Acta Mechanica Sinica (2013) 29(2):196–201
DOI 10.1007/s10409-013-0022-5

RESEARCH PAPER

Analytical solution of axisymmetric contact problem about indentation of a circular indenter into a soft functionally graded elastic layer

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Received: 3 December 2012 / Revised: 14 January 2013 / Accepted: 4 February 2013

Abstract The paper addresses a contact problem of the theory of elasticity, i.e., the penetration of a circular indenter with a flat base into a soft functionally graded elastic layer. The elastic properties of a functionally graded layer arbitrarily vary with depth, and the foundation is assumed to be elastic, yet much harder than a layer. Approximated analytical solution is constructed, and it is shown that the solutions are asymptotically exact both for large and small values of characteristic dimensionless geometrical parameter of the problem. Numerical examples are analyzed for the cases of monotonic and nonmonotonic variations of elastic properties. Numerical results for the case of homogeneous layer are compared with the results for nondeformable foundation.

Keywords Contact problems · Indentation · Functionally graded layer

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1 Introduction

The method described in the paper allows obtaining solutions of the contact problem of indentation of a circular indenter into a soft multilayered or functionally graded elastic layer. The substrate is assumed to be elastic, yet much harder than the layer (cases of 100 and 1 000 times harder substrates are considered). The solution is asymptotically exact for small and large values of the relative layer thickness (the ratio of indenter radius to layer thickness). Implementation of the method is based on analytical approximations of high accuracy of kernel transforms by specific expressions [1–4].

In classical models, elastic properties of the substrate are not taken into account. First results on the contact problems for homogeneous layers were obtained by Lebedev et al. [5] and Vorovich et al. [6]. In Ref. [6] the solution was given in the form of series expansion with respect to λ, where λ is the retio of the punch radius to the thickness of the layer. This solution was constructed using regular asymptotical method and is effective for $\lambda > 1$. Singular asymptotical method was used for the solution of axisymmetric contact problems for an elastic layer of small thickness ($\lambda < 0.5$) by Alexandrov et al. [7, 8]. For construction of approximate analytical solution of the contact problems, Popov [9] and Alexandrov [10] used expansion of main part of the kernel of integral equation in orthogonal polynomials. It was traditionally used for constructing solutions in the neighborhood of $\lambda = 1$.

One of the disadvantages of these solutions is that it is necessary to use different mathematical models to construct the solution for different layer thickness and radius of the contact area between indenter and layer.

Most of the known solutions to the contact problems for inhomogeneous materials are obtained under specific assumptions about the variation of elastic properties (exponential or power law, for example).

Gibson and Sills [11] studied the elastic layer, the elastic modulus of which grows linearly with depth. In the work of Awojobi [12], the shear modulus of the elastic layer varied by hyperbolic law. Kassir [13] considered inhomogeneous media with a power law of elastic properties variation. Aizikovich et al. [1, 14, 15] constructed mathematical model of a contact between a stamp and an elastic half-space with inhomogeneous coating for the entire range of values of characteristic geometric parameters of the problem and for arbitrary variation of elastic modulus of coating with the depth, in absence of any jump at the layer-substrate boundary.

Axisymmetric and plane contact problems for an inhomogeneous layer on an elastic half-space and/or a half-plane for arbitrary variation of elastic properties were considered in Refs. [16–19]. Wang et al. [17, 18] used piecewise linear approximation of elastic moduli to reduce the problem to the solution of an integral equation, which was then solved by Krenk's method [20].

In the solution of axisymmetric contact problems for an inhomogeneously varied elastic half-space in contact with a layer (with exponential variation of elastic properties) in Ref. [20], a modified method of collocation (Multhopp–Kalandiya method) was used.

2 Problem statement

Nondeformable circular indenter is pressed into the upper bound Γ of an elastic half-space with force P. Cylindrical coordinate system r, φ, z is associated with the half-space. Friction between the indenter and the half-space is assumed to be absent. The half-space is not loaded outside the indenter. The indenter is an axisymmetric body with a cross section Ω ($r \leqslant a$) and a flat base (Fig. 1).

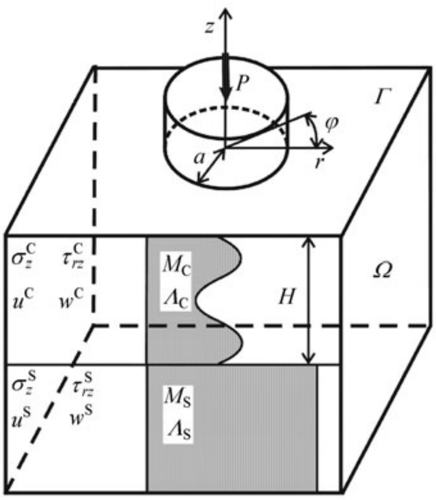

Fig. 1 Statement of the problem

Lame coefficients Λ and M of the half-space vary with depth according to the following rule (a substarte with a coating layer) (1) $\Lambda = \Lambda_C(z)$, $M = M_C(z)$, $-H \leqslant z \leqslant 0$; (2) $\Lambda = \Lambda_S$, $M = M_S$, $-\infty < z < -H$.

Under the action of a central applied force P the indenter will move a distance δ. The boundary conditions are

$$z = 0: \quad \tau_{zr} = \tau_{z\varphi} = 0, \quad \begin{cases} \sigma_z = 0, & r > a, \\ w = -\delta, & r \leqslant a. \end{cases}$$

The coating and the substrate are assumed to be coupled without sliding, so that the continuity conditions

$$z = -H: \quad \begin{cases} \tau_{zr}^C = \tau_{zr}^S, & \sigma_z^C = \sigma_z^S, \\ w^C = w^S, & u^C = u^S \end{cases}$$

are satisfied. Hereafter, superscripts "C" and "S" correspond to the coating and the substrate, respectively. The stresses vanish as $r \to \infty$ or $z \to -\infty$.

It is required to determine the contact normal stresses under the indenter

$$\sigma_z^C|_{z=0} = -q_a(r), \quad r \leqslant a,$$

as well as the relation between the applied force P and movement of the indenter δ.

3 Solution of the problem

Using integral transformation technique, the problem is reduced to the solution of the following integral equation

$$\int_0^1 q(\rho)\rho \mathrm{d}\rho \int_0^\infty L(u) \mathrm{J}_0(ur\lambda^{-1}) \mathrm{J}_0(u\rho\lambda^{-1}) \mathrm{d}u = \lambda \Theta_C(0) f(r), \qquad r \leqslant 1, \tag{1}$$

where $r = r'a$; $q(\rho' a) = q_a(a)$; $\lambda = H/a$ is the geometric parameter of the problem indicating the relative thickness of the layer, $L(u)$ is the kernel transform of the integral equation, $\Theta_C(z) = 2M_C(z)(\Lambda_C(z) + M_C(z))/(\Lambda_C(z) + 2M_C(z))$, J_0 is the Bessel function.

Kernel transform $L(u)$ is constructed numerically using the method of modeling functions [21]. It is shown [22] that the kernel transform has the following properties

$$L(u) = \Theta_C(0)/\Theta_S + \beta u + \chi u^2 + O(u^3), \quad u \to 0, \tag{2}$$

$$L(u) = 1 + \eta u^{-1} + \phi u^{-2} + O(u^{-3}), \quad u \to \infty, \tag{3}$$

where $\Theta_S = 2M_S(\Lambda_S + M_S)/(\Lambda_S + 2M_S)$; β, χ, η, ϕ are some constants; $\mathrm{O}(u^3)$ is infinitesimal of order 3.

Due to the assumption of deformability of the substrate the kernel transform is not equal to zero for all non negative values of argument that allows to use bilateral asymptotical method [1–3] to construct the solution.

Function $L(u)$ with properties (2) and (3) can be represented [22] in the form of

$$L(u) = L_\Pi^N(u) + L_\Sigma^\infty(u),$$

where

$$
\begin{aligned}
L_{\Pi}^{N}(u) &= \prod_{i=1}^{N} \frac{u^2 + A_i^2}{u^2 + B_i^2}, \\
L_{\Sigma}^{\infty}(u) &= \sum_{k=1}^{\infty} \frac{p_k |u|}{u^2 + D_k^2},
\end{aligned} \tag{4}
$$

where $A_i,\ B_i,\ D_k \in C;\ p_k \in R;\ A_i \neq A_j,\ B_i \neq B_j$ are some constants.

For the kernel transform $L(u)$ of the form of Eq. (4), analytical solution of the integral equation (1) is obtained as [1]

$$
q(r) = \frac{2}{\pi}\Theta_{\mathrm{C}}(0)\delta\left[\frac{1}{L_N(0)\sqrt{1-r^2}} + \sum_{i=1}^{N} C_i \Psi(r, A_i\lambda^{-1})\right], \tag{5}
$$

where the constants C_i can be determined from the system of linear algebraic equations of

$$
\sum_{i=1}^{N} C_i \alpha\left(\frac{A_i}{\lambda}, \frac{B_k}{\lambda}\right) + \frac{B_k^{-1}\lambda}{L_N(0)} = 0, \quad k = 1, 2, \cdots, N.
$$

Here

$$
\alpha(x, y) = \frac{x\mathrm{ch}y + y\mathrm{sh}y}{x^2 - y^2},
$$

$$
\Psi(r, y) = \frac{\mathrm{ch}y}{\sqrt{1-r^2}} - y\int_r^1 \frac{\mathrm{sh}yt\mathrm{d}t}{\sqrt{t^2 - r^2}}.
$$

Relation between the applied force and the movement of the indenter has the form

$$
P = 4a\delta\Theta_{\mathrm{C}}(0)\left(L^{-1}(0) + \sum_{i=1}^{N} C_i A_i^{-1}\lambda \mathrm{sh} A_i \lambda^{-1}\right).
$$

It is proved [1] that the solution is asymptotically exact for both large and small values of the λ. Error of the solution depends on the error in approximating the kernel transform of the integral equation $L(u)$ by expression (4).

4 Numerical examples for homogeneous layer

Let us consider a soft homogeneous layer lying on a hard elastic foundation and compare the results with previously ones. We assume that the Young modulus of the layer differ in 2, 5, 10, 100, 1 000 times from the Young moduli of the substrate, the Poisson ratio of both the substrate and the coating is 0.3. We denote Young's moduli of the coating and the substrate as E_{C} and E_{S} respectively.

Table 1 makes comparison between the expression $\frac{q(r)}{\Theta_{\mathrm{C}}(0)\delta}\sqrt{1-r^2}$ for the cases of $E_{\mathrm{S}}/E_{\mathrm{C}} = 2, 5, 10, 100,$ 1 000 and nondeformable substrate [10]. Results for nondeformable substrate were obtained earlier for $\lambda = 0.25$ using singular asymptotic method [8], for $\lambda = 1$ using expansion in orthogonal polynomials [9], and for $\lambda = 4$ using regular asymptotic method [6].

From analysis of these results, it can be concluded that in the case $E_{\mathrm{S}}/E_{\mathrm{C}} = 1\,000$ the values of contact stresses are close to the values obtained by other methods for nondeformable rigid foundation. The maximum difference is observed in the case of $\lambda = 0.25$ and does not exceed 3%. For $\lambda = 1$ and $\lambda = 4$ the difference is less than 0.8%.

Table 1 Contact stresses for elastic and nondeformable substrates

	$E_{\mathrm{C}}/E_{\mathrm{S}}$	2	5	10	100	1 000	Nondeformable substrate [10]
$\lambda = 0.25$	$r = 0$	1.357	3.150	5.026	9.254	10.051	9.780
	$r = 0.2$	1.384	3.182	5.047	9.233	10.021	9.756
	$r = 0.4$	1.469	3.282	5.107	9.162	9.918	9.704
	$r = 0.6$	1.634	3.461	5.197	9.007	9.717	9.600
	$r = 0.8$	1.970	3.781	5.361	8.764	9.444	9.492
	$r = 0.95$	3.117	5.061	6.674	10.069	10.738	11.043
$\lambda = 1$	$r =0$	1.029	1.533	1.796	2.113	2.177	2.183
	$r = 0.2$	1.042	1.542	1.804	2.119	2.183	2.185
	$r = 0.4$	1.085	1.578	1.837	2.144	2.207	2.204
	$r = 0.6$	1.190	1.673	1.932	2.229	2.288	2.285
	$r = 0.8$	1.490	1.994	2.268	2.577	2.624	2.643
	$r = 0.95$	2.702	3.459	3.871	4.339	4.391	4.432
$\lambda = 4$	$r =0$	0.717	0.775	0.801	0.819	0.821	0.821
	$r = 0.2$	0.732	0.790	0.817	0.835	0.837	0.837
	$r = 0.4$	0.781	0.842	0.870	0.890	0.892	0.892
	$r = 0.6$	0.891	0.961	0.991	1.014	1.016	1.018
	$r = 0.8$	1.183	1.274	1.313	1.343	1.347	1.347
	$r = 0.95$	2.265	2.439	2.510	2.569	2.582	2.572

It should be noted that all given numerical results, both obtained in this work and those taken for comparison, are obtained using approximate methods and have calculation errors less than 3%.

Table 2 expresses the correlation between the applied force P and the movement of the indenter δ. The maximum difference in the results does not exceed 1%.

Table 2 Correlation between the applied force and the movement of the indenter

E_C/E_S	2	5	10	100	1 000	Nondeformable substrate [10]
$\lambda = 0.25$	6.98	13.09	18.66	30.75	33.08	32.86
$\lambda = 1$	5.60	7.50	8.52	9.70	9.88	9.93
$\lambda = 4$	4.46	4.80	4.95	5.06	5.08	5.07

5 Numerical results for soft functionally graded layer

We consider that the Young moduli of the soft functionally-gradient layer varies with depth as

$$E(z) = \frac{1}{100} E_S f_j(z), \qquad -H \leqslant z \leqslant 0,$$

where f_j are given by one of the following relations (see Fig. 2)

(1) $f_1(z) = f_0 + (f_0 - 1)z/H$;

(2) $f_2(z) = \dfrac{1}{f_0} - \dfrac{f_0 - 1}{f_0}\dfrac{z}{H}$;

(3) $f_3(z) = \dfrac{f_0 + 1}{2f_0} + \dfrac{f_0 - 1}{2f_0}\cos\left(2\pi\dfrac{z}{H}\right)$;

(4) $f_4(z) = \dfrac{f_0 + 1}{2} - \dfrac{f_0 - 1}{2}\cos\left(2\pi\dfrac{z}{H}\right)$.

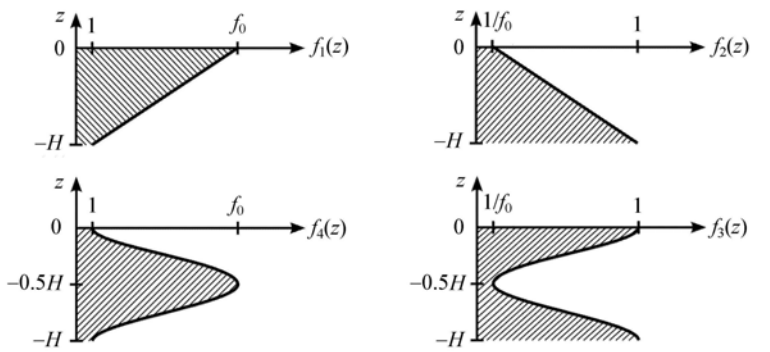

Fig. 2 Variation of Young's modulus with depth in layers 1–4

Functions f_j describe the variation of elastic properties with depth in the layer. In materials 1, 2, Young's modulus vary linearly, while materials 3, 4 correspond to the materials with nonmonotonic (trigonometric) variation of Young's modulus. Parameter f_0 is the maximum ratio of elastic properties in the layer.

Value of f_0 is considered to be 3.5, which refers to the case of coating-substrate metal combinations used in engineering practice. For example, when the coating is a soft metal (Ag, $E = 80$ GPa or Al, $E = 72$ GPa), and the substrate is a hard metal (Fe, $E = 217$ GPa or Cr, $E = 240$ GPa or Mo, $E = 340$ GPa), f_0 changes from 2.7 to 4.7.

The Young modulus of a hard elastic foundation is

$$E(z) = E_S, \qquad z \leqslant -H.$$

It is 100 times greater than Young's modulus on the lower boundary of the layer.

Kernel transforms corresponding to coating 1–4 are illustrated in Fig. 3 (both axes are of logarithmic scale). The maximum error in approximating the kernel transforms with expressions (4) does not exceed 3%.

Figure 3 illustrate the validity of properties (2), (3) of kernel transform: $L(0) = \Theta_C(0)/\Theta_S = E(0)/E_S$ (3.5/100 for materials 1, 1/100 for materials 3, 4, and 1/350 for materials 2).

Figures 4–7 show relative contact stresses

$$q_{\mathrm{rel}}(r, \lambda) = q(r, \lambda)/q_{\mathrm{hom}}(r, \lambda),$$

where $q_{\mathrm{hom}}(r, \lambda)$ are contact stresses for a homogeneous layer, constructed for $E_S/E_C = 100$ and $\Theta_C^{\mathrm{hom}}(0) = \Theta_C(0)$. In all figures $r \in [0, 0.98]$, $\lambda \in [0.05, 1\,000]$.

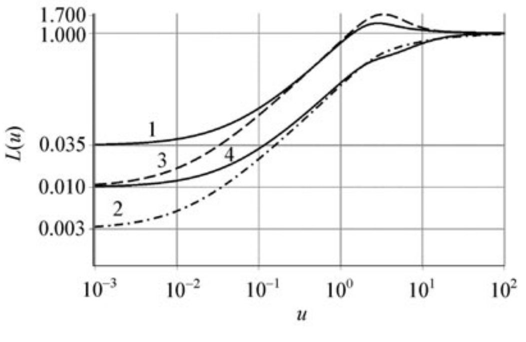

Fig. 3 Kernel transforms of coatings 1–4

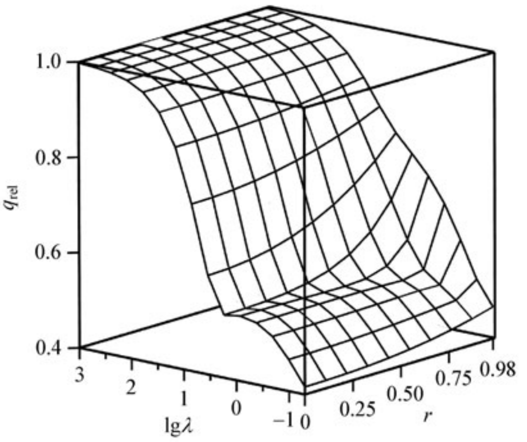

Fig. 4 Relative contact stresses for coating 1 (linear law)

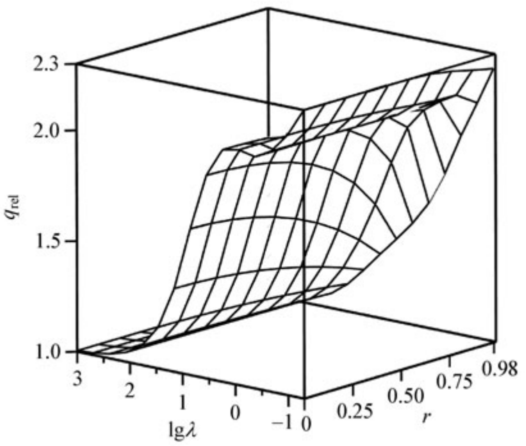

Fig. 5 Relative contact stresses for coating 2 (linear law)

It can be concluded that the inhomogeneity of a layer significantly changes the distribution of contact stresses under the indenter. Only for $\lambda > 32$ (big layer thickness) the contact stresses of coatings 1–4 are close to the contact stresses of a homogeneous layer.

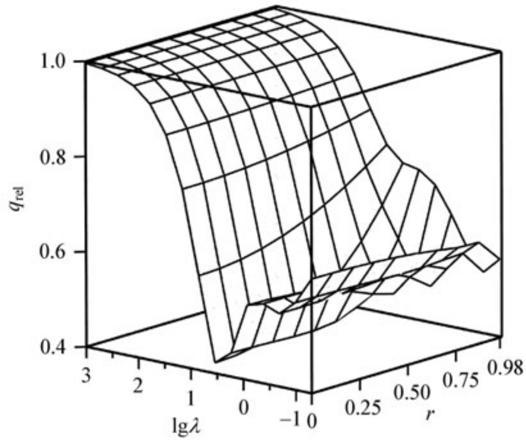

Fig. 6 Relative contact stresses for coating 3 (trigonometric law)

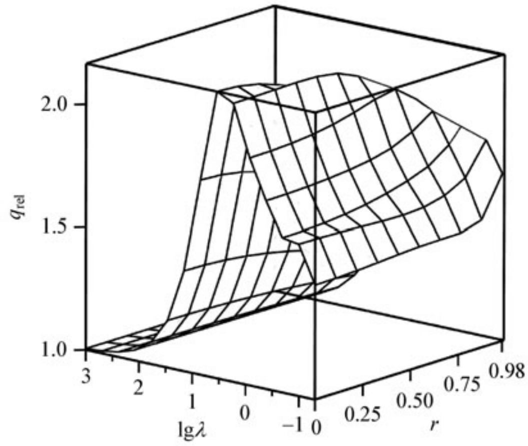

Fig. 7 Relative contact stresses for coating 4 (trigonometric law)

6 Conclusion

Mathematical model of a contact problem is developed for penetration of circular indenter into a soft elastic functionally graded layer, in which elastic properties of the foundation are taken into account. Numerical examples are analyzed for monotonic and nonmonotonic (trigonometric) variations of elastic properties.

Bilateral asymptotically exact solution is constructed for layers of complicated structure which is effective for any layer thickness. High accuracy of the constructed solution follows from high accuracy of kernel transform approximation for inhomogeneous layers, and the numerical results for homogeneous layer are compared with the results for nondeformable foundation obtained by other methods [10].

The method presented in this work allows obtaining an approximate solution of the contact problem in single form for arbitrary variation of elastic properties in the coating and for all range of applicable values of λ, in which the jump of elastic properties on the boundary between coating and substrate is taken into account.

Acknowledgements The work is completed under supports of the Ministry of Education and Science of Russia (11.519.11.3028, 14.B37.21.1131, 14.B7.21.1632) and Russian Foundation of Basic Research (11-08-91168-GFEN_a).

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