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RESEARCH ARTICLE

Generalized volatility-stabilized processes

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Abstract We consider systems of interacting diffusion processes which generalize the volatility-stabilized market models introduced in Fernholz and Karatzas (Ann Finance 1(2):149–177, 2005). We show how to construct a weak solution of the underlying system of stochastic differential equations. In particular, we express the solution in terms of time changed squared-Bessel processes, and discuss sufficient conditions under which one can show that this solution is unique in distribution (respectively, does not explode). Sufficient conditions for the existence of a strong solution are also provided. Moreover, we discuss the significance of these processes in the context of arbitrage relative to the market portfolio within the framework of Stochastic Portfolio Theory.

Keywords Stochastic differential equations · Time-change · Stochastic portfolio theory · Arbitrage

JEL Classification G10

1 Introduction

Let us consider a vector process $X(t) = \big(X_1(t), \ldots, X_n(t)\big),\ t \in [0, \infty)$ with values in the state space $(0, \infty)^n$, that solves the following system of stochastic differential equations

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$$\mathrm{d}\big(\log X_i(t)\big) = \frac{\alpha_i}{2\big(\mu_i(t)\big)^{2\beta}}\Big[K\big(X(t)\big)\Big]^2 \mathrm{d}t + \frac{\sigma}{\big(\mu_i(t)\big)^{\beta}} K\big(X(t)\big)\mathrm{d}W_i(t),$$
$$X_i(0) = x_i > 0\,, \qquad i = 1, \ldots, n. \tag{1}$$

Here $\alpha_i \geq 0, \sigma > 0, \beta > 0$ are given real numbers, $\mu(\cdot) = \big(\mu_1(\cdot), \ldots, \mu_n(\cdot)\big)$ is the vector of market weights

$$\mu_i(t) = \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}\,, \qquad i = 1, \ldots, n, \tag{2}$$

the given function $K(\cdot) : (0, \infty)^n \to (0, \infty)$ is measurable, and $W_1(\cdot), \ldots, W_n(\cdot)$ are independent Brownian motions. Sufficient conditions on $K(\cdot)$ so that the system in (1) has a weak solution which is unique in distribution (and does not explode in finite time) will be provided in Sect. 3 below. Moreover, sufficient conditions for path-wise uniqueness and the existence of a strong solution for the system of equations (1) are stated in Sect. 4. Introducing the function $\mathcal{T}(\cdot) : (0, \infty)^n \to (0, \infty)$ given by

$$\mathcal{T}(x) := \Big(\sum_{i=1}^{n} x_i\Big)^{\beta} K(x)\,, \qquad x \in (0, \infty)^n, \tag{3}$$

we shall seek a solution to the equivalent system of stochastic differential equations

$$\mathrm{d}X_i(t) = \frac{\alpha_i + \sigma^2}{2}\big[X_i(t)\big]^{1-2\beta}\Big[\mathcal{T}\big(X(t)\big)\Big]^2 \mathrm{d}t \;+\; \sigma\,\big[X_i(t)\big]^{1-\beta}\mathcal{T}\big(X(t)\big)\,\mathrm{d}W_i(t) \tag{4}$$

for $i = 1, \ldots, n$, with state space $(0, \infty)^n$ and independent Brownian motions $W_1(\cdot), \ldots, W_n(\cdot)$.

Notice two special cases: first, if $K(\cdot) \equiv 1$ and if we allow $\beta = 0$, then the system of equations (1) corresponds to the setting, where

$$X_i(t) = x_i\, \mathrm{e}^{(\alpha/2)t + \sigma W_i(t)}, \qquad i = 1, \ldots, n$$

are independent Geometric Brownian motions; secondly, the case of $K(\cdot) \equiv 1$ and $\beta = 1/2$ corresponds to the volatility-stabilized market models, introduced and studied by Fernholz and Karatzas (2005) and studied in further detail by Goia (2009) and Pal (2011).

If $K(\cdot) \equiv 1$ (or any other positive real constant) and $\beta > 0$ is arbitrary, it is possible to use the theory of degenerate differential equations developed by Bass and Perkins (2002) and show that the system of equations (1) in this case has a weak solution, unique in the sense of the probability distribution.

With more general (possibly discontinuous) drift and volatility coefficients, the system in (1) fails to satisfy the conditions required in Bass and Perkins (2002). However, as we will discuss in the following sections (especially in Sects. 2 and 3), it is still possible to construct a weak solution from first principles, and express it in terms of time-changed squared-Bessel processes. We shall describe this construction below and argue that under certain assumptions (see Sect. 3) this weak solution is unique in the sense of the probability distribution and does not explode in finite time.

Sufficient assumptions on $K(\cdot)$ so that the system in (1) has a weak solution which is unique in distribution are provided in Sect. 3, and are relatively weak, in the sense that the coefficients of the system in (1) for these choices of $K(\cdot)$ would not satisfy the classical sufficient conditions for existence and uniqueness of general systems of stochastic differential equations stated in most well-known theorems. Namely, the function $K(\cdot)$ is not even assumed to be continuous; thus, the coefficients of the systems in (1) need not be continuous, or bounded, and therefore results of many classical theorems would not apply. For instance, the foundational results of Itô require Lipschitz continuity of coefficients (see Theorem 2.9 in Karatzas and Shreve 1991), Skorokhod's theorem also requires continuous, bounded coefficients (see Theorem 23.5 in Rogers and Williams 2000), so does Stroock and Varadhan (see Theorem 4.22 in Karatzas and Shreve 1991), whereas Krylov (1969) does not require continuity but does assume bounded coefficients.

The remainder of this paper is organized as follows. In Sect. 2, a weak solution to the system in (1) is constructed following the steps first of Analysis and consequently of Synthesis. Under certain conditions on the function $K(\cdot)$, this approach also allows to argue that the constructed solution is unique in distribution (respectively, does not explode in finite time). Section 3 discusses these conditions on the function $K(\cdot)$ that are sufficient for the existence of a weak solution which is unique in distribution (respectively, for the existence of a non-exploding solution), whereas Sect. 4 focuses on conditions on the function $K(\cdot)$ that lead to pathwise uniqueness, and hence to the existence of a strong solution, for the system in (1). Section 5 reviews briefly the basic concepts of Stochastic Portfolio Theory and discusses arbitrage opportunities in the context of a financial market, where the underlying model follows the system in (1).

2 Construction of a weak solution

In this section, we shall show that it is possible to construct a weak solution of the system in (1) using appropriately scaled and time-changed squared-Bessel processes. We shall also argue that under certain conditions the solution is unique in distribution and does not explode in finite time. Indeed, whether these conditions are satisfied depends on the particular choice of the function $K(\cdot)$, which will be discussed in detail in Sect. 3.

2.1 Analysis

Suppose we have constructed a weak solution of the system (4); in other words, suppose that on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbf{F} = \{\mathcal{F}(t)\}_{0\le t<\infty}$ we have constructed independent Brownian motions $\big(W_1(\cdot), \ldots, W_n(\cdot)\big)$ and continuous, strictly positive and adapted processes $\big(X_1(\cdot), \ldots, X_n(\cdot)\big)$, such that the integral version of

(4) is satisfied, namely, for $0 \le t < \infty,\ i = 1, \dots n$,

$$X_i(t) \ = \ x_i + \frac{\alpha_i + \sigma^2}{2} \int_0^t \frac{\left[\mathcal{T}\big(X(s)\big)\right]^2}{\big(X_i(s)\big)^{2\beta-1}}\, \mathrm{d}s + \sigma \int_0^t \frac{\mathcal{T}\big(X(s)\big)}{\big(X_i(s)\big)^{\beta-1}}\, \mathrm{d}W_i(s).$$

Consider the continuous, strictly increasing process $A(\cdot)$ defined as follows

$$A(t) \triangleq \int_0^t \left[\mathcal{T}\big(X(s)\big)\right]^2 \mathrm{d}s, \qquad 0 \le t < \infty. \tag{5}$$

This process $A(\cdot)$ is clearly adapted to the filtration $\mathbf{F}^X = \big\{\mathcal{F}^X(t)\big\}_{0 \le t < \infty}$, where

$$\mathcal{F}^X(t) \triangleq \sigma\big(X_i(s) :\ 0 \le s \le t,\ i = 1, \dots, n\big), \ \ 0 \le t < \infty, \quad \text{and}$$
$$\mathcal{F}^X(\infty) \triangleq \sigma\Big(\bigcup_{0 \le t < \infty} \mathcal{F}^X(t)\Big)$$

We have $A(0) = 0$, let us denote $A(\infty) := \lim_{t \to \infty} A(t)$, and assume that

$$A(t) < \infty \ \text{ a.s., } \ \text{for } t \in (0, \infty). \tag{6}$$

Sufficient conditions for (6) to be satisfied are discussed in Sect. 3. Let us also denote by

$$\Upsilon(\theta) \triangleq \inf\big\{t \ge 0 :\ A(t) > \theta\big\}\,, \qquad 0 \le \theta < \infty \tag{7}$$

the inverse of this strictly increasing process, with the standard convention that $\inf \varnothing = \infty$; thus, if $A(\infty) < \infty$, then $\Upsilon(\theta) = \infty$ for all $\theta \ge A(\infty)$. We note that each $\Upsilon(\theta)$ is an $\mathbf{F}^X$-stopping time, so that

$$\mathbf{H} = \big\{\mathcal{H}(\theta)\big\}_{0 \le \theta < A(\infty)}, \ \text{ with } \ \mathcal{H}(\theta) \triangleq \mathcal{F}^X\big(\Upsilon(\theta)\big),\ 0 \le \theta < A(\infty) \tag{8}$$

defines another filtration on this space. Note also that, if we define

$$\begin{aligned} N_i(\theta) &\triangleq X_i\big(\Upsilon(\theta)\big), \qquad 0 \le \theta < A(\infty)\,,\ i = 1, \dots, n \\ N(\theta) &= \big(N_1(\theta), \dots, N_n(\theta)\big) = X\big(\Upsilon(\theta)\big), \quad 0 \le \theta < A(\infty) \end{aligned} \tag{9}$$

and

$$G(\theta) \triangleq \left[\mathcal{T}\big(N(\theta)\big)\right]^2, \qquad 0 \le \theta < A(\infty),$$

we have by virtue of $\Upsilon\big(A(t)\big) = t$, respectively, $A\big(\Upsilon(\theta)\big) = \theta \wedge A(\infty)$, the representation $A(\cdot) = \int_0^{\cdot} G\big(A(t)\big)\,\mathrm{d}t$, as well as

$$\Upsilon(\theta) = \int_0^{\theta} \frac{1}{A'\big(\Upsilon(\xi)\big)}\,\mathrm{d}\xi = \int_0^{\theta} \frac{1}{\Big[\mathcal{T}\Big(X\big(\Upsilon(\xi)\big)\Big)\Big]^2}\,\mathrm{d}\xi,$$

hence we obtain

$$\Upsilon(\theta) = \int_0^{\theta} \frac{1}{\Big[\mathcal{T}\big(N(\xi)\big)\Big]^2}\,\mathrm{d}\xi = \int_0^{\theta} \frac{1}{G(\xi)}\,\mathrm{d}\xi, \tag{10}$$

for $0 \le \theta < A(\infty)$. In particular, with $\mathbf{F}^N = \big\{\mathcal{F}^N(\theta)\big\}_{0 \le \theta < A(\infty)}$ where

$$\mathcal{F}^N(\theta) \triangleq \sigma\big(N_j(\xi) :\ 0 \le \xi \le \theta,\ j = 1, \ldots, n\big), \qquad 0 \le \theta < A(\infty),$$

we see that the processes $G(\cdot)$, $\Upsilon(\cdot)$ are $\mathbf{F}^N$-adapted.

Consider now for $i = 1, \ldots, n$ and $0 \le \theta < A(\infty)$ the continuous local martingales

$$V_i(\theta) \triangleq \int_0^{\Upsilon(\theta)} \sqrt{A'(t)}\,\mathrm{d}W_i(t) = \int_0^{\Upsilon(\theta)} \mathcal{T}\big(X(t)\big)\,\mathrm{d}W_i(t) \tag{11}$$

of the filtration $\mathbf{H}$ in (8). They satisfy, for $0 \le \theta < A(\infty)$

$$\langle V_i, V_j\rangle(\theta) = \delta_{ij} \int_0^{\Upsilon(\theta)} A'(t)\,\mathrm{d}t = \delta_{ij} A\big(\Upsilon(\theta)\big) = \delta_{ij}\theta,$$

therefore $V_1(\cdot), \ldots, V_n(\cdot)$ are independent stopped Brownian motions by the P. Lévy theorem (see for instance Theorem 3.16 in Karatzas and Shreve 1991) of the filtration $\mathbf{H}$ on the interval $\theta \in [0, A(\infty))$. Note, that in order to extend the definition of the processes $V_1(\cdot), \ldots, V_n(\cdot)$ of (11) on the interval $\theta \in [0, \infty)$ one would need to introduce a set of n independent Brownian motions that are independent of $(W_1(\cdot), \ldots, W_n(\cdot))$ on a possibly extended probability space. However, this is not necessary since we can construct the solution only considering $\theta \in [0, A(\infty))$ below.

In terms of the processes $V_1(\cdot), \ldots, V_n(\cdot)$ of (11), and in conjunction with the consequence

$$X_i(t) = N_i\big(A(t)\big), \qquad 0 \le t < \infty,\ i = 1, \ldots, n \tag{12}$$

of (9), we may rewrite the system of equations (4) as

$$\mathrm{d}X_i(t) = \frac{\alpha_i + \sigma^2}{2\big(X_i(t)\big)^{2\beta-1}}\,\mathrm{d}A(t) + \frac{\sigma}{\big(X_i(t)\big)^{\beta-1}}\,\mathrm{d}V_i\big(A(t)\big)$$
$$= \frac{\alpha_i + \sigma^2}{2\Big(N_i\big(A(t)\big)\Big)^{2\beta-1}}\,\mathrm{d}A(t) + \frac{\sigma}{\Big(N_i\big(A(t)\big)\Big)^{\beta-1}}\,\mathrm{d}V_i\big(A(t)\big),$$

or equivalently as

$$N_i\big(A(t)\big) = x_i + \frac{\alpha_i + \sigma^2}{2}\int_0^{A(t)} \frac{1}{\big(N_i(\xi)\big)^{2\beta-1}}\mathrm{d}\xi + \sigma \int_0^{A(t)} \frac{1}{\big(N_i(\xi)\big)^{\beta-1}}\mathrm{d}V_i(\xi). \tag{13}$$

This leads us to the system of stochastic differential equations

$$\begin{aligned} \mathrm{d}N_i(\theta) &= \frac{\alpha_i + \sigma^2}{2}\big(N_i(\theta)\big)^{1-2\beta}\mathrm{d}\theta + \sigma\big(N_i(\theta)\big)^{1-\beta}\mathrm{d}V_i(\theta),\ 0 \le \theta < A(\infty) \\ N_i(0) &= x_i \in (0,\infty)\ , \quad i = 1,\dots,n \end{aligned} \tag{14}$$

for the processes of (9).

Next, we define

$$Z_i(\theta) \triangleq \frac{1}{(\beta\sigma)^2}\big(N_i(\theta)\big)^{2\beta}\ , \qquad 0 \le \theta < A(\infty), \tag{15}$$

and note from (14) that this process satisfies the stochastic differential equation

$$\begin{aligned} \mathrm{d}Z_i(\theta) &= m_i\ \mathrm{d}\theta + 2\sqrt{Z_i(\theta)}\,\mathrm{d}V_i(\theta)\ , \qquad 0 \le \theta < A(\infty) \\ Z_i(0) &= \frac{1}{(\beta\sigma)^2}x_i^{2\beta} =: z_i > 0 \end{aligned} \tag{16}$$

for a squared-Bessel process in "dimension" $m_i \triangleq 2 + \alpha_i/(\beta\sigma^2) \ge 2$, for each $i = 1,\dots,n$.

It is well known that the squared-Bessel SDE of (16) with dimension $m_i \ge 2$ admits a pathwise unique, strong and strictly positive solution (see, for instance Revuz and Yor 1999, p. 439); in other words, we have

$$\mathcal{F}_i^N(\theta) = \mathcal{F}_i^Z(\theta) = \mathcal{F}_i^V(\theta)\ , \qquad 0 \le \theta < A(\infty)\ ,\ i = 1,\dots,n \tag{17}$$

where we have defined the filtrations $\mathcal{F}_i^N(\theta) \triangleq \sigma\big(N_i(\xi)\ :\ 0 \le \xi \le \theta\big)$, $\mathcal{F}_i^Z(\theta) \triangleq \sigma\big(Z_i(\xi)\ :\ 0 \le \xi \le \theta\big)$, and $\mathcal{F}_i^V(\theta) \triangleq \sigma\big(V_i(\xi)\ :\ 0 \le \xi \le \theta\big)$ for each $i = 1,\dots,n$ and every $0 \le \theta < A(\infty)$. Since the processes $V_1(\cdot),\dots,V_n(\cdot)$ are independent, (17) implies that the squared-Bessel processes $Z_1(\cdot), \ldots, Z_n(\cdot)$ of (15) are also independent; and thus so are the processes $N_1(\cdot), \ldots, N_n(\cdot)$ of (14).

It follows also from (10), (12) and (15) that the inverse of the time-change $A(\cdot)$ of (5) is given for $0 \le \theta < A(\infty)$ as

$$\Upsilon(\theta) = \inf\{t \ge 0;\ A(t) > \theta\} = \int_0^\theta \Big[\mathcal{T}\big(N(\xi)\big)\Big]^{-2} \mathrm{d}\xi$$
$$= \int_0^\theta \Big[\mathcal{T}\Big((\beta\sigma)^{\frac{1}{\beta}}\big(Z_1(\xi)\big)^{\frac{1}{2\beta}}, \ldots, (\beta\sigma)^{\frac{1}{\beta}}\big(Z_n(\xi)\big)^{\frac{1}{2\beta}}\Big)\Big]^{-2} \mathrm{d}\xi. \tag{18}$$

Now it is clear, recalling (12) and (15) once more, that the processes

$$X_i(t) = N_i\big(A(t)\big) = (\beta\sigma)^{\frac{1}{\beta}}\Big(Z_i\big(A(t)\big)\Big)^{\frac{1}{2\beta}}, \quad 0 \le t < \infty,\ i = 1, \ldots, n \tag{19}$$

are all $\mathcal{F}^Z\big(A(\infty)\big)$-measurable, since the process $A(\cdot)$ is the inverse of the $\mathbf{F}^Z$-adapted process $\Upsilon(\cdot)$ in (18).

In conclusion, we see that, if (6) is satisfied and if the vector processes $X(\cdot)$ and $W(\cdot)$ are parts of a weak solution of the equation (1) or (4), then $X(.)$ is necessarily of the form (19), expressible in terms of some appropriate independent squared-Bessel processes $Z_1(.), \ldots, Z_n(.)$ as in (16), in dimensions $m_1, \ldots, m_n$, respectively. In particular, since the paths of $(X_1(\cdot), \ldots, X_n(\cdot))$ are determined uniquely from the paths of $(Z_1(\cdot), \ldots, Z_n(\cdot))$, the joint distributions of $(X_1(\cdot), \ldots, X_n(\cdot))$ are determined uniquely. In other words, uniqueness in distribution holds for the system of equations (1), as well as for the system of equations (4).

Remark For a specific choice of the function $K(\cdot)$, and the corresponding function $\mathcal{T}(\cdot)$ as in (3), one can use the representation in (18) and the properties of squared Bessel processes to decide whether this choice of $K(\cdot)$ implies $\lim_{\theta\to\infty}\Upsilon(\theta) = \infty$ a.s. (thus (6) also holds). An example of a sufficient condition on $K(\cdot)$ for this to be satisfied is provided in Sect. 3.

2.2 Synthesis

Let us follow now this same thread in reverse, in an effort actually to construct a weak solution to the system of (4). On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}), \mathbf{F} = \{\mathcal{F}(t)\}_{0\le t<\infty}$ rich enough to carry n independent standard Brownian motions $V_1(\cdot), \ldots, V_n(\cdot)$, we construct the squared-Bessel processes described by stochastic equations of the form

$$\mathrm{d}Z_i(\theta) = m_i\ \mathrm{d}\theta + 2\sqrt{Z_i(\theta)}\,\mathrm{d}V_i(\theta)\,, \qquad Z_i(0) = \frac{1}{(\beta\sigma)^2}x_i^{2\beta} > 0$$

with $m_i = 2 + \alpha_i/(\beta\sigma^2) \geq 2$ for $i = 1, \ldots, n$ as in (16). These equations admit pathwise unique, strong and strictly positive solutions, so

$$\mathcal{F}_i^Z(\theta) = \mathcal{F}_i^V(\theta)\,, \qquad 0 \leq \theta < \infty\,,\ i = 1, \ldots, n,$$

where $\mathcal{F}_i^Z(\theta) \triangleq \sigma\big(Z_i(\xi):\ 0 \leq \xi \leq \theta\big)$ and $\mathcal{F}_i^V(\theta) \triangleq \sigma\big(V_i(\xi):\ 0 \leq \xi \leq \theta\big)$. Let us also denote $\mathbf{F}^Z = \big\{\mathcal{F}^Z(\theta)\big\}_{0 \leq \theta < \infty}$, where

$$\mathcal{F}^Z(\theta) \triangleq \sigma\big(Z_i(\xi):\ 0 \leq \xi \leq \theta,\ i = 1, \ldots, n\big)\,, \qquad 0 \leq \theta < \infty.$$

In terms of the squared-Bessel processes $Z(\cdot) = \big(Z_1(\cdot), \ldots, Z_n(\cdot)\big)$ and by analogy with (18), we define then the continuous, strictly increasing and $\mathbf{F}^Z$-adapted time change

$$\Upsilon(\theta) \triangleq \int_0^\theta \Big[\mathcal{T}\Big((\beta\sigma)^{\frac{1}{\beta}}\big(Z_1(\xi)\big)^{\frac{1}{2\beta}}, \ldots, (\beta\sigma)^{\frac{1}{\beta}}\big(Z_n(\xi)\big)^{\frac{1}{2\beta}}\Big)\Big]^{-2}\, \mathrm{d}\xi \tag{20}$$

for $0 \leq \theta < \infty$, as in (18). The function $\mathcal{T}(\cdot)$ is defined in (3). Obviously we have $\Upsilon(0) = 0$ a.s.

Let us now assume that the process $\Upsilon(\cdot)$ satisfies the following property

$$(NE) \qquad \lim_{\theta\to\infty} \Upsilon(\theta) = \infty \ \text{ a.s.} \tag{21}$$

Next, we define the process $A(\cdot)$ as the inverse of $\Upsilon(\cdot)$, that is

$$A(t) \triangleq \inf\big\{\theta \geq 0:\ \Upsilon(\theta) > t\big\}\,, \qquad 0 \leq t < \infty,$$

and denote $A(\infty) := \lim_{t\to\infty} A(t) = \inf\big\{\theta \geq 0:\ \Upsilon(\theta) = \infty\big\} \in (0, \infty]$. Notice that the process $A(\cdot)$ is strictly increasing, continuous and satisfies $A(0) = 0$ and $A(t) < \infty$, a.s for all $t \in (0, \infty)$. Note that, in fact, we have $A(t) < A(\infty)$, a.s for all $t \in (0, \infty)$.

Moreover, each $A(t)$ is a stopping time of the filtration $\mathbf{F}^Z$, therefore

$$\mathbf{G} = \big\{\mathcal{G}(t)\big\}_{0 \leq t < \infty}\,, \qquad \text{where } \mathcal{G}(t) \triangleq \mathcal{F}^Z\big(A(t)\big)\,,\ 0 \leq t < \infty \tag{22}$$

is also a filtration. The processes

$$N_i(\theta) \triangleq (\beta\sigma)^{\frac{1}{\beta}}\big(Z_i(\theta)\big)^{\frac{1}{2\beta}}\,, \qquad 0 \leq \theta < A(\infty) \tag{23}$$

$$X_i(t) \triangleq N_i\big(A(t)\big)\,, \qquad 0 \leq t < \infty \tag{24}$$

defined for $i = 1, \ldots, n$ according to (15) and (12), are respectively $\mathbf{F}^Z$-adapted and $\mathbf{G}$-adapted. Furthermore, $X(\cdot) = \big(X_1(\cdot), \ldots, X_n(\cdot)\big)$ is $\mathcal{F}^Z(\infty)$-measurable since the process $A(\cdot)$ is the inverse of the $\mathbf{F}^Z$-adapted process $\Upsilon(\cdot)$. This means that the paths of $X(\cdot)$ are determined uniquely from those of $Z(\cdot)$.

Note furthermore, that we have

$$A(t) = \int_0^t \frac{1}{\Upsilon'\big(A(s)\big)}\, \mathrm{d}s = \int_0^t \Big[\mathcal{T}\Big(N\big(A(s)\big)\Big)\Big]^2 \mathrm{d}s = \int_0^t \Big[\mathcal{T}\big(X(s)\big)\Big]^2 \mathrm{d}s \tag{25}$$

in accordance with (5); this means that the process $A(\cdot)$ is adapted to the filtration $\mathbf{F}^X = \big\{\mathcal{F}^X(t)\big\}_{0\le t<\infty}$, where

$$\mathcal{F}^X(t) \triangleq \sigma\big(X_i(s) :\ 0 \le s \le t,\ i = 1, \dots, n\big)\,, \qquad 0 \le t < \infty.$$

The processes $N_i(\cdot)$ of (23) are themselves independent one-dimensional diffusions with state-space $I = (0, \infty)$ and dynamics

$$\begin{aligned} \mathrm{d}N_i(\theta) &= \frac{\alpha_i + \sigma^2}{2}\big(N_i(\theta)\big)^{1-2\beta}\,\mathrm{d}\theta + \sigma\big(N_i(\theta)\big)^{1-\beta}\,\mathrm{d}V_i(\theta)\,, \quad 0 \le \theta < A(\infty) \\ N_i(0) &= x_i > 0\,, \qquad i = 1, \dots, n \end{aligned}$$

as in (14). Hence, for the processes $X_i(\cdot)$ defined in (24) we obtain the following equations

$$\begin{aligned} X_i(t) &= N_i\big(A(t)\big) \\ &= x_i + \frac{\alpha_i + \sigma^2}{2}\int_0^t \Big(N_i\big(A(s)\big)\Big)^{1-2\beta} A'(s)\mathrm{d}s + \sigma \int_0^t \big(N_i(\theta)\big)^{1-\beta}\mathrm{d}V_i\big(A(s)\big) \end{aligned} \tag{26}$$

Consider now the continuous local martingales

$$W_i(t) \triangleq \int_0^t \frac{\mathrm{d}V_i\big(A(s)\big)}{\sqrt{A'(s)}} = \int_0^t \frac{\mathrm{d}V_i\big(A(s)\big)}{\mathcal{T}\Big(N\big(A(s)\big)\Big)} = \int_0^{A(t)} \frac{\mathrm{d}V_i(\xi)}{\mathcal{T}\big(N(\xi)\big)}, \quad 0 \le t < \infty$$

of the filtration $\mathbf{G}$ defined in (22), for $i = 1, \dots, n$. Their (cross-)variations are given as

$$\langle W_i, W_j\rangle(t) = \delta_{ij} \int_0^{A(t)} \frac{1}{\Big[\mathcal{T}\big(N(\xi)\big)\Big]^2}\,\mathrm{d}\xi = \delta_{ij}\int_0^{A(t)} \Upsilon'(\xi)\,\mathrm{d}\xi = \delta_{ij} t\,, \quad t \ge 0,$$

thus $W_1(\cdot), \dots, W_n(\cdot)$ are independent Brownian motions. Moreover, in terms of these processes and using the representation in (25), we can write the equations in (26) as

$$X_i(t) = x_i + \frac{\alpha_i + \sigma^2}{2} \int_0^t \frac{\left[\mathcal{T}\Big(N\big(A(s)\big)\Big)\right]^2}{\Big(N_i\big(A(s)\big)\Big)^{2\beta-1}} \mathrm{d}s + \sigma \int_0^t \frac{\mathcal{T}\Big(N\big(A(s)\big)\Big)}{\Big(N_i\big(A(s)\big)\Big)^{\beta-1}} \mathrm{d}W_i(s)$$

$$= x_i + \frac{\alpha_i + \sigma^2}{2} \int_0^t \frac{\left[\mathcal{T}\big(X(s)\big)\right]^2}{\big(X_i(s)\big)^{2\beta-1}} \mathrm{d}s + \sigma \int_0^t \frac{\mathcal{T}\big(X(s)\big)}{\big(X_i(s)\big)^{\beta-1}} \mathrm{d}W_i(s),$$

which is precisely (4). Note also that each

$$W_i(t) = \int_0^t \left[\frac{\big(X_i(s)\big)^{\beta-1}}{\sigma \mathcal{T}\big(X(s)\big)} \mathrm{d}X_i(s) - \frac{\alpha_i + \sigma^2}{2} \frac{\mathcal{T}\big(X(s)\big)}{\big(X_i(s)\big)^{\beta}} \mathrm{d}s \right]$$

is $\mathcal{F}^X(t)$-measurable, so the independent Brownian motions $W_1(\cdot), \ldots, W_n(\cdot)$ are $\mathbf{F}^X$-adapted.

In other words, $\big(\Omega, \mathcal{F}, \mathbb{P}\big), \mathbf{F}^X, \big(X(\cdot), W(\cdot)\big)$ constitutes a weak solution of the system of equations (4), which is equivalent to the system in (1). According to our discussion in the Analysis section, uniqueness in distribution holds for this system assuming that (6) is satisfied, i.e. the property (NE) stated in (21) is satisfied. Let us summarize the results of this section in the following proposition.

Proposition *Assume $K(\cdot) : (0, \infty)^n \to (0, \infty)$ is a measurable function chosen so that the non-explosiveness property (NE) stated in (21) is satisfied. Then there exists a unique in distribution weak solution of the system of equations (1) which does not explode in finite time.*

Remark The non-explosiveness condition (NE) stated in (21) is equivalent to the condition (6), which ensures that the process $A(\cdot)$ does not explode in finite time, and is sufficient to argue that there exists a weak solution which is unique in distribution (respectively, does not explode). We will discuss sufficient conditions for this in more detail in the following section.

Remark It is possible to extend the results of this section also for path-dependent function $K(\cdot)$; one would need to introduce slightly more complicated notation, but the whole construction would still hold and the solution will be unique in distribution, once the appropriate version of condition (NE) is satisfied.

3 Sufficient conditions on $K(\cdot)$ to avoid explosions

In the following proposition, we state conditions on $K(\cdot)$ that are sufficient so that the process $A(\cdot)$ does not explode in finite time. In other words, the stated conditions imply that the time-change process $\Upsilon(\cdot)$ defined in (20), with $\mathcal{T}(\cdot)$ defined in (3), satisfies property (NE) in (21), that is $\Upsilon(\theta) \to \infty$ a.s. as $\theta \to \infty$. The main tool in proving the statement below is finding bounds in terms of integral functionals of one-dimensional squared-Bessel processes (respectively, functionals of one-dimensional Bessel processes), and applying results known for these functionals.

Proposition *Assume $K(\cdot)\ :\ (0,\infty)^n \to (0,\infty)$ is a measurable function and there exists a measurable function $f\ :\ (0,\infty)\to(0,\infty)$ such that for all $x = (x_1,\dots,x_n)\in(0,\infty)^n$*

$$K(x) \le f(||x||_{2\beta}), \quad \text{and} \quad \int_a^\infty [uf^2(u^{1/\beta})]^{-1}\,\mathrm{d}u = \infty, \tag{27}$$

where $a := \sum_{i=1}^n Z_i(0) = 1/(\beta\sigma)^2 \sum_{i=1}^n x_i^{2\beta} > 0$, and we denoted $||x||_{2\beta} := \big(x_1^{2\beta}+\cdots+x_n^{2\beta}\big)^{1/2\beta}$ for any $\beta>0$. Then the time-change process $\Upsilon(\cdot)$ defined in (20) satisfies the non-explosiveness property (NE) in (21).

Proof Recalling the definition of the process $\Upsilon(\cdot)$ in (20), with $\mathcal{T}(\cdot)$ defined in (3), we have

$$\begin{aligned}\Upsilon(\theta) &= \int_0^\theta \Big[\mathcal{T}\Big((\beta\sigma)^{\frac{1}{\beta}}\big(Z_1(\xi)\big)^{\frac{1}{2\beta}},\dots,(\beta\sigma)^{\frac{1}{\beta}}\big(Z_n(\xi)\big)^{\frac{1}{2\beta}}\Big)\Big]^{-2}\,\mathrm{d}\xi\\ &= C_{\beta,\sigma,n}\int_0^\theta \Big[K\Big(\big(Z_1(\xi)\big)^{\frac{1}{2\beta}},\dots,\big(Z_n(\xi)\big)^{\frac{1}{2\beta}}\Big)\Big]^{-2}\Big[\sum_{j=1}^n\big(Z_j(\xi)\big)^{\frac{1}{2\beta}}\Big]^{-2\beta}\,\mathrm{d}\xi\end{aligned}$$

where $C_{\beta,\sigma,n}$ is a scaling constant depending only on β,σ, and n. Noticing the following inequalities

$$\begin{aligned}\Big[\sum_{j=1}^n\big(Z_j(\xi)\big)^{\frac{1}{2\beta}}\Big]^{-2\beta} &\ge \Big(n\max_{1\le j\le n}\Big\{\big(Z_j(\xi)\big)^{\frac{1}{2\beta}}\Big\}\Big)^{-2\beta}\\ &= n^{-2\beta}\Big(\max_{1\le j\le n}\{Z_j(\xi)\}\Big)^{-1} \ge n^{-2\beta}\Big(\sum_{j=1}^n Z_j(\xi)\Big)^{-1},\end{aligned}$$

we obtain from (27)

$$\begin{aligned}\frac{\Upsilon(\theta)}{C_{\beta,\sigma,n}} &\ge \int_0^\theta \Big[f\Big(\big|\big|\big(\big(Z_1(\xi)\big)^{\frac{1}{2\beta}},\dots,\big(Z_n(\xi)\big)^{\frac{1}{2\beta}}\big)\big|\big|_{2\beta}\Big)\Big]^{-2}\Big[\sum_{j=1}^n\big(Z_j(\xi)\big)^{\frac{1}{2\beta}}\Big]^{-2\beta}\mathrm{d}\xi\\ &\ge n^{-2\beta}\int_0^\theta\Big[f\Big(\big|\big|\big(Z_1(\xi),\dots,\big(Z_n(\xi)\big)\big|\big|_1^{\frac{1}{2\beta}}\Big)\Big]^{-2}\Big(\sum_{j=1}^n Z_j(\xi)\Big)^{-1}\mathrm{d}\xi\\ &= n^{-2\beta}\int_0^\theta\Big[R(\xi)\,f\Big(\big(R(\xi)\big)^{\frac{1}{\beta}}\Big)\Big]^{-2}\mathrm{d}\xi.\end{aligned}$$

where we have noted that $R(\cdot) := \sqrt{Z(\cdot)} = \sqrt{Z_1(\cdot) + \cdots + Z_n(\cdot)}$ is a Bessel process in dimension $m = m_1 + \cdots + m_n > 2$ starting from $R(0) = \sqrt{a}$. The claim follows from Theorem 2 in Engelbert and Schmidt (1987), along with the assumption in (27).

Remark It is easy to see that if $K(\cdot)$ is bounded from above, i.e. if there exists a real constant $K_{\max} > 0$ such that $K(x) \leq K_{\max}$ for all $x \in (0, \infty)^n$, then the condition (27) is trivially satisfied. Therefore, if $K(\cdot)$ is bounded, the system of equations in (1) has a weak solution which is unique in the sense of the probability distribution. Notice that $K(\cdot)$ need not be continuous.

4 Pathwise uniqueness and strength

After constructing a weak solution, a natural question arises: is the constructed solution strong? In other words, one would like to know if the processes $X_1(\cdot), \ldots, X_n(\cdot)$ are adapted to the filtration $\mathbf{F}^W = \{\mathcal{F}^W(t)\}_{0 \leq t < \infty}$ of the driving Brownian motion $W(\cdot)$ in (1), where we have denoted

$$\mathcal{F}^W(t) \triangleq \sigma\big(W_i(s) :\ 0 \leq s \leq t,\ i = 1, \ldots, n\big)\,, \qquad 0 \leq t < \infty.$$

In this section, we argue that under certain additional conditions, pathwise uniqueness holds for the system of equations (1) in the state space $(0, \infty)^n$. As a consequence, we obtain strength thanks to the results of Yamada and Watanabe (1971).

We will use the following notation for the (Euclidean) L_2-norm $||\cdot||_2$, resp. the L_1-norm $||\cdot||_1$,

$$||u||_1 \triangleq \sum_{\nu=1}^{n} |u_\nu|\,, \qquad ||u||_2 \triangleq \Big(\sum_{\nu=1}^{n} u_\nu^2\Big)^{1/2}\,, \qquad u \in \mathbb{R}^n.$$

Assume that $K(\cdot)$ is continuous and bounded from above, i.e., assume there exists a constant $K_{\max} > 0$ such that

$$K(x) \leq K_{\max}\,, \quad \forall\, x \in (0, \infty)^n. \tag{28}$$

Then the system of stochastic differential equations in (1) has a non-exploding weak solution which is unique in distribution (according to the results of the previous section), and is equivalent to the system

$$\begin{aligned} \mathrm{d}X_i(t) = {} & \frac{\alpha_i + \sigma^2}{2}\big[X_i(t)\big]^{1-2\beta}\Big(\sum_{\nu=1}^{n} X_\nu(t)\Big)^{2\beta}\Big[K\big(X(t)\big)\Big]^2\,\mathrm{d}t \\ & + \sigma\,\big[X_i(t)\big]^{1-\beta}\Big(\sum_{\nu=1}^{n} X_\nu(t)\Big)^{\beta} K\big(X(t)\big)\,\mathrm{d}W_i(t), \end{aligned} \tag{29}$$

for $i = 1, \ldots, n$, with the state process $X(\cdot) = \big(X_1(\cdot), \ldots, X_n(\cdot)\big)$ taking values in the strictly positive orthant $(0, \infty)^n$. If we define

$$Y_i(t) \triangleq \log X_i(t)\ ,\ 0 \le t < \infty, \quad i = 1, \ldots, n, \tag{30}$$

we can rewrite the system of stochastic differential equations in (29) as

$$\begin{aligned} \mathrm{d}Y_i(t) = &\frac{\alpha_i}{2}\, \mathrm{e}^{-2\beta Y_i(t)} \Big(\sum_{\nu=1}^{n} \mathrm{e}^{Y_\nu(t)}\Big)^{2\beta} \Big[K\Big(\xi\big(Y(t)\big)\Big)\Big]^2\, \mathrm{d}t \\ &+ \sigma\, \mathrm{e}^{-\beta Y_i(t)} \Big(\sum_{\nu=1}^{n} \mathrm{e}^{Y_\nu(t)}\Big)^{\beta} K\Big(\xi\big(Y(t)\big)\Big)\, \mathrm{d}W_i(t), \end{aligned} \tag{31}$$

for $i = 1, \ldots, n$, where we have defined the continuously differentiable (i.e. C^∞) function $\xi(\cdot) : \mathbb{R}^n \mapsto (0, \infty)^n$ as $\xi(y) := (\mathrm{e}^{y_1}, \ldots, \mathrm{e}^{y_n})$, for all $y \in \mathbb{R}^n$, and the state process $Y(\cdot) = \big(Y_1(\cdot), \ldots, Y_n(\cdot)\big)$ takes values in $\mathbb{R}^n$.

In addition to the assumption that $K(\cdot)$ is bounded, assume that $K(\cdot)$ is differentiable in the strictly positive orthant $(0, \infty)^n$, and all of its partial derivatives are locally bounded. Then, for any positive integer k there exists a constant D_k such that

$$||\nabla K\big(\xi(y)\big)||_1 \le D_k,\ \forall\, y \in B_k, \tag{32}$$

where we denoted

$$B_k := \big\{u = (u_1, \ldots, u_n) \in \mathbb{R}^n \mid ||u||_1 \le k\big\},\ k \ge 1. \tag{33}$$

We claim that under assumptions (28) and (32), namely if the function $K(\cdot)$ is bounded and has locally bounded partial derivatives, pathwise uniqueness holds for the system of equations (31) in the state space $\mathbb{R}^n$, thus also for (29) in the strictly positive orthant $(0, \infty)^n$ thanks to the definition in (30).

We shall show that the coefficients in (31) are locally Lipschitz in the state space. First, fix an arbitrary $j \in \{1, \ldots, n\}$ and $p > 0$, and consider a function $g_j^p(\cdot) : \mathbb{R}^n \mapsto (0, \infty)$ defined as follows

$$g_j^p(y) \triangleq \mathrm{e}^{-py_j} \Big(\sum_{\nu=1}^{n} \mathrm{e}^{y_\nu}\Big)^{p}, \quad y \in \mathbb{R}^n. \tag{34}$$

It is easy to see that all partial derivatives of the function $g_j^p(\cdot)$ are bounded on compact sets in $\mathbb{R}^n$. Therefore, for any positive integer k and $u, v \in B_k$, where B_k is defined as in (33), there exist a constant $C_{p,k}$ (which depends only on k and p) such that

$$\big|g_j^p(u) - g_j^p(v)\big| \ \le\ C_{p,k}\, ||u - v||_1\ , \quad \forall\, u, v \in B_k. \tag{35}$$

The constant $C_{p,k}$ can be chosen as $C_{p,k} := p\mathrm{e}^{pk}\big[n^{|p-1|}\mathrm{e}^{k(|p-1|+1)} + (nk)^p\big]$.

The drift vector $b(\cdot) = \{b_i(\cdot)\}_{1\le i\le n}$ and the dispersion matrix $\sigma(\cdot) = \{\sigma_{ij}(\cdot)\}_{1\le i,j\le n}$ in (31) are given by

$$b_i(y) = \frac{\alpha_i}{2}\, g_i^{2\beta}(y)\left[K\big(\xi(y)\big)\right]^2, \qquad s_{ij}(y) = \sigma\, g_i^{\beta}(y)\, K\big(\xi(y)\big)\delta_{ij}\,, \tag{36}$$

respectively, for $y \in \mathbb{R}^n$, $1 \le i, j \le n$, recalling the definition in (34). Thanks to the bounds in (32) and (35), and since for any positive integer k and $p > 0$ we have $\left|g_j^p(y)\right| \le \mathrm{e}^{pk}(n\mathrm{e}^k)^p$ for $y \in B_k$, all the partial derivatives of the functions in (36) are locally bounded, in particular for any $1 \le i, j \le n$, any positive integer k and any $y \in B_k$, we have

$$\left|\frac{\partial}{\partial y_j}\, b_i(y)\right| \le \frac{\alpha_i}{2}\left[C_{2\beta,k}\cdot K_{\max}^2 \;+\; \mathrm{e}^{2\beta}(n\mathrm{e}^k)^{2\beta}\cdot 2K_{\max}\cdot D_k\cdot \mathrm{e}^k\right],$$
$$\left|\frac{\partial}{\partial y_j}\, s_{ii}(y)\right| \le \sigma\left[C_{\beta,k}\cdot K_{\max} \;+\; \mathrm{e}^{\beta}(n\mathrm{e}^k)^{\beta}\cdot D_k\cdot \mathrm{e}^k\right].$$

Therefore, there exists a constant $\widetilde{K}_{k,\alpha,\beta,\sigma,n}$ which depends only on the values of $k, \alpha_{\max}, \beta, \sigma$ and n, such that for any positive integer k and any $u, v \in B_k$

$$||b(u) - b(v)||_2 + ||s(u) - s(v)||_2 \;\le\; \widetilde{K}_{k,\alpha,\beta,\sigma,n}\; ||u - v||_2.$$

In other words, the coefficients in (31) are locally Lipschitz in the state space $\mathbb{R}^n$. Hence, pathwise uniqueness holds for (31), thanks to the Itô theory (see for instance Theorem 5.2.5 in Karatzas and Shreve 1991), which, in conjunction with the existence of a weak solution, implies strength (thanks to the results of Yamada and Watanabe 1971). In conclusion, the system in (1) admits a pathwise unique strong solution under the above stated assumptions on $K(\cdot)$. Let us summarize this result in the following proposition.

Proposition *Assume $K(\cdot) : (0,\infty)^n \to (0,\infty)$ is a continuous and differentiable function such that* (28) *and* (32) *are satisfied (namely, it is bounded and has locally bounded partial derivatives). Then the system in* (1) *admits a pathwise unique strong solution.*

Remark The assumptions on $K(\cdot)$ can be further relaxed. In particular, if $K(\cdot)$ is bounded and locally Lipschitz, then again the coefficients in (31) are locally Lipschitz in the state space $\mathbb{R}^n$, and pathwise uniqueness holds for (31), respectively (1).

5 Applications to stochastic portfolio theory

Let us start this section with a brief overview of the basic concepts of stochastic portfolio theory, introduce some definitions and statements that we will use in the following subsection. For more details we refer the reader to the monograph Fernholz (2002) and to the survey paper Fernholz and Karatzas (2009), as well as the references mentioned there.

5.1 Basic concepts of stochastic portfolio theory

Consider a model $\mathcal{M}$ for a financial market consisting of n stocks with capitalizations $X_1(\cdot)\ldots, X_n(\cdot)$

$$\begin{aligned} \mathrm{d}X_i(t) &= X_i(t)\Big(b_i(t)\,\mathrm{d}t + \sum_{\nu=1}^{d} s_{i\nu}(t)\,\mathrm{d}W_\nu(t)\Big), \\ X_i(0) &= x_i > 0, \qquad i = 1, \ldots, n, \end{aligned} \tag{37}$$

driven by the d-dimensional Brownian motion $W(\cdot) = \big(W_1(\cdot), \ldots, W_d(\cdot)\big)$, with $d \geq n \geq 2$, on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbf{F} = \big\{\mathcal{F}(t)\big\}_{0\leq t<\infty}$ with $\mathcal{F}(0) = \{\emptyset, \Omega\}$. We shall assume that the vector-valued process $X(\cdot) = \big(X_1(\cdot), \ldots, X_n(\cdot)\big)'$ of capitalizations, the vector-valued process $b(\cdot) = \big(b_1(\cdot), \ldots, b_n(\cdot)\big)'$ of rates of return, and the $(n \times d)$-matrix-valued process $s(\cdot) = \big(s_{i\nu}\big)_{1\leq i\leq n, 1\leq \nu\leq d}$ of stock-price volatilities are all $\mathbf{F}$-progressively measurable, where the filtration $\mathbf{F}$ (which represents the "flow of information" in the market), is part of a weak solution to the system of stochastic differential equations in (37) and satisfies the usual conditions of right-continuity and augmentation by $\mathbb{P}$-negligible sets. Note, that it does not necessarily have to be the filtration generated by the Brownian motion itself.

Elementary stochastic calculus allows us to rewrite the system in (37) in the equivalent form

$$\begin{aligned} \mathrm{d}\big(\log X_i(t)\big) &= \gamma_i(t)\,\mathrm{d}t + \sum_{\nu=1}^{d} s_{i\nu}(t)\,\mathrm{d}W_\nu(t), \\ X_i(0) &= x_i > 0, \qquad i = 1, \ldots, n, \end{aligned} \tag{38}$$

where we have introduced

$$\gamma_i(t) := b_i(t) - \frac{1}{2}a_{ii}(t), \qquad a_{ij}(t) := \sum_{\nu=1}^{d} s_{i\nu}(t)s_{j\nu}(t) = \big(s(t)s'(t)\big)_{ij}. \tag{39}$$

Here $a(\cdot) = \big(a_{ij}(\cdot)\big)_{1\leq i,j\leq n}$ is the nonnegative definite matrix-valued covariance process of the stocks in the market, and $\gamma_i(\cdot)$ will further be referred to as the growth rate of the ith stock.

Next, we define a *long-only portfolio* rule $\pi(\cdot) = \big(\pi_1(\cdot), \ldots, \pi_n(\cdot)\big)$, that is, an $\mathbf{F}$-progressively measurable process, with values in the simplex

$$\Delta^n = \big\{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 \geq 0, \ldots, x_n \geq 0 \text{ and } x_1 + \cdots + x_n = 1\big\}. \tag{40}$$

The quantity $\pi_i(t)$ is interpreted as the proportion of wealth invested in the ith stock at time t.

The wealth process $V^{\omega,\pi}(t)$, which corresponds to a portfolio rule $\pi(\cdot)$ and some initial capital $\omega > 0$, satisfies the stochastic differential equation

$$\frac{\mathrm{d}V^{\omega,\pi}(t)}{V^{\omega,\pi}(t)} = \sum_{i}^{n} \pi_i(t) \frac{\mathrm{d}X_i(t)}{X_i(t)} = b_\pi(t)\,\mathrm{d}t + \sum_{\nu=1}^{d} s_{\pi\nu}(t)\,\mathrm{d}W_\nu(t), \quad V^{\omega,\pi}(0) = \omega, \tag{41}$$

where

$$b_\pi(t) := \sum_{i}^{n} \pi_i(t) b_i(t), \quad s_{\pi\nu}(t) := \sum_{i}^{n} \pi_i(t) s_{i\nu}(t), \quad \text{for } \nu = 1, \ldots, d, \tag{42}$$

are, respectively, the rate of return and the volatility coefficients associated with the portfolio $\pi(\cdot)$.

Using elementary stochastic calculus as in (38), we can write the dynamics for the wealth process in the equivalent form

$$\mathrm{d}\big(\log V^{\omega,\pi}(t)\big) = \gamma_\pi(t)\,\mathrm{d}t + \sum_{\nu=1}^{d} s_{\pi\nu}(t)\,\mathrm{d}W_\nu(t), \quad V^{\omega,\pi}(0) = \omega, \tag{43}$$

where

$$\gamma_\pi(t) := \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \gamma_\pi^*(t)$$

is the growth rate of the portfolio rule $\pi(\cdot)$, and

$$\gamma_\pi^*(t) = \frac{1}{2}\Big(\sum_{i=1}^{n} \pi_i(t) a_{ii}(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t) a_{ij} \pi_j(t)\Big) \tag{44}$$

is the *excess growth rate* of the portfolio $\pi(\cdot)$. The excess growth rate is always nonnegative for any long-only portfolio (see Lemma 3.3 in Fernholz and Karatzas 2009, and the alternative expression (45) below). Under certain conditions on the market (see Remark 3.2 in Fernholz and Karatzas 2009), the excess growth rate is strictly positive for portfolios that do not concentrate their holdings in just one stock (that is if $\pi_i(t) > 0$ holds a.s. for all $i = 1, \ldots, n$ and all $t \geq 0$).

Alternatively, the excess growth rate (44) can be written as

$$\gamma_\pi^*(t) = \frac{1}{2}\sum_{i=1}^{n} \pi_i(t)\tau_{ii}^\pi(t), \tag{45}$$

where we have denoted by $\tau_{ij}^\pi(\cdot)$ the individual stocks' covariance rates relative to the portfolio $\pi(\cdot)$,

$$\tau_{ij}^\pi(t) := \sum_{k=1}^{n} \big(s_{ik}(t) - s_{\pi k}(t)\big)\big(s_{jk}(t) - s_{\pi k}(t)\big), \qquad 1 \leq i, j \leq n. \tag{46}$$

It is of key interest in mathematical finance, whether it is possible to outperform a given strategy. The assumption that such outperformance is not possible is common in classical mathematical finance, and one is usually interested in finding what conditions on the underlying model would prevent such "arbitrage". Stochastic portfolio theory, on the contrary, does not rule out arbitrage, and studies the market characteristics that allow for the possibility of outperformance. We say that a portfolio rule $\pi(\cdot)$ is an arbitrage opportunity relative to (equivalently, outperforms) the portfolio rule $\rho(\cdot)$ over the time horizon $[0, T]$, with $T > 0$ a given real number, if

$$\mathbb{P}\big[V^{\omega,\pi}(T) \geq V^{\omega,\rho}(T)\big] = 1 \quad \text{and} \quad \mathbb{P}\big[V^{\omega,\pi}(T) > V^{\omega,\rho}(T)\big] > 0. \tag{47}$$

Moreover, if we have

$$\mathbb{P}\big[V^{\omega,\pi}(T) > V^{\omega,\rho}(T)\big] = 1, \tag{48}$$

we say that $\pi(\cdot)$ is a strong arbitrage opportunity relative to $\rho(\cdot)$ (equivalently, strongly outperforms $\rho(\cdot)$). The notion of relative arbitrage was introduced by Fernholz (2002). Under certain conditions on the market model $\mathcal{M}$, Fernholz and Karatzas (2009) show that the existence of relative arbitrage implies the absence of equivalent martingale measure in the market model $\mathcal{M}$. In the following, we shall use the notation $V^{\pi}(t) := V^{1,\pi}(t)$ whenever we start with initial capital $\omega = 1$.

An important long-only portfolio (and also a natural choice for a reference portfolio) is the market portfolio, which invests in all stocks in proportion to their relative weights

$$\mu_i(t) := \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}, \qquad i = 1, \ldots n. \tag{49}$$

It is obvious from (41) that

$$\frac{\mathrm{d}V^{\omega,\mu}(t)}{V^{\omega,\mu}(t)} = \frac{\mathrm{d}\big(X_1(t) + \cdots + X_n(t)\big)}{X_1(t) + \cdots + X_n(t)},$$

and hence

$$V^{\omega,\mu}(t) = \frac{\omega}{x}\big(X_1(t) + \cdots + X_n(t)\big), \qquad 0 \leq t < \infty$$

where $x = X_1(0) + \cdots + X_n(0)$. Therefore, holding the market portfolio amounts to owning the entire market in proportion to the initial capital.

The excess growth rate

$$\gamma_\mu^*(\cdot) = \frac{1}{2}\sum_{i=1}^{n} \mu_i(\cdot)\tau_{ii}^{\mu}(\cdot)$$

of the market portfolio measures the average relative variance rate of stocks in the market at any given time, as it is the average of relative market capitalization of the individual stocks' relative variance rates $\tau_{ii}^{\mu}(\cdot)$ with respect to the market. If it is bounded away from zero over a period of time, i.e., if there exists a constant $\zeta \in (0, \infty)$ such that

$$\gamma_{\mu}^{*}(t) \geq \zeta, \qquad \forall\, 0 \leq t \leq T$$

holds with probability one, then certain types of portfolios outperform the market portfolio over the fixed time horizon $[0, T]$, with $T \in (0, \infty)$ a given real number, as was shown in Proposition 3.1 in Fernholz and Karatzas (2005). Another way to construct arbitrage opportunities is using the functionally generated portfolios (see Chapter III in Fernholz and Karatzas 2009). In the next subsection, we will provide examples of arbitrage opportunities in a particular financial market, the Generalized Volatility-stabilized market (GVSM) which assumes the dynamics in (1) for stocks' capitalizations.

5.2 Arbitrage opportunities in generalized volatility-stabilized markets

As we have already mentioned, the special case of the system in (1) with $\beta = 1/2$ and $K(\cdot) \equiv 1$ corresponds to the volatility-stabilized market models which were introduced in Fernholz and Karatzas (2005). These markets exhibit one of the features observed in the real-life equity markets, in particular, the fact that small stocks tend to have bigger growth rates and are more volatile than the largest stocks in the markets. Fernholz and Karatzas (2005) discuss arbitrage opportunities that are present in these markets which we will now extend to the more general system in (1).

Let us first consider the case of the system in (1) with $K(\cdot) \equiv 1$ but $\beta > 0$, not necessarily 1/2, that is the following system of stochastic differential equations

$$\mathrm{d}\big(\log X_i(t)\big) = \frac{\alpha_i}{2\big(\mu_i(t)\big)^{2\beta}}\,\mathrm{d}t + \frac{\sigma}{\big(\mu_i(t)\big)^{\beta}}\,\mathrm{d}W_i(t)\,, \qquad i = 1, \ldots, n \tag{50}$$

or equivalently

$$\mathrm{d}X_i(t) = \frac{\alpha_i + \sigma^2}{2}\big[X_i(t)\big]^{1-2\beta}\big[S(t)\big]^{2\beta}\,\mathrm{d}t + \sigma\,\big[X_i(t)\big]^{1-\beta}\big[S(t)\big]^{\beta}\,\mathrm{d}W_i(t)\,,$$

where $\alpha_i \geq 0, \sigma > 0, \beta > 0$ are given constants, $\mu(\cdot) = \big(\mu_1(\cdot), \ldots, \mu_n(\cdot)\big)$ is the vector of market weights

$$\mu_i(t) = \frac{X_i(t)}{S(t)} = \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}\,, \qquad i = 1, \ldots, n$$

and $\big(W_1(\cdot), \ldots, W_n(\cdot)\big)$ is n-dimensional Brownian motion.

5.2.1 Excess growth rate of the market portfolio and the diversity weighted portfolio

Assuming that the dynamics of the processes $X_i(\cdot)$ are described by the system of equations (50), the corresponding growth rates and volatilities are given by

$$\gamma_i(t) = \frac{\alpha_i}{2}\left(\mu_i(t)\right)^{-2\beta}, \qquad s_{i\nu}(t) = \sigma\left(\mu_i(t)\right)^{-\beta}\delta_{i\nu}$$

respectively. The covariance matrix is given by

$$a_{ij}(t) = \left(s(t)s^T(t)\right)_{ij} = \sigma^2\left(\mu_i(t)\right)^{-2\beta}\delta_{ij}. \tag{51}$$

Therefore, we have for this model

$$a_{\mu\mu}(t) := \mu(t)a(t)\mu^T(t) = \sigma^2\sum_{\nu=1}^{n}\left(\mu_\nu(t)\right)^{2-2\beta},$$

as well as

$$\gamma_\mu^*(t) = \frac{1}{2}\Big(\sum_{i=1}^{n}\mu_i(t)a_{ii}(t) - a_{\mu\mu}(t)\Big) = \frac{\sigma^2}{2}\Big(\sum_{i=1}^{n}\left(\mu_i(t)\right)^{1-2\beta} - \sum_{i=1}^{n}\left(\mu_i(t)\right)^{2-2\beta}\Big).$$

Hence, the excess growth rate of the market portfolio in the model (50) is given by

$$\gamma_\mu^*(t) = \frac{\sigma^2}{2}\sum_{i=1}^{n}\left(\mu_i(t)\right)^{1-2\beta}\left(1-\mu_i(t)\right)$$

Let us show that the excess growth rate $\gamma_\mu^*(t)$ of the market portfolio is bounded away from zero, if $1/2 \le \beta < \infty$; indeed, since all market weights are smaller than 1, we have then

$$\gamma_\mu^*(t) \ge \frac{\sigma^2}{2}\sum_{i=1}^{n}1\left(1-\mu_i(t)\right) = \frac{\sigma^2}{2}(n-1) > 0 \tag{52}$$

for $n \ge 2$. Therefore, in this case, the condition (3.2) in Proposition 3.1 in Fernholz and Karatzas (2005) is satisfied with $\Gamma(t) = t\sigma^2\,(n-1)/2$, and the model of (50) admits relative arbitrage opportunities, namely there exist a sufficiently large real number $c > 0$ such that the portfolio rule

$$\pi_i(t) \; := \; \frac{c\mu_i(t) - \mu_i(t)\log\mu_i(t)}{c - \sum_{j=1}^{n}\mu_j(t)\log\mu_j(t)}, \qquad j = 1, \ldots n$$

outperforms the market portfolio at least on the time-horizons $[0, T]$ with $T > 2\log(n)/\left[\sigma^2(n-1)\right]$ (for the proof we refer the reader to Proposition 3.1 in Fernholz and Karatzas 2005).

If $\beta \in (0, 1/2)$, then $\gamma^*_\mu(t)$ can get arbitrarily close to zero whenever $\mu_{(1)} = \max_{i=1,\ldots,n}\{\mu_i\}$ approaches one. Hence, condition (3.2) in Fernholz and Karatzas (2005) is not satisfied in this case.

However, one can construct a simple example of an arbitrage relative to the market portfolio that works for any value of $\beta > 0$, as follows. With $0 < p < 1$ and $p \le 2\beta$, let us consider the so-called *diversity-weighted portfolio*

$$\mu_i^{(p)}(t) := \frac{\big(\mu_i(t)\big)^p}{\sum_{j=1}^n \big(\mu_j(t)\big)^p}, \qquad i = 1, \ldots n. \tag{53}$$

In model (50) we have $a_{ij}(t) = \sigma^2\big(\mu_i(t)\big)^{-2\beta}\delta_{ij}$ for the elements of the variance/covariance matrix as in (51), so the excess growth rate of the portfolio $\mu^{(p)}(\cdot)$ is given by

$$\begin{aligned}
2\gamma^*_{\mu^{(p)}}(t) &= \sum_{i=1}^n \mu_i^{(p)}(t)\big(1-\mu_i^{(p)}(t)\big)a_{ii}(t) \\
&= \sum_{i=1}^n \frac{\big(\mu_i(t)\big)^p}{\sum_{j=1}^n \big(\mu_j(t)\big)^p}\big(1-\mu_i^{(p)}(t)\big)\sigma^2\big(\mu_i(t)\big)^{-2\beta} \\
&\ge \sigma^2 \sum_{i=1}^n \frac{\big(\mu_i(t)\big)^p}{\sum_{j=1}^n \big(\mu_j(t)\big)^p}\big(1-\mu_i^{(p)}(t)\big)\big(\mu_i(t)\big)^{-p} \\
&= \sigma^2 \frac{\sum_{i=1}^n \big(1-\mu_i^{(p)}(t)\big)}{\sum_{j=1}^n \big(\mu_j(t)\big)^p} \;=\; \sigma^2 \frac{n-1}{\sum_{i=1}^n \big(\mu_i(t)\big)^p}
\end{aligned}$$

where the inequality is only valid if $p \le 2\beta$. Since the function $\Delta^n \ni \pi \mapsto \sum_{i=1}^n (\pi_i)^p$ attains its maximum, namely n^{1-p}, over the simplex Δ^n defined in (40), at the point $(1/n, \ldots, 1/n)$, we further have $\sum_{i=1}^n \big(\mu_i(t)\big)^p \le n^{1-p}$, and therefore

$$\gamma^*_{\mu^{(p)}}(t) \;\ge\; \frac{\sigma^2}{2}\frac{n-1}{\sum_{i=1}^n \big(\mu_i(t)\big)^p} \;\ge\; \frac{\sigma^2}{2}\frac{n-1}{n^{1-p}}. \tag{54}$$

If we introduce $\mathfrak{D}(\pi) := \left(\sum_{i=1}^n \pi_i^p\right)^{\frac{1}{p}}$, $\pi \in \Delta^n$, we can derive the following expression

$$\log\left(\frac{V^{\mu^{(p)}}(T)}{V^\mu(T)}\right) = \log\left(\frac{\mathfrak{D}\big(\mu(T)\big)}{\mathfrak{D}\big(\mu(0)\big)}\right) + (1-p)\int_0^T \gamma^*_{\mu^{(p)}}(t)\,\mathrm{d}t\,, \qquad \text{a.s.} \tag{55}$$

for the wealth process $V^{\mu^{(p)}}(\cdot)$ of the diversity-weighted portfolio $\mu_i^{(p)}(\cdot)$ in (53) (see (7.5) in Fernholz and Karatzas 2009). Notice that there is no stochastic integral term on the right hand side of the expression (55); this will allow us to make pathwise comparisons as follows: Since the function $\mathfrak{D}(\pi)$ takes values in $[1, n^{(1-p)/p}]$ for all $\pi \in \Delta^n$, and thanks to the lower bound on $\gamma^*_{\mu^{(p)}}(t)$ in (54) we obtain

$$\log\left(\frac{V^{\mu^{(p)}}(T)}{V^{\mu}(T)}\right) = \log\left(\frac{\mathfrak{D}\big(\mu(T)\big)}{\mathfrak{D}\big(\mu(0)\big)}\right) + (1-p)\int_0^T \gamma^*_{\mu^{(p)}}(t)\,\mathrm{d}t$$
$$\geq (1-p)\left[\frac{(n-1)T\sigma^2}{2n^{1-p}} - \frac{\log(n)}{p}\right] > 0\,, \qquad \text{a.s.} \qquad (56)$$

provided that

$$T > \frac{2}{p\sigma^2} \cdot \frac{\log(n)}{n^p(1-1/n)} =: T^*(\beta, \sigma, n).$$

In other words, the diversity-weighted portfolio of (53) outperforms the market portfolio over sufficiently large time horizons $[0, T]$, namely with $T > T^*(\beta, \sigma, n)$. Notice, that the threshold $T^*(\beta, \sigma, n)$ depends on the choice of parameter β through the requirement $p \leq 2\beta$. If the parameter $\beta \to 0$, we also need to choose $p \leq 2\beta \to 0$, and then the threshold $T^*(\beta, \sigma, n) \to \infty$ (which means one needs to wait longer for the arbitrage). On the other hand, if either the volatility parameter σ or the number of stocks n increases to infinity, then $T^*(\beta, \sigma, n) \to 0$.

5.2.2 Generalized excess growth rate of the market portfolio

We can construct a similar example of an arbitrage that is valid for any value of $\beta > 0$ using the notion of generalized excess growth rate and Proposition 3.8 in Fernholz and Karatzas (2005).

Notice that in model (50) we have $s_{\mu\nu}(t) = \sum_{i=1}^n \mu_i(t)s_{i\nu}(t) = \sigma\big(\mu_\nu(t)\big)^{1-\beta}$ for the quantities of (42), and

$$\tau^{\mu}_{ii}(t) = \sum_{\nu=1}^n \big(s_{i\nu}(t) - s_{\mu\nu}(t)\big)^2 = \sum_{\nu\neq i}\big(s_{\mu\nu}(t)\big)^2 + \big(s_{ii}(t) - s_{\mu i}(t)\big)^2$$
$$= \sigma^2\sum_{\nu\neq i}\big(\mu_\nu(t)\big)^{2(1-\beta)} + \sigma^2\big(\mu_i(t)\big)^{-2\beta}\big(1-\mu_i(t)\big)^2$$
$$= \sigma^2\sum_{\nu=1}^n\big(\mu_\nu(t)\big)^{2(1-\beta)} + \sigma^2\big(\mu_i(t)\big)^{-2\beta}\big(1-2\mu_i(t)\big)$$

for those of (46). Therefore, the generalized excess growth rate

$$\gamma^*_{\mu,p}(t) := \frac{1}{2}\sum_{i=1}^n \big(\mu_i(t)\big)^p \tau^{\mu}_{ii}(t), \qquad 0 < p < 1$$

for this market, introduced in (3.24) of Fernholz and Karatzas (2005), now takes the form

$$\frac{2}{\sigma^2}\gamma^*_{\mu,p}(t) = \sum_{i=1}^{n}\big(\mu_i(t)\big)^p \cdot \sum_{\nu=1}^{n}\big(\mu_\nu(t)\big)^{2(1-\beta)} + \sum_{i=1}^{n}\big(\mu_i(t)\big)^{p-2\beta}\big(1-2\mu_i(t)\big). \tag{57}$$

Assume now that we choose $p \in (0, 1)$ so that $0 < p \leq 2\beta$. Then we also have $2(1-\beta) \leq 2-p$, and since obviously $\mu_i(t) < 1$, we also have $\big(\mu_i(t)\big)^{2(1-\beta)} \geq \big(\mu_i(t)\big)^{2-p}$ for all $i = 1, \dots, n$ and all t. Thus, for the first term on the right-hand side in (57) we have

$$\sum_{i=1}^{n}\big(\mu_i(t)\big)^p \cdot \sum_{\nu=1}^{n}\big(\mu_\nu(t)\big)^{2(1-\beta)} \geq \sum_{i=1}^{n}\big(\mu_i(t)\big)^p \cdot \sum_{i=1}^{n}\big(\mu_i(t)\big)^{2-p} \geq 1, \tag{58}$$

where the last inequality follows from Cauchy–Schwarz, namely

$$1 = \sum_{i=1}^{n}\big(\mu_i(t)\big)^{\frac{p}{2}} \cdot \big(\mu_i(t)\big)^{1-\frac{p}{2}} \leq \left(\sum_{i=1}^{n}\big(\mu_i(t)\big)^p \cdot \sum_{i=1}^{n}\big(\mu_i(t)\big)^{2-p}\right)^{\frac{1}{2}}.$$

For the second term on the right-hand side in (57) and for every fixed t, we need to consider two cases:

First, if all market weights are smaller than $1/2$, i.e., $0 < \mu_i(t) \leq 1/2$ for $i = 1, \dots, n$, then we have $\big(\mu_i(t)\big)^{p-2\beta} \geq (1/2)^{p-2\beta} = 2^{2\beta-p} \geq 1$, therefore also

$$\sum_{i=1}^{n}\big(\mu_i(t)\big)^{p-2\beta}\big(1-2\mu_i(t)\big) \geq \sum_{i=1}^{n} 2^{2\beta-p}\big(1-2\mu_i(t)\big) = 2^{2\beta-p}(n-2) \geq n-2.$$

Secondly, if one of the market weights is bigger than $1/2$, i.e., there exist an integer $1 \leq j \leq n$ such that $1/2 < \mu_j(t) \leq 1$, the remaining market weights must then all be strictly less than $1/2$, i.e., $0 < \mu_i(t) < 1/2$ for $i \neq j$. In this case we have $1 \leq \big(\mu_j(t)\big)^{p-2\beta} \leq 2^{2\beta-p}$, and $\big(\mu_i(t)\big)^{p-2\beta} \geq 2^{2\beta-p}$ for $i \neq j$, and moreover we have $-1 \leq \big(1-2\mu_j(t)\big) < 0$, and $\big(1-2\mu_i(t)\big) > 0$ for $i \neq j$. We obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\big(\mu_i(t)\big)^{p-2\beta}\big(1-2\mu_i(t)\big) \\
&\quad = \big(\mu_j(t)\big)^{p-2\beta}\big(1-2\mu_j(t)\big) + \sum_{i\neq j}\big(\mu_i(t)\big)^{p-2\beta}\big(1-2\mu_i(t)\big) \\
&\quad \geq -2^{2\beta-p} + \sum_{i\neq j} 2^{2\beta-p}\big(1-2\mu_i(t)\big) = 2^{2\beta-p}\big[-1+(n-1)-2(1-\mu_j(t))\big] \\
&\quad = 2^{2\beta-p}(n-4+2\mu_j(t)) > 2^{2\beta-p}(n-3) > n-3\,;
\end{aligned}$$

Thus, in either case, the second term on the right-hand side of (57) satisfies

$$\sum_{i=1}^{n} \big(\mu_i(t)\big)^{p-2\beta}\big(1-2\mu_i(t)\big) > n-3. \tag{59}$$

If we combine (57), (58) and (59) together, still under the assumption that $0 < p \leq 2\beta$, we obtain

$$\frac{2}{\sigma^2}\gamma^*_{\mu,p}(t) > 1+n-3 = n-2, \tag{60}$$

and $\gamma^*_{\mu,p}(t) > 0$ for $n \geq 2$ and all t. Now, Proposition 3.8 in Fernholz and Karatzas (2005) guarantees that, over sufficiently long time-horizons $[0,T]$ (in particular, with $T > \frac{2}{p\sigma^2}\cdot\frac{\log(n)\cdot n^{1-p}}{n-2}$), there exist arbitrages relative to the market portfolio $\mu(\cdot)$. More precisely, it is shown there that the portfolio rule

$$\pi_i(t) = p\,\frac{\big(\mu_i(t)\big)^p}{\sum_{j=1}^{n}\big(\mu_j(t)\big)^p} + (1-p)\mu_i(t), \qquad i=1,\ldots n \tag{61}$$

is a strong arbitrage opportunity relative to the market portfolio $\mu(t)$ in the sense of definition (48).

Notice that the portfolio of (61) is a convex combination, with fixed weights $1-p$ and p, of the market portfolio and of the diversity-weighted portfolio of (53), respectively. Note also that if $\beta \geq 1/2$ one can choose $p \in (0,1)$ arbitrarily, but if $0 < \beta < 1/2$ one needs to choose $p \in (0, 2\beta]$, in order to get the inequality in (58).

5.2.3 Arbitrage in the general model

Let us return to the model of (1), in which the stocks' volatilities are given by

$$s_{i\nu}(t) = \sigma\big(\mu_i(t)\big)^{-\beta} K\big(X(t)\big)\delta_{i\nu},$$

therefore we have

$$s_{\mu\nu}(t) = \sum_{i=1}^{n}\mu_i(t)s_{i\nu}(t) = \sigma\big(\mu_\nu(t)\big)^{1-\beta} K\big(X(t)\big)$$

and the variance relative to the market of the ith stock is

$$\begin{aligned}\tau^{\mu}_{ii}(t) &= \sum_{\nu=1}^{n}\big(s_{i\nu}(t)-s_{\mu\nu}(t)\big)^2 = \sum_{\nu\neq i}\big(s_{\mu\nu}(t)\big)^2 + \big(s_{ii}(t)-s_{\mu i}(t)\big)^2\\ &= \sigma^2\Big[K\big(X(t)\big)\Big]^2\Big[\sum_{\nu=1}^{n}\big(\mu_\nu(t)\big)^{2(1-\beta)} + \big(\mu_i(t)\big)^{-2\beta}\big(1-2\mu_i(t)\big)\Big].\end{aligned} \tag{62}$$

Consider the case of $K(\cdot)$ bounded away from zero, that is, there exists $K_{\min} > 0$ such that $K_{\min} \leq K(u)$ for all $u \in (0, \infty)^n$. Then the excess growth rate of the market portfolio $\mu(\cdot)$ is bounded away from zero as well, namely

$$\gamma_\mu^*(t) \geq \frac{\sigma^2}{2}(n-1)\Big[K\big(\mu(t)\big)\Big]^2 \geq \frac{\sigma^2}{2}(n-1)K_{\min}^2 > 0,$$

whenever $\beta \in [1/2, \infty)$, thanks to (52).

Moreover, for any value of $\beta \in (0, \infty)$ and arbitrary $p \leq 2\beta$, the excess growth rate of the diversity-weighted portfolio, defined in (53), is bounded away from zero. Indeed, using the inequality in (54), we obtain

$$\gamma_{\mu^{(p)}}^*(t) \geq \frac{\sigma^2}{2}\frac{n-1}{n^{1-p}}\Big[K\big(X(t)\big)\Big]^2 \geq \frac{\sigma^2}{2}\frac{n-1}{n^{1-p}}K_{\min}^2 > 0.$$

Therefore, recalling the formula in (55), and the computations in (56), there exist strong arbitrage opportunities relative to the market portfolio over sufficiently large time horizons for any value of $\beta > 0$. For instance, with $p \in (0, \min\{1, 2\beta\})$ and

$$T > \frac{2}{p\sigma^2 K_{\min}^2} \cdot \frac{\log(n) \cdot n^{1-p}}{n-1} =: T^*(\beta, \sigma, n, K_{\min})$$

the diversity-weighted portfolio $\mu^{(p)}(\cdot)$ outperforms the market over $[0, T]$.

If in addition to the assumption that $K(\cdot)$ is bounded away from zero, we assume that $\beta \geq 1/2$, then we obtain from (62) a lower bound on the individual stocks' covariances relative to the market portfolio $\mu(\cdot)$, namely

$$\tau_{ii}^\mu(t) \geq \sigma^2 K_{\min}^2\Big(\frac{1}{\mu_i(t)} - 1\Big).$$

This allows to use the same approach as in Proposition 2 in Section 5 of Banner and Fernholz (2008), and construct a portfolio which is guaranteed to outperform the market portfolio over arbitrarily short time horizon ("short-term arbitrage"). Note, that if $\sigma^2 K_{\min}^2 \geq 1$, then we can use exactly the same construction (and the same portfolio rule) as in Proposition 2 in Section 5 of Banner and Fernholz (2008). If $\sigma^2 K_{\min}^2 < 1$, then only minor adjustments are needed. Hence, if $K(\cdot)$ is bounded and $\beta \geq 1/2$, then short-term arbitrage exists in the model of (1).

Example Let us conclude with a simple example of systems that lead to markets in which both long-term and short-term arbitrage opportunities are present. It is easy to see that if $K(\cdot)$ is chosen to be the reciprocal of the L_p-norm of the market weights [defined in (2)], with $p \geq 1$, then $K(\cdot)$ is bounded on the state space and has locally bounded partial derivatives. Therefore, the corresponding system of stochastic differential equations in (1), which with this choice takes the form

$$\begin{aligned} d\big(\log X_i(t)\big) &= \frac{\alpha_i}{2\big(\mu_i(t)\big)^{2\beta}\,\big|\big|\mu(t)\big|\big|_p^2}\,dt + \frac{\sigma}{\big(\mu_i(t)\big)^{\beta}\,\big|\big|\mu(t)\big|\big|_p}\,dW_i(t), \\ X_i(0) &= x_i > 0, \qquad i = 1, \ldots, n, \end{aligned} \tag{63}$$

has a unique in distribution weak solution, and it also admits a pathwise unique, strong solution. Moreover, according to the results above, if $\beta \geq 1/2$, then there exist arbitrage opportunities over any given time horizon in the model described by (63).

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