Annals of Finance (2023) 19:383–400
https://doi.org/10.1007/s10436-022-00422-x

RESEARCH ARTICLE

Analysis of fair fee in guaranteed lifelong withdrawal and Markovian health benefits

Guglielmo D'Amico[1] · **Shakti Singh**[2] · **Dharmaraja Selvamuthu**[2]

Received: 13 September 2022 / Accepted: 20 December 2022 / Published online: 17 January 2023

Abstract

This study proposed and evaluated a new insurance product, i.e., the variable annuity product, accompanied by the health status and the guaranteed lifelong withdrawal benefit (GLWB). Due to specific problems, the insurance sector is now one of the riskiest industries. The aging of the population and rising medical service costs as a result of technological advancements are to blame for this. Thus one of the most basic needs in the health insurance sector is to design an innovative product. In this article, a mixed discrete-continuous time model is proposed to calculate the fair fee of the product, calculated using equilibrium condition between premium and benefits. We considered constant volatility and rate of interest along with health status benefits and hospitalization coverage. For an illustration of the capability of this product and some possible improvements in the product, a numerical study, and sensitivity analysis have been conducted. The results showed that the withdrawal amount and age have a significant impact on the cost. A rise in the initial insured age and withdrawal amount increases the fair fee of the product. The GLWB rider's guaranteed amount and medical expenses are included in the withdrawal amount.

Keywords Variable annuity · Guaranteed lifetime withdrawal benefit · Hospitalization coverage · Health status

---

Guglielmo D'Amico, Shakti Singh and Dharmaraja Selvamuthu have contributed equally to this work.

✉ Guglielmo D'Amico
g.damico@unich.it

Shakti Singh
singh.shakti182@gmail.com

Dharmaraja Selvamuthu
dharmar@maths.iitd.ac.in

1 Department of Economics, University G. d'Annunzio of Chieti-Pescara, Viale Pindaro, 65127 Pescara, Italy

2 Mathematics, IIT Delhi, New Delhi 110016, India

JEL Classification C02 · G22 · G23

List of symbols

y	Initial insured age, i.e., insured age at time period 0
t	Time in years
r	Rate of interest
h_0	Premium paid as lump sum
S_t	Value of the stock price at time t
s_0	Initial value of the stock price
γ	Rate of the fee charged by the insurance company
W	Periodic random withdrawal % of h_0

1 Introduction

The popularity of healthcare insurance products and the ever-increasing demand for such products can be attributed to increased awareness of longevity risk and advancements in medical technology. Because of decreasing mortality and fertility rates, the population of many wealthy countries is rapidly aging (Cristea et al. 2020). Longevity increases contribute to rising medical expenditures and increased demand in the health insurance industry. Levantesi et al. (2020) used a machine learning technique to predict mortality and improve longevity. Individuals' financial responsibilities are reduced by health insurance in the case of an illness or injury that necessitates hospitalization or results in a loss of income. The private health insurance market covers most of the medical expenses in nations without National Health Insurance, such as the United States. The Health Insurance Portability and Accountability Act of 1996 also requires practically all individual insurance policies to be guaranteed renewable. However, their market share remains limited due to the lack of appeal and variety in these products. Looking at the health insurance industry, we can see that spending has increased dramatically over the last 50 years (Dieleman et al. 2017). Growth and aging of the human population, increased demand for health care owing to economic development, and rising healthcare expenditures are all factors that contribute to the rise in health insurance premiums. Health policymakers in the public sector and management of private health insurance companies are concerned about the rise of health insurance and its financing.

This research suggests a new variable annuity product embedded with GLWB benefit with a dynamic withdrawal strategy dependent on the policyholder's health status. We assume that $X(t)$ is a continuous-time Markov chain. This choice is due to the flexibility of this class of stochastic processes in describing systems that randomly change state in time, see, e.g., D'Amico and Villani (2021) and De Blasis (2020) and due to its primary benefits of simplicity and out-of-sample forecasting accuracy. Moreover the value of health state is influenced only by its current state, and not by any prior activity. In essence, it can be predicted based solely upon the current circumstances surrounding the variable. Other main reason of using Markovian models is because they can enrich the point estimates with error estimates or even provide the whole probability distribution. A large body of literature has shown the ability of Markov chains to describe disability states (see, e.g., Pitacco 1995) or hospitalization conditions, and mode of discharge (see, e.g., Jones et al. (2019)). The works by Manton et al. (1993), Haberman and Pitacco (2018), Pritchard (2006), Baione and Levantesi (2014) and Yang et al. (2016) have used continuous-time Markov processes to model the health status transitions.

The research on variable annuities with health status influences is significantly less. The Life Care Annuity-Guaranteed Lifetime Withdrawal Benefit was firstly proposed by Hsieh et al. (2018). Fard and Rong (2014) proposed a model for valuing ruin contingent life annuities under the regime-switching variance gamma process. In the direction of hospitalization coverage based on the claims of families and people covered by the policy, Tessera (2007) developed two probabilistic models for claim size in health insurance. A statistical model to predict the incidence and cost of hospitalizations for a given chronic disease was developed by Rosenberg and Farrell (2008). D'Amato et al. (2013) analyzed the performance of a portfolio of participating variable annuities focusing on the minimum income level.

Moosavi and Payandeh (2021) developed a model for the valuation of variable long-term care annuities with static guaranteed lifetime withdrawal and limited hospitalization coverage benefits. They used two investment funds in which the Geometric Brownian Motion (GBM) model is used with constant volatility. Moreover, they considered only limited health status states.

The contribution of this article is to present a modeling framework that gives the policyholder the flexibility to withdraw different amounts according to his/her health status. A one-time lump sum is invested in a risky fund that follows a GBM. The expected life and death benefits are calculated, and the fair rate of the fee charged by the insurance company is determined as the solution of the equilibrium equation between expected benefits and premium. The remainder of the paper is laid out as follows: Sect. 2 includes the description of the continuous-discrete time model and the mathematical formulas needed to determine the fair fee. Section 3 includes data description along with numerical results of calculating the fee for $S\&P$ 500 index and the effect of age and guaranteed withdrawal amount on the fee. Section 4 summarizes our contribution and results with some potential further work.

2 Model description

This section presents the random processes involved in analyzing the GLWB with Markovian health states. The model is mixed discrete-continuous in time because while the policyholder's health state and the fund's value evolve continuously in time, the payment of the benefits occurs according to a discrete-time scale. This choice agrees with Piscopo and Haberman (2011), who presented a continuous and discrete-time model with death and GLWB benefit without any health status benefit but using mortality rates. The health status changes in time according to a stochastic process $\{X(t), t \geq 0\}$ with state-space $E = \{1, 2, 3, \ldots, D\}$. The elements $1, 2, 3 \ldots$ represent different health states of the policyholder and D is the death of the insured. Moosavi and Payandeh (2021) provided with a continuous-time health model with

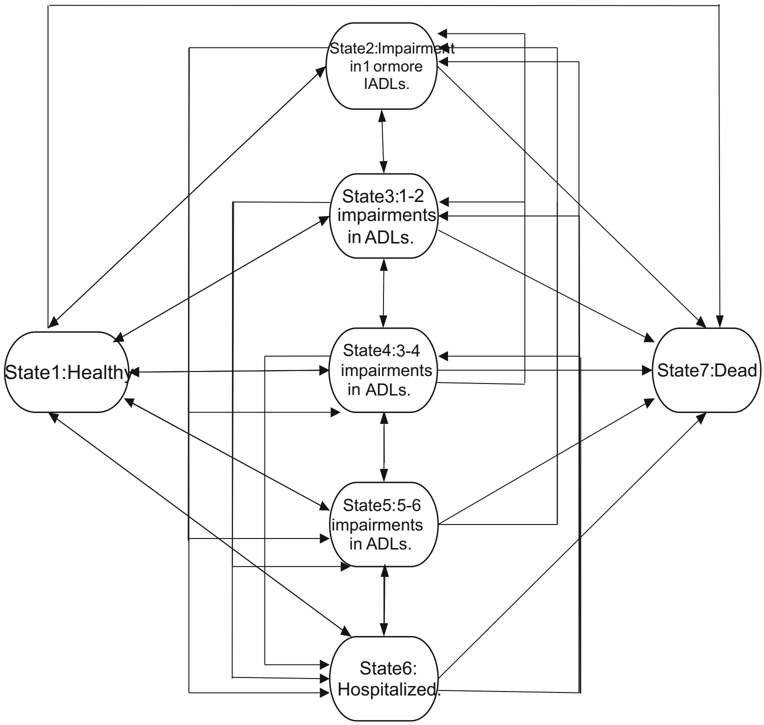

Fig. 1 The health status of the policyholder according to Moosavi and Payandeh (2021)

GLWB with limited health states i.e. $\{1, 2, \ldots, 7\}$, with 7 as death (see Fig. 1). In this section, we will describe the policyholder's health status as a continuous-time Markov chain with health states $E = \{1, 2, \ldots, D\}$. The benefits are calculated based on the policyholder's health evolution, and the fair fee is computed according to a balance condition between premium and expected benefits. This will provide the actuarial sector with a better product that combines the benefit of GLWB and includes health insurance features.

This paper uses capital letters to denote random variables, and the following notations are considered.

With the notations mentioned above and for simplicity, let us make some assumptions that the model's description will follow.

Assumption 1. We analyze a single index and assume that the premium is a one-time lump sum investment of h_0 made by the insured at the start of the contract, which makes $\frac{h_0}{s_0}$ as the number of initial stocks.
2. The insurance company's fee is deducted from the funds value when fund units are canceled at a rate equal to γ.
3. We will assume the fee is charged continuously, and the guarantee (g % of h_0) is deducted by the policyholder over a discrete-time set.
4. Death benefits are paid when the insured dies.

Let us consider $\{X(t), t \geq 0\}$ be a stochastic process describing the health status of the policyholder at time t. Let T_a be the set of times where the payments of benefits occur, i.e.,

$$T_a = \{t \geq 0 : t = ak \text{ with a } \in \mathbb{R}^+ \text{ and k } \in \mathbb{N}\}. \tag{1}$$

For example if time is measured in months, and benefits are paid annually, then $a = 12$ so that

$$k = 1, t = 12 \text{ months } = 1 \text{ year },$$
$$k = 2, t = 24 \text{ months } = 2 \text{ year },$$

and so on.

At a first stage $\{X(t), t \geq 0\}$ is considered to be a continuous-time Markov chain with finite state space $E = \{1, 2, \ldots, D\}$. For $t > s$, let us define $p_{ij}(t-s) = \mathbb{P}(X(t) = j | X(s) = i)$ be the transition probability from state i at time s to state j at time t and let $\mathbf{Q}$ be the generator matrix for $\{X(t), t \geq 0\}$ such that

$$q_{ij} = \lim_{t \to 0} \frac{p_{ij}(t)}{t} \quad \forall i \neq j \quad \text{and} \quad q_{ii} = -\sum_{j \neq i} q_{ij}.$$

Then, the following formula holds true

$$p_{ij}(t-s) = \mathbb{P}(X(t) = j | X(s) = i) = (e^{\mathbf{Q}(t-s)})_{(i,j)} = \left(\sum_{l=0}^{\infty} \frac{(t-s)^l \mathbf{Q}^l}{l!} \right)_{(i,j)}. \tag{2}$$

Let $g(t)$ be the health expenditure at time t with health state $X(t)$ such that for $t \in T_a$

$$g(t) = g(X(t)). \tag{3}$$

It is important to note that the wealth and income of an individual, as well as the likelihood or probability of getting sick, can all affect health expenditures. It is possible to take these variables into account as covariates in both the generator matrix $\mathbf{Q}$ and the function $g(t)$. We do not go into further details about this possibility because, while it is simple to model mathematically, it is not realistic given the lack of actual information regarding the social conditions of the policyholders.

Let $G(t)$ be the guaranteed amount at time t then specifically, we consider $G(t) = G(X(t))\, h_0$ as a certain percentage of the initial investment to express the random guaranteed withdrawal rate paid by the insurer to the policyholder at time $t = ka, k \in \mathbb{N}$ as

$$G(X(t)) = W(X(t)) + g(X(t)), \tag{4}$$

where $W(X(t))$ is the random amount offered by the GLWB rider. Policyholder will get a variable amount $W(X(t))$ through out his/her life and along with it he/she can

withdraw amounts according to his/her health status $\{X(t), t \geq 0\}$, which may depends on the policyholder's will also.
Now, to find out the break-even fee (γ^*), we equate the outflow which consists of lump sum premium h_0 to the expected present value (EPV) of the inflows, which consists of discounted expected life benefit (ELB) and discounted expected death benefit (EDB) to that of outflows. Hence,

$$h_0 = ELB + EDB. \tag{5}$$

It is worth noting that a safety margin could be added to the break-even fee to ensure profits for the insurance company, or loaded premiums could be used instead, see, e.g., Furman and Zitikis (2008).

2.1 Computation of life benefit

Benefits are paid as a random amount $G(X(t))h_0$ according to the set of times T_a periodically. The accumulated benefits over the life of the policyholder are given by

$$LB(0) = \sum_{t \in T_a} G(X(t))h_0 e^{-rt} \mathbb{1}_{\{X(t) \neq D\}}. \tag{6}$$

Now, take the conditional expectation of Eq. (6), that is

$$ELB_i = \mathbb{E}[LB(0)|X(0) = i] = \mathbb{E}_i[LB(0)]. \tag{7}$$

It denotes the conditional expected life benefits given health state i of the policyholder at initial time zero. The ELB_i can be obtained as follows

$$ELB_i = \sum_{t \in T_a} \mathbb{E}_i[G(X(t))\mathbb{1}_{\{X(t) \neq D\}}]h_0 e^{-rt}, \tag{8}$$

where $ELB_D = 0$. This expected value can be evaluated according to the time scale of benefit payments

$$ELB_i = \sum_{k \geq 1} \mathbb{E}_i[G(X(ka))\mathbb{1}_{\{X(ka) \neq D\}}]h_0 e^{-rka}, \tag{9}$$

$$= \sum_{k \geq 1} \sum_{j \neq D} G(j)\mathbb{P}(X(ka) = j|X(0) = i)h_0 e^{-rka}, \tag{10}$$

$$= \sum_{k \geq 1} \sum_{j \neq D} G(j)(e^{\mathbf{Q}ka})_{(i,j)} h_0 e^{-rka}. \tag{11}$$

Formula (11) calls for the evaluation of transition probabilities at various points in time and entails an infinite summation. A more efficient way to compute the ELB_i is by establishing a system of linear equations for them and solving it using matrix representation. The computation are in line with Markov reward formalism, see, e.g., D'Amico et al. (2010, 2015).

Proposition 1 *The expected life benefits can be calculated according to the following matrix formula:*

$$\boldsymbol{ELB} = \Big(I - e^{-ra} \cdot \boldsymbol{P}(a)\Big)^{-1} * \Big(e^{-ra} \cdot \boldsymbol{P}(a) * \boldsymbol{G}\Big), \tag{12}$$

where

$$\boldsymbol{ELB} = \big[ELB_1,\ ELB_2,\ \ldots,\ ELB_D\big]^T \quad \textit{with} \quad ELB_D = 0,$$

$$\boldsymbol{G} = \big[G(1),\ G(2),\ \ldots,\ G(D)\big]^T \quad \textit{with} \quad G(D) = 0.$$

and $\boldsymbol{P}(a) = e^{Qa}$ *represents the transition probability matrix at time a, with (*) as standard matrix product operator.*

Proof According to Equation (8) we have that

$$ELB_i = \mathbb{E}_i\Big[\sum_{k\geq 1} G(X(ka)\mathbb{1}_{\{X(t)\neq D\}} h_0 e^{-rka}\Big].$$

The application of the tower property of conditional expectation gives

$$\begin{aligned}
&= \mathbb{E}_i\Big[\mathbb{E}[\sum_{k\geq 1} G(X(ka))\mathbb{1}_{\{X(ka)\neq D\}} h_0 e^{-rka}/X(a)]\Big],\\
&= \mathbb{E}_i\Big[G(X(a))\mathbb{1}_{\{X(a)\neq D\}} h_0 e^{-ra}\\
&\quad + \mathbb{E}\Big[\sum_{k\geq 2} G(X(ka))\mathbb{1}_{\{X_{ak}\neq D\}} h_0 e^{-rka}/X(a)\Big]\Big].
\end{aligned}$$

Now we change variable by setting $s = k - 1$ in the former expectation to get

$$ELB_i = \mathbb{E}_i\Big[G(X(a))\mathbb{1}_{\{X(a)\neq D\}} h_0 e^{-ra} \tag{13}$$

$$+ e^{-ra}\mathbb{E}\Big[\sum_{s\geq 1} G(X(sa+a))\mathbb{1}_{\{X(sa+a)\neq D\}} h_0 e^{-rsa}/X(a)\Big]\Big]. \tag{14}$$

Observe now that

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{s\geq 1} G(X(sa+a))\mathbb{1}_{\{X(sa+a)\neq D\}} h_0 e^{-rsa}/X(a)\Big],\\
&= \sum_{s\geq 1}\sum_{j\neq D} G(j)P(X(sa+s) = j/X(a))h_0 e^{-rsa},\\
&= \sum_{s\geq 1}\sum_{j\neq D} G(j)(e^{\mathbf{Q}(sa)})_{(X(a),j)} h_0 e^{-rsa},
\end{aligned}$$

$$= ELB_{X(a)}.$$

Then,

$$ELB_i = \mathbb{E}_i\Big[G(X(a))\mathbb{1}_{\{X(a)\neq D\}}h_0e^{-ra} + e^{-ra}ELB_{X(a)}\Big],$$
$$= \sum_{j\in E}(e^{(\mathbf{Q}a)})_{(i,j)}\Big[G(j)h_0e^{-ra} + e^{-ra}ELB_j\Big].$$

The previous equation can be written in matrix form by using the column-vectors **ELB** and **G** as follows

$$\mathbf{ELB} = e^{-ra}\cdot\mathbf{P}(a) * \Big(\mathbf{G} + \mathbf{ELB}\Big).$$

Algebraic manipulations give

$$\Big(I - e^{-ra}\cdot\mathbf{P}(a)\Big) * \mathbf{ELB} = e^{-ra}\cdot\mathbf{P}(a) * \mathbf{G}, \quad (15)$$

which in turn provides

$$\mathbf{ELB} = \Big(I - e^{-ra}\cdot\mathbf{P}(a)\Big)^{-1} * e^{-ra}\cdot\mathbf{P}(a) * \mathbf{G}, \quad (16)$$

provided that $\Big(I - e^{-ra}\cdot\mathbf{P}(a)\Big)^{-1}$ exists.
In any case, $\mathbf{P}(a)$ is a stochastic matrix, thus its spectral radius $\rho(\mathbf{P}(a)) = 1$ as a consequence of Perron-Frobenius theorem.
Moreover, observe that the matrix $I - e^{-ra}\cdot\mathbf{P}(a)$ can be rewritten as follows:

$$I - e^{-ra}\cdot\mathbf{P}(a) = \frac{1}{e^{ra}}\Big(e^{ra}\cdot I - \mathbf{P}(a)\Big). \quad (17)$$

The matrix $e^{ra}\cdot I - \mathbf{P}(a)$ belongs to the class of M-matrix (see Le and Tsatsomeros 2021) and is non-singular if and only if $e^{ra} \geq \rho(\mathbf{P}(a)) = 1$ which is trivially satisfied being $e > 1$ and $a \geq 0$. □

Remark Proposition 1 provides an exact representation of the ELB in terms of the model's parameters. As shown, there is no need to introduce any maximal allowed age for the policyholder as done in Piscopo and Haberman (2011) or to implement Monte Carlo simulations as developed by Moosavi and Payandeh (2021).

2.2 Computation of death benefit

According to Piscopo and Haberman (2011), we adopt the GBM hypothesis under the risk-neutral measure $\tilde{\mathbb{P}}$. The conservative GBM model assumes that return volatility remains constant throughout time. Under a risk-neutral metric, the stock price dynamic for a GBM is

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad s_0 > 0, \tag{18}$$

here μ is the drift coefficient, σ^2 is the variance and $\{B_t, t \geq 0\}$ is a standard Wiener process. Then the stock price at time t can be represented as

$$S_t = s_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t}. \tag{19}$$

Moreover, as usual, we denote by γ (in basis points) the insurance fee paid as a function of the asset. Furthermore we have a sequence of random variables $\{G(X(t))\}_{t \in T_a}$ denoting the time-varying withdrawal from the fund at time t. Note that we do not consider a rate of withdrawal, as rather a sequence of withdrawal of random amounts depending on the health state of the policyholder. Under these assumptions, the dynamic of the fund can be described as follows (Fig. 2)

$$\begin{cases} dH_t = (\mu - \gamma) S_t dt + \sigma S_t dB_t, \; \forall\, t : ak < t < a(k+1) \\ H_t = H_{t^-} - G(X(t))h_0, \qquad \forall\, t \in T_a. \end{cases} \tag{20}$$

Here the present value of expected death benefit can be calculated. Let τ be the time of death of the policyholder, that is

$$\tau = \inf\{t \in \mathbb{R} : X(t) = D\}.$$

Let $DB_\beta(\gamma)$ be the death benefit conditional on the death at time $\tau = \beta \in \mathbb{R}_+$ then,

$$DB_\beta(\gamma) = \max\{H_\beta, 0\}.$$

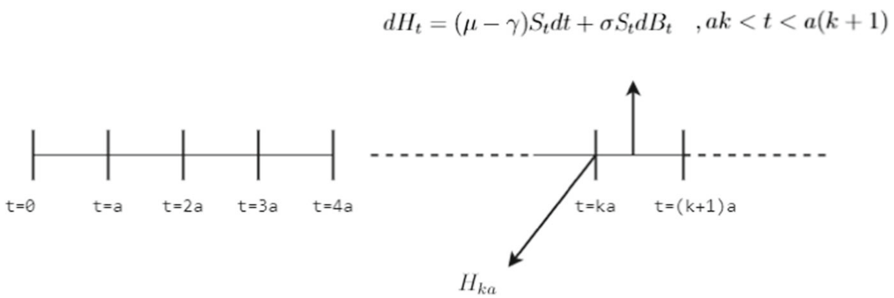

Fig. 2 Dynamic of fund

It results that

$$DB_\beta(\gamma) = e^{(\mu-\gamma-\sigma^2/2)\beta+\sigma B_\beta} \max[0, h_0 - \sum_{v=1}^{\lfloor\beta\rfloor} G(X(va))h_0 e^{-(\mu-\gamma-\sigma^2/2)va-\sigma B_{va}}], \tag{21}$$

where $\lfloor\beta\rfloor = \max\{k \in \mathbb{N} : k \cdot a \leq \beta\}$.
Now the expected death benefit is just the discounted expectation of the conditional death benefit with respect to the random time of death, i.e.,

$$EDB(\gamma) = \int_0^\infty e^{-r\beta}\mathbb{E}[DB_\beta(\gamma)]\lambda(\beta)d\beta,$$

where

$$\lambda(\beta) = \lim_{h\to 0} \frac{\mathbb{P}(\beta < \tau \leq \beta + h|\tau > \beta)}{h},$$

is the so-called failure rate function. For an n-state continuous-time Markov chain, it can be expressed according to Sadek and Limnios (2005)

$$\lambda(\beta) = \begin{cases} \frac{-1}{R(\beta}) \cdot \frac{dR(\beta)}{d\beta} = -\frac{\alpha_1 e^{\beta \mathbf{Q}_{(n-1,n-1)}} \cdot \mathbf{Q}_{(n-1,n-1)} \mathbb{1}_{(n-1)}}{\alpha_1 e^{\beta \mathbf{Q}_{(n-1,n-1)}} \mathbb{1}_{(n-1)}} & \text{if } R(\beta) \neq 0, \\ 0 & \text{if } R(\beta) = 0, \end{cases} \tag{22}$$

where $\mathbf{Q}$ is the generator matrix of the Markov chain, α is the initial probability distribution, $R(\beta)$ the reliability function and $\mathbb{1}_{(n-1)}$ is the unitary vector of the dimension $n-1$. We then have

$$LB_0 + DB_0 = \sum_{i=1}^{D} \alpha_i \cdot ELB_i + \int_0^\infty e^{-r\beta}\mathbb{E}[DB_\beta(\gamma)]\lambda(\beta)d\beta, \tag{23}$$

hence, we get the final equation as

$$h_0 = \alpha^T * \mathbf{ELB} + EDB(\gamma). \tag{24}$$

The above equation in γ is solved numerically to obtain the break-even value for the fee (γ^*) charged by the insurer.

3 Numerical results and sensitivity analysis

In this section description of the data taken for the numerical calculation is mentioned, along with the numerical results for the given model.

3.1 Data description

In this paper, $S\&P$ 500 composite index prices from the US market are considered for the numerical analysis. The index was chosen based on data availability and popularity among the people of the respective country. We considered information from January 1, 1970, until October 31, 2019. The data for $S\&P$ 500 index is taken from Thomson and Reuters Datastream. Furthermore, annual interest rate data for the US market for the previous 50 years is taken from https://fred.stlouisfed.org/.
In evaluating an insurance health product, the health state of policyholders and their corresponding transition probabilities play critical roles. According to Moosavi and Payandeh (2021), a seven-state continuous-time Markov process has been used to classify an individual's health status based on his/her ability to perform daily activities. Successfully performing instrumental activities of daily living (IADL) such as light housework, laundry, grocery shopping, meal preparation, getting around outside, money management, and using the telephone and activities of daily living (ADL) such as eating classified as (level 1), bathing (level 2), dressing (level 3), moving around (level 4), doing personal hygiene (level 5) and going to the toilet (level 6), see also Pritchard (2006), Haberman and Pitacco (2018) for more details.
The state of each policyholder begins from state 1 (healthy) and moves to state 2 (impairment in at least one IADLs), state 3 (ADL level = 1 and 2), state 4 (ADL level = 3 and 4), state 5 (ADL level = 5 and 6), state 6 (stay at a hospital) and finally state 7 (death). The state transition diagram for the health status of the policyholder is shown in Fig. 1. This study follows two-year transition probabilities between disability states calculated from the 1982 and 1984 National Long Term Care Survey, as a percentage, for males and females mentioned in Pritchard (2006). Table 1 shows the two-year transition probabilities. For example, the $(1, 7)$-th entry (i.e., 4.04) of the matrix represents the chances of dying of a healthy person in percentage within two years. Parameters μ and σ are first estimated from real data using statistical method using R software in order to calculate the expected death benefit EDB. Estimated value for μ is 4.21 and σ is 1.17. For the numerical calculation RStudio 2021.09.0 Build 351 is used in the system with configuration as follows: AMD Ryzen 5 3500U with Radeon Vega Mobile Gfx 2.10 GHz, 64-bit operating system, x64-based processor, and 8GB RAM.

3.2 Numerical calculation

For the numerical analysis the following assumptions are considered:

1. Premium paid is a lump sum amount of 100, i.e., $h_0 = \$100$ and a is considered to be 24 months.
2. The range of break-even fee (γ^*) is considered to be 0 to 1000 basis points (bp).
3. Initially, the State of each policyholder begins with state 1 (healthy) i.e., $\alpha = (1, 0, 0, 0, 0, 0, 0)$.
4. Withdrawal is according to the health status of the patient such that for the health states 1 and 2, the withdrawal amount is assumed to be 2\$, for states 3 and 4, it is assumed to be 3\$, and for 5th and 6th state it is assumed to be 4\$. That is

Table 1 Two-year transition probabilities between disability states calculated from the 1982 and 1984 NLTCS, as a percentage, for males and females according to Pritchard (2006)

1984 status Age	1982 status	Healthy	IADL only	1–2 ADLs	3–4 ADLs	5–6 ADLs	Inst'd	Dead
65	Healthy	90.01	2.41	1.92	0.59	0.56	0.47	4.04
71		83.73	3.60	2.41	0.95	0.93	1.11	7.28
76		76.74	5.40	4.08	1.30	1.24	1.94	9.30
65	IADL Only	30.67	31.61	15.67	4.55	4.27	2.19	11.05
71		25.27	32.53	17.48	3.64	2.77	4.93	13.38
76		15.39	30.65	18.73	5.01	5.31	7.53	17.38
65	1–2 ADLs	16.29	12.84	36.49	10.47	6.08	2.95	14.89
71		10.52	12.85	35.00	11.99	5.78	6.94	16.92
76		9.22	12.06	35.08	10.33	6.73	6.09	20.05
65	3–4 ADLs	7.15	6.27	26.77	23.56	12.68	4.73	18.83
71		8.31	4.64	18.09	29.12	18.18	3.98	17.67
76		3.73	4.37	17.57	20.78	16.35	10.97	26.23
65	5–6 ADLs	7.03	5.54	8.85	9.01	33.15	5.86	30.57
71		6.15	4.71	9.13	10.16	28.68	8.03	33.15
76		3.97	3.64	4.98	7.71	34.73	11.56	33.42
65	Inst'd	7.53	0.94	1.59	2.01	0.48	72.24	15.21
71		2.97	0.75	1. 81	1.24	2.36	52.40	38.48
76		1.74	1.48	0.88	0.74	1.48	56.87	36.81
65	Dead	0	0	0	0	0	0	1
71		0	0	0	0	0	0	1
76		0	0	0	0	0	0	1

$$\mathbf{G} = \left[2,\ 2\ ,3\ ,3\ ,4\ ,4\right]^T .$$

In order to move further with applying the model, the generator matrix $\mathbf{Q}$ of the Markov chain must be estimated. We use the results shown in Table 1 due to rough data unavailability. This table, as previously stated, provides 2-year transition probability matrices. From the knowledge of these matrices, it is possible, in some cases, to recover the generator matrix.

Indeed, the maximum likelihood estimator $\hat{\mathbf{Q}}$ of the generator matrix $\mathbf{Q}$ satisfies the relation $\hat{\mathbf{P}} = \exp(a\hat{\mathbf{Q}})$ and it can be obtained as the logarithm matrix,

$$\hat{\mathbf{Q}} = \frac{\log(\hat{\mathbf{P}})}{a}, \tag{25}$$

where $\hat{\mathbf{P}}$ is the estimate of transition probability matrix with data observed periodically at every integer multiple of the real number a. The matrix $\hat{\mathbf{P}} = (\hat{p}_{ij})_{i,j\in E}$ can be estimated according to

$$\hat{p}_{ij} = \frac{K_{ij}}{K_i},$$

where K_{ij} is the number of transitions from state i to state j, and $K_i = \sum_{j=1}^{D} K_{ij}$ is the total number of times households have been allocated to state i. The period of observation is $a = 2$ years in our application. It should be remarked that the maximum likelihood estimator of $\mathbf{Q}$ under this observational scheme is not guaranteed to exist or to be unique (see, e.g., Bladt and Sørensen 2005; Regnault 2012). It was proved in D'Amico and Regnault (2018) that estimator $\hat{\mathbf{Q}}$ exists and is unique whenever the Markov chain is irreducible, and the transition probability matrix has positive eigenvalues. The estimate of transition probability matrices is given in Table 1. Thus, the estimated generator matrix for ages 65, 71 and 76 estimated are given by $\hat{\mathbf{Q}}_{65}$, $\hat{\mathbf{Q}}_{71}$ and $\hat{\mathbf{Q}}_{76}$:

$$\hat{\mathbf{Q}}_{65} = \begin{pmatrix} -0.1157 & 0.0396 & 0.0237 & 0.0054 & 0.0057 & 0.0044 & 0.0367 \\ 0.5195 & -1.2726 & 0.4570 & 0.0689 & 0.0876 & 0.0275 & 0.1122 \\ 0.1952 & 0.3982 & -1.2717 & 0.3801 & 0.0983 & 0.0356 & 0.1638 \\ 0.0174 & 0.00008 & 1.0290 & -1.7539 & 0.4354 & 0.0749 & 0.1976 \\ 0.0608 & 0.1349 & 0.1039 & 0.3188 & -1.1832 & 0.1036 & 0.4548 \\ 0.0883 & 0.0135 & 0.01204 & 0.0453 & 0.000004 & -0.3273 & 0.1702 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{\mathbf{Q}}_{71} = \begin{pmatrix} -0.1913 & 0.0629 & 0.0265 & 0.0110 & 0.0117 & 0.0137 & 0.0676 \\ 0.4540 & -1.2719 & 0.5823 & 0.0258 & 0.0585 & 0.0754 & 0.1276 \\ 0.0940 & 0.4279 & -1.3158 & 0.4342 & 0.0408 & 0.1412 & 0.1764 \\ 0.1069 & 0.0002 & 0.5720 & -1.4850 & 0.6984 & 0.0047 & 0.1145 \\ 0.0707 & 0.0850 & 0.1921 & 0.3470 & -1.3976 & 0.1870 & 0.5019 \\ 0.0382 & 0.0074 & 0.0305 & 0.0176 & 0.0538 & -0.6540 & 0.5065 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{\mathbf{Q}}_{76} = \begin{pmatrix} -0.2799 & 0.1012 & 0.0507 & 0.0144 & 0.0110 & 0.0183 & 0.0865 \\ 0.2782 & -1.3390 & 0.6011 & 0.0628 & 0.1035 & 0.1340 & 0.0872 \\ 0.1279 & 0.4041 & -1.3039 & 0.4069 & 0.0797 & 0.0643 & 0.2218 \\ 0.0185 & 0.0147 & 0.7587 & -1.8751 & 0.6317 & 0.2237 & 0.2774 \\ 0.0540 & 0.0900 & 0.0340 & 0.2974 & -1.1518 & 0.2267 & 0.4483 \\ 0.0212 & 0.0321 & 0.0054 & 0.0140 & 0.0275 & -0.5715 & 0.4718 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

According to Eq. (24), the break-even fee is the value of γ for which the expected discounted benefits equal the premium h_0. The break-even fee corresponding to different guarantee amounts for ages 65, 71 and 76 is shown in Table 2. In the table's first column, we consider a constant variation ϵ, which is added to all the elements of the vector $\mathbf{G}$. Thus, for example when ϵ is 1.12 the $\mathbf{G}$ modifies to

$$\mathbf{G} = \left[3.12,\ 3.12\ ,4.12\ ,4.12\ ,5.12\ ,5.12\right]^T .$$

Results from Table 2 illustrate that with the increase in ϵ, the fee associated with the contract also increases. Furthermore, there is no need for a break-even charge corresponding to each guarantee percentage. Even if the cost is 0 bp, a very low guarantee will not allow the contract's present value to match the premium. Similarly,

Table 2 Break even fee (γ^*) for age 65, 71 and 76

ϵ	γ for Age 65	γ for Age 71	γ for Age 76
1.12	76.00062	0	0
1.13	178.37623	0	0
1.14	314.39606	0	0
1.15	517.65370	0	0
1.16	930.53027	0	0
1.17	0	0	0
1.18	0	0.40650	13.7546
1.19	0	45.97255	39.09075
1.2	0	97.14045	66.06790
1.21	0	155.48348	94.91331
1.22	0	223.34696	125.90514
1.23	0	304.45578	159.38880
1.24	0	405.26541	195.80020
1.25	0	538.52462	235.70084
1.26	0	735.61659	279.83102
1.27	0	0	329.19493
1.28	0	0	385.20313

a very high guarantee cannot make the contract's current value equal to the premium, even if the cost is 3000 bp. Therefore the value of ϵ has been selected such that the value of the break-even fee exists between 0 and 1000 basis points.
Additionally, in Fig. 3, plots are drawn of fee vs. value of shift parameter (ϵ) for different ages to support our results graphically. Increases in ϵ enhance the guaranteed withdrawal amounts while decreasing the death benefit value. The increase in the value of the life benefit overcomes the reduction in the value of the death benefit as ϵ increases. As a result, as the value of ϵ rises, the value of GLWB as a whole rises with it. The effect is more significant for younger policyholders, as the difference in GLWB values corresponding to different ϵ values decreases and eventually vanishes as the policyholder's age at inception rises. Senior policyholders might thus benefit from a larger guarantee value without having to pay a higher charge.

3.3 Sensitivity analysis

In this section, we will study the change in the expected present value of the contract with respect to the different parameters.

1. **Age:** From Fig. 3, it is observed that the initial insured age also impacts the fee, i.e., with the increment of initial age, GLWB value increases, and hence γ^* increases. We examine the influence on the death and living benefits of a change in age at inception y on the GLWB value. As the initial insured age increases, contract length decreases but also increases the level of the health status, resulting in fewer withdrawals with an increment in the withdrawal amount and a lower living benefit

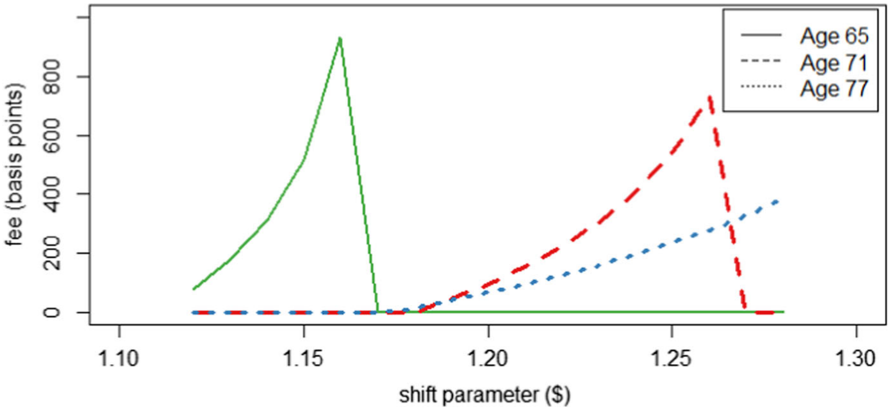

Fig. 3 Break even fee for different age groups

amount. Furthermore, when the initial insured age decreases, the time it takes to discount the death benefit value is reduced, resulting in a more considerable death benefit. An increase in initial insured age increases the overall GLWB value since the loss in living benefit value is lesser than the death benefit value for younger policyholders. However, for senior policyholders, this gap between the death and living benefits keeps decreasing.

2. **Guaranteed withdrawal rate:** In Fig. 4, the plot of EPV vs. ϵ is drawn to show the change in the behavior of the GLWB value with the change in the shift parameter ϵ. An increase in ϵ enhances the overall withdrawal amounts while decreasing the death benefit value. The increase in the value of the life benefit overcomes the reduction in the value of the death benefit as ϵ rises. As a result, as the value of ϵ rises, the value of the GLWB as a whole rises with it. The effect is greater for the senior policyholders, as the GLWB value increases with an increase in age and ϵ. Younger policyholders might thus have some benefit from low charges with a larger guarantee amount.

4 Conclusion and future work

This study proposed and evaluated a new product that is the variable annuity which is accompanied by the health status and hospitalization coverage benefit along with the guaranteed lifelong withdrawal benefit. In this article, a mixed continuous-discrete time model with constant volatility and rate of interest is proposed to calculate the fair value of the product.

People approaching retirement must be insured from outliving their assets and protected from aging problems. This product may provide a more comprehensive solution to all these needs. In this article, expected life and death benefits are evaluated, and the fair fee is determined.

These products can be made more attractive if the health benefits can be modeled better to provide an increase in medical and hospitalization costs to the policyholder.

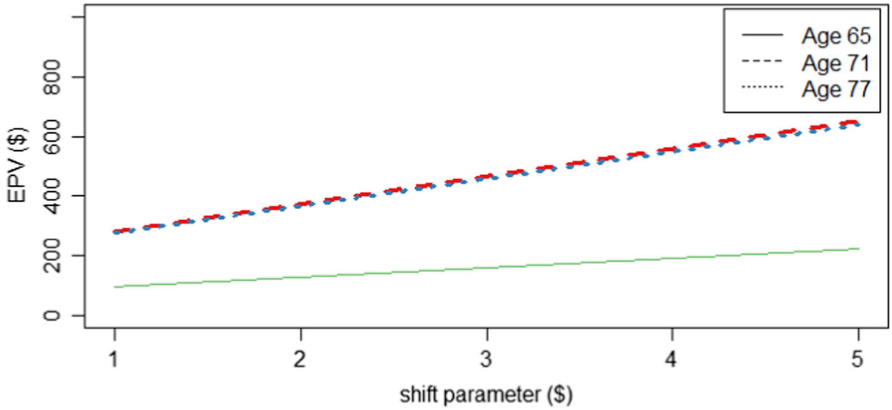

Fig. 4 EPV versus shift parameter (ϵ)

As discussed in the sensitivity analysis section, changes in age and Guaranteed withdrawal amount affect the fair fee (γ^*) of the product effectively. Future works include the application of more sophisticated models of health evolution like semi-Markov processes, and the consideration of path-dependent health-related rewards (see, e.g., D'Amico 2011). Generalizations in the actuarial variables, such as the valuation of surrender benefits using a suitable penalty process, can also be included in future work. The removal of various financial presumptions, the subsequent application of stochastic volatility, and the interest rate as considered for a different financial problem in Bufalo et al. (2022) is also of interest. An interesting challenge is to consider a multi-dimensional scenario where the fund value depends on a basket of correlated assets. In this case, n-dimensional diffusion equations should be considered as the process generating the fund value according to the share invested in each asset.

Acknowledgements We thank the editor and the anonymous reviewers for their insightful comments, which helped in improving the paper. This research work is supported by the Department of Science and Technology, India.

Funding Open access funding provided by Università degli Studi G. D'Annunzio Chieti Pescara within the CRUI-CARE Agreement.

References

Baione, F., Levantesi, S.: A health insurance pricing model based on prevalence rates: application to critical illness insurance. Insur Math Econ **58**(C), 174–184 (2014)

Bladt, M., Sørensen, M.: Statistical inference for discretely observed Markov jump processes. J R Stat Soc Ser B (Stat Methodol) **67**(3), 395–410 (2005)
Bufalo, M., Di Bari, A., Villani, G.: Multi-stage real option evaluation with double barrier under stochastic volatility and interest rate. Ann Finance **18**(2), 247–266 (2022)
Cristea, M., Noja, G.G., Stefea, P., Sala, A.L.: The impact of population aging and public health support on EU labor markets. Int J Environ Res Public Health **17**(4), 1439 (2020)
D'Amato, V., Di Lorenzo, E., Orlando, A., Russolillo, M., Sibillo, M.: Profit participation annuities: a business profitability analysis within a demographic risk sensitive approach. Invest Manag Financial Innov **10**(1), 155–165 (2013)
D'Amico, G.: Nonparametric estimation of a path-dependent health-related quality of life index. Model Assist Stat Appl **6**(2), 99–110 (2011)
D'Amico, G., Regnault, P.: Dynamic measurement of poverty: modeling and estimation. Sankhya B **80**(2), 305–340 (2018)
D'Amico, G., Villani, G.: Valuation of R&D compound option using Markov chain approach. Ann Finance **17**(3), 379–404 (2021)
D'Amico, G., Janssen, J., Manca, R.: Discrete time Markov reward processes a motor car insurance example. Technol Investig **1**(2), 135 (2010)
D'Amico, G., Gismondi, F., Janssen, J., Manca, R.: Discrete time homogeneous markov processes for the study of the basic risk processes. Methodol Comput Appl Probab **17**(4), 983–998 (2015)
De Blasis, R.: The price leadership share: a new measure of price discovery in financial markets. Ann Finance **16**(3), 381–405 (2020)
Dieleman, J.L., Squires, E., Bui, A.L., Campbell, M., Chapin, A., Hamavid, H., Horst, C., Li, Z., Matyasz, T., Reynolds, A., et al.: Factors associated with increases in us health care spending, 1996–2013. JAMA **318**(17), 1668–1678 (2017)
Fard, F.A., Rong, N.: Pricing and managing risks of ruin contingent life annuities under regime switching variance gamma process. Ann Finance **10**(2), 315–332 (2014)
Fred Economic Data of annual interest rate for US market for the previous 50 years. https://fred.stlouisfed.org/
Furman, E., Zitikis, R.: Weighted risk capital allocations. Insur Math Econ **43**(2), 263–269 (2008)
Haberman, S., Pitacco, E.: Actuarial Models for Disability Insurance. Abingdon: Routledge (2018)
Hsieh, M.-H., Wang, J.L., Chiu, Y.-F., Chen, Y.-C.: Valuation of variable long-term care annuities with guaranteed lifetime withdrawal benefits: a variance reduction approach. Insur Math Econ **78**(C), 246–254 (2018)
Jones, B., McClean, S., Stanford, D.: Modelling mortality and discharge of hospitalized stroke patients using a phase-type recovery model. Health Care Manag Sci **22**(4), 570–588 (2019)
Le, H.V., Tsatsomeros, M.: Matrix analysis for continuous-time Markov chains. Special Matrices **10**(1), 219–233 (2021)
Levantesi, S., Nigri, A., Piscopo, G.: Longevity risk management through machine learning: state of the art. Insur Mark Comp **11**(1), 11–20 (2020)
Manton, K.G., Corder, L.S., Stallard, E.: Estimates of change in chronic disability and institutional incidence and prevalence rates in the us elderly population from the 1982, 1984, and 1989 national long term care survey. J Gerontol **48**(4), 153–166 (1993)
Moosavi, S., Payandeh, A.T.: Valuation of variable long-term care annuities with guaranteed lifetime withdrawal and limited hospitalization coverage benefits and two investment funds. Soft Comput **26**(7), 3227–3235 (2021)
Piscopo, G., Haberman, S.: The valuation of guaranteed lifelong withdrawal benefit options in variable annuity contracts and the impact of mortality risk. N Am Actuar J **15**(1), 59–76 (2011)
Pitacco, E.: Actuarial models for pricing disability benefits: towards a unifying approach. Insur Math Econ **16**(1), 39–62 (1995)
Pritchard, D.: Modeling disability in long-term care insurance. N Am Actuar J **10**(4), 48–75 (2006)
Regnault, P.: Entropy estimation for m/m/1 queueing systems. In: AIP Conference Proceedings 31st, vol. 1443, pp. 330–337. American Institute of Physics (2012)
Rosenberg, M.A., Farrell, P.M.: Predictive modeling of costs for a chronic disease with acute high-cost episodes. N Am Actuar J **12**(1), 1–19 (2008)
Sadek, A., Limnios, N.: Nonparametric estimation of reliability and survival function for continuous-time finite Markov processes. J Stat Plan Inference **133**(1), 1–21 (2005)

Tessera, A.: Probabilistic models for medical insurance claims. Appl Stoch Model Bus Ind **23**(4), 307–317 (2007)
Yang, S.-Y., Wang, C.-W., Huang, H.-C.: The valuation of lifetime health insurance policies with limited coverage. J Risk Insur **83**(3), 777–800 (2016)