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RESEARCH PAPER

Fuzzy finite difference method for heat conduction analysis with uncertain parameters

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Abstract A new numerical technique named as fuzzy finite difference method is proposed to solve the heat conduction problems with fuzzy uncertainties in both the physical parameters and initial/boundary conditions. In virtue of the level-cut method, the difference discrete equations with fuzzy parameters are equivalently transformed into groups of interval equations. New stability analysis theory suited to fuzzy difference schemes is developed. Based on the parameter perturbation method, the interval ranges of the uncertain temperature field can be approximately predicted. Subsequently, fuzzy solutions to the original difference equations are obtained by the fuzzy resolution theorem. Two numerical examples are given to demonstrate the feasibility and efficiency of the presented method for solving both steady-state and transient heat conduction problems.

Keywords Heat conduction · Fuzzy uncertainties · Finite difference method · Parameter perturbation · Stability analysis

1 Introduction

Thermal analysis has undergone a rapid development in engineering, especially in the field of aeronautics and astronautics, where the coupled interaction of structure and heat conduction is playing a more and more significant role. Traditional thermal analysis has been conducted by solving the heat conduction equations under the assumption that the physical properties and initial/boundary conditions are deterministic. But in actual engineering projects, due to model inaccuracies, physical imperfections and system complexities, uncertainties in material properties, geometric dimensions and initial/boundary conditions are unavoidable, which will lead to the uncertainty of the structural temperature field [1–3].

The main approaches to model above uncertainties include three categories: stochastic analysis, fuzzy set and interval model [4]. When there exists substantial statistical information, the probability theory offers a powerful mathematical framework to represent such uncertainties. The probabilistic approaches, such as Monte Carlo simulation, stochastic perturbation method and stochastic spectral method, can be designated as the most valuable solution strategies [5–7]. Reliable application of probabilistic methods requires enough information to construct the probability density functions of uncertain parameters, which are not easily available for many complex practical problems. In such situations, non-probabilistic approaches, such as interval algebra [8], convex models [9] and fuzzy set [10] can be used. In this paper, the system uncertainties are described as fuzzy variables. The fuzzy set theory, introduced by Zadeh [11] in 1965, is very suitable for representing the uncertain parameter whose subjective probability based on the expert opinions is available. Combined with the finite element method, the fuzzy numerical analysis has achieved many research results regarding non-deterministic models [12–14].

Over last several decades, the uncertain analysis of heat conduction problems has not developed to the same level as structural mechanics, but it still gained some valuable results. Hien and Kleiber [15] suggested the stochastic variational principle and stochastic finite element method for transient heat problem with random parameters. Kaminski and Hien [16] tested the stochastic finite element method with a laminated composite plate and examined the impact on the

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uncertain temperature response made by every single random variable. By combining boundary element method with orthogonal expansion theory, a new numerical method was proposed for solving stochastic heat transfer problems [17]. On the basis of generalized polynomial chaos, Xiu [18] presented a random spectral decomposition method for the solution of transient heat conduction subjected to random inputs. From the current research on uncertain heat conduction we can see that the development of numerical computing techniques are mainly concentrated in the field of stochastic analysis with finite element method. However, the finite difference method [19], with the unique advantages of intuitive mathematical concepts, optional precision and easy programming, should be emphasized in the uncertain thermal analysis.

Hence, deriving the fuzzy difference method with broader applicability is significant in both theory and engineering practice. In view of this need, we take the heat conduction problems with kinds of fuzzy parameters as an example to propose the new method in this paper. Besides the computing formulae of the fuzzy temperature field, we will develop the stability theory applicable to fuzzy difference schemes.

2 Basis of fuzzy mathematics

Definition 1: Given the definition domain U, the mapping

$$A: \quad U \to [0,1], \quad u| \to A(u) \tag{1}$$

is called a fuzzy set. $A(u)$ is denoted as the membership function, whose values locate in a closed interval $[0,1]$. If only the point values 0 and 1 are suitable, the fuzzy set A degenerates into a normal one, which means that the normal set is a special form of the fuzzy sets.

Definition 2: Given a fuzzy set A, for any $\lambda \in [0,1]$, the normal set

$$A_\lambda = \{u | u \in U, A(u) \geqslant \lambda\} \tag{2}$$

is defined as the λ cut set, where λ is the cut level. Usually, the cut sets are considered as intervals of confidence, since in the case of convex fuzzy sets, they are closed intervals associated with a gradation of confidence between $[0,1]$.

Definition 3: For any $\lambda \in [0,1]$, the mathematical operation

$$(\lambda A)(u) = \lambda \wedge A(u) = \min\{\lambda, A(u)\} \tag{3}$$

is called the multiplying operator of real number λ and fuzzy set A. Obviously, the product λA is also a fuzzy set.

Decomposition Theorem: In virtue of the cut sets, any fuzzy set A defined in the domain U can be visually expressed by normal sets

$$A = \bigcup_{\lambda \in [0,1]} (\lambda A_\lambda). \tag{4}$$

3 Steady-state heat conduction problem

The governing equation of a two-dimensional steady-state heat conduction problem with a heat source can be written as

$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + f(x,y) = 0, \quad (x,y) \in G, \tag{5}$$

where G is a bounded domain, $u = u(x,y)$ stands for the temperature field, k represents the heat conductivity of the material, and $f(x,y)$ is the intensity of the heat source. In order to determine the solution to the elliptic equation (5), here we provide three commonly used types of boundary conditions as follows.

Dirichlet Condition: the temperature u_{w} on the boundary is given as

$$u|_\Gamma = u_{\mathrm{w}}. \tag{6}$$

Neumann Condition: the heat flux $\boldsymbol{q}$ is given as

$$-k\frac{\partial u}{\partial \boldsymbol{n}}\bigg|_\Gamma = \boldsymbol{q}, \tag{7}$$

where $\boldsymbol{n}$ is the unit normal vector of the boundary.

Robin Condition: the temperature u_{f} of the fluid circumstance and the heat transfer coefficient $\boldsymbol{h}$ are known

$$-k\frac{\partial u}{\partial \boldsymbol{n}}\bigg|_\Gamma = \boldsymbol{h}(u - u_{\mathrm{f}}). \tag{8}$$

In the finite difference method, derivatives in control equations are usually replaced by difference schemes. Here we just take the five-point scheme with second-order accuracy into account and gradually introduce the matrix expression of steady-state heat conduction equation with fuzzy parameters. Firstly, choose space steps h_1, h_2 along the x-axis and y-axis respectively, and then establish two groups of straight lines parallel to the axis to divide the bounded region G into finite units. Subsequently, denote the temperature at grid node (x_i, y_j) by $u_{i,j}$, adopt second-order central difference schemes to replace the derivative terms $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ in Eq. (5), and obtain the following discrete expression

$$k\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h_1^2} + k\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h_2^2} + f_{i,j} = 0, \tag{9}$$

which can be rewritten as

$$\frac{k}{h_2^2}u_{i,j-1} + \frac{k}{h_1^2}u_{i-1,j} - \left(\frac{2k}{h_1^2} + \frac{2k}{h_2^2}\right)u_{i,j} + \frac{k}{h_1^2}u_{i+1,j} + \frac{k}{h_2^2}u_{i,j+1} = -f_{i,j}. \tag{10}$$

By means of matrices and vectors, the commonly used finite difference formats including Eq. (10) can always be expressed as an algebraic equation

$$\boldsymbol{KU} = \boldsymbol{F}, \tag{11}$$

where $\boldsymbol{U}$ is the temperature vector at all nodes; $\boldsymbol{K}$ represents the thermal stiffness matrix; $\boldsymbol{F}$ stands for the thermal load vector.

In this paper, fuzziness in both system parameters and boundary conditions, such as heat conductivity k, intensity of the heat source f, heat flux $\boldsymbol{q}$, boundary temperature u_{w}, ambient temperature u_{f} and heat transfer coefficient $\boldsymbol{h}$, are all taken into account simultaneously. From various approaches to deal with boundary conditions [20], we know that fuzziness in every parameter will lead to certain uncertainties in coefficient matrix $\boldsymbol{K}$ or vector $\boldsymbol{F}$ in Eq. (11), so that the temperature field $\boldsymbol{U}$ will present certain fuzzy uncertainty.

Supposed that all the uncertain parameters can be denoted as a fuzzy vector

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_m)^{\mathrm{T}}, \tag{12}$$

the difference equation of the steady-state heat conduction problem with fuzzy parameters can be expressed as

$$\boldsymbol{K}(\boldsymbol{\alpha})\boldsymbol{U} = \boldsymbol{F}(\boldsymbol{\alpha}). \tag{13}$$

Similarly, for the high-dimensional fuzzy heat conduction equation, by using difference schemes to approximate the spatial derivative in every direction, we can still derive its matrix expression like Eq. (13).

4 Transient heat conduction problem

For simplicity, here we just take the following one-dimensional transient heat conduction equation as an instance to introduce the common difference discrete schemes

$$\rho c \frac{\partial u}{\partial t} + k \frac{\partial^2 u}{\partial x^2} + f(x) = 0, \quad 0 < t \leqslant T, \tag{14}$$

where $u = u(x, t)$ is the unsteady temperature field; ρ and c are the density, specific heat capacity of the material, respectively.

Besides the boundary conditions as shown in Eqs. (6)–(8), an initial condition is necessary for solving transient problems. We assume that the distribution law $\phi(x)$ of the initial temperature field is known, i.e.,

$$u(x, 0) = \phi(x). \tag{15}$$

Space step Δx and time step Δt are used to divide the spatial domain and time domain, respectively. Then denote the temperature at grid node (x_j, t_k) by u_j^k, and adopt forward difference scheme in time and central difference scheme in space to compose a discrete equation

$$\rho c \frac{u_j^{k+1} - u_j^k}{\Delta t} + k \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{\Delta x^2} + f_j = 0. \tag{16}$$

By introducing a transition parameter $r = k\Delta t/(\rho c \Delta x^2)$, Eq. (16) can be simplified as

$$u_j^{k+1} = -r u_{j-1}^k + (1 + 2r) u_j^k - r u_{j+1}^k - \frac{\Delta t}{\rho c} f_j, \tag{17}$$

which is called the forward difference scheme.

Similarly, by adopting backward difference in time and central difference in space at grid node (x_j, t_{k+1}), the so-called backward difference scheme is presented to be

$$r u_{j-1}^{k+1} + (1 - 2r) u_j^{k+1} + r u_{j+1}^{k+1} = u_j^k - \frac{\Delta t}{\rho c} f_j. \tag{18}$$

Combining the above two schemes to extract the arithmetic mean, one can get the six-node symmetrical difference format as follows

$$\begin{aligned} &\frac{r}{2} u_{j-1}^{k+1} + (1 - r) u_j^{k+1} + \frac{r}{2} u_{j+1}^{k+1}, \\ &\quad = -\frac{r}{2} u_{j-1}^k + (1 + r) u_j^k - \frac{r}{2} u_{j+1}^k - \frac{\Delta t}{\rho c} f_j, \end{aligned} \tag{19}$$

In all, the commonly used two-layer finite difference formats such as Eqs. (17)–(19) can be expressed in the form of matrices and vectors, i.e.,

$$\boldsymbol{A}\boldsymbol{U}^{k+1} = \boldsymbol{B}\boldsymbol{U}^k + \boldsymbol{F}, \tag{20}$$

where $\boldsymbol{U}^{\boldsymbol{k}} = (u_1^k, u_2^k, \cdots, u_N^k)^{\mathrm{T}}$, $\boldsymbol{F} = -\dfrac{\Delta t}{\rho c}(f_1, f_2, \cdots, f_N)^{\mathrm{T}}$; $\boldsymbol{A}$, $\boldsymbol{B}$ are $N \times N$-order matrices.

In virtue of the fuzzy vector, we can denote the recurrence equation (20) with fuzzy parameters as

$$\boldsymbol{A}(\boldsymbol{\alpha})\boldsymbol{U}^{k+1} = \boldsymbol{B}(\boldsymbol{\alpha})\boldsymbol{U}^k + \boldsymbol{F}(\boldsymbol{\alpha}). \tag{21}$$

Besides, by extending the coefficient matrices and related vectors, such an expression as Eq. (21) could be derived for high-dimensional fuzzy transient heat conduction problems.

5 Stability analysis

As we know, fuzzy variables can be viewed as a generalization of interval variables. Considering values of interval variables lie within the lower and upper bounds, the fuzzy approach generalizes this concept by introducing a membership function. Thus, for any $\lambda \in [0, 1]$, the fuzzy parameter vector can be visually expressed by an interval vector, i.e.,

$$\begin{aligned} \boldsymbol{\alpha}_\lambda &= ((\alpha_1)_\lambda, (\alpha_2)_\lambda, \cdots, (\alpha_m)_\lambda)^{\mathrm{T}} \\ &= (\alpha_{1,\lambda}^{\mathrm{I}}, \alpha_{2,\lambda}^{\mathrm{I}}, \cdots, \alpha_{m,\lambda}^{\mathrm{I}})^{\mathrm{T}} = \boldsymbol{\alpha}_\lambda^{\mathrm{I}}. \end{aligned} \tag{22}$$

Therefore, given one fixed cut level, the original discrete equations (13) and (21) with fuzzy parameters can be transformed into the following formats with interval parameters, respectively,

$$\boldsymbol{K}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}})\boldsymbol{U} = \boldsymbol{F}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}), \tag{23}$$

$$\boldsymbol{A}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}})\boldsymbol{U}^{k+1} = \boldsymbol{B}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}})\boldsymbol{U}^k + \boldsymbol{F}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}). \tag{24}$$

Beside the interval vector $\boldsymbol{\alpha}_\lambda^{\mathrm{I}}$, it is obvious that coefficient matrices $\boldsymbol{A}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}})$, $\boldsymbol{B}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}})$ and right vector $\boldsymbol{F}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}})$ in Eq. (24) also depend on the time step Δt and space step Δx. Supposing that for any $\boldsymbol{\alpha}^* \in \boldsymbol{\alpha}_\lambda^{\mathrm{I}}$, the matrix $\boldsymbol{A}(\boldsymbol{\alpha}^*, \Delta t, \Delta x)$ is always nonsingular, thus we can introduce another parameter matrix

$$\boldsymbol{C}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}, \Delta t, \Delta x) = \boldsymbol{A}^{-1}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}, \Delta t, \Delta x) \times \boldsymbol{B}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}, \Delta t, \Delta x). \tag{25}$$

For a discrete equation with an initial condition, if it can be ensured that the error introduced in one time layer would not be constantly amplified in the later time, then the difference discrete format is said stable [21]. Accordingly, given the initial condition $\boldsymbol{U}^0 = \boldsymbol{\Psi}$, Eq. (24) can be rewritten as

$$\begin{aligned}\boldsymbol{U}^{k+1} &= \boldsymbol{C}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}, \Delta t, \Delta x)\boldsymbol{U}^k \\ &\quad + \boldsymbol{A}^{-1}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}, \Delta t, \Delta x) \times \boldsymbol{F}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}, \Delta t, \Delta x), \\ \boldsymbol{U}^0 &= \boldsymbol{\Psi}.\end{aligned} \tag{26}$$

In this paper, the error $\boldsymbol{\varepsilon}^0$ in initial condition is taken into account to investigate the interval stability. Denoting the corresponding solution by $\tilde{\boldsymbol{U}}$, one can get

$$\begin{aligned}\tilde{\boldsymbol{U}}^{k+1} &= \boldsymbol{C}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}, \Delta t, \Delta x)\tilde{\boldsymbol{U}}^k \\ &\quad + \boldsymbol{A}^{-1}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}, \Delta t, \Delta x) \times \boldsymbol{F}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}, \Delta t, \Delta x), \\ \tilde{\boldsymbol{U}}^0 &= \boldsymbol{\Psi} + \boldsymbol{\varepsilon}^0.\end{aligned} \tag{27}$$

By using $\boldsymbol{\varepsilon}^{k+1}$ to express the error in every time layer, and subtracting Eq. (27) from Eq. (26), we obtain

$$\begin{aligned}\boldsymbol{\varepsilon}^{k+1} &= \tilde{\boldsymbol{U}}^{k+1} - \boldsymbol{U}^{k+1} = \boldsymbol{C}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}, \Delta t, \Delta x)(\tilde{\boldsymbol{U}}^k - \boldsymbol{U}^k) \\ &= \boldsymbol{C}^{k+1}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}, \Delta t, \Delta x)\boldsymbol{\varepsilon}^0, \\ \boldsymbol{\varepsilon}^0 &= \tilde{\boldsymbol{U}}^0 - \boldsymbol{U}^0.\end{aligned} \tag{28}$$

If there exists a constant $K > 0$, for any $\boldsymbol{U}^0 \in \boldsymbol{R}^N$, $0 < \Delta t \leqslant \Delta t_0$, $0 < \Delta x \leqslant \Delta x_0$, $\boldsymbol{\alpha}^* \in \boldsymbol{\alpha}_\lambda^{\mathrm{I}}$, the following inequation is always satisfied,

$$\left\|\boldsymbol{\varepsilon}^{k+1}\right\| = \left\|\boldsymbol{C}^{k+1}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}, \Delta t, \Delta x)\boldsymbol{\varepsilon}^0\right\| \leqslant K\left\|\boldsymbol{\varepsilon}^0\right\|, \tag{29}$$

and then the interval difference format is considered to be stable.

In other words, the necessary and sufficient condition to ensure the stability is

$$\left\|\boldsymbol{C}^{k+1}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}, \Delta t, \Delta x)\right\| \leqslant K, \tag{30}$$

which means that the interval matrix family

$$\left\{\boldsymbol{C}^{k+1}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}, \Delta t, \Delta x)\middle| 0 < \Delta t \leqslant \Delta t_0, 0 < \Delta x \leqslant \Delta x_0\right\} \tag{31}$$

is uniformly bounded.

Based on the traditional Von Neumann qualification, for any $\boldsymbol{\alpha}^* \in \boldsymbol{\alpha}_\lambda^{\mathrm{I}}$, the spectral radius of the interval coefficient matrix $\boldsymbol{C}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}, \Delta t, \Delta x)$ is required to satisfy

$$\rho(\boldsymbol{C}(\boldsymbol{\alpha}^*, \Delta t, \Delta x)) \leqslant 1 + O(\Delta t, \Delta x), \quad \forall \boldsymbol{\alpha}^* \in \boldsymbol{\alpha}_\lambda^{\mathrm{I}}, \tag{32}$$

which is called the interval Von Neumann condition to analyze the stability of interval difference scheme.

For any cut level $\lambda \in [0, 1]$, if the stability condition such as inequation (32) is always satisfied to the corresponding interval difference format,

$$\max_{\boldsymbol{\alpha}^* \in \boldsymbol{\alpha}_\lambda^{\mathrm{I}}} \rho(\boldsymbol{C}(\boldsymbol{\alpha}^*, \Delta t, \Delta x)) \leqslant 1 + O(\Delta t, \Delta x), \quad \forall \lambda \in [0, 1], \tag{33}$$

then the original fuzzy difference equation (21) is claimed to be stable regarding the initial values.

6 Parameter perturbation method

Parameter perturbation technique, combined with the finite element method, has gained many research results in structural mechanics with uncertain parameters [22], while it is yet unexplored in thermal analysis. In this section, we will focus on the application of parameter perturbation in the fuzzy finite difference method to deal with uncertain heat conduction problems.

In Eq. (23), expand the coefficient matrix $\boldsymbol{K}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}})$ and right vector $\boldsymbol{F}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}})$ at the midpoints of the interval parameters, ignore the terms higher than the second order ones and obtain the following approximate expressions according to the interval expansion principle [23].

$$\begin{aligned}\boldsymbol{K}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}) &= \boldsymbol{K}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}}) + \sum_{i=1}^{m} \frac{\partial \boldsymbol{K}}{\partial \alpha_{i,\lambda}}\bigg|_{\boldsymbol{\alpha}_\lambda^{\mathrm{c}}} (\alpha_{i,\lambda}^{\mathrm{I}} - \alpha_{i,\lambda}^{\mathrm{c}}) \\ &= \boldsymbol{K}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}}) + \sum_{i=1}^{m} \frac{\partial \boldsymbol{K}}{\partial \alpha_{i,\lambda}}\bigg|_{\boldsymbol{\alpha}_\lambda^{\mathrm{c}}} \Delta\alpha_{i,\lambda}\delta_i \\ &= \boldsymbol{K}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}}) + \Delta\boldsymbol{K}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}), \\ \boldsymbol{F}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}) &= \boldsymbol{F}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}}) + \sum_{i=1}^{m} \frac{\partial \boldsymbol{F}}{\partial \alpha_{i,\lambda}}\bigg|_{\boldsymbol{\alpha}_\lambda^{\mathrm{c}}} (\alpha_{i,\lambda}^{\mathrm{I}} - \alpha_{i,\lambda}^{\mathrm{c}}) \\ &= \boldsymbol{F}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}}) + \sum_{i=1}^{m} \frac{\partial \boldsymbol{F}}{\partial \alpha_{i,\lambda}}\bigg|_{\boldsymbol{\alpha}_\lambda^{\mathrm{c}}} \Delta\alpha_{i,\lambda}\delta_i \\ &= \boldsymbol{F}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}}) + \Delta\boldsymbol{F}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}),\end{aligned} \tag{34}$$

where $\alpha_{i,\lambda}^{\mathrm{c}} = (\overline{\alpha}_{i,r} + \underline{\alpha}_{i,r})/2$ and $\Delta\alpha_{i,r} = (\overline{\alpha}_{i,r} - \underline{\alpha}_{i,r})/2$ are the midpoint and the radius of the interval number $\alpha_{i,\lambda}^{\mathrm{I}}$, respectively; the transition parameter δ_i belongs to a fixed interval, i.e. $\delta_i = [-1, 1]$.

The steady-state interval temperature field can be denoted as the sum of midpoint and radius under the λ cut level

$$\boldsymbol{U}_\lambda^{\mathrm{I}} = \boldsymbol{U}_\lambda^{\mathrm{c}} + \boldsymbol{\Delta}(\boldsymbol{U}_\lambda)^{\mathrm{I}}. \tag{35}$$

Substituting Eqs. (34) and (35) into Eq. (23)

$$(\boldsymbol{K}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}}) + \Delta\boldsymbol{K}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}))(\boldsymbol{U}_\lambda^{\mathrm{c}} + \boldsymbol{\Delta}(\boldsymbol{U}_\lambda)^{\mathrm{I}}) = \boldsymbol{F}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}}) + \Delta\boldsymbol{F}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}) \tag{36}$$

and ignoring the cross second-order small terms, one can obtain an approximate solution

$$\begin{aligned}\boldsymbol{U}_\lambda^{\mathrm{c}} &= \boldsymbol{K}^{-1}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}})\boldsymbol{F}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}}), \\ \boldsymbol{\Delta}(\boldsymbol{U}_\lambda)^{\mathrm{I}} &= \boldsymbol{K}^{-1}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}})(\Delta\boldsymbol{F}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}) - \Delta\boldsymbol{K}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}})\boldsymbol{U}_\lambda^{\mathrm{c}}) \\ &= \boldsymbol{K}^{-1}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}})\Bigg(\sum_{i=1}^{m} \frac{\partial \boldsymbol{F}}{\partial \alpha_{i,\lambda}}\bigg|_{\boldsymbol{\alpha}_\lambda^{\mathrm{c}}} \Delta\alpha_{i,\lambda}\delta_i \\ &\quad - \sum_{i=1}^{m} \frac{\partial \boldsymbol{K}}{\partial \alpha_{i,\lambda}}\bigg|_{\boldsymbol{\alpha}_\lambda^{\mathrm{c}}} \Delta\alpha_{i,\lambda}\delta_i\boldsymbol{U}_\lambda^{\mathrm{c}}\Bigg) \\ &= \sum_{i=1}^{m} \boldsymbol{K}^{-1}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}})\Bigg(\frac{\partial \boldsymbol{F}}{\partial \alpha_{i,\lambda}}\bigg|_{\boldsymbol{\alpha}_\lambda^{\mathrm{c}}} - \frac{\partial \boldsymbol{K}}{\partial \alpha_{i,\lambda}}\bigg|_{\boldsymbol{\alpha}_\lambda^{\mathrm{c}}} \boldsymbol{U}_\lambda^{\mathrm{c}}\Bigg)\Delta\alpha_{i,\lambda}\delta_i \\ &= \boldsymbol{\Delta}(\boldsymbol{U}_\lambda)\delta = \boldsymbol{\Delta}(\boldsymbol{U}_\lambda) \cdot [-1, 1],\end{aligned} \tag{37}$$

where

$$\boldsymbol{\Delta}(\boldsymbol{U}_\lambda) = \sum_{i=1}^{m} \left|\boldsymbol{K}^{-1}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}})\Bigg(\frac{\partial \boldsymbol{F}}{\partial \alpha_{i,\lambda}}\bigg|_{\boldsymbol{\alpha}_\lambda^{\mathrm{c}}} - \frac{\partial \boldsymbol{K}}{\partial \alpha_{i,\lambda}}\bigg|_{\boldsymbol{\alpha}_\lambda^{\mathrm{c}}} \boldsymbol{U}_\lambda^{\mathrm{c}}\Bigg)\Delta\alpha_{i,\lambda}\right|. \tag{38}$$

The interval of the steady-state fuzzy temperature field under the λ cut level can be calculated by

$$\boldsymbol{U}_\lambda^{\mathrm{I}} = [\boldsymbol{U}_\lambda^{\mathrm{c}} - \boldsymbol{\Delta}(\boldsymbol{U}_\lambda), \boldsymbol{U}_\lambda^{\mathrm{c}} + \boldsymbol{\Delta}(\boldsymbol{U}_\lambda)]. \tag{39}$$

Similarly, in terms of the Taylor expansion series, the interval coefficient matrices and vector in Eq. (24) can be expressed as the sum of midpoint and radius. Subsequently, based on the matrix perturbation theory, the approximate calculations of the transient interval temperature field can be obtained by

$$\begin{aligned}
(\boldsymbol{U}_\lambda^{k+1})^{\mathrm{c}} &= \boldsymbol{A}^{-1}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}})(\boldsymbol{B}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}})(\boldsymbol{U}_\lambda^{k})^{\mathrm{c}} + \boldsymbol{F}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}})),\\
\boldsymbol{\Delta}(\boldsymbol{U}_\lambda^{k+1})^{\mathrm{I}} &= \boldsymbol{A}^{-1}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}})(\Delta\boldsymbol{F}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}}) + \Delta\boldsymbol{B}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}})(\boldsymbol{U}_\lambda^{k})^{\mathrm{c}}\\
&\quad + \boldsymbol{B}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}})\boldsymbol{\Delta}(\boldsymbol{U}_\lambda^{k})^{\mathrm{I}} - \Delta\boldsymbol{A}(\boldsymbol{\alpha}_\lambda^{\mathrm{I}})(\boldsymbol{U}_\lambda^{k+1})^{\mathrm{c}})\\
&= \boldsymbol{A}^{-1}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}})\Big(\sum_{i=1}^{m}\frac{\partial\boldsymbol{F}}{\partial\alpha_{i,\lambda}}\Big|_{\boldsymbol{\alpha}_\lambda^{\mathrm{c}}}\Delta\alpha_{i,\lambda}\delta_i\\
&\quad + \sum_{i=1}^{m}\frac{\partial\boldsymbol{B}}{\partial\alpha_{i,\lambda}}\Big|_{\boldsymbol{\alpha}_\lambda^{\mathrm{c}}}\Delta\alpha_{i,\lambda}\delta_i(\boldsymbol{U}_\lambda^{k})^{\mathrm{c}} + \boldsymbol{B}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}})\boldsymbol{\Delta}(\boldsymbol{U}_\lambda^{k})\delta\\
&\quad - \sum_{i=1}^{m}\frac{\partial\boldsymbol{A}}{\partial\alpha_{i,\lambda}}\Big|_{\boldsymbol{\alpha}_\lambda^{\mathrm{c}}}\Delta\alpha_{i,\lambda}\delta_i(\boldsymbol{U}_\lambda^{k+1})^{\mathrm{c}}\Big)\\
&= \sum_{i=1}^{m}\boldsymbol{A}^{-1}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}})\Big(\frac{\partial\boldsymbol{F}}{\partial\alpha_{i,\lambda}}\Big|_{\boldsymbol{\alpha}_\lambda^{\mathrm{c}}} + \frac{\partial\boldsymbol{B}}{\partial\alpha_{i,\lambda}}\Big|_{\boldsymbol{\alpha}_\lambda^{\mathrm{c}}}(\boldsymbol{U}_\lambda^{k})^{\mathrm{c}}\\
&\quad - \frac{\partial\boldsymbol{A}}{\partial\alpha_{i,\lambda}}\Big|_{\boldsymbol{\alpha}_\lambda^{\mathrm{c}}}(\boldsymbol{U}_\lambda^{k+1})^{\mathrm{c}}\Big)\Delta\alpha_{i,\lambda}\delta_i\\
&\quad + \boldsymbol{A}^{-1}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}})\boldsymbol{B}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}})\boldsymbol{\Delta}(\boldsymbol{U}_\lambda^{k})\delta\\
&= \boldsymbol{\Delta}(\boldsymbol{U}_\lambda^{k+1})\delta\\
&= \boldsymbol{\Delta}(\boldsymbol{U}_\lambda^{k+1})\cdot[-1,1],
\end{aligned} \tag{40}$$

where

$$\begin{aligned}
\boldsymbol{\Delta}(\boldsymbol{U}_\lambda^{k+1}) &= \sum_{i=1}^{m}\Big|\boldsymbol{A}^{-1}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}})\Big(\frac{\partial\boldsymbol{F}}{\partial\alpha_{i,\lambda}}\Big|_{\boldsymbol{\alpha}_\lambda^{\mathrm{c}}} + \frac{\partial\boldsymbol{B}}{\partial\alpha_{i,\lambda}}\Big|_{\boldsymbol{\alpha}_\lambda^{\mathrm{c}}}(\boldsymbol{U}_\lambda^{k})^{\mathrm{c}}\\
&\quad - \frac{\partial\boldsymbol{A}}{\partial\alpha_{i,\lambda}}\Big|_{\boldsymbol{\alpha}_\lambda^{\mathrm{c}}}(\boldsymbol{U}_\lambda^{k+1})^{\mathrm{c}}\Big)\Delta\alpha_{i,\lambda}\Big|\\
&\quad + \Big|\boldsymbol{A}^{-1}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}})\boldsymbol{B}(\boldsymbol{\alpha}_\lambda^{\mathrm{c}})\boldsymbol{\Delta}(\boldsymbol{U}_\lambda^{k})\Big|.
\end{aligned} \tag{41}$$

Therefore, the interval range of the transient fuzzy temperature field at every time layer under the λ cut level can be expressed as

$$(\boldsymbol{U}_\lambda^{k+1})^{\mathrm{I}} = [(\boldsymbol{U}_\lambda^{k+1})^{\mathrm{c}} - \Delta(\boldsymbol{U}_\lambda^{k+1}), (\boldsymbol{U}_\lambda^{k+1})^{\mathrm{c}} + \Delta(\boldsymbol{U}_\lambda^{k+1})]. \tag{42}$$

Finite cut levels $\lambda_i \in [0,1]$ $i = 1, 2, \cdots, n$ are chosen to express the character of the fuzzy parameter vector. For any fixed cut level λ_i, the interval temperature field can be approximately calculated by using the above perturbation method. Thus, based on the fuzzy decomposition theorem, the final fuzzy temperature field of steady-state and transient thermal conduction problems can be expressed as

$$\boldsymbol{U} = \bigcup_{i=1,2,\cdots,n}(\lambda_i\boldsymbol{U}_{\lambda_i}^{\mathrm{I}}),\quad \boldsymbol{U}^{k+1} = \bigcup_{i=1,2,\cdots,n}(\lambda_i(\boldsymbol{U}_{\lambda_i}^{k+1})^{\mathrm{I}}). \tag{43}$$

7 Numerical examples

7.1 Steady-state fuzzy temperature field of a circular plate

In order to test the performance of the proposed method, here we consider a circular plate with the thickness of 0.1m, as shown in Fig. 1, where the left and right borders are coated with heat-insulating layers. Due to the errors in manufacture and measurement, some parameters present fuzzy uncertainties, where the membership functions of heating ambient temperature T_{uf}(°C), internal working temperature T_{df}(°C), heat conductivity k (W/(m·°C)), heat transfer coefficient h (W/(m^2·°C)) and heat source f (W/m^3) all satisfy mid-trapezoid distribution as follows

$$\mu(T_{\mathrm{uf}}) = \begin{cases} 0, & T_{\mathrm{uf}} \leqslant 85,\\ (T_{\mathrm{uf}}-85)/10, & 85 < T_{\mathrm{uf}} < 95,\\ 1, & 95 \leqslant T_{\mathrm{uf}} \leqslant 105,\\ (110-T_{\mathrm{uf}})/10, & 105 < T_{\mathrm{uf}} < 115,\\ 0, & T_{\mathrm{uf}} \geqslant 115,\end{cases}$$

$$\mu(T_{\mathrm{df}}) = \begin{cases} 0, & T_{\mathrm{df}} \leqslant 10,\\ (T_{\mathrm{df}}-10)/5, & 10 < T_{\mathrm{df}} < 15,\\ 1, & 15 \leqslant T_{\mathrm{df}} \leqslant 25,\\ (30-T_{\mathrm{df}})/5, & 25 < T_{\mathrm{df}} < 30,\\ 0, & T_{\mathrm{df}} \geqslant 30,\end{cases}$$

$$\mu(k) = \begin{cases} 0, & k \leqslant 15,\\ (k-15)/3, & 15 < k < 18,\\ 1, & 18 \leqslant k \leqslant 22,\\ (25-k)/3, & 22 < k < 25,\\ 0, & k \geqslant 25,\end{cases}$$

$$\mu(h) = \begin{cases} 0, & h \leqslant 70,\\ (h-70)/5, & 70 < h < 75,\\ 1, & 75 \leqslant h \leqslant 85,\\ (90-h)/5, & 85 < h < 90,\\ 0, & h \geqslant 90,\end{cases}$$

$$\mu(f) = \begin{cases} 0, & f \leqslant 90,\\ (f-90)/5, & 90 < f < 95,\\ 1, & 95 \leqslant f \leqslant 105,\\ (110-f)/5, & 105 < f < 110,\\ 0, & f \geqslant 110.\end{cases}$$

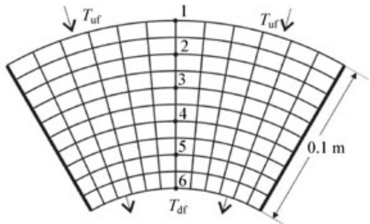

Fig. 1 Circular plate model and difference elements

Divide the plate uniformly along the thickness direction as shown in Fig. 1, and establish the algebraic equation based on the five-point difference scheme. Under the eleven cut levels $\lambda_k = (k-1) \times 0.1$, $k = 1, 2, \cdots, 11$, the lower and upper bounds of the interval temperature field can be efficiently calculated by the proposed parameter perturbation method.

From the results listed in Table 1, we can see that the ranges of interval temperatures at reference points gradually decrease when the cut level increases. Furthermore, in virtue of the decomposition theorem, corresponding membership functions of the fuzzy temperature field are plotted in Fig. 2.

Table 1 Bounds of the temperature response under different cut levels

Cut level λ	Bounds	Reference points 1	2	3	4	5	6
0	Lower	53.27	50.98	48.69	46.15	43.31	40.46
	Upper	80.61	77.40	74.19	71.26	68.67	66.07
0.3	Lower	55.73	53.36	50.99	48.41	45.60	42.79
	Upper	77.98	74.92	71.86	69.02	66.42	63.83
0.5	Lower	57.38	54.95	52.52	49.92	47.12	44.33
	Upper	76.25	73.28	70.31	67.52	64.92	62.33
0.7	Lower	59.03	56.54	54.05	51.43	48.65	45.86
	Upper	74.52	71.65	68.76	66.02	63.43	60.83
1.0	Lower	61.53	58.94	56.35	53.68	50.92	48.16
	Upper	71.96	69.21	66.45	63.77	61.17	58.58

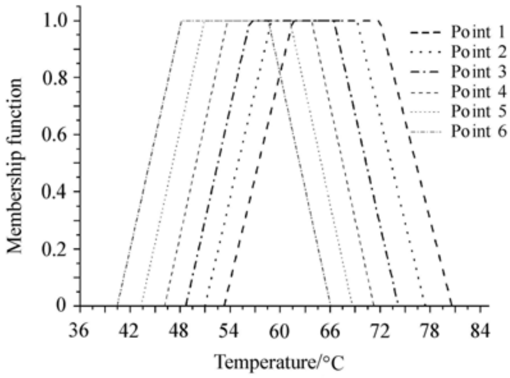

Fig. 2 Fuzzy temperature responses at reference points

It is illustrated that fuzzy uncertainties in physical parameters and boundary condition could cause non-negligible impact on the temperature field, whose membership function still satisfies trapezoid distribution.

7.2 Transient fuzzy temperature field of a square-sectioned column

Figure 3 depicts a 20 mm × 10 mm square-sectioned infinitely long column with adiabatic lower and upper borders. It is supposed that the physical parameters, initial value and boundary conditions are all fuzzy, where the membership functions of heat conductivity k (W/(m·°C)), specific heat capacity c (J/(kg·°C)), material density ρ (kg/m^3), initial temperature $T|_{t=0}$(°C),ambient temperature $T_{\rm f}$(°C) and heat transfer coefficient h (W/(m^2·°C)) all satisfy the following normal distribution

$$\mu(x) = \begin{cases} \exp[-(x-a_\alpha)^2/b_\alpha^2], & |x-a_\alpha| \leqslant c_\alpha, \\ 0, & |x-a_\alpha| > c_\alpha. \end{cases} \quad (44)$$

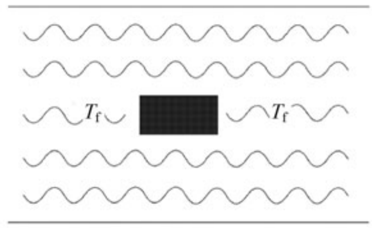

Fig. 3 Square-sectioned column

For every fuzzy parameter, the specific values of characteristic parameters a_α, b_α, c_α in Eq. (44) are listed in Table 2.

Table 2 Specific values of characteristic parameters

Characteristic parameters	Fuzzy parameters k	c	ρ	$T\|_{t=0}$	$T_{\rm f}$	h
a_α	16	200	5 000	20	100	1 600
b_α	0.9	4.7	23.3	0.9	2.3	23.3
c_α	2	10	50	2	5	50

In order to calculate the structural transient temperature field in the heating process, space steps $\Delta x = \Delta y = 1$ mm are selected to divide the model into 20 × 10 square elements by two groups of parallel lines as shown in Fig. 4. To ensure the fuzzy stability of the six-node symmetrical difference format with second-order accuracy, we set the time step to be $\Delta t = 0.01$ s. It should be noted that owing to the symmetry of structure about $x = 10$ mm, the analysis needs to cover only half of the structure in the calculating process, which can effectively reduce the computational cost.

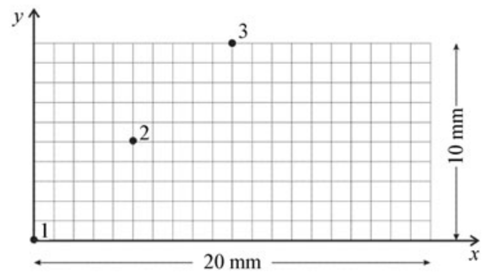

Fig. 4 Discrete diagram of difference elements

Given the uniform cut level for all the fuzzy parameters, by using the proposed interval parameter perturbation method we can easily obtain the time evolutions of the temperature bounds at reference points 1, 2, 3, which are plotted in Figs. 5–7, respectively. It is visible that the node temperature always changes in a certain interval range for any fixed cut level and the range gradually decreases with the increase of cut level. After the time $t = 40\,\mathrm{s}$, the upper and lower bounds at the three reference points all tend to be stationary. The same case happens to other points, which means that the transient temperature field enters gradually into a steady-state.

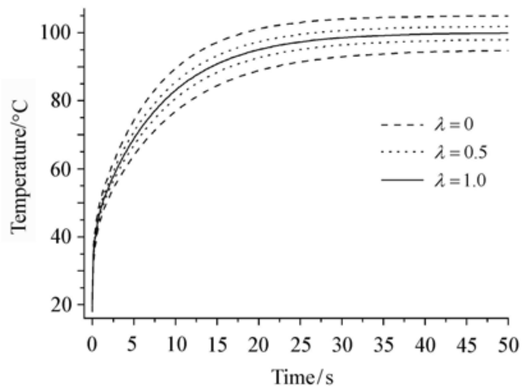

Fig. 5 Time evolution of the temperature at reference point 1 under different cut levels

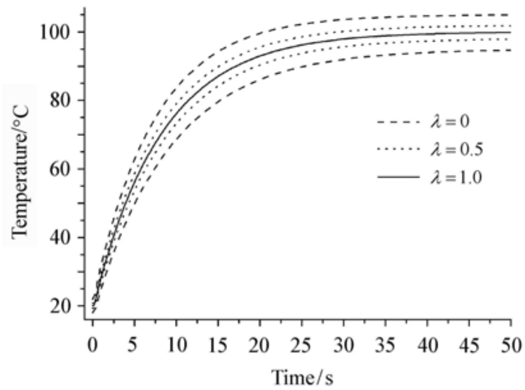

Fig. 6 Time evolution of the temperature at reference point 2 under different cut levels

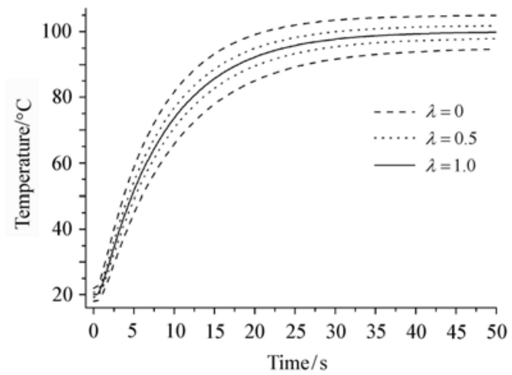

Fig. 7 Time evolution of the temperature at reference point 3 under different cut levels

At different time instants, based on the interval temperature results and decomposition theorem, we can plot in Figs. 8 and 9 the membership functions of the fuzzy temperature responses at the three reference points. It is illustrated that the fuzzy uncertainties in structural parameters and initial/boundary conditions bring certain fuzziness to the transient temperature field, whose membership function still follows the normal distribution.

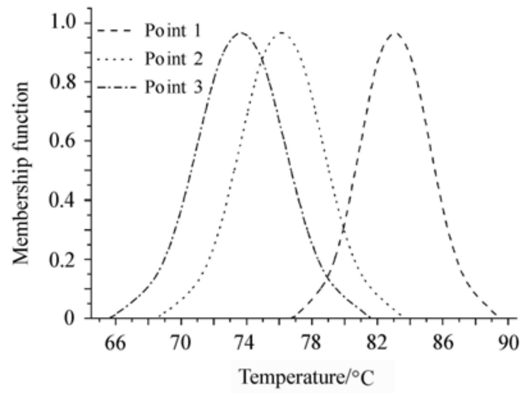

Fig. 8 Membership function of the fuzzy temperature at time $t = 10\,\mathrm{s}$

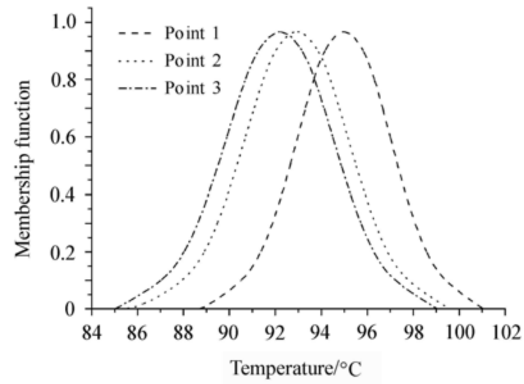

Fig. 9 Membership function of the fuzzy temperature at time $t = 20\,\mathrm{s}$

8 Conclusion

In this study, we have combined the fuzzy theory with the finite difference method, and proposed a new technique, named as fuzzy finite difference method, to analyze the uncertainty in the heat conduction problems with fuzzy parameters. The fuzzy uncertainties in structural parameters and initial/boundary conditions are all fully considered, which makes the calculation more objective. By introducing the cut level, the fuzzy parameters are transformed into a series of interval parameter vectors. Subsequently new stability theory suited to transient fuzzy difference schemes is developed. On the basis of parameter perturbation theory, the upper and lower bounds of the temperature field can be obtained more effectively. Then the membership function of the fuzzy temperature field is derived by means of the decomposition theorem. Compared with the existing uncertain finite element method, application of the proposed fuzzy finite difference method avoids the complex discrete process and greatly improves the efficiency of numerical calculation. For steady state and transient state, two numerical examples fully validate the feasibility and superiority of the suggested model and method for dealing with heat conduction problems with kinds of fuzziness. Furthermore, the research in this paper aims at providing a potential tool for uncertainty analysis in the thermal material engineering.

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