therefore,

$$\begin{aligned}
&\text{ess sup}_{\mathcal{F}_t}(X) - \text{ess sup}_{\mathcal{F}_t}(X_n) \le \text{ess sup}_{\mathcal{F}_t}(X - X_n)^+,\\
&\text{ess sup}_{\mathcal{F}_t}((\text{ess sup}_{\mathcal{F}_t}(X) - \text{ess sup}_{\mathcal{F}_t}(X_n))^+) \le \text{ess sup}_{\mathcal{F}_t}(X - X_n)^+,\\
&d_t^+(\text{ess sup}_{\mathcal{F}_t}(X), \text{ess sup}_{\mathcal{F}_t}(X_n)^+) \le E(\text{ess sup}_{\mathcal{F}_t}((X - X_n)^+) \wedge 1).
\end{aligned}$$

The conclusion follows. □

Remark 8 If a sequence $(X_n)_n$ converges to X with respect to $\widehat{\mathcal{T}}$ or $\mathcal{T}$ it does not imply that $(-X_n)_n$ converges to $-X$. Take for example the sequence $(-1)^n$. We have $(-1-(-1)^n)^+ = 0$ for any $n \in \mathbb{N}$. Then, $(-1)^n$ converges to -1 for $\widehat{\mathcal{T}}$ and $\mathcal{T}$. But $(1-(-1)^{n+1})^+ \wedge 1 = 1$ when n is even. Then $(1-(-1)^{n+1})^+$ does not converge to 0 in probability. So, $-(-1)^n$ does not converge to -1 for $\mathcal{T}$ nor for $\widehat{\mathcal{T}}$. ◊

Lemma 24 *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of elements in $L^0(\mathbf{R}, \mathcal{F}_T)$ that converge to $X \in L^0(\mathbf{R}, \mathcal{F}_T)$ with respect to $\mathcal{T}$. Then, for every random subsequence $(n_k)_{k\ge1}$, $(X_{n_k})_k$ converges to X with respect to $\mathcal{T}$. The same holds with respect to $\widehat{\mathcal{T}_t}$ if the random subsequence $(n_k)_{k\ge1}$ is $\mathcal{F}_t$-measurable.*

Proof Note that $(X - X_{n_k})^+ = \sum_{j=k}^{\infty}(X - X_j)^+ 1_{n_k=j}$. Therefore,

$$\begin{aligned}
\mathbb{P}((X - X_{n_k})^+ \ge \varepsilon) &= \mathbb{P}\left(\sum_{j=k}^{\infty}\{(X - X_j)^+ \ge \varepsilon\} \cap \{n_k = j\}\right),\\
&\le \sum_{j=k}^{\infty}\mathbb{P}\left(\{(X - X_j)^+ \ge \varepsilon\} \cap \{n_k = j\}\right).
\end{aligned}$$

Let $\alpha > 0$. Consider M such that $\sum_{j=M+1}^{\infty}\mathbb{P}(n_k = j) \le \alpha/2$ and k_0 such that, for every $k \ge k_0$, we have $\mathbb{P}((X - X_k)^+ \ge \varepsilon) \le \alpha/2\,M$. Then,

$$\begin{aligned}
\mathbb{P}((X - X_{n_k})^+ \ge \varepsilon) &\le \sum_{j=k}^{M\vee k}\mathbb{P}(\{(X - X_j)^+ \ge \varepsilon) + \sum_{j=M+1}^{\infty}\mathbb{P}(n_k = j)\\
&\le M\alpha/2M + \alpha/2 \le \alpha.
\end{aligned}$$

So $(X - X_{n_k})^+$ converges to zero in probability hence $(X_{n_k})_k$ converges to X with respect to $\mathcal{T}$.

For the second statement, it suffices to observe that, when $(n_k)_{k\geq 1}$ is $\mathcal{F}_t$-measurable, we have:

$$(X - X_{n_k})^+ \leq \sum_{j=k}^{\infty} \text{ess sup}_{\mathcal{F}_t}(X - X_j)^+ 1_{n_k=j},$$
$$\text{ess sup}_{\mathcal{F}_t}(X - X_{n_k})^+ \leq \sum_{j=k}^{\infty} \text{ess sup}_{\mathcal{F}_t}(X - X_j)^+ 1_{n_k=j}.$$

It is then possible to repeat the previous reasoning, replacing $(X - X_j)^+$ by $\text{ess sup}_{\mathcal{F}_t}(X - X_j)^+$, $j \geq 1$. □

Proposition 25 *A sequence $(X_n)_{n\in\mathbb{N}}$ of elements in $L^0(\mathbf{R}, \mathcal{F}_T)$ converges with respect to $\widehat{\mathcal{T}_t}$ (respectively $\mathcal{T}$) if and only if*

$$\inf_n (X_n) > -\infty.$$

Moreover, $\inf_n(X_n)$ *is a limit of $(X_n)_{n\in\mathbb{N}}$ for $\widehat{\mathcal{T}_t}$ and $\mathcal{T}_t$.*

Proof Suppose that $(X_n)_{n\in\mathbb{N}}$ converges to X with respect to $\mathcal{T}$ and suppose that $\inf_n(X_n) = -\infty$ on a non null set. Then, on this set, there exists a random subsequence X_{n_k} that converges to $-\infty$ almost surely. By Lemma 24, $(X_{n_k})_{n\in\mathbb{N}}$ converges to X with respect to $\mathcal{T}$. In other words, $(X - X_{n_k})^+$ converges to zero in probability. Therefore, there exits a subsequence $X_{n_{k_j}}$ such that $(X - X_{n_{k_j}})^+$ converges to zero almost surely. This is in contradiction with the fact that $X_{n_{k_j}}$ converges to $-\infty$.

Now suppose that $\inf_n(X_n) > -\infty$. We have $X_n \geq \inf_n(X_n) > -\infty$. So $(\inf_n(X_n) - X_n)^+ = 0$. This implies that $\text{ess sup}_{\mathcal{F}_t}(\inf_n(X_n) - X_n)^+ = 0$ hence $(X_n)_{n\geq 1}$ converges to $\inf_n(X_n)$ with respect to $\widehat{\mathcal{T}_t}$. □

Corollary 26 *A sequence $(X_n)_{n\in\mathbb{N}}$ of elements in $L^0(\mathbf{R}, \mathcal{F}_T)$ is such that $(X_n)_{n\in\mathbb{N}}$ and $(-X_n)_{n\in\mathbb{N}}$ converge with respect to $\widehat{\mathcal{T}_t}$ (respectively $\mathcal{T}$) if and only if* $\sup_n(|X_n|) < \infty$ *almost surely.*

Corollary 27 *A sequence $(X_n)_{n\in\mathbb{N}}$ of elements in $L^0(\mathbf{R}, \mathcal{F}_T)$ converges with respect to $\widehat{\mathcal{T}_t}$ if and only if $(X_n)_{n\in\mathbb{N}}$ converges with respect to $\mathcal{T}$(not necessarily with the same limits).*

Lemma 28 *A sequence $(X_n)_{n\in\mathbb{N}}$ of elements in $L^0(\mathbf{R}, \mathcal{F}_T)$ is such that $(X_n)_{n\in\mathbb{N}}$ converges to X and $(-X_n)_{n\in\mathbb{N}}$ converges to $-X$ with respect to $\widehat{\mathcal{T}_t}$ if and only if* $\text{ess sup}_{F_t}(|X - X_n|)$ *converges to* 0 *in probability.*

Proposition 29 *If a sequence $(X_n)_{n\in\mathbb{N}}$ of elements in $L^0(\mathbf{R}, \mathcal{F}_T)$ converges to $X \in L^0(\mathbf{R}, \mathcal{F}_T)$, with respect to $\widehat{\mathcal{T}_t}$ (resp. $\mathcal{T}$), then there exists a deterministic subsequence $(n_k)_{k\geq 1}$ such that*

$$X \leq \liminf_k (X_{n_k}).$$

Proof Recall that a sequence $(X_n)_{n\in\mathbb{N}}$ of elements in $L^0(\mathbf{R},\mathcal{F}_T)$ converges to $X\in L^0(\mathbf{R},\mathcal{F}_T)$ if and only if $\text{ess sup}_{\mathcal{F}_t}(X-X_n)^+$ converges to 0 in probability. Therefore, there exists a subsequence $(n_k)_{k\geq 1}$ such that $\text{ess sup}_{\mathcal{F}_t}(X-X_{n_k})^+$ converges to 0 almost surely. As

$$X-X_{n_k}\leq \text{ess sup}_{\mathcal{F}_t}(X-X_{n_k})^+$$

then $\liminf_k[X-\text{ess sup}_{\mathcal{F}_t}(X-X_{n_k})^+]\leq \liminf_k(X_{n_k})$. So, we deduce that

$$X\leq \liminf_k(X_{n_k}).$$

The same reasoning holds for $\mathcal{T}$. □

Definition 18 For a converging sequence $X=(X_n)_n$ we denote by $\widehat{\mathcal{L}}(X)$ (resp. $\mathcal{L}(X)$) the set of all limits with respect to $\widehat{\mathcal{T}_t}$ and $\mathcal{T}_t$ respectively.

Lemma 30 *If a sequence $(X_n)_n$ converges to X in probability then $(X_n)_n$ converges to X for the topology $\mathcal{T}$ and $\mathcal{L}(X)=L^0((-\infty,X],\mathcal{F}_T)$.*

Proof If $|X_n-X|$ converges to zero in probability then the same holds for $(X_n-X)^+$. Indeed, $(X_n-X)^+\leq|X_n-X|$. Therefore, $(X_n)_n$ converges to X for the topology $\mathcal{T}$. Moreover, there exists a subsequence $(n_k)_{k\geq1}$ such that $(X_{n_k})_{k\geq1}$ converges to X a.s. but also in $\mathcal{T}$ by the first part. By Proposition 29, any $Z\in\mathcal{L}(X)$ satisfies $Z\leq X$. The conclusion follows. □

Remark 9 The convergence almost surely to a limit X does not imply the convergence for $\widehat{\mathcal{T}}$ to X. Also the convergence for $\widehat{\mathcal{T}}$ and $\mathcal{T}$ does not necessarily imply the almost surely convergence. To see it, let us consider the two following examples.

(1) We consider $\Omega=[0,1]$ equipped with the Lebesgue measure. Take the sequence $X_n(\omega)=-1$ on $[0,1/n]$ and $X_n(\omega)=1/2^n$ on $(1/n,1]$, $n\geq1$. It is clear that $(X_n)_n$ converges to $X_0=0$ almost surely. But observe that $\text{ess sup}_{\mathcal{F}_0}(X_0-X_n)^+=1$. So, X_n does not converge to 0 for $\widehat{\mathcal{T}_0}$. Note that X_n converges to -1 for $\widehat{\mathcal{T}_0}$ and $\mathcal{T}$.
(2) We consider $\Omega=\mathbf{R}_+$ equipped with the Lebesgue measure. Consider $X_n(\omega)=cos(n\omega)$ for any $\omega\in\mathbf{R}$ and $Y_n(\omega)=(-1)^n$, $n\geq0$. Then, $(X_n)_n$ and $(Y_n)_n$ do not converge almost surely but $(X_n)_n$ and $(Y_n)_n$ converge for $\mathcal{T}$ and $\widehat{\mathcal{T}}$ towards -1. ♢

Definition 19 (*Cauchy sequence*) A sequence $(X_n)_n$ is said a Cauchy sequence for the pseudo-distance d if:

$$\forall\varepsilon>0,\exists n_0,\forall n,m\geq n_0,d(X_n,X_m)\leq\varepsilon.$$

Remark 10 If a sequence $(X_n)_n$ is convergent for $\widehat{\mathcal{T}}$ (or $\mathcal{T}$) it is not necessarily a Cauchy sequence. Take the sequence $X_n=(-1)^n$. It converges but it is not a Cauchy one. In fact

$$d_t^+(X_{2n}, X_{2n+1}) = 1, \forall n \in \mathbb{N}.$$

$\diamondsuit$

Proposition 31 *Every Cauchy sequence for d_t^+ is convergent in probability.*

Proof Let $(X_n)_n$ be a Cauchy sequence for d_t^+:

$$\forall \varepsilon > 0, \exists n_0, \forall n, m \geq n_0, d_t^+(X_n, X_m) \leq \varepsilon.$$

So, we also have $d_t^+(X_m, X_n) \leq \varepsilon$. In other terms $E((X_n - X_m)^+ \wedge 1) \leq \varepsilon$ and $E((X_m - X_n)^+ \wedge 1) \leq \varepsilon$. Then $E(|X_n - X_m| \wedge 1) \leq \varepsilon$. Then $(X_n)_n$ is a Cauchy sequence for the convergence in probability. Consequently it is convergent for the convergence in probability. $\square$

Example 4 Let $C \in \mathbf{R}$. Consider the sequence $X = (X_n)_n$ of elements in $L^0(\mathbf{R}, \mathcal{F}_T)$ such that $X_n = C$ for every $n \in \mathbb{N}$. Consider any $Z \in \widehat{\mathcal{L}}(X)$. By Proposition 29, $Z \leq C$. On the other hand, $(C - X_n)^+ = 0$ hence (X_n) converges to C in $\widehat{\mathcal{T}}_t$. By similar arguments, we finally deduce that $\widehat{\mathcal{L}}(X) = \mathcal{L}(X) = L^0((-\infty, C], \mathcal{F}_T)$. $\diamondsuit$

Example 5 Consider the sequence $X = (X_n)_n$ of elements in $L^0(\mathbf{R}, \mathcal{F}_T)$ such that $X_n = (-1)^n$ for every $n \in \mathbb{N}$. We have $\mathcal{L}(X) = L^0((-\infty, -1], \mathcal{F}_T)$. Indeed, as $E[(-1 - (-1)^n) \wedge 1] = 0$, -1 is a limit of X for $\mathcal{T}$. So for any $Z \leq -1$, Z is a limit for X. Now consider any $Z \in \mathcal{L}(X)$. Let us show that, $Z \leq -1$. We know that $(Z - (-1)^n)^+$ converges to zero in probability. Then, if $A_n = \{(Z - (-1)^n)^+ \leq \varepsilon\}$, $\mathbb{P}(A_n)$ converges to 1 when $n \to \infty$. On A_n, $Z - (-1)^n \leq \varepsilon$ hence $Z \leq \varepsilon - 1$ when n is odd. As n goes to ∞ we deduce that $Z \leq \varepsilon - 1$ almost surely. To see it, suppose by contradiction that $\mathbb{P}(B) > 0$ where $B = \{Z > \varepsilon - 1\}$. Therefore, there exists n_0 such that $\mathbb{P}(B \cap A_n) > 0$ for any $n \geq n_0$. If not, there exists a subsequence (A_{n_k}) such that $\mathbb{P}(B \cap A_{n_k}) = 0$. Hence, $\mathbb{P}(A_{n_k}) = \mathbb{P}(B^c \cap A_{n_k}) \leq \mathbb{P}(B^c) < 1$, in contradiction with $\lim_{k\to\infty} \mathbb{P}(A_{n_k}) = 1$. Finally, $\mathbb{P}(B \cap A_n) > 0$ for any $n \geq n_0$ in contradiction with the inequality $Z \leq \varepsilon - 1$ on A_n, when n is odd. We conclude that $Z \leq \varepsilon - 1$ a.s. and the result follows. We also deduce that $\widehat{\mathcal{L}}(X) = \mathcal{L}(X)$. $\diamondsuit$

Example 6 Consider the sequence $X = (X_n)_n$ of elements in $L^0([0, 1], \mathcal{F}_T)$, equipped with the Lebesgue measure, such that $X_n(\omega) = -1_{[0,1/n]}$ for every $n \geq 1$. We suppose that $\mathcal{F}_0$ is trivial. We know by Lemma 30 that $\mathcal{L}(X) = L^0((-\infty, 0], \mathcal{F}_T)$ but $\widehat{\mathcal{L}}(X) \subset L^0((-\infty, 0], \mathcal{F}_T)$. Indeed, 0 is not a limit for $\widehat{\mathcal{T}}_0$ as $\operatorname{ess\,sup}_{\mathcal{F}_0}(0 - X_n)^+ = 1$.

Moreover, consider $\widehat{X}_\infty \in \widehat{\mathcal{L}}(X)$. Observe that the deterministic sequence $\alpha_n = \operatorname{ess\,sup}_{\mathcal{F}_0}(\widehat{X}_\infty - X_n)^+$ converges to 0 and $\widehat{X}_\infty - X_n \leq (\widehat{X}_\infty - X_n)^+ \leq \alpha_n$. We finally conclude that $\widehat{\mathcal{L}}(X)$ is the family of all random variables $\widehat{X}_\infty$ that satisfies $\widehat{X}_\infty \leq \inf_n(X_n + \alpha_n)$ for some non negative deterministic sequence $(\alpha_n)_{n\geq 1}$ with $\lim_{n\to\infty} \alpha_n = 0$. For example, take $\alpha_n = 1$ if $n < n_0$, $n_0 > 0$ is fixed, and $\alpha_n = 0$ otherwise. Then, $Z_{n_0} = \inf_{n\geq n_0} X_n \in \widehat{\mathcal{L}}(X)$. $\diamondsuit$

Proposition 32 *If a sequence $X = (X_n)_n$ of elements in $L^0(\mathbf{R}, \mathcal{F}_T)$ converges in $\widehat{\mathcal{T}}$, then the set $\widehat{\mathcal{L}}(X)$ coincides with the family of all $\widehat{X}_\infty$ such that $\widehat{X}_\infty \leq \inf_n(X_n + \alpha_n)$*

for some sequence $(\alpha_n)_{n\geq 1}$ *in* $L^0(\mathbf{R}_+, \mathcal{F}_t)$ *that converges to zero in probability. If a sequence* $X = (X_n)_n$ *of elements in* $L^0(\mathbf{R}, \mathcal{F}_T)$ *converges in* $\mathcal{T}$, *then the set* $\mathcal{L}(X)$ *coincides with the family of all* X_∞ *such that* $X_\infty \leq \inf_n(X_n+\alpha_n)$ *for some sequence* $(\alpha_n)_{n\geq 1}$ *in* $L^0(\mathbf{R}_+, \mathcal{F}_T)$ *that converges to zero in probability.*

Proof Consider a sequence $X = (X_n)_n$ of elements in $L^0(\mathbf{R}, \mathcal{F}_T)$ converging for $\widehat{\mathcal{T}}$. Let $\widehat{X}_\infty \in \widehat{\mathcal{L}}(X)$. By definition, $\alpha_n = \text{ess sup}_{\mathcal{F}_t}(\widehat{X}_\infty - X_n)^+$ converges to 0 in probability. As $\widehat{X}_\infty - X_n \leq \text{ess sup}_{\mathcal{F}_t}(\widehat{X}_\infty - X_n)^+ \leq \alpha_n$, then we deduce that $\widehat{X}_\infty \leq \inf_n(X_n + \alpha_n)$. Conversely, if $\widehat{X}_\infty \leq \inf_n(X_n + \alpha_n)$, then $\widehat{X}_\infty \leq X_n + \alpha_n$. Therefore, $\text{ess sup}_{\mathcal{F}_t}(\widehat{X}_\infty - X_n)^+ \leq \alpha_n$ and the conclusion follows. For the second statement it suffices to consider $\alpha_n = (X_\infty - X_n)^+$. □

Appendix A: Proof of Proposition 10

The proof of Proposition 10 is deduced from Lemma 37. To get it, we first show intermediate steps such as the following Lemmas 33, 34, 35 and 36.

Lemma 33 *Let* $(M_t)_{t\in[0,T]}$ *be a sub-maxingale. Let* τ *be a stopping time such that* $\tau(\Omega) = \{t_1, t_2, \cdots, t_n\}$ *where* $(t_i)_{i=1}^n$ *is an increasing sequence of discrete dates. Then, for all* $i = 1, \cdots, n$, *we have* $\text{ess sup}_{\mathcal{F}_{t_i}}(M_\tau) \geq M_{\tau\wedge t_i}$.

Proof We have:

$$\begin{aligned}\text{ess sup}_{\mathcal{F}_{t_i}}(M_{\tau\wedge t_{i+1}}) &= \text{ess sup}_{\mathcal{F}_{t_i}}(M_{\tau\wedge t_{i+1}}1_{\{\tau\leq t_i\}}) + \text{ess sup}_{\mathcal{F}_{t_i}}(M_{\tau\wedge t_{i+1}}1_{\{\tau> t_i\}}),\\ &= 1_{\{\tau>t_i\}}\,\text{ess sup}_{\mathcal{F}_{t_i}}(M_{t_{i+1}}) + 1_{\{\tau\leq t_i\}}\,\text{ess sup}_{\mathcal{F}_{t_i}}(M_{\tau\wedge t_i}),\\ &\geq 1_{\{\tau>t_i\}}M_{t_i} + 1_{\{\tau\leq t_i\}}M_{\tau\wedge t_i} = M_{\tau\wedge t_i}.\end{aligned}$$

If $j > i+1$, argue by induction. By the tower property, we first have $\text{ess sup}_{\mathcal{F}_{t_i}}(M_{\tau\wedge t_j}) = \text{ess sup}_{\mathcal{F}_{t_i}}(\text{ess sup}_{\mathcal{F}_{t_{j-1}}}(M_{\tau\wedge t_j}))$. Therefore, by the first step above, $\text{ess sup}_{\mathcal{F}_{t_i}}(M_{\tau\wedge t_j}) \geq \text{ess sup}_{\mathcal{F}_{t_i}}(M_{\tau\wedge t_{j-1}})$ and we conclude by induction. □

Lemma 34 *Let* τ *be a stopping time such that* $\tau(\Omega) = \{t_1, t_2, \cdots, t_n\}$ *where* $(t_i)_{i=1}^n$ *is an increasing sequence of discrete dates. Then, for any random variable* X, *we have*

$$\text{ess sup}_{\mathcal{F}_\tau}(X1_{\{\tau=t_i\}}) = \text{ess sup}_{\mathcal{F}_{t_i}}(X)1_{\{\tau=t_i\}}.$$

Proof As $1_{\{\tau=t_i\}}$ is $\mathcal{F}_\tau$-mesurable, then we get that

$$\text{ess sup}_{\mathcal{F}_\tau}(X1_{\{\tau=t_i\}}) = \text{ess sup}_{\mathcal{F}_\tau}(X)1_{\{\tau=t_i\}}.$$

Since $X1_{\{\tau=t_i\}} \leq \text{ess sup}_{\mathcal{F}_{t_i}}(X)1_{\{\tau=t_i\}}$, we deduce that

$$\text{ess sup}_{\mathcal{F}_\tau}(X1_{\{\tau=t_i\}}) \leq \text{ ess sup}_{\mathcal{F}_\tau}(\text{ess sup}_{\mathcal{F}_{t_i}}(X)1_{\{\tau=t_i\}}).$$

We claim that $Z = \text{ess sup}_{\mathcal{F}_{t_i}}(X)1_{\{\tau=t_i\}}$ is $\mathcal{F}_\tau$-mesurable. For any $k \in \mathbf{R}$,

$$\{Z \leq k\} = \{0 \leq k\} \cap \{\tau \neq t_i\} \cup \{\tau = t_i\} \cap \{\text{ess sup}_{\mathcal{F}_{t_i}}(X) \leq k\}.$$

Note that $\{0 \leq k\} = \emptyset$ or Ω and $\{\tau \neq t_i\} \in \mathcal{F}_\tau$ hence $\{0 \leq k\} \cap \{\tau \neq t_i\} \in \mathcal{F}_\tau$. Now let us show that $B = \{\tau = t_i\} \cap \{\text{ess sup}_{\mathcal{F}_{t_i}}(X) \leq k\} \in \mathcal{F}_\tau$. To do so, we evaluate $B \cap \{\tau \leq t\}$ for $t \geq 0$. Note that $t_j \leq t < t_{j+1}$ for some $t_j \in \{t_0, \cdots, t_n, t_{n+1}\}$, where $t_{n+1} = \infty$. So, we deduce that $B \cap \{\tau \leq t\}$ coincides with $B \cap \{\tau \leq t_j\} = \emptyset$ if $t_j < t_i$. Otherwise, we obtain that $B \cap \{\tau \leq t\} = B \in F_{t_i} \subseteq F_{t_j} \subseteq F_t$. Therefore, $B \cap \{\tau \leq t\} \in \mathcal{F}_t$, for all $t \in \mathbf{R}$, hence $B \in \mathcal{F}_\tau$. Finally, Z is $\mathcal{F}_\tau$-mesurable and the inequality $\text{ess sup}_{\mathcal{F}_\tau}(X1_{\{\tau=t_i\}}) \leq \text{ess sup}_{\mathcal{F}_{t_i}}(X)1_{\{\tau=t_i\}}$ holds. For the reverse inequality it suffices to show that $Y = \text{ess sup}_{\mathcal{F}_\tau}(X1_{\{\tau=t_i\}})$ is $\mathcal{F}_{t_i}$-measurable. Since $\{\tau \neq t_i\} \in \mathcal{F}_\tau$, we get that $Y1_{\{\tau\neq t_i\}} = 0$ and

$$\{Y \leq k\} = (\{0 \leq k\} \cap \{\tau \neq t_i\}) \cup (A \cap \{\tau = t_i\}),$$

with $A = \{\text{ess sup}_{\mathcal{F}_\tau}(X1_{\{\tau=t_i\}}) \leq k\}$. As $A \in \mathcal{F}_\tau$, $A \cap \{\tau \leq t_i\} \in \mathcal{F}_{t_i}$ and, finally, $A \cap \{\tau = t_i\} = A \cap \{\tau = t_i\} \cap \{\tau \leq t_i\} \in \mathcal{F}_{t_i}$. Therefore, for all $k \in \mathbf{R}$, $\{Y \leq k\} \in \mathcal{F}_{t_i}$, i.e. Y is $\mathcal{F}_{t_i}$-measurable. At last, notice that $\text{ess sup}_{\mathcal{F}_\tau}(X1_{\{\tau=t_i\}}) \geq X1_{\{\tau=t_i\}}$ and, since Y is $\mathcal{F}_{t_i}$-measurable, we get that $\text{ess sup}_{\mathcal{F}_\tau}(X1_{\{\tau=t_i\}}) \geq \text{ess sup}_{\mathcal{F}_{t_i}}(X1_{\{\tau=t_i\}})$. The conclusion follows. □

Lemma 35 *Let $(M_t)_{t\in[0,T]}$ be a sub-maxingale. Let τ, S be two stopping times. Suppose that $S(\Omega) = \{t_1, t_2, \cdots, t_n\}$ where $(t_i)_{i=1}^n$ is an increasing sequence of discrete dates and suppose that $\tau(\Omega)$ is also a finite set. Then* $\text{ess sup}_{\mathcal{F}_S}(M_\tau) \geq M_{\tau\wedge s}$.

Proof By Lemma 34, we obtain $\text{ess sup}_{\mathcal{F}_S}(M_\tau) = \sum_{i=1}^n \text{ess sup}_{\mathcal{F}_{t_i}}(M_\tau)1_{\{S=t_i\}}$. By Lemma 33, we deduce that

$$\text{ess sup}_{\mathcal{F}_S}(M_\tau) \geq \sum_{i=1}^n M_{\tau\wedge t_i}1_{\{S=t_i\}} = \sum_{i=1}^n M_{\tau\wedge S}1_{\{S=t_i\}} = M_{\tau\wedge S}.$$

□

Lemma 36 *Let $\tau \in [0, T]$ be a stopping time. Suppose that the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is right-continuous. There exists a non increasing sequence $(\tau_n)_n$ of stopping times converging to τ such that, for any $X \in L^0(\mathbf{R}, \mathcal{F}_T)$,*

$$\text{ess sup}_{\mathcal{F}_\tau}(X) = \lim_n \uparrow \text{ess sup}_{\mathcal{F}_{\tau_n}}(X).$$

Moreover, $\tau^n(\Omega)$ is finite for all $n \geq 1$.

Proof Let τ be a stopping time taking values in $[0, T]$. For any $n \geq 1$, we define $\tau^n(\omega) = T(i+1)/2^n$ where $i = i(\omega)$ is uniquely defined such that $Ti/2^n < \tau(\omega) \leq T(i+1)/2^n$ for $i \geq 1$ or $0 \leq \tau(\omega) \leq T/2^n$ when $i = 0$. Note that $\tau^n(\Omega)$ is finite and $\tau^n \geq \tau$. It is easily seen that $(\tau_n)_n$ is non increasing, positive and $\lim_n \tau_n = \tau$.

Moreover, τ^n is a stopping time. Indeed, for any fixed $t \in [0, T)$, there exists $i \in \mathbb{N}$ such that $Ti/2^n \le t < T(i+1)/2^n$. Then $\{\tau^n \le t\} = \{\tau \le Ti/2^n\} \in \mathcal{F}_{Ti/2^n} \subset \mathcal{F}_t$ and the conclusion follows.

As $(\tau^n)_n$ is non increasing, then $(\mathcal{F}_{\tau^n})_n$ non increasing. As we know that $\text{ess sup}_{\mathcal{F}_{\tau^{n+1}}}(X) \ge X$ and $\text{ess sup}_{\mathcal{F}_{\tau^{n+1}}}(X)$ is $\mathcal{F}_{\tau^n}$-measurable $(\tau_{n+1} \le \tau_n)$, we deduce that $\text{ess sup}_{\mathcal{F}_{\tau^n}}(X) \le \text{ess sup}_{\mathcal{F}_{\tau^{(n+1)}}}(X)$, i.e. $(\text{ess sup}_{\mathcal{F}_{\tau^n}}(X))_n$ is non decreasing.

Similarly, $\tau^n \ge \tau$ implies that $\text{ess sup}_{\mathcal{F}_{\tau^n}}(X) \le \text{ess sup}_{\mathcal{F}_\tau}(X)$. Therefore, $\lim_n \uparrow \text{ess sup}_{\mathcal{F}_{\tau^n}}(X) \le \text{ess sup}_{\mathcal{F}_\tau}(X)$. To obtain the reverse inequality, we consider the sequence $(\text{ess sup}_{\mathcal{F}_{\tau+T/n}}(X))_n$. Since $\tau + T/n \ge \tau^n$, then

$$\lim_n \uparrow \text{ess sup}_{\mathcal{F}_{\tau+T/n}}(X) \le \lim \uparrow \text{ess sup}_{\mathcal{F}_{\tau^n}}(X) \le \text{ess sup}_{\mathcal{F}_\tau}(X).$$

It suffices to see that $Z = \lim_n \uparrow \text{ess sup}_{\mathcal{F}_{\tau+T/n}}(X)$ is $\mathcal{F}_\tau$-measurable to conclude. Indeed, $Z \ge X$ hence $Z \ge \text{ess sup}_{\mathcal{F}_\tau}(X)$ and inequalities above are equalities. For all $k \in \mathbf{R}$, $t \ge 0$, and any $n_0 \ge 1$

$$\begin{aligned}\{Z \le k\} \cap \{\tau \le t\} &= \bigcap_{n\ge 1} \{\text{ess sup}_{\mathcal{F}_{\tau+T/n}}(X) \le k\} \cap \{\tau \le t\}, \\ &= \bigcap_{n \ge n_0} \{\text{ess sup}_{\mathcal{F}_{\tau+T/n}}(X) \le k\} \cap \{\tau + T/n \le t + T/n\}.\end{aligned}$$

Notice that $\text{ess sup}_{\mathcal{F}_{\tau+T/n}}(X)$ is $\mathcal{F}_{\tau+T/n}$-measurable. We deduce that:

$$\begin{aligned}&\{\text{ess sup}_{\mathcal{F}_{\tau+T/n}}(X) \le k\} \in \mathcal{F}_{\tau+T/n}, \\ &\{\text{ess sup}_{\mathcal{F}_{\tau+T/n}}(X) \le k\} \cap \{\tau + T/n \le t + T/n\} \in \mathcal{F}_{t+T/n}.\end{aligned}$$

Therefore, for any $\epsilon > 0$ and $n_0 \ge 1$ such that $t + T/n \le t + \epsilon$, we have $\mathcal{F}_{t+T/n} \subseteq \mathcal{F}_{t+\epsilon}$ and, finally, $\{Z \le k\} \cap \{\tau \le t\} \in \cap_{\epsilon>0}\mathcal{F}_{t+\epsilon} = \mathcal{F}_{t+} = \mathcal{F}_t$. We deduce that $\{Z \le k\} \in \mathcal{F}_\tau$, for all $k \in \mathbf{R}$, i.e. Z is $\mathcal{F}_\tau$-measurable. □

Lemma 37 *Suppose that the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is right-continuous. Let $(M_t)_{t\in[0,T]}$ be a right-continuous sub-maxingale. Let τ, S be two stopping times such that $\tau(\Omega)$ is a finite set. Then, we have* $\text{ess sup}_{\mathcal{F}_S}(M_\tau) \ge M_{\tau\wedge S}$.

Proof Let $(S_n)_n$ be a sequence of stopping times decreasing to S as given in Lemma 36. Recall that $S_n(\Omega)$ is finite for all n. Moreover, we have $\text{ess sup}_{\mathcal{F}_s}(M_\tau) = \lim_n \uparrow \text{ess sup}_{\mathcal{F}_{s_n}}(M_\tau)$. By Lemma 35, we deduce that $\text{ess sup}_{\mathcal{F}_S}(M_\tau) \ge \lim \uparrow M_{\tau\wedge S_n}$. As $(\tau \wedge S_n)_n$ decreases to $\tau \wedge S$ and M is right-continuous, we conclude that $\text{ess sup}_{\mathcal{F}_s}(M_\tau) \ge M_{\tau\wedge s}$. □

Appendix B: Auxiliary results

Lemma 38 *Suppose that, at time $t \le T$, $\mathcal{O}_t$ is the pseudo-distance topology defined by (8) and $\mathcal{V}^c_{t,T} = \mathcal{V}^c_{t,T}(\mathcal{O}_t)$. If $\mathcal{V}_{t,T}$ is $\mathcal{F}_t$-decomposable, then $\mathcal{V}^c_{t,T}$ is $\mathcal{F}_t$-decomposable.*

Proof Consider $V^{c,i}_{t,T} \in \mathcal{V}^c_{t,T}$, $i = 1, 2$, and $F_t \in \mathcal{F}_t$. By Proposition 32, $V^{c,i}_{t,T} \leq V^{n,i}_{t,T} + \alpha^{n,i}_t$ where $V^{n,i}_{t,T} \in \mathcal{V}_{t,T}$ and $\alpha^{n,i}_t$ converges to 0 in probability as $n \to \infty$, for $i = 1, 2$. We set

$$V^n_{t,T} = V^{n,1}_{t,T} 1_{F_t} + V^{n,2}_{t,T} 1_{\Omega \setminus F_t}, \quad \alpha^n_t = \alpha^{n,1}_t 1_{F_t} + \alpha^{n,2}_t 1_{\Omega \setminus F_t}.$$

Note that $V^n_{t,T} \in \mathcal{V}_{t,T}$ by assumption and α^n_t converges to 0 in probability. Moreover, $V^{c,1}_{t,T} 1_{F_t} + V^{c,2}_{t,T} 1_{\Omega \setminus F_t} \leq V^n_{t,T} + \alpha^n_t$. Therefore, Proposition 32 implies that $V^{c,1}_{t,T} 1_{F_t} + V^{c,2}_{t,T} 1_{\Omega \setminus F_t} \in \mathcal{V}^c_{t,T}$ and the conclusion follows. □

Lemma 39 *Let $h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$ be a payoff. If $\mathcal{V}_{t,T}$ (resp. $\mathcal{V}^c_{t,T}$) is $\mathcal{F}_t$-decomposable (resp. infinitely $\mathcal{F}_t$-decomposable), then $\mathcal{P}_{t,T}(h_T)$ (resp. $\mathcal{P}^c_{t,T}(h_T)$) is $\mathcal{F}_t$-decomposable (resp. infinitely $\mathcal{F}_t$-decomposable).*

Proof Suppose that $\mathcal{V}_{t,T}$ is $\mathcal{F}_t$-decomposable and consider $p^1_t, p^2_t \in \mathcal{P}_{t,T}(h_T)$ and $F_t \in \mathcal{F}_t$. Then, $p^i_t + V^i_{t,T} \geq h_T$ for some $V^i_{t,T} \in \mathcal{V}_{t,T}$, $i = 1, 2$. By assumption, we have $V_{t,T} = V^1_{t,T} 1_{F_t} + V^2_{t,T} 1_{\Omega \setminus F_t} \in \mathcal{V}_{t,T}$ by assumption and $p^1_t 1_{F_t} + p^2_t 1_{\Omega \setminus F_t} + V_{t,T} \geq h_T$. We deduce that $p^1_t 1_{F_t} + p^2_t 1_{\Omega \setminus F_t} \in \mathcal{P}_{t,T}(h_T)$. By the same reasoning, the property holds for $\mathcal{V}^c_{t,T}$ and the infinite $\mathcal{F}_t$-decomposability is obtained similarly. The conclusion follows. □

Lemma 40 *Let $h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$ be a payoff. If $\mathcal{V}_{t,T}$ is infinitely $\mathcal{F}_t$-decomposable, then for any $\gamma_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ such that $\gamma_t > \pi_{t,T}(h_T)$, there exists a price $p_t \in \mathcal{P}_{t,T}(h_T)$ such that $p_t < \gamma_t$. In particular, $\gamma_t \in \mathcal{P}_{t,T}(h_T)$.*

Proof Since $\mathcal{V}_{t,T}$ is infinitely $\mathcal{F}_t$-decomposable, $\mathcal{P}_{t,T}(h_T)$ is infinitely $\mathcal{F}_t$-decomposable by Lemma 39. Therefore, $\mathcal{P}_{t,T}(h_T)$ is directed downward and we deduce that $\pi_{t,T}(h_T) \lim_n \downarrow p^n_t$ where $p^n_t \in \mathcal{P}_{t,T}(h_T)$, see Kabanov and Safarian (2009, Sect. 5.3.1). Then, a.s.(ω), there exits $n(\omega)$ such that $p^n_t(\omega) < \gamma_t(\omega)$. We then define

$$N_t = \inf\{n \geq 1 : \; p^n_t < \gamma_t\} \in L^0(\mathbb{N}, \mathcal{F}_t),$$
$$p_t = \sum_{j=1}^{\infty} p^j_t 1_{\{N_t = j\}}.$$

By assumption $p_t \in \mathcal{P}_{t,T}(h_T)$ and $p_t < \gamma_t$. The conclusion follows. □

References

Baptiste, J., Carassus, L., Lépinette, E.: Pricing without martingale measure. Preprint, https://hal.archives-ouvertes.fr/hal-01774150, (2020)

Barron, E.N., Cardaliaguet, P., Jensen, R.: Conditional essential suprema with applications. Appl Math Optim **48**, 229–253 (2003)

Bayraktar, E., Sayit, H.: No arbitrage conditions for simple trading strategies. Ann Finance **6**(1), 147–156 (2010)

Bender, C., Sottinen, T., Valkeila, E.: Pricing by hedging and no-arbitrage beyond semi-martingale. Finance Stochast **12**(4), 441–468 (2008)
Carassus, L., Lépinette, E.: Pricing without no-arbitrage condition in discrete-time. J Math Anal Appl **505**(1), 125441 (2021)
Cheredito, P.: Arbitrage in fractional Brownian motion models. Finance Stochast **7**(4), 533–553 (2003)
Dalang, E.C., Morton, A., Willinger, W.: Equivalent martingale measures and no-arbitrage in stochastic securities market models. Stochast Stochast Rep **29**, 185–201 (1990)
Delbaen, F., Schachermayer, W.: A general version of the fundamental theorem of asset pricing. Math Ann **300**, 463–520 (1994)
El Mansour, M., Lépinette, E.: Conditional interior and conditional closure of a random sets. J Optim Theory Appl **187**, 356–369 (2020)
Filipovic, D., Kupper, M., Vogelpoth, N.: Separation and duality in locally L^0 convex modules. J Funct Anal **256**(12), 3996–4029 (2009)
Guasoni, P.: No arbitrage with transaction costs, with fractional Brownian motion and beyond. Math Financ **16**(2), 469–588 (2006)
Jacod, J., Shiryaev, A.N.: Limit theorems for stochastic processes. Grundlehren der mathematischen Wissenschaften 288, a series of comprehensive studies in mathematics. Springer: Berlin (2003)
Jarrow, R., Protter, P., Sayit, H.: No arbitrage without semimartingales. Ann Appl Probab **19**(2), 596–616 (2009)
Kabanov, Y., Stricker, C.: A Teachers' note on no-arbitrage criteria. In: Séminaire de Probabilités, XXXV, volume 1755 of Lecture Notes in Mathematics, Springer: Berlin, 149–152 (2001)
Kabanov, Y., Safarian, M.: Markets with Transaction Costs. Mathematical Theory. Springer, Berlin (2009)
Karatzas, I., Kardaras, C.: The numéraire portfolio in semimartingale financial models. Finance Stochast **11**, 447–493 (2007)
Kreps, D.M.: Arbitrage and equilibrium in economies with infinitely many commodities. J Math Econ **8**, 15–35 (1981)
Lépinette, E., Molchanov, I.: Conditional cores and conditional convex hulls of random sets. J Math Anal Appl **478**(2), 368–392 (2019)
Lo, A.W.: Long-term memory in stock market prices. Econometrica **59**(5), 1279–1313 (1991)
Pakkanen, M.: Stochastic integrals and conditional full support. J Appl Probab **47**(3), 650–667 (2010)
Rockafellar, R.T., Wets, R.J.B.: Variational Analysis, Grundlehren der mathematischen Wissenschaften, vol. 317. Springer, Berlin (1998)
Rogers, L.C.G.: Arbitrage with fractional Brownian motion. Math Financ **7**, 95–105 (1997)
Sayit, H.: Absence of arbitrage in a general framework. Ann Finance **9**, 611–624 (2013)
Sottinen, T.: Fractional Brownian motion, random walks and binary market models. Finance Stochast **5**(3), 343–355 (2001)