Ann Finance (2013) 9:611–624
DOI 10.1007/s10436-012-0207-0

RESEARCH ARTICLE

Absence of arbitrage in a general framework

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Received: 6 January 2011 / Accepted: 9 August 2012 / Published online: 28 August 2012

Abstract Cheridito (Finance Stoch 7:533–553, 2003) studies a financial market that consists of a money market account and a risky asset driven by a fractional Brownian motion. It is shown that arbitrage possibilities in such markets can be excluded by suitably restricting the class of allowable trading strategies. In this note, we show an analogous result in a multi-asset market where the discounted risky asset prices follow more general non-semimartingale models. In our framework, investors are allowed to trade between a risk-free asset and multiple risky assets by following simple trading strategies that require a minimal deterministic waiting time between any two trading dates. We present a condition on the discounted risky asset prices that guarantee absence of arbitrage in this setting. We give examples that satisfy our condition and study its invariance under certain transformations.

Keywords Simple trading strategies · Absence of arbitrage · Conditional full support · Non-semimartingale models

JEL Classification G10

1 Introduction

Absence of arbitrage is a minimal requirement for an equilibrium financial market. In Delbaen and Schachermayer (1994) it is shown that in frictionless markets any process that does not have an equivalent local martingale measure admits free lunch with vanishing risk, which is a weak form of arbitrage. However, models without such

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local martingale measure appear regularly in the empirical literature estimating stock prices. A typical example is a fractional Brownian motion (fBm), a process which has been suggested to model long-range dependence observed in empirical data, see Willinger et al. (1999) and Lo (1991), and the references therein. Since fBm is not a semimartingale for most parameter values, the results in Delbaen and Schachermayer (1994) show that frictionless markets driven by fBm admit free lunch. In fact, several authors have shown that fractional Brownian motion models actually admit arbitrage, see Rogers (1997) and Sottinen (2001).

Recently, arbitrage problems for non-semimartingale models have been extensively studied in the literature by introducing various types of frictions into the market. The recent papers Guasoni (2006), Guasoni et al. (2008, 2010), studied absence of arbitrage conditions in markets with proportional transaction costs. The authors introduced an interesting condition, *conditional full support (CFS)*, for the asset prices and showed that it is sufficient for absence of arbitrage in such markets. They proved the CFS property for fBm and continuous Markov processes with the full support property. In the subsequent papers Cherny (2008), Pakkanen (2010), and Gasbarra et al. (2008), the CFS property was shown to hold for more general non-semimartingale models. Also see Maris et al. (2011) and Maris and Sayit (2012) for recent results on the CFS property.

Without imposing transaction costs, absence of arbitrage was also shown to hold for certain non-semimartingale models by restricting the class of admissible trading strategies. This was studied in the recent papers Bayraktar and Sayit (2010), Bender (2010), Bender et al. (2008, 2011), Cheridito (2003), Jarrow et al. (2009). More specifically, these papers considered a market that consists of a money market account and one risky asset and provided absence of arbitrage conditions within the class of simple trading strategies. Especially, the paper Cheridito (2003) introduced a class of simple trading strategies that require a minimal deterministic waiting time between any two consecutive trading dates and showed that fBm driven models are arbitrage free within such class of trading strategies. This class of trading strategies is called Cheridito's class of trading strategies in Jarrow et al. (2009), where more general processes are shown to satisfy absence of arbitrage in Cheridito's class of simple trading strategies.

The purpose of this paper is to study absence of arbitrage conditions within the class of simple trading strategies in a market that consists of a risk-free asset and multiple risky assets. We will present a condition on the discounted risky asset prices and show that it is sufficient for absence of arbitrage within Cheridito class of trading strategies. We study the invariance of our condition under certain transformations and provide examples that satisfy it.

The market has one risk-free asset used as a numéraire and hence assumed identically equal to one and d risky assets whose price processes are given by adapted, càdlàg processes $X_t^1, X_t^2, \ldots, X_t^d$. We assume all the trading takes place in a finite time horizon $[0, T]$ and all the price processes are defined on a filtered probability space $(\Omega, \mathscr{F}, P, \mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]})$ satisfying the "usual hypotheses" (i.e., the filtration $\mathbb{F}$ is right continuous, and $\mathscr{F}_0$ contains all of the P null sets of $\mathscr{F}$).

In the framework of Delbaen and Schachermayer (1994), price processes $X_t = (X_t^1, X_t^2, \ldots, X_t^d)$ are semimartingales and the class of trading strategies is described by predictable and $X-$ integrable processes that are admissible. A trading strategy

$H = (H_t^1, H_t^2, \ldots, H_t^d)$ is admissible if $(H \cdot X) \geq -a$ for some $a \in \mathbb{R}_+$. Denote by L^0 the space of equivalent classes of finite valued random variables. Define the following convex cone in L^0,

$$K_0 = \{(H \cdot X)_T | H \text{is admissible}\}.$$

Let $C_0 = K_0 - L_+^0$ be the cone of random variables dominated by elements of K_0. Let $K = K_0 \cap L^\infty$ and $C = C_0 \cap L^\infty$, where L^∞ is the space of bounded random variables. Let $\bar{C}$ denote the closure of C with respect to the norm topology of L^∞. We say that the market $(1, X_t^1, \ldots, X_t^d)$ is arbitrage-free with the general class of admissible trading strategies if $C \cap L_+^\infty$ and it satisfies the no free lunch with vanishing risk (NFLVR) property if $\bar{C} \cap L_+^\infty = \{0\}$.

The following fundamental theorem is proved in Delbaen and Schachermayer (1994).

Theorem 1 *A locally bounded semimartingale X_t satisfies the NFLVR property if and only if X admits an equivalent local martingale measure.*

This theorem shows that if we permit all the admissible trading strategies in the market, all the price processes that are consistent with NFLVR admit local martingale measures. Next we introduce the class of trading strategies that we study in this note.

For each $h > 0$, let $\mathscr{J}^h(\mathbb{F})$ denote the class of finite sequences of stopping times $\tau = (\tau_1, \ldots, \tau_n)$, where $n \geq 1$ is any positive integer, $\tau_1 = 0$, and $\tau_{j+1} \geq \tau_j + h$ a.s. for all $1 \leq j \leq n-1$. For any $h > 0$, let

$$\mathscr{S}^h = \{(H_s^1, H_s^2, \ldots, H_s^d) : H_s^i = h_0^i 1_{\{0\}}(s) + \sum_{j=1}^{n-1} h_j^i 1_{(\tau_j, \tau_{j+1}]}(s), 1 \leq i \leq d\}$$

where $n \geq 1$ is any integer, $\tau = (\tau_1, \ldots, \tau_n) \in \mathscr{J}^h$, for each $1 \leq i \leq d$, h_0^i is a real number, for each $1 \leq j \leq n$ and $1 \leq i \leq d$, h_j^i is a $\mathscr{F}_{\tau_j}$-measurable random variable.

Definition 1 The Cheridito class of simple trading strategies is given by

$$\mathscr{L}(\mathbb{F}) =: \cup_{h>0} \mathscr{S}^h(\mathbb{F}). \tag{1}$$

The class of general simple trading strategies is given by $\mathscr{S}(\mathbb{F}) =: \mathscr{S}^0(\mathbb{F})$.

For any given trading strategy $H_s = (H_s^1, H_s^2, \ldots, H_s^d)$ in $\mathscr{L}(\mathbb{F})$ or in $\mathscr{S}(\mathbb{F})$ with the representation

$$H_s^i = h_0^i 1_0 + \sum_{j=1}^{n-1} h_j^i 1_{(\tau_j, \tau_{j+1}]}(s), 1 \leq i \leq n$$

the corresponding gains process with zero initial cost is given by

$$G_t(H) = \sum_{i=1}^{d} \sum_{j=1}^{n-1} h_j^i (X_{\tau_{j+1} \wedge t}^i - X_{\tau_j \wedge t}^i). \tag{2}$$

Let $K = \{G_T(H) : H \in \mathscr{L}(\mathbb{F})\}$ and $K^s = \{G_T(H) : H \in \mathscr{S}(\mathbb{F})\}$ denote the corresponding sets of total gains with zero initial cost.

Definition 2 We say that the market $(1, X_t^1, X_t^2, \ldots, X_t^d)$ is arbitrage-free with the Cheridito class of trading strategies or with $\mathscr{L}(\mathbb{F})$, if $K \cap L_0^+ = \emptyset$. Also we say that the market $(1, X_t^1, X_t^2, \ldots, X_t^d)$ is arbitrage-free with the class of general simple trading strategies if $K^s \cap L_0^+ = \emptyset$.

We remark that here we do not require the gains process to be bounded from below. Since the class of doubling strategies is not a subset of the class of simple trading strategies. Here we state our main result of this paper.

Theorem 2 *Let $S_t^1, S_t^2, \ldots, S_t^d$ be a sequence of independent continuous semimartingales defined in the time horizon $[0, +\infty)$. Assume $[S^i, S^i]_t = [S^j, S^j]_t$ a.s. for all $t \geq 0$ and all $i, j \in \{1, 2, \ldots, d\}$ and $[S^i, S^i]_t$ is bounded a.s. for each $t \geq 0$. Further assume that the vector-valued semimartingale $(S_t^1, S_t^2, \ldots, S_t^d)$ admits a local martingale measure. If*

$$[S^i, S^i]_{t+h} - [S^i, S^i]_t \geq p(h) \;\; a.s. \tag{3}$$

for some deterministic function $p(\cdot) : [0, +\infty) \to [0, +\infty)$, with $p(0) = 0$ and $p(h) > 0$ for each $h > 0$, then for any sequence $V_t^1, V_t^2, \ldots, V_t^2$ of adapted and càdlàg processes that are bounded in $[0, T]$, the market $(1, S_t^1 + V_t^1, S_t^2 + V_t^2, \ldots, S_t^d + V_t^d)$, $t \in [0, T]$, is arbitrage-free within Cheridito class of simple trading strategies.

We remark that there is no any assumption on the bounded processes $V_t^1, V_t^2, \ldots, V_t^2$ except that they be càdlàd and adapted. The following example directly follows from the above theorem.

Example 1 Let $(B_t^1, B_t^2, \ldots, B_t^d)$ be a $d-$dimensional Brownian motion. It is clear that the market $(1, |B_t^1|, |B_t^2|, \ldots, |B_t^d|)$ is not arbitrage-free within Cheridito class of trading strategies. The process $H_t = (H_t^1, H_t^2, \ldots, H_t^d)$, where $H_t^i = 1_{[0,T]}(t)$, $1 \leq i \leq d$, is clearly an arbitrage strategy for this market. next, we modify the risky asset price processes slightly to get an arbitrage-free market. By Tanaka's formula we have

$$|B_t^i| = \int_0^t sign(B_s^i) dB_s^i + L_t^i, \quad 1 \leq i \leq d.$$

For any finite real number $m > 0$, let $\tau = inf\{t \geq 0 : L_t^i \geq m, 1 \leq i \leq d\}$. Define the following processes

$$D_t^i = \int_0^t sign(B_s^i) dB_s^i + L_{t \wedge \tau}^i, \quad 1 \leq i \leq d.$$

Then the market $(1, D_t^1, D_t^2, \ldots, D_t^d)$ is arbitrage-free within the class $\mathscr{L}(\mathbb{F})$ of simple trading strategies, thanks to Theorem 2. Note that the vector-valued semimartingale

$$\left(\int_0^t sign(B_s^1)dB_s^1, \int_0^t sign(B_s^2)dB_s^2, \ldots, \int_0^t sign(B_s^d)dB_s^d\right)$$

satisfies the conditions of Theorem 2.

We remark that the market $(1, D_t^1, D_t^2, \ldots, D_t^d)$ defined in the above example admits arbitrage within the class of general admissible trading strategies discussed in Delbaen and Schachermayer (1994). Clearly, the trading strategy $H_t = (H_t^1, H_t^2, \ldots, H_t^d)$, where $H_t^i = 1_{[0,\tau\wedge T]}(t)$, $1 \leq i \leq d$, and τ is the stopping time defined in the same example, is an admissible arbitrage strategy for the market. To see this, observe that each process D_t^i equals to $|B_t^i|$ in the stochastic interval $[0, \tau]$. Therefore, the corresponding gains process for H_t is given by $G_t(H) = \sum_1^d |B^i_{t\wedge\tau\wedge T}|$, which is a non-negative process that starts from zero at $t = 0$.

We should mention that, while absence of arbitrage holds for a large class of models beyond semimartingales in our framework, the Cheridito class of trading strategies is very restrictive. For example, as pointed out in the paper Bender et al. (2011), the stopping time $\tau = \inf\{t \geq 0 : |S_t - S_0| \geq 1\}$, where S_t is a geometric Brownian motion, does not belong to the Cheridito class.

The remainder of the paper is organized as follows: in the next section we introduce a condition, which we call *joint conditional up and down* ($\mathbb{F} - CUD$) *condition,* and show that it is sufficient for absence of arbitrage in our framework, see Definition 3 and Proposition 1. A similar condition for the one dimensional case is first introduced in Jarrow et al. (2009) and it is called conditional up and down condition in Bender et al. (2011). In the same section we also show that the joint $\mathbb{F} - CUD$ condition is invariant under composition with strictly monotone functions, see Proposition 2. In Sect. 3, we introduce a condition which we call *joint conditional high up and deep down* ($\mathbb{F} - CHUDD$) *condition*, on the discounted risky asset price processes and show that it implies the joint $\mathbb{F} - CUD$ condition, see Definition 4 and Proposition 3. In Propositions 4 and 5, we study the invariance of the joint $\mathbb{F} - CHUDD$ condition under composition with certain class of continuous functions and also under addition with bounded processes, respectively. In Sect. 4, we present the Proof of Theorem 2 and give examples of arbitrage-free models with the Cheridito class of trading strategies.

2 A sufficient condition for absence of arbitrage

Let $X_t^1, X_t^2, \ldots, X_t^d$ be a finite sequence of càdlàg processes adapted to the filtration $\mathbb{F}$. For any two stopping times $\tau_1 \leq \tau_2$ a.s. of $\mathbb{F}$ and any $1 \leq i \leq d$, let

$$A_i^+ = \{X_{\tau_1}^i < X_{\tau_2}^i\}, \quad A_i^- = \{X_{\tau_1}^i > X_{\tau_2}^i\} \tag{4}$$

Definition 3 We say that an adapted càdlàg process $X_t = (X_t^1, X_t^2, \ldots, X_t^d)$ satisfies joint $\mathbb{F} - CUD$ condition with respect to $\mathscr{L}(\mathbb{F})$, if for any $h \in (0, T)$ and any two stopping times $\tau_1 \leq \tau_2$ with $\tau_2 \geq \tau_1 + h$ a.s., and any $B \in \mathscr{F}_{\tau_1}$ with $P(B) > 0$, the following holds

$$P([\cap_{i \in I^B} A_i^{\alpha_i}] \cap B) > 0, \tag{5}$$

whenever I^B is not empty, where $\alpha_i \in \{+, -\}$, $A_i^{\alpha_i}$ are defined as in (4), and I^B is the set of all $k \in \{1, 2, \ldots, d\}$ such that $P(\{X_{\tau_1}^k \neq X_{\tau_2}^k\} \cap B) > 0$. Similarly, we say that X_t satisfies joint $\mathbb{F} - CUD$ condition with respect to $S(\mathbb{F})$, if the condition (5) holds for any pair of stopping times τ_1 and τ_2 as long as $\tau_1 \leq \tau_2$ a.s.

The next proposition shows that joint $\mathbb{F} - CUD$ condition implies absence of arbitrage.

Proposition 1 *If the adapted càdlàg process $X_t = (X_t^1, X_t^2, \ldots, X_t^d)$ satisfies the joint $\mathbb{F} - CUD$ condition with respect to $\mathscr{L}(\mathbb{F})$, then the market $(1, X_t^1, X_t^2, \ldots, X_t^d)$ is arbitrage-free with $\mathscr{L}(\mathbb{F})$. If X_t satisfies the joint $\mathbb{F} - CUD$ condition with respect to $S(\mathbb{F})$, then the market $(1, X_t^1, X_t^2, \ldots, X_t^d)$ is arbitrage-free with $S(\mathbb{F})$.*

Proof We prove the first part of the proposition, the second part follows similarly. By the way of contradiction, assume $H_s = (H_s^1, H_s^2, \ldots, H_s^d) \in \mathscr{L}(\mathbb{F})$ is an arbitrage strategy. Then there is a $h > 0$, such that $H \in \mathscr{S}^h(\mathbb{F})$. By the definition of $\mathscr{S}^h(\mathbb{F})$, we can assume that there is a $\tau = (\tau_1, \tau_2, \ldots, \tau_n) \in \mathscr{J}^h$ such that $H_s^i = h_0^i 1_0 + \sum_{j=1}^{n-1} h_j^i 1_{(\tau_j, \tau_{j+1}]}(s)$, $1 \leq i \leq d$. The corresponding gains process is given by

$$G_t(H) = \sum_{j=1}^{n-1} \sum_{i=1}^{d} h_j^i (X_{\tau_{j+1} \wedge t}^i - X_{\tau_j \wedge t}^i).$$

For each $1 \leq j \leq n$ denote $S_j = (X_{\tau_j}^1, X_{\tau_j}^2, \ldots, X_{\tau_j}^d)$ and for each $1 \leq j \leq n-1$ denote $h_j = (h_j^1, h_j^2, \ldots, h_j^d)$. We can write $G_T(H) = \sum_{j=1}^n h_j \cdot (S_{j+1} - S_j) = (h \cdot S)_n$. Since H is an arbitrage strategy, we conclude that the $\mathbb{R}^d-$valued discrete time process $(S_j, \mathscr{F}_{\tau_i})_{j=1}^n$ admits arbitrage. Therefore from proposition 2.1 of Rogers (1994), we conclude that for some $j \in \{1, 2, \ldots n\}$ there is a $\mathscr{F}_{\tau_j}-$measurable $\mathbb{R}^d-$valued random variable $\theta_j = (\theta_j^1, \theta_j^2, \ldots, \theta_j^d)$ such that

$$\theta_j \cdot (S_{j+1} - S_j) \geq 0 \quad \text{a.s and} \quad P[\theta_j \cdot (S_{j+1} - S_j) > 0] > 0. \tag{6}$$

In the following, we shall show that (6) contradicts with the joint $\mathbb{F} - CUD$ condition. To see this, note that the latter inequality in (6) implies the existence of a $k \in \{1, 2, \ldots, d\}$ such that $P(\{\theta_j^k \neq 0\} \cap \{X_{\tau_{j+1}}^k \neq X_{\tau_j}^k\}) > 0$. Without loss of generality, we assume $P(\{\theta_j^1 \neq 0\} \cap \{X_{\tau_{j+1}}^1 \neq X_{\tau_j}^1\}) > 0$, and further we can assume $P(\{\theta_j^1 > 0\} \cap \{X_{\tau_{j+1}}^1 \neq X_{\tau_j}^1\}) > 0$. Denote $C = \{\theta_j^1 > 0\} \cap \{X_{\tau_{j+1}}^1 \neq X_{\tau_j}^1\}$. Let $L_i^+ = \{\theta_j^i \leq 0\}$ and $L_i^- = \{\theta_j^i \geq 0\}$. Observe that

$$\cup_{\beta_2, \beta_3, \ldots, \beta_d} [\cap_{i=2}^d L_i^{\beta_i}] = \Omega$$

where the union is taken for all $\beta_2, \beta_3, \ldots, \beta_d \in \{+, -\}$. Therefore, we can assume that there are

$$\alpha_2, \alpha_3, \ldots, \alpha_d \in \{+, -\} \tag{7}$$

such that $P(C \cap [\cap_{i=2}^{d} L_i^{\alpha_i}]) > 0$. Let $D = \{\theta_j^1 > 0\} \cap [\cap_{i=2}^{d} L_i^{\alpha_i}]$. It is clear that $D \in \mathscr{F}_{\tau_j}$ and $P(D) > 0$. Also note that $P(D \cap \{X_{\tau_{j+1}}^1 \neq X_{\tau_j}^1\}) > 0$.

Now, let I^D be the set of all $k \in \{2, 3, \ldots, d\}$ such that $P(D \cap \{X_{\tau_{j+1}}^k \neq X_{\tau_j}^k) > 0$. For each $k \in I^D$ (If I^D is not empty set), define

$$A_k^+ = \{X_{\tau_{j+1}}^k > X_{\tau_j}^k\}, \quad A_k^- = \{X_{\tau_{j+1}}^k < X_{\tau_j}^k\}.$$

Then from the $(\star)$ condition, we have

$$P(D \cap \{X_{\tau_{j+1}}^1 < X_{\tau_j}^1\} \cap [\cap_{k \in I^D} A_k^{\alpha_k}]) > 0,$$

here $\{\alpha_k\}_{k \in I^D}$ is given by (7). Denote $F = D \cap \{X_{\tau_{j+1}}^1 < X_{\tau_j}^1\} \cap [\cap_{k \in I^D} A_k^{\alpha_k}]$. From the definition of I^D, it follows that on the set F we have

$$\theta_j \cdot (S_{j+1} - S_j) = \theta_j^1 (X_{\tau_{j+1}}^1 - X_{\tau_j}^1) + \sum_{k \in I^D} \theta_j^k (X_{\tau_{j+1}}^k - X_{\tau_j}^k).$$

Note that $F \subset L_k^{\alpha_k} \cap A_k^{\alpha_k}$ for each $k \in I^D$, therefore on F we have $\sum_{k \in I^D} \theta_j^k (X_{\tau_{j+1}}^k - X_{\tau_j}^k) \le 0$. Also since $\theta_j^1 < 0$ and $X_{\tau_{j+1}}^1 < X_{\tau_j}^1$ on F, we conclude that $\theta_j \cdot (S_{j+1} - S_j) < 0$ on F. This contradicts with (6). □

We should mention that the joint $\mathbb{F} - CUD$ condition is not a necessary condition for absence of arbitrage. To see this, let M_t be any real-valued martingale and let $X_t^i = M_t, 1 \le i \le d$. Then the market $(1, X_t^1, X_t^2, \ldots, X_t^d)$ satisfies the absence of arbitrage condition with $S(\mathbb{F})$ (and with $\mathscr{L}(\mathbb{F})$). However, it is clear that the process $(X_t^1, X_t^2, \ldots, X_t^d)$ does not satisfy the joint $\mathbb{F} - CUD$ condition with respect to $S(\mathbb{F})$ or with respect to $\mathscr{L}(\mathbb{F})$.

The next proposition shows that the joint $\mathbb{F} - CUD$ condition is invariant under componentwise composition with strictly monotone functions. The following notations will be used in the proof of this proposition and also in the rest of the paper: for any $\alpha \in \{+, -\}$ $-\alpha$ is understood "+" if $\alpha = -$ and "−" if $\alpha = +$.

Proposition 2 *The joint $\mathbb{F} - CUD$ condition on $X_t = (X_t^1, X_t^2, \ldots, X_t^d)$ with respect to $\mathscr{L}(\mathbb{F})$ (or to $S(\mathbb{F})$) is equivalent to the joint $\mathbb{F} - CUD$ condition on the process*

$$(f_1(X_t^1), f_2(X_t^2), \ldots, f_d(X_t^d)),$$

with respect to $\mathscr{L}(\mathbb{F})$ (to $S(\mathbb{F})$), where $f_1, f_2, \ldots, f_d$ are strictly monotone functions.

Proof For any $h \in (0, T)$, and any two stopping times $\tau_2 \geq \tau_1$ with $\tau_2 \geq \tau_1 + h$ a.s., define

$$F_i^+ = \{f_i(X^i_{\tau_1}) < f_i(X^i_{\tau_2})\}, \quad F_i^- = \{f_i(X^i_{\tau_1}) > f_i(X^i_{\tau_2})\}$$

for each $1 \leq i \leq d$. For any $B \in \mathscr{F}_{\tau_1}$ with $P(B) > 0$, let J^B the set of all $k \in \{1, 2, \ldots, d\}$ for which $P(\{f_k(X^k_{\tau_1}) \neq f_k(X^k_{\tau_2})\} \cap B) > 0$. Also define $A_i^{\pm}$ in terms of X^i_t as in (4) for each $1 \leq i \leq d$ and let I^B be the set of all $k \in \{1, 2, \ldots, d\}$ for which $P(\{X^k_{\tau_1} \neq X^k_{\tau_2}\} \cap B) > 0$. Since $f_1, f_2, \ldots, f_d$ are strictly monotone, we have $J^B = I^B$. For any sequence $\beta_1, \beta_2, \ldots, \beta_d \in \{+, -\}$, let $\alpha_i =: \beta_i$ if f_i is strictly increasing and let $\alpha_i =: -\beta_i$ if f_i is strictly decreasing. Now the claim follows from

$$[\cap_{i \in J^B} F_i^{\beta_i}] \cap B = [\cap_{i \in I^B} A_i^{\alpha_i}] \cap B$$

which in turn follows from

$$F_i^{\beta_i} = A_i^{\alpha_i}, \;\; i \in J^B.$$

□

3 Models that satisfy the conditional up and down condition

In this section, we present a condition on the process $X_t = (X^1_t, X^2_t, \ldots, X^d_t)$ that is sufficient for $\mathbb{F} - CUD$. We call this new condition joint conditional high up and deep down ($\mathbb{F} - CHUDD$) condition. We study the invariance of this new condition under certain transformations. We first introduce a few notations. For any $0 < \delta < T$ and any stopping time τ with values in $[0, T - \delta)$ and any $c > 0$, let

$$\begin{aligned} &(i) \;\; B_i^+(X^i, \tau, \delta, c) = \{\inf_{t \in [\delta, T-\tau]}(X^i_{\tau+t} - X^i_\tau) > c\} \\ &(ii) \; B_i^-(X^i, \tau, \delta, c) = \{\sup_{t \in [\delta, T-\tau]}(X^i_{\tau+t} - X^i_\tau) < -c\} \end{aligned} \tag{8}$$

Definition 4 We say that an adapted càdlàg process $X_t = (X^1_t, X^2_t, \ldots, X^d_t)$ satisfies joint $\mathbb{F} - CHUDD$ condition for $c > 0$, if for any $\delta \in (0, T)$ and any stopping time τ with values in $[0, T - \delta)$, the following holds

$$P(\cap_i^d B_i^{\alpha_i}(X^i, \tau, \delta, c) | \mathscr{F}_\tau) > 0 \;\; \text{a.s.} \tag{9}$$

where $\alpha_i \in \{+, -\}$, and $B_i^{\alpha_i}(X^i, \tau, \delta, c)$, $1 \leq i \leq d$ are defined as in (8).

Proposition 3 *The joint $\mathbb{F} - CHUDD$ condition on $X_t = (X^1_t, X^2_t, \ldots, X^d_t)$ for some $c > 0$ implies the joint $\mathbb{F} - CUD$ condition on X.*

Proof Assume CHUDD holds for some $c > 0$ and for any $\delta \in (0, T)$ and any $[0, T-\delta)$ valued stopping time τ. Let $\tau_2 \geq \tau_1$ be any two stopping times with $\tau_2 \geq \tau_1 + h$ for some $h > 0$. For each $1 \leq i \leq d$ and any $\alpha_i \in \{+, -\}$, define $A_i^{\alpha_i}$ as in (4) and

$B_i^{\alpha_i}(X^i, \tau_1, \frac{h}{2}, c)$ as in (8) (note that $\tau_1 \leq \tau_2 - h < T - \frac{h}{2}$). By the assumption we have $P(\cap_i^d B_i^{\alpha_i}(X^i, \tau_1, \frac{h}{2}, c)|\mathscr{F}_\tau) > 0$. Therefore $P(\cap_i^d A_i^{\alpha_i}|\mathscr{F}_\tau) > 0$ follows from

$$B_i^{\alpha_i}(X^i, \tau_1, \frac{h}{2}, c) \subset A_i^{\alpha_i}, \quad 1 \leq i \leq d.$$

□

Next, we study the invariance of the joint $\mathbb{F} - CHUDD$ condition under composition with continuous functions. Let $\mathscr{C}$ be the class of real-valued continuous functions $f(x)$ that satisfy either (a) or (b) in the following

$$\begin{aligned} &(a) \quad lim_{x\to+\infty} f(x) = +\infty, \quad lim_{x\to-\infty} f(x) = -\infty \\ &(b) \quad lim_{x\to-\infty} f(x) = +\infty, \quad lim_{x\to+\infty} f(x) = -\infty. \end{aligned} \tag{10}$$

Proposition 4 *Let $X_t = (X_t^1, X_t^2, \ldots, X_t^d)$ be an adapted càdlàg process that satisfies the joint $\mathbb{F} - CHUDD$ condition for any $c > 0$. Then, for any sequence $f_1, f_2, \ldots, f_d \in \mathscr{C}$, the process*

$$Y_t = (Y_t^1, Y_t^2, \ldots, Y_t^d),$$

where $Y_t^i = f_i(X_t^i)$, $1 \leq i \leq d$, also satisfies the joint $\mathbb{F} - CHUDD$ condition for any $c > 0$.

Proof Fix any $c > 0$. For any $\delta \in (0, T)$ and any $[0, T - \delta)$ valued stopping time τ, define $B_i^{\pm}(Y^i, \tau, \delta, c)$ for each $1 \leq i \leq d$ as in (8). For any fixed sequence $\alpha_1, \alpha_2, \ldots, \alpha_d \in \{+, -\}$, we need to show

$$P(\cap_i^d B_i^{\alpha_i}(Y^i, \tau, \delta, c)|\mathscr{F}_\tau) > 0 \text{ a.s.} \tag{11}$$

and this is equivalent to showing

$$P(A \cap [\cap_i^d B_i^{\alpha_i}(Y^i, \tau, \delta, c)]) > 0 \tag{12}$$

for any $A \in \mathscr{F}_\tau$ with $P(A) > 0$.

Fix $A \in \mathscr{F}_\tau$ with $P(A) > 0$. Also fix a finite real-number $m > 0$ with $P(A \cap [\cap_i^d \{|X_\tau^i| \leq m\}]) > 0$. Note that such a number m exists since X_τ^i, $1 \leq i \leq d$, are finite valued random variables. Denote $E = A \cap [\cap_i^d \{|X_\tau^i| \leq m\}]$ and observe that $E \in \mathscr{F}_\tau$. Let $M = max\{|f_1(x)|, |f_2(x)|, \ldots, |f_d(x)| : x \in [-m, m]\}$ and observe that $|Y_\tau^i| \leq M$ for each $1 \leq i \leq d$ on E. Since each $f_i(x)$ satisfies either (a) or (b) in (10), we can find a real number $H > 0$ large enough, such that for each $1 \leq i \leq d$:

- (O1) If $f_i(x)$ satisfies (a) of (10), then $f_i(x) > M + c$ for $x \geq H$, and $f_i(x) < -M - c$ for $x \leq -H$.
- (O2) If $f_i(x)$ satisfies (b) of (10), then $f_i(x) < -M - c$ for $x \geq H$, and $f_i(x) > M + c$ for $x \leq -H$.

For each $1 \leq i \leq d$, let

$$\beta_i = \begin{cases} \alpha_i & \text{if } f_i(x) \text{ satisfies } (a)\ in(10), \\ -\alpha_i & \text{if } f_i(x) \text{ satisfies } (b)\ in(10). \end{cases}$$

In the following we will show that

$$E \cap [\cap_{i=1}^d B_i^{\beta_i}(X^i, \tau, \delta, H+m)] \subset E \cap [\cap_{i=1}^d B_i^{\alpha_i}(Y^i, \tau, \delta, c)]. \tag{13}$$

Since the set in the left-hand side of (13) has positive probability by the assumption on X, (13) implies (12) and this completes the proof.

To show (13), it is sufficient to show

$$E \cap B_i^{\beta_i}(X^i, \tau, \delta, H+m) \subset E \cap B_i^{\alpha_i}(Y^i, \tau, \delta, c), \tag{14}$$

for each $1 \leq i \leq d$.

Case1: Assume f_i satisfies (a) of (10) and $\alpha_i = +$ or it satisfies (b) of (10) and $\alpha_i = -$. In both of these cases we have $\beta_i = +$. Therefore on the set $E \cap B_i^{\beta_i}(X^i, \tau, \delta, H+m)$, we have

$$inf_{t\in[\delta,T-\tau]} X_{\tau+t}^i > X_\tau^i + H + m \geq H.$$

If f_i satisfies (a) of (10) and $\alpha_i = +$, then from $(O1)$ we conclude that on $E \cap B_i^{\beta_i}(X^i, \tau, \delta, H+m)$ we have

$$inf_{t\in[\delta,T-\tau]}(Y_{\tau+t}^i - Y_\tau^i) = inf_{t\in[\delta,T-\tau]} f_i(X_{\tau+t}^i) - f_i(X_\tau^i)$$

$$> M + c - M = c.$$

This shows that (14) holds. If f_i satisfies (b) of (10) and $\alpha_i = -$, then from $(O2)$ we conclude that

$$sup_{t\in[\delta,T-\tau]}(Y_{\tau+t}^i - Y_\tau^i) = sup_{t\in[\delta,T-\tau]} f(X_{\tau+t}^i) - f(X_\tau^i)$$

$$< -M - c + M = c$$

on $E \cap B_i^{\beta_i}(X^i, \tau, \delta, H+m)$ and this again shows that (14) holds.

Case2: Assume f_i satisfies (a) of (10) and $\alpha_i = -$ or it satisfies (b) of (10) and $\alpha_i = +$. In these cases $\beta_i = -1$. Therefore, on the set $E \cap B_i^{\beta_i}(X^i, \tau, \delta, H+m)$, we have

$$sup_{t\in[\delta,T-\tau]}(X_{\tau+t}^i) < X_\tau^i - H - m < -H.$$

If f_i satisfies (a) of (10) and $\alpha_i = -$, then from $(O1)$ we conclude that on $E \cap B_i^{\beta_i}(X^i, \tau, \delta, H+m)$ we have

$$sup_{t\in[\delta,T-\tau]}(Y^i_{\tau+t} - Y^i_\tau) = sup_{t\in[\delta,T-\tau]} f(X^i_{\tau+t}) - f(X^i_\tau)$$

$$< -M - c + M = -c.$$

Therefore in this case (14) holds. If it satisfies (b) of (10) and $\alpha_i = +$, then from $(O2)$ we conclude that on $E \cap B_i^{\beta_i}(X^i, \tau, \delta, H+m)$ we have

$$inf_{t\in[\delta,T-\tau]}(Y^i_{\tau+t} - Y^i_\tau) = inf_{t\in[\delta,T-\tau]} f(X^i_{\tau+t}) - f(X^i_\tau)$$

$$> M + c - M = c$$

and this again implies (14). □

The next proposition shows that the joint $\mathbb{F}-$*CHUDD* condition remains unchanged under componentwise addition with bounded processes. Note that there is no any assumption on the bounded processes except that they be adapted and càdlàg.

Proposition 5 *Let $X_t = (X^1_t, X^2_t, \ldots, X^d_t)$ be an adapted càdlàg process that satisfy the joint $\mathbb{F} - CHUDD$ condition for any $c > 0$. Then for any sequence of adapted, càdlàg, and bounded processes $V^1_t, V^2_t, \ldots, V^d_t$, the process $Y_t = (Y^1_t, Y^2_t, \ldots, Y^d_t)$, where $Y^i_t = X^i_t + V^i_t$, $1 \le i \le d$, also satisfies the joint $\mathbb{F} - CHUDD$ condition for any $c > 0$.*

Proof Since $V^1_t, V^2_t, \ldots, V^d_t$ are bounded, there is a $M > 0$ such that $sup_{t\in[0,T]}|V^i_t| \le M$ a.s. for all $1 \le i \le d$. Fix $c > 0$. For any $\delta \in (0, T)$ and any $[0, T-\delta)$ valued stopping time τ, define $B_i^{\pm}(Y^i, \tau, \delta, c)$ for each $1 \le i \le d$ as in (8). For any sequence $\alpha_1, \alpha_2, \ldots, \alpha_d \in \{+, -\}$, we need to show

$$P(\cap_i^d B_i^{\alpha_i}(Y^i, \tau, \delta, c)|\mathscr{F}_\tau) > 0 \ \text{ a.s.} \tag{15}$$

and this is equivalent to showing

$$P(A \cap [\cap_i^d B_i^{\alpha_i}(Y^i, \tau, \delta, c)]) > 0 \tag{16}$$

for any $A \in \mathscr{F}_\tau$ with $P(A) > 0$. By the assumption on X we have

$$P(A \cap [\cap_i^d B_i^{\alpha_i}(X^i, \tau, \delta, c+2M)]) > 0,$$

therefore it is sufficient to show

$$A \cap B_i^{\alpha_i}(X^i, \tau, \delta, c+2M) \subset A \cap B_i^{\alpha_i}(Y^i, \tau, \delta, c) \tag{17}$$

for each $1 \leq i \leq d$, and this follows easily from the following relation

$$X^i_{\tau+t} - X^i_\tau - 2M \leq Y^i_{\tau+t} - Y^i_\tau \leq X^i_{\tau+t} - X^i_\tau + 2M, \quad 1 \leq i \leq d.$$

4 Proof of Theorem 2 and examples

In this section we present the proof of the main Theorem 2 and provide examples of arbitrage-free models within the class $\mathscr{L}(\mathbb{F})$ of simple trading strategies.

Proof of Theorem 2 Because of Propositions 5 and 3, we only need to check the joint $\mathbb{F} - CHUDD$ condition on the process $(S^1_t, S^2_t, \ldots, S^d_t)$. Since the joint $\mathbb{F} - CHUDD$ condition is invariant under equivalent changes of measure, we can assume that $(S^1_t, S^2_t, \ldots, S^d_t)$ is a local martingale under the original measure. In the following we use N to denote sets of measure zero (Although these sets might differ we use the same letter to avoid notational crowding). Fix any $c > 0$, $T > 0$, $\delta \in [0, T)$, and any sequence $\alpha_1, \alpha_2, \ldots, \alpha_d \in \{+, -\}$. Also fix any $[0, T - \delta)$ valued stopping time τ and $A \in \mathscr{F}_\tau$ with $P(A) > 0$. We need to show

$$P(A \cap [\cap^d_i B^{\alpha_i}_i(S^i, \tau, \delta, c)]) > 0, \tag{18}$$

where $B^{\pm}_i(S^i, \tau, \delta, c)$ is defined as in (8) for each $1 \leq i \leq d$.

Let $\eta^i_s = \inf\{t > 0 : [S, S]^i_t \geq s\}, 1 \leq i \leq d$. Since $[S^i, S^i]_t = [S^j, S^j]_t$, $1 \leq i, j \leq d$ on Ω/N, we have $\eta^i_t = \eta^j_t, 1 \leq i, j \leq d$ on Ω/N. For notational simplicity, in the following we denote $\eta_s =: \eta^i_s, 1 \leq i \leq d$ and $[S, S]_t =: [S^i, S^i]_t, 1 \leq i \leq d$. Note that η_s is a stopping time for $\mathbb{F}$ for each $s \geq 0$. Let $W^i_s = S^i_{\eta_s}$ and $\mathscr{G}_s = \mathscr{F}_{\eta_s}$. The condition (3) on the quadratic variation implies that $\lim_{t\to\infty}[S, S]^i_t = \infty, 1 \leq i \leq d$. Thus from theorem 42 of chapter II of Protter (2005) and also from the independence assumption on $S^1_t, S^2_t, \ldots, S^d_t$, we conclude that $(W^1_s, W^2_s, \ldots, W^d_s)$ is a $d-$dimensional Brownian motion with respect to the filtration $\mathbb{G} = (\mathscr{G}_s)_{s\geq 0}$, where $W^i_s = S^i_{\eta_s}, 1 \leq i \leq d$. Observe that $(S^1_t, S^2_t, \ldots, S^d_t) = (W^1_{[S,S]_t}, W^2_{[S,S]_t}, \ldots, W^d_{[S,S]_t})$. Since $[S, S]_t$ has continuous and strictly increasing paths we have $\{[S, S]_\tau \leq s\} = \{\eta_s \geq \tau\} \in \mathscr{F}_{\eta_s}$. Therefore $[S, S]_\tau$ is a stopping time for the filtration $\mathbb{G}$. Also observe that $\mathscr{F}_\tau \subset \mathscr{G}_{[S,S]_\tau}$. This shows that $A \in \mathscr{G}_{[S,S]_\tau}$. Let L be a real number such that $L > [S, S]_T$ a.s. Since $T - \tau \geq h$, from (8) we have $[S, S]_T - [S, S]_\tau \geq p(h)$. Therefore $[S, S]_\tau$ is a $[0, L - [S, S]_\tau)$ valued stopping time. Define $B^{\pm}_i(W^i, [S, S]_\tau, p(\delta), c)$ for each $1 \leq i \leq d$ as in (8) with T replaced by L. In the following we will show that

$$B^{\alpha_i}_i(W^i, [S, S]_\tau, p(\delta), c) \subset B^{\alpha_i}_i(S^i, \tau, \delta, c), \quad 1 \leq i \leq d. \tag{19}$$

Since $(W^1_s, W^2_s, \ldots, W^d_s)$ satisfies the joint $\mathbb{G} - CHUHD$ condition for any $c > 0$, we have

$$P(A \cap [\cap^d_i B^{\alpha_i}_i(W^i, [S, S]_\tau, p(\delta), c)]) > 0,$$

and therefore (19) implies (18).

To show (19) for $\alpha_i = -$, note that $S_t^i = W_{[S,S]_t}^i$. Therefore we have

$$B_i^{\alpha_i}(S^i, \tau, \delta, c) = \{sup_{t\in[\delta,T]}(S_{\tau+t}^i - S_\tau^i) < -c\}$$

$$= \{sup_{t\in[\delta,T]}(W_{[S,S]_{\tau+t}}^i - W_{[S,S]_\tau}^i) < -c\} \supset \{sup_{t\in[p(\delta),L]}(W_{t+[S,S]_\tau}^i - W_{[S,S]_\tau}^i) < -c\}$$

$$= B_i^{\alpha_i}(W^i, [S,S]_\tau, p(\delta), c).$$

In the case $\alpha_i = +$, (19) can be shown similarly.

The following example follows from Theorem 2 and Proposition 2. □

Example 2 Let $(W_t^1, W_t^2, \ldots, W_t^d)$ be a d–dimensional Brownian motion and

$$V_t^1, V_t^2, \ldots, V_t^d$$

be a sequence of adapted, càdlàg, and bounded processes. For any real-valued function $g(x)$ with

$$0 < inf_{x\in(-\infty,+\infty)}|g(x)| \leq \sup_{x\in(-\infty,+\infty)} |g(x)| < +\infty,$$

the market $(1, Y_t^1, Y_t^2, \ldots, Y_t^d)$, where $Y_t^i = e^{\int_0^t g(W_s^i)dW_s^i + V_t^i}$ for each $1 \leq i \leq d$, is arbitrage-free within the class $\mathscr{L}(\mathbb{F})$ of trading strategies. Note that the semimartingale

$$(S_t^1, S_t^2, \ldots, S_t^d),$$

where $S_t^i = \int_0^t g(W_s^i)dW_s^i$, $1 \leq i \leq d$, satisfies the conditions of Theorem 2 and e^x is a strictly monotone function.

Before presenting our next example, we first recall the CFS property introduced in Guasoni et al. (2008). A $\mathbb{R}^d$-valued continuous process $X_t = (X_t^1, X_t^d, \ldots, X_t^d)$ has the CFS property if

$$Supp P(X|_{[t,T]}|\mathscr{F}_t) = C_{X_t}[t, T] \text{ a.s.}$$

where $C_x[t, T]$ denotes the space of $\mathbb{R}^d-$valued continuous functions that start from x at t, $P(X|_{[t,T]}|\mathscr{F}_t)$ denotes the $\mathscr{F}_t$-conditional distribution of the $C_x[t, T]$-valued random variable $X|_{[t,T]}$, and "Supp" denotes the support (i.e., the smallest closed set of probability one).

It is clear that any process with the CFS property satisfies the joint $\mathbb{F} - CHUDD$ condition for any $c > 0$. So far, in multi-dimension, the CFS property was shown for continuous Markov processes with the full support property, see Guasoni et al. (2008), and also for vector valued processes with independent fBm components with possibility different Hurst parameters, see Sayit and Viens (2011).

Example 3 Let $X_t = (X_t^1, X_t^d, \dots, X_t^d)$ be a continuous process with the CFS property. Let $V_t^1, V_t^2, \dots, V_t^d$ be any sequence of adapted, cádlág, bounded processes. Let $Y_t^i = e^{X_t^i - \sqrt{|X_t^i|} + V_t^i}$, $1 \le i \le d$. Then the market $(1, Y_t^1, Y_t^2, \dots, Y_t^d)$ is arbitrage-free within the class $\mathscr{L}(\mathbb{F})$ of trading strategies. Note that the function $x - \sqrt{|x|}$ belongs to $\mathscr{C}$ and e^x is a strictly monotone function. Therefore the claim follows from Propositions 1, 2, 3, 4, 5.

Acknowledgments I would like to express my gratitude to Philip Protter, Christian Bender, and the two anonymous referees for their valuable comments from which the manuscript greatly benefited.

References

Bayraktar, E., Sayit, H.: No arbitrage conditions for simple trading strategies. Ann Finance **6**(1), 147–156 (2010)

Bender, C.: Simple arbitrage. Ann Appl Probab (2010) (to appear). http://www.math.uni-sb.de/ag-bender/bender_publikationen.html

Bender, C., Sottinen, T., Valkeila, E.: Fractional processes as models in stochastic finance. In: Di Nunno, G., Øskendal, B. (eds.) AMaMef: Advanced Mathematical Methods for Finance. Berlin: Springer (2011). doi:10.1007/978-3-642-18412-3-3

Bender, C., Sottinen, T., Valkeila, E.: Pricing by hedging an no-arbitrage beyond semimartingales. Finance Stoch **12**(4), 441–468 (2008)

Cheridito, P.: Arbitrage in fractional brownian motion models. Finance Stoch **7**(4), 533–553 (2003)

Cherny, A.S.: Brownian moving averages have conditional full support. Ann Appl Probab **18**(5), 1825–1830 (2008)

Delbaen, F., Schachermayer, W.: A general version of the fundamental theorem of asset pricing. Math Ann **300**, 463–520 (1994)

Gasbarra, D., Sottinen, T., Van Zanten, H.: Conditional full support of gaussian processes with stationary increments. J Appl Probab **48**(2), 561–568 (2008)

Guasoni, P.: No arbitrage with transaction costs, with fractional brownian motion and beyond. Math Finance **16**(2), 469–588 (2006)

Guasoni, P., Rasonyi, M., Schachermayer, W.: Consistent price systems and face-lifting pricing under transaction costs. Ann Appl Probab **18**(2), 491–520 (2008)

Guasoni, P., Rasonyi, R., Schachermayer, W.: The fundamental theorem of asset pricing for continuous processes under small transaction costs. Ann Finance **6**(2), 157–191 (2010)

Jarrow, R., Protter, P., Sayit, H.: No arbitrage without semimartingales. Ann Appl Probab **19**(2), 596–616 (2009)

Lo, A.W.: Long-term memory in stock market prices. Econometrica **59**(5), 1279–1313 (1991)

Maris, F., Mbakop, E., Sayit, H.: Consistent price systems for bounded processes. Commun Stoch Anal **5**(4), 616–679 (2011)

Maris, F., Sayit, H.: Consistent price systems in multi-asset markets. Int J Stoch Anal (2012) (to appear)

Pakkanen, M.: Stochastic integrals and conditional full support. J Appl Probab **47**(3), 650–667 (2010)

Protter, P.: Stochastic Integration and Differential Equations Second Edition, Version 2.1. Heidelberg: Springer (2005)

Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion 3rd edn. Heidelberg: Springer (1999)

Rogers, L.C.G.: Arbitrage with fractional brownian motion. Math Finance **7**, 95–105 (1997)

Rogers, L.C.G.: Equivalent martingale measures and no-arbitrage. Stoch Stoch Rep **51**(1&2), 41–49 (1994)

Sayit, H., Viens, F.: Arbitrage-free models in markets with transaction costs. Electron Commun Probab **16**, 614–622 (2011)

Sottinen, T.: Fractional brownian motion, random walks and binary market models. Finance Stoch **5**(3), 343–355 (2001)

Willinger, W., Taqqu, M.S., Teverovsky, V.: Stock market prices and long-range dependence. Finance Stoch **3**(1), 1–13 (1999)