Acta Mechanica Sinica (2013) 29(2):267–283
DOI 10.1007/s10409-013-0019-0

RESEARCH PAPER

Parametric characteristic of the random vibration response of nonlinear systems

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Received: 23 July 2012 / Revised: 19 October 2012 / Accepted: 19 November 2012

Abstract Volterra series is a powerful mathematical tool for nonlinear system analysis, and there is a wide range of nonlinear engineering systems and structures that can be represented by a Volterra series model. In the present study, the random vibration of nonlinear systems is investigated using Volterra series. Analytical expressions were derived for the calculation of the output power spectral density (PSD) and input-output cross-PSD for nonlinear systems subjected to Gaussian excitation. Based on these expressions, it was revealed that both the output PSD and the input-output cross-PSD can be expressed as polynomial functions of the nonlinear characteristic parameters or the input intensity. Numerical studies were carried out to verify the theoretical analysis result and to demonstrate the effectiveness of the derived relationship. The results reached in this study are of significance to the analysis and design of the nonlinear engineering systems and structures which can be represented by a Volterra series model.

Keywords Volterra series · Nonlinear system · Random vibration · Power spectrum density · Generalized frequency response functions

The project was supported by the National Science Fund for Distinguished Young Scholars (11125209), the National Natural Science Foundation of China (10902068, 51121063 and 10702039), the Shanghai Pujiang Program (10PJ1406000) and the Opening Project of State Key Laboratory of Mechanical System and Vibration (MSV201103).

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Nomenclature

m, c, k	Mass, damping and stiffness coefficients respectively
k_2, k_3	Quadratic and cubic nonlinear stiffness parameters respectively
x, y	System input and output respectively
$X(\cdot)$, $Y(\cdot)$	Fourier spectra of system input and output respectively
$S_{yy}(\cdot)$, $S_{xy}(\cdot)$	Power spectral density (PSD) and cross PSD of system output respectively
$H(\cdot)$	Frequency response function (FRF)
$h(t)$	Impulse response function
$y_n(t)$	The n-th order Volterra output
$h_n(\tau_1, \tau_2, \cdots, \tau_n)$	The n-th order Volterra kernel
$E\{\cdot\}$	Expected value operator
$H_n(\Omega_1, \Omega_2, \cdots, \Omega_n)$	The n-th order generalized frequency response function
$\mathcal{F}[\cdot]$	Fourier transform operator
$\delta(\cdot)$	Dirac delta function
$\otimes$	Kronecker product

1 Introduction

It is of great significance for engineering practices to establish a functional relationship between system input and output from observations of the in- and out-going signals. For a linear system, the relationship can be characterized uniquely by its impulse response function in time domain or its frequency response function (FRF) in frequency domain [1]. Especially, the FRF has greatly facilitated the analysis and design of linear systems. However, it is well known that some dynamic behaviors would uniquely happen to nonlinear systems [2], i.e., frequency distortion, the generation of sub- and super-harmonic components, the occurrence of sub-resonance, limit cycle oscillation, bifurcation, chaos, et al., and all of them can not be explained with linear system theory. To understand these dynamic behaviors of nonlinear systems, various theories and methods have been developed. Unfortunately, none of them can encompass all conceivable nonlinear systems. The Volterra series [3], which is essentially an extension of the standard one-dimensional convolution operator of linear systems by a series of multi-dimensional integral operators with increasing degree of nonlinearity, is among the earliest approaches to achieve a systematic characterization of nonlinear systems. In spirit, the Volterra series expansion for nonlinear systems is similar to the Taylor series expansion [4] for analytic functions and is particularly appropriate for systems with smooth nonlinearity [5], which can be described by a polynomial form differential equation. Nevertheless, as the Weierstrass approximation theorem [6] guarantees that any continuous function on a closed and bounded interval can be uniformly approximated on that interval by a polynomial to any degree of accuracy, and therefore there actually exist a wide class of nonlinear systems that can be well represented by a Volterra series, and thereby it is no surprise to see that the applications of Volterra series range widely from neuroscience [7], biomedical engineering [8], fluid dynamics [9] and aerodynamics engineering [10] to ocean engineering [11] and mechanical engineering [12–15], et al.

Based on the Volterra series, some concepts have been developed to facilitate the analysis of nonlinear systems in frequency domain. The generalized frequency response functions (GFRFs) [16] that is defined as the Fourier transforms of the Volterra kernels is usually regarded as an extension of the linear FRF to the nonlinear case. Billings and his colleagues [17, 18] proposed an algorithm to determine the GFRFs for the nonlinear systems described by differential equations or discrete-time models. However, due to the multidimensional nature of the GFRFs [19, 20], they are much more complicated than the FRFs of linear systems and therefore it is difficult to measure, display and interpret the GFRFs in practice. Feijoo, et al. [21–23] demonstrated that a Volterra series can be decomposed into a series of associated linear equations (ALEs) whose FRFs are one dimensional and so are easier to analyze and interpret than the GFRFs. Also to overcome the difficulties associated with the GFRFs, Lang and Billings [24] proposed another concept, the nonlinear output frequency response functions (NOFRFs), which are one-dimensional functions of frequency. With the NOFRF concept, it is possible to implement the nonlinear systems analysis in a manner similar to the linear systems analysis. However, neither the GFRFs nor the NOFRFs can provide a clear explicit relationship for the system output spectrum and the nonlinear characteristics parameters. To address this problem, Lang and his colleagues [25] have recently developed another new concept—output frequency response function (OFRF), through which it was revealed that the system output spectrum can be expressed as a polynomial function of the nonlinear characteristics parameters of the systems. This greatly facilitates the analysis and design of the output dynamics of nonlinear systems in frequency domain. Now these frequency domain concepts are frequently applied to study the nonlinear systems subjected to deterministic inputs, e.g., the sinusoidal excitation. On the other hand, the problem of nonlinear systems subjected to non-deterministic inputs, e.g., random inputs, has attracted relatively little attention and relevant analyses are relatively few. For the deterministic input cases, the frequency spectrum is sufficient to characterize the feature of the system output; on the contrary, for the non-deterministic input cases, the power spectrum rather then frequency spectrum is needed to specify the system response.

It is well known that, for linear systems, by using the FRF, the relationship between the output and input power spectra can be described in a form similar to the relationship between the output and input frequency spectra. But, for the nonlinear systems, there are very few analytical works dedicated to deducing an explicit expression for the output power spectrum by using the GFRFs. It is mainly because, as indicated by this study, although the GFRF expression of the usual output spectrum for nonlinear systems is very complicated, even more so is the GFRF expression of the output power spectrum.

In this paper, the Volterra nonlinear systems subjected to random Gaussian inputs are studied, and a general expression for the power spectrum is derived, through which the effects of the random Gaussian input intensity and the nonlinear characteristic parameters on the output power spectrum are investigated.

2 Nonlinear systems

2.1 Volterra series representation

According to the linear system theory, the system output response $y(t)$ can be expressed in the following convolution integral form

$$y(t) = \int h_1(\tau)x(t-\tau)\mathrm{d}\tau, \tag{1}$$

where $x(t)$ is the system input and $h_1(\tau)$ is a linear convolution kernel, also called as impulse response function. The Volterra series [5] extends this familiar convolution integral for linear systems to a series of multidimensional convolution integrals

$$y(t) = y_1(t) + y_2(t) + \cdots + y_n(t), \tag{2}$$

in which

$$y_n(t) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1, \tau_2, \cdots, \tau_n) \times \prod_{i=1}^{n} x(t-\tau_i)\mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\cdots\,\mathrm{d}\tau_n. \tag{3}$$

In the above multidimensional integral, h_n $(\tau_1, \tau_2, \cdots, \tau_n)$ are the Volterra kernels. Like the well-known Taylor series, the Volterra series is in essence a polynomial function approximation of the nonlinear system. The difference between them is that the Taylor series is memoryless but the Volterra series is of memory. Therefore, the Taylor series can only represent systems in which the output depends only on the current input, but the Volterra series [4] can characterize systems in which the output also depends on past inputs. The system memory is determined by the support of the Volterra kernel.

Generally, the Volterra series is particularly appropriate for systems with smooth nonlinearity, that is, the nonlinear systems which can be described by a polynomial form differential equation model [17, 18] as follows

$$\sum_{m=1}^{M} \sum_{\substack{p=0 \\ p+q=m}}^{m} \sum_{l_1,l_2,\cdots,l_{p+q}=0}^{Z} c_{pq}\left(l_1, l_2, \cdots, l_{p+q}\right) \times \prod_{i=1}^{p} \mathrm{D}^{l_i} y(t) \prod_{i=p+1}^{p+q} \mathrm{D}^{l_i} x(t) = 0, \tag{4}$$

where the differential operator "D" is defined by $\mathrm{D}^l y(t) = \mathrm{d}^l y(t)/\mathrm{d}t^l$, and M is the maximum degree of nonlinearity and Z is the maximum order of derivative. According to the Weierstrass approximation theorem [6], on a closed and bounded interval any continuous function can be uniformly approximated by a polynomial to any degree of accuracy; therefore, there actually exist a wide class of nonlinear systems that can be well represented by a Volterra series, for example, the nonlinear oscillators with bilinear stiffness or other kinds of piecewise linear stiffness.

As indicated by Eqs. (2) and (3), the Volterra series is an infinite power series, and this raises convergence problem. Generally, a Volterra series is convergent only for a limited region of the input amplitude. However, it is a challenging problem to determine the convergence region for a Volterra series representation. Although efforts [26–32] have been made to address this problem, there are still no general criteria or methods available to determine the convergence region, or the available criteria are often very conservative and can only provide a rough estimation for the real convergence region. The only exception may be for the case of the Duffing oscillator subjected to a harmonic excitation [26, 28]. When a Volterra series representation is convergent, a truncated series can be adopted to approximate a nonlinear system, i.e.

$$y(t) = \sum_{n=1}^{Q} y_n(t). \tag{5}$$

As there is no criteria available to determine the convergence region for the input amplitude, and consequently the criteria to determine the value of Q and the residual error is not available as well. However, some applications have demonstrated that, in the convergent case, a Volterra series of three order is usually enough to model a nonlinear system. For example, Worden and Manson [33] used a three order Volterra series to model a Duffing oscialltor, and Chatterjee [34] adopted a Volterra series of the same order to represent a bilinear oscillator.

2.2 Generalized frequency response functions (GFRFs)

For the linear system given as Eq. (1), in frequency domain the relationship between $y(t)$ and $x(t)$ can be expressed by an FRF, $H_1(\omega)$, i.e.

$$Y(\omega) = H_1(\omega)X(\omega), \tag{6}$$

where $Y(\omega)$ and $X(\omega)$ are the spectra of the system output and input, respectively, and $H_1(\omega)$ is the Fourier transform of the linear convolution kernel, $h_1(t)$. Similarly, the output frequency response of the nonlinear systems to a general input can be expressed in the following form [35]

$$Y(\omega) = \sum_{n=1}^{Q} Y_n(\omega),$$

$$Y_n(\omega) = \frac{n^{-1/2}}{(2\pi)^{n-1}} \int_{\substack{R^n \\ \omega_1+\omega_2+\cdots+\omega_n=\omega}} H_n(\omega_1, \omega_2, \cdots, \omega_n) \times \prod_{i=1}^{n} X(\omega_i)\mathrm{d}\omega_{1\to n}, \tag{7}$$

where $\omega_{1\to n}$ are dummy variables of integrations, and $Y_n(\omega)$ is the n-th order output frequency response of the system, and $H_n(\bullet)$ is the generalized frequency response function (GFRF), defined as

$$H_n(\omega_1, \omega_2, \cdots, \omega_n) = \int_{R^n} h_n(\tau_1, \tau_2, \cdots, \tau_n) \times \exp[-(\omega_1\tau_1 + \omega_2\tau_2 + \cdots + \omega_n\tau_n)\mathrm{j}]\mathrm{d}\tau_{1\to n}, \tag{8}$$

where $\mathrm{j} = \sqrt{-1}$. Equation (7) is a natural extension of the well-known linear relationship expressed by Eq. (6) to the nonlinear case.

The concept of the GFRF first appeared in a research report [16] by George from the MIT in 1959, in which it was named as the nonlinear frequency response function (NFRF). Afterwards, this concept has been further developed by many researchers from different aspects. By using the GFRF, Bedrosian and Rice [36] have investigated the frequency responses of the Volterra nonlinear systems driven by harmonic signals and Gaussian noise signals, where a method named as harmonic probing algorithm was proposed to determine the GFRF for the nonlinear systems whose differential equations of motion are available. Bussgang and his colleagues [37] have extended this concept for the study of nonlinear systems subjected to multiple inputs. Victor and Knight [38] have given a more rigorous formulation to the Volterra frequency kernel. Billings and his colleague [39] have extended the harmonic probing method to the discrete time nonlinear systems. Moreover, Billings and Peyton Jones [17, 18] have further developed the harmonic probing method by putting forward an effective method which could recursively determine the GFRFs from low to high orders.

Later, Billings and his colleagues [40] have extended this recursive method to multiple inputs case. The algorithm can be easily implemented by computer using symbolic operation method and can facilitate the application of the GFRFs to a certain extent. Using this method the GFRFs of the polynomial nonlinear system (4) can be recursively determined as follows

$$
\begin{aligned}
&H_n(\omega_1,\omega_2,\cdots,\omega_n)\\
&= -\frac{1}{\left[\sum_{l_1=0}^{Z} c_{10}(l_1)(\mathrm{j}\omega_1+\mathrm{j}\omega_2+\cdots+\mathrm{j}\omega_n)^{l_1}\right]}\\
&\times\Bigg[\sum_{l_1,l_2,\cdots,l_n=0}^{Z} c_{0n}(l_1,l_2,\cdots,l_n)(\mathrm{j}\omega_1)^{l_1}(\mathrm{j}\omega_2)^{l_2}\cdots(\mathrm{j}\omega_n)^{l_n}\\
&+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{l_1,l_2,\cdots,l_n=0}^{Z} c_{pq}(l_1,l_2,\cdots,l_n)\\
&\times\left(\mathrm{j}\omega_{n-q+1}\right)^{l_{p+1}}\cdots(\mathrm{j}\omega_n)^{l_n} H_{n-q,p}\left(\omega_1,\omega_2,\cdots,\omega_{n-q}\right)\\
&+\sum_{p=2}^{n}\sum_{l_1,l_2,\cdots,l_p=0}^{Z} c_{p0}\left(l_1,l_2,\cdots,l_p\right)\\
&\times H_{np}(\omega_1,\omega_2,\cdots,\omega_n)\Bigg],
\end{aligned}
\tag{9}
$$

where

$$
\begin{aligned}
H_{np}(\cdot) = &\sum_{i=1}^{n-p+1} H_i(\omega_1,\omega_2,\cdots,\omega_i)H_{n-i,p-1}(\omega_{i+1},\cdots,\omega_n)\\
&\times(\mathrm{j}\omega_1+\mathrm{j}\omega_2+\cdots+\mathrm{j}\omega_i)^{l_p},
\end{aligned}
\tag{10}
$$

with

$$
\begin{aligned}
H_{n1}(\omega_1,\omega_2,\cdots,\omega_n) = &H_n(\omega_1,\omega_2,\cdots,\omega_n)\\
&\times(\mathrm{j}\omega_1+\mathrm{j}\omega_2+\cdots+\mathrm{j}\omega_n)^{l_1}.
\end{aligned}
\tag{11}
$$

For example, consider the nonlinear oscillator shown in Fig. 1, its governing equation of motion is

$$
m\ddot{y}+(c_1+c_2y^2)\dot{y}+(k_1+k_2y^2)y = x(t). \tag{12}
$$

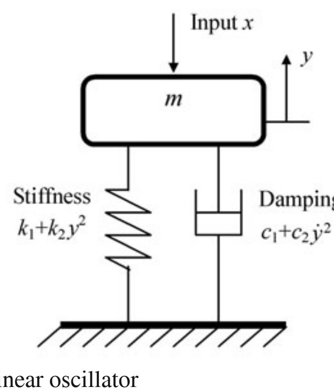

Fig. 1 A nonlinear oscillator

It is easy to know that the above oscillator actually represents a specific instance of polynomial nonlinear systems expressed by Eq. (4) with $c_{0,1}(0) = -1$, $c_{1,0}(2) = m$, $c_{1,0}(0) = k_1$, $c_{1,0}(1) = c_1$, $c_{3,0}(0,0,1) = c_2$, $c_{3,0}(0,0,0) = k_2$ else $c_{p,q}(\bullet) = 0$. The first five order GFRFs of oscillator (12) can be calculated recursively using algorithm (9) as follows

$$
H_1(\omega_1) = \frac{1}{-m\omega_1^2+\mathrm{j}c_1\omega_1+k_1}, \tag{13}
$$

$$
H_2(\omega_1,\omega_2) = 0, \tag{14}
$$

$$
\begin{aligned}
H_3(\omega_1,\omega_2,\omega_3) = &-\left(k_2+\mathrm{j}\frac{c_2}{3}\sum_{i=1}^{3}\omega_i\right)H_1(\omega_1+\omega_2+\omega_3)\\
&\times H_1(\omega_1)H_1(\omega_2)H_1(\omega_3),
\end{aligned}
\tag{15}
$$

$$
H_4(\omega_1,\omega_2,\omega_3,\omega_4) = 0, \tag{16}
$$

$$
\begin{aligned}
&H_5(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5)\\
&= -\frac{3}{10}\left(k_2+\mathrm{j}\frac{c_2}{3}\sum_{i=1}^{5}\omega_i\right)H_1(\omega_1+\omega_2+\omega_3+\omega_4+\omega_5)\\
&\times\sum_{\substack{(i1,i2,\cdots,i5)\text{ is all}\\ \text{permutations}(1,2,\cdots,5)}} H_1(\omega_{i1})H_1(\omega_{i2})H_3(\omega_{i3},\omega_{i4},\omega_{i5}).
\end{aligned}
\tag{17}
$$

The GFRFs has been used to nonlinear systems subjected to deterministic inputs, e.g., the sinusoidal excitation. On the other hand, the analysis of nonlinear systems subjected to non-deterministic inputs by the GFRFs has attracted relatively few attentions. In the following section, using the GFRFs, we are going to establish an explicit expression for the power spectra density (PSD) for the nonlinear systems subjected to random Gaussian excitations.

3 The PSD and cross-PSD of nonlinear systems

The Gaussian input model is perhaps the most commonly used in random vibration as there are a large number of random processes can be effectively modeled as joint normally distributed. With the concept of GFRFs, Bedrosian and Rice [36] have developed calculational expressions for the PSD of the nonlinear systems subjected to Gaussian input. In their study, special attention was paid to the case in which the Volterra series consists of only the linear and quadratic terms. Later, by making use of the Hermite expansion, Barrett [41] has derived similar formula for the output spectrum of a time-invariant nonlinear system. Worden and Manson [42] studied the random vibration of Duffing oscillator and developed respective formula for the calculations of cross-PSD and coherence [33]. Marzocca and his colleagues [43] have studied the random vibration of quadratic nonlinear systems. In this study, general polynomial nonlinear systems, which consist of arbitrary nonlinear terms and are modeled as Eq. (4), are considered. The input signal is assumed to be Gaussian distributed with mean zero and standard deviation λ, i.e., $x(t) \sim N(0,\lambda)$.

3.1 The PSD of the system output

The auto-covariance of the output $y(t)$ is calculated as

$$E\{y(t)y(t-\tau)\} = E\left\{\sum_{z=1}^{Q} y_z(t)\left[\sum_{d=1}^{Q} y_d(t-\tau)\right]\right\}$$
$$= \sum_{z=1}^{Q}\sum_{d=1}^{Q} E\{y_z(t)y_d(t-\tau)\}. \quad (18)$$

By using Eq. (3), it can be known that

$$E\{y_z(t)y_d(t-\tau)\}$$
$$= \int_{R^{z+d}} h_z(\tau_1,\tau_2,\cdots,\tau_z)h_d(\tau_{z+1},\tau_{z+2},\cdots,\tau_{z+d})$$
$$\times E\left\{\prod_{i=1}^{z} x(t-\tau_i)\prod_{l=z+1}^{z+d} x(t-\tau-\tau_l)\right\}\mathrm{d}\tau_{1\to z+d}. \quad (19)$$

Denote $u_i = x(t-\tau_i)$ $(i = 1,2,\cdots,z)$, and $u_i = x(t-\tau-\tau_i)$ $(i = z+1, z+2,\cdots,z+d)$, then expectation of the production of the input series in Eq. (19) can be written as a concise form as follows

$$E\left\{\prod_{i=1}^{z} x(t-\tau_i)\prod_{l=z+1}^{z+d} x(t-\tau-\tau_l)\right\} = E\left\{\prod_{i=1}^{z+d} u_i\right\}.$$

Moreover, according to Isserlis' theorem [44], if the input $x(t)$ is a zero mean Gaussian signal (it does not assume that the input is spectrally white), when $z+d$ is odd, then

$$E\left\{\prod_{i=1}^{z+d} u_i\right\} = 0, \quad (20)$$

and when $z+d$ is even, then

$$E\left\{\prod_{i=1}^{z+d} u_i\right\} = \sum\prod E\left\{u_p u_q\right\}, \quad (21)$$

where the notation $\sum\prod$ means summing over all distinct ways of portioning u_i $(i = 1,2,\cdots,z+d)$ into pairs. Equations (20) and (21) indicate that only if z and d are all odd or are all even, then $E\{y_z(t)y_d(t-\tau)\}$ is non-zero.

Consider the case where both z and d are odd. Without loss of generality, it is assumed that $z \leqslant d$, then all the terms in Eq. (21) can be classified into $(z+1)/2$ categories, i.e., in Category L $(L = 1,3,\cdots,z)$, there are L elements selected from $\{x(t-\tau_i), i = 1,2,\cdots,z\}$ paired with the elements from $\{x(t-\tau-\tau_i), i = z+1, z+2,\cdots,z+d\}$. Denote Category L as $C_{\mathrm{I}}\{x : z,d,L\}$, then the production of the input series in Eq. (19) can be expressed as

$$E\left\{\prod_{i=1}^{z} x(t-\tau_i)\prod_{l=z+1}^{z+d} x(t-\tau-\tau_l)\right\}$$
$$= \sum_{J=1}^{(z+1)/2} C_{\mathrm{I}}\{x : z,d,2J-1\}. \quad (22)$$

In addition, it is not difficult to know that the number of the terms included in Category $C_{\mathrm{I}}\{x : z,d,L\}$ can be calculated as

$$\Xi_{\mathrm{I}}(z,d,L) = \begin{cases} \dfrac{z!d!}{4L![(z-L)/2]![(d-L)/2]!}, & \text{if } z \neq L, \\ \dfrac{d!}{2((d-L)/2)!}, & \text{if } z = L. \end{cases} \quad (23)$$

In Category L, there is one term which can be expressed as

$$\hat{T}\{x : z,d,L\}$$
$$= \prod_{i=1}^{(z-L)/2} E\{x(t-\tau_{(2i-1)})x(t-\tau_{(2i)})\}$$
$$\times\prod_{p=1}^{L} E\left\{x(t-\tau_{(z-L+p)})x(t-\tau_{(z+p)}-\tau)\right\}$$
$$\times\prod_{r=[(z+L)/2+1]}^{(z+d)/2} E\{x(t-\tau_{(2r-1)}-\tau)x(t-\tau_{(2r)}-\tau)\}. \quad (24)$$

Moreover, the expectation of two random variables can be calculated by using the power spectrum, i.e.

$$E\{x(t-\tau_a)x(t-\tau_b)\} = \frac{1}{2\pi}\int_R S_{xx}(\omega)\mathrm{e}^{\mathrm{j}\omega(\tau_a-\tau_b)}\mathrm{d}\omega, \quad (25)$$

where $S_{xx}(\omega)$ is the power spectra density of $x(t)$. Making use of Eq. (25), Eq. (24) can be rewritten as

$$\hat{T}(x : z,d,L)$$
$$= \frac{1}{(2\pi)^{(z+d)/2}}\times\int_{R^{(z+d)/2}}\left(\prod_{i=1}^{(z+d)/2} S_{xx}(\omega_i)\right.$$
$$\times\prod_{q=1}^{(z-L)/2}\exp[\mathrm{j}\omega_q(\tau_{(2q-1)}-\tau_{2q})]$$
$$\times\prod_{p=1}^{L}\exp[\mathrm{j}\omega_{[(z-L)/2+p]}(\tau_{(z-L+p)}-\tau_{(z+p)}-\tau)]$$
$$\left.\times\prod_{r=[(z+L)/2+1]}^{(z+d)/2}\exp[\mathrm{j}\omega_r(\tau_{(2r-1)}-\tau_{2r})]\right)\mathrm{d}\omega_{1\to(z+d)/2}. \quad (26)$$

Define

$$E\{y,\hat{T}(x : z,d,L)\}$$
$$= \int_{R^{z+d}} h_z(\tau_1,\tau_2,\cdots,\tau_z)\times h_d(\tau_{z+1},\tau_{z+2},\cdots,\tau_{z+d})$$
$$\times\hat{T}(x : z,d,L)\,\mathrm{d}\tau_{1\to(z+d)}. \quad (27)$$

Substituting Eq. (26) into Eq. (27) gives

$$E\{y,\hat{T}(x : z,d,L)\}$$
$$= \frac{1}{(2\pi)^{(z+d)/2}}\int_{R^{(z+d)/2}} H_{z,d,L}(\omega_1,\omega_2,\cdots,\omega_{(z+d)/2})$$

$$\times\left(\prod_{i=1}^{(z+d)/2} S_{xx}(\omega_i)\prod_{p=1}^{L}\exp(-\mathrm{j}\tau\omega_{[(z-L)/2+p]})\right)$$
$$\times \mathrm{d}\omega_{1\to(z+d)/2}, \tag{28}$$

with

$$H_{z,d,L}(\omega_1,\omega_2,\cdots,\omega_{(z+d)/2})$$
$$=\int_{R^{(z+d)}} h_z(\tau_1,\tau_2,\cdots,\tau_z)h_d(\tau_{(z+1)},\tau_{(z+2)},\cdots,\tau_{(z+d)})$$
$$\times\left(\prod_{q=1}^{(z-L)/2}\exp[\mathrm{j}\omega_q(\tau_{(2q-1)}-\tau_{2q})]\right.$$
$$\times\prod_{p=1}^{L}\exp[(\mathrm{j}\omega_{[(z-L)/2+p]}(\tau_{(z-L+p)}-\tau_{(z+p)})]$$
$$\left.\times\prod_{r=[(z+L)/2+1]}^{(z+d)/2}\exp[\mathrm{j}\omega_r(\tau_{(2r-1)}-\tau_{2r})]\right)\mathrm{d}\tau_{1\to(z+d)}. \tag{29}$$

Moreover, from the definition of GFRFs, i.e., Eq. (8), $H_{z,d,L}(\omega_1,\omega_2,\cdots,\omega_{(z+d)/2})$ can be simplified as follows

$$H_{z,d,L}(\omega_1,\omega_2,\cdots,\omega_{(z+d)/2})$$
$$=H_z(-\omega_1,\omega_1,\cdots,-\omega_{(z-L)/2},\omega_{(z-L)/2},$$
$$-\omega_{[(z-L)/2+1]},\cdots,-\omega_{[(z+L)/2]})$$
$$\times H_d(\omega_{[(z-L)/2+1]},\cdots,\omega_{(z+L)/2},-\omega_{[(z+L)/2+1]},$$
$$\omega_{[(z+L)/2+1]},\cdots,-\omega_{(z+d)/2},\omega_{(z+d)/2}). \tag{30}$$

By setting

$$H^x_{z,d,L}(\omega_1,\omega_2,\cdots,\omega_{(z+d)/2})$$
$$=H_{z,d,L}(\omega_1,\omega_2,\cdots,\omega_{(z+d)/2})\prod_{i=1}^{(z+d)/2}S_{xx}(\omega_i), \tag{31}$$

$E\{y,\hat{T}(x:z,d,L)\}$ can be rewritten in the following concise form

$$E\{y,\hat{T}(x:z,d,L)\}$$
$$=\frac{1}{(2\pi)^{(z+d)/2}}\int_{R^{(z+d)/2}}H^x_{z,d,L}(\omega_1,\omega_2,\cdots,\omega_{(z+d)/2})$$
$$\times\prod_{p=1}^{L}\exp(-\mathrm{j}\tau\omega_{[(z-L)/2+p]})\mathrm{d}\omega_{1\to(z+d)/2}. \tag{32}$$

Performing the Fourier transform at both sides of Eq. (32) yields

$$\mathcal{F}[E\{y,\hat{T}(x:z,d,L)\}]$$
$$=\frac{1}{(2\pi)^{(z+d)/2}}\int_{R^{(z+d)/2}}H^x_{z,d,L}(\omega_1,\omega_2,\cdots,\omega_{(z+d)/2})$$
$$\left[\int_R\exp\left[-\mathrm{j}\tau\left(\omega+\sum_{p=1}^{L}\omega_{[(z-L)/2+p]}\right)\right]\mathrm{d}\tau\right]\mathrm{d}\omega_{1\to(z+d)/2}$$
$$=\frac{1}{(2\pi)^{(z+d-2)/2}}\left[\int_{R^{(z+d)/2}}H^x_{z,d,L}(\omega_1,\omega_2,\cdots,\omega_{(z+d)/2})\right.$$
$$\left.\times\delta\left(\omega+\sum_{p=1}^{L}\omega_{[(z-L)/2+p]}\right)\mathrm{d}\omega_{1\to(z+d)/2}\right]$$
$$=\frac{1}{(2\pi)^{(z+d-2)/2}}\int_{\substack{R^{(z+d)/2}\\ \sum_{p=1}^{L}\omega_{[(z-L)/2+p]}=-\omega}}H^x_{z,d,L}$$
$$\times(\omega_1,\omega_2,\cdots,\omega_{(z+d)/2})\mathrm{d}\omega_{1\to(z+d)/2}, \tag{33}$$

where $\delta(\Delta)$ is the Dirac delta function.

Define

$$E\{y,C_{\mathrm{I}}(x:z,d,L)\}$$
$$=\int_{R^{z+d}}h_z(\tau_1,\tau_2,\cdots,\tau_z)h_d(\tau_{z+1},\tau_{z+2},\cdots,\tau_{z+d})$$
$$\times C_{\mathrm{I}}(x:z,d,L)\,\mathrm{d}\tau_{1\to(z+d)}, \tag{34}$$

then it can be determined that the Fourier spectrum for all the terms in $E\{y,C_{\mathrm{I}}(x:z,d,L)\}$ can be expressed in exactly the same form as Eq. (33) due to the symmetry of the GFRFs. Therefore, the Fourier spectrum of $E\{y,C_{\mathrm{I}}(x:z,d,L)\}$, simply denoted as $S_y^{C_{\mathrm{I}}(z,d,L)}(\omega)$, can be calculated as

$$S_y^{C_{\mathrm{I}}(z,d,L)}(\omega)=\Xi_{\mathrm{I}}(z,d,L)\mathcal{F}[E\{y,T(x:z,d,L)\}]$$
$$=\frac{\Xi_{\mathrm{I}}(z,d,L)}{(2\pi)^{(z+d-2)/2}}\int_{\substack{R^{(z+d)/2}\\ \sum_{p=1}^{L}\omega_{[(z-L)/2+p]}=-\omega}}H^x_{z,d,L}$$
$$\times(\omega_1,\omega_2,\cdots,\omega_{(z+d)/2})\mathrm{d}\omega_{1\to(z+d)/2}. \tag{35}$$

From Eqs. (19) and (22), it can be known that

$$\mathcal{F}[E\{y_z(t)y_d(t-\tau)\}]=\sum_{J=1}^{(z+1)/2}S_y^{C_{\mathrm{I}}(z,d,2J-1)}(\omega). \tag{36}$$

Similarly, it can be derived that, when z and d are both even number, the Fourier spectrum of $E\{y_z(t)y_d(t-\tau)\}$ can be calculated as

$$\mathcal{F}[E\{y_z(t)y_d(t-\tau)\}]=\sum_{J=1}^{z/2+1}S_y^{C_{\mathrm{II}}(z,d,2J-2)}(\omega), \tag{37}$$

with

$$S_y^{C_{\mathrm{II}}(z,d,L)}(\omega)=\frac{\Xi_{\mathrm{II}}(z,d,L)}{(2\pi)^{(z+d-2)/2}}\int_{\substack{R^{(z+d)/2}\\ \sum_{p=1}^{L}\omega_{[(z-L)/2+p]}=-\omega}}H^x_{z,d,L}$$
$$\times(\omega_1,\omega_2,\cdots,\omega_{(z+d)/2})\mathrm{d}\omega_{1\to(z+d)/2}, \tag{38}$$

and

$$\Xi_{\mathrm{II}}(z,d,L)=\begin{cases}\dfrac{z!d!}{4L!((z-L)/2)!((d-L)/2)!}, & \text{if } z\neq L,\\ \dfrac{d!}{2((d-L)/2)!}, & \text{if } z=L,\\ \dfrac{z!d!}{4(z/2)!(d/2)!}, & \text{if } L=0.\end{cases}\quad(39)$$

Moreover, following the same procedure, it can be easily derived that

$$\mathcal{F}\left[E\left\{y_z(t)y_d(t-\tau)\right\}\right]=\mathrm{conj}\left(\mathcal{F}\left[E\left\{y_d(t)y_z(t-\tau)\right\}\right]\right),\quad(40)$$

and

$$\mathcal{F}\left[E\left\{y_z(t)y_d(t-\tau)\right\}+E\left\{y_d(t)y_z(t-\tau)\right\}\right]=2\mathrm{Re}\left(\mathcal{F}\left[E\left\{y_z(t)y_d(t-\tau)\right\}\right]\right).\quad(41)$$

According to the results described by Eqs. (20) and (21), the auto-covariance of the output $y(t)$, i.e., $E\{y(t)y(t-\tau)\}$ can be calculated as

$$E\{y(t)y(t-\tau)\}=\underbrace{\sum_{z=1}^{[(Q+1)/2]}\sum_{d=1}^{[(Q+1)/2]}E\left\{y_{(2z-1)}(t)y_{(2d-1)}(t-\tau)\right\}}_{E_{\mathrm{I}}\{y(t)y(t-\tau)\}}+\underbrace{\sum_{z=1}^{[Q/2]}\sum_{d=1}^{[Q/2]}E\left\{y_{2z}(t)y_{2d}(t-\tau)\right\}}_{E_{\mathrm{II}}\{y(t)y(t-\tau)\}},\quad(42)$$

where [•] means to take the integer part. Carrying out the Fourier transform at both sides of Eq. (42), then the power spectra density of the Volterra nonlinear system, $y(t)$ can be given as

$$S_{yy}(\omega)=S^{\mathrm{I}}_{yy}(\omega)+S^{\mathrm{II}}_{yy}(\omega),\quad(43)$$

with

$$S^{\mathrm{I}}_{yy}(\omega)=\mathcal{F}\left[E_{\mathrm{I}}\left\{y(t)y(t-\tau)\right\}\right],\quad S^{\mathrm{II}}_{yy}(\omega)=\mathcal{F}\left[E_{\mathrm{II}}\left\{y(t)y(t-\tau)\right\}\right].\quad(44)$$

Using the results given by Eqs. (35), (37) and (41), $S^{\mathrm{I}}_{yy}(\omega)$ and $S^{\mathrm{II}}_{yy}(\omega)$ can be calculated as

$$S^{\mathrm{I}}_{yy}(\omega)=\sum_{z=1}^{[(Q+1)/2]}\sum_{d=z+2}^{[(Q+1)/2]}\sum_{J=1}^{z}2\mathrm{Re}\left(S_y^{C_{\mathrm{I}}(2z-1,2d-1,2J-1)}(\omega)\right)+\sum_{z=1}^{[(Q+1)/2]}\sum_{J=1}^{z}\mathrm{Re}\left(S_y^{C_{\mathrm{I}}(2z-1,2z-1,2J-1)}(\omega)\right),\quad(45)$$

and

$$S^{\mathrm{II}}_{yy}(\omega)=\sum_{z=1}^{[Q/2]}\sum_{d=z+2}^{[Q/2]}\sum_{J=1}^{z+1}2\mathrm{Re}\left(S_y^{C_{\mathrm{II}}(2z,2d,2J-2)}(\omega)\right)+\sum_{z=1}^{[Q/2]}\sum_{J=1}^{z+1}\mathrm{Re}\left(S_y^{C_{\mathrm{II}}(2z,2z,2J-2)}(\omega)\right).\quad(46)$$

To illustrate the calculation of the PSD, the method is applied to the nonlinear oscillator (12), and the following result is obtained,

$$\begin{aligned}S_{yy}(\omega)&=H_1(\omega)H_1(-\omega)S_{xx}(\omega)+\frac{6}{\pi}\mathrm{Re}\Bigg(\int_{\substack{R^2\\ \omega_1=\omega}}H_1(-\omega_1)\\&\times H_3(\omega_1,\omega_2,-\omega_2)\prod_{i=1}^{2}S_{xx}(\omega_i)\mathrm{d}\omega_{1\to2}\Bigg)\\&+\frac{60}{\pi^2}\mathrm{Re}\Bigg(\int_{\substack{R^3\\ \omega_1=\omega}}H_1(-\omega_1)H_5(\omega_1,\omega_2,-\omega_2,\omega_3,-\omega_3)\\&\times\prod_{i=1}^{3}S_{xx}(\omega_i)\mathrm{d}\omega_{1\to3}\Bigg)\\&+\frac{9}{4\pi^2}\mathrm{Re}\Bigg(\int_{\substack{R^3\\ \omega_2=\omega}}H_3(-\omega_1,\omega_1,-\omega_2)H_3(\omega_2,\omega_3,-\omega_3)\\&\times\prod_{i=1}^{3}S_{xx}(\omega_i)\mathrm{d}\omega_{1\to3}\Bigg)\\&+\frac{1}{4\pi^2}\mathrm{Re}\Bigg(\int_{\substack{R^3\\ \omega_1+\omega_2+\omega_3=\omega}}H_3(-\omega_1,-\omega_2,-\omega_3)\\&\times H_3(\omega_1,\omega_2,\omega_3)\prod_{i=1}^{3}S_{xx}(\omega_i)\mathrm{d}\omega_{1\to3}\Bigg)\\&+\frac{45}{\pi^3}\mathrm{Re}\Bigg(\int_{\substack{R^4\\ \omega_2=\omega}}H_3(-\omega_1,\omega_1,-\omega_2)\\&\times H_5(\omega_2,\omega_3,-\omega_3,\omega_4,-\omega_4)\prod_{i=1}^{4}S_{xx}(\omega_i)\mathrm{d}\omega_{1\to4}\Bigg)\\&+\frac{15}{\pi^3}\mathrm{Re}\Bigg(\int_{\substack{R^4\\ \omega_1+\omega_2+\omega_3=\omega}}H_3(-\omega_1,-\omega_2,-\omega_3)\\&\times H_5(\omega_1,\omega_2,\omega_3,\omega_4,-\omega_4)\prod_{i=1}^{4}S_{xx}(\omega_i)\mathrm{d}\omega_{1\to4}\Bigg)\\&+\cdots.\end{aligned}\quad(47)$$

3.2 The cross-PSD of the system input and output

The cross-covariance of the output and the input is

$$E\{y(t)x(t-\tau)\}=E\left\{x(t-\tau)\sum_{z=1}^{Q}y_z(t)\right\}=\sum_{z=1}^{Q}E\{y_z(t)x(t-\tau)\}.\quad(48)$$

Substituting Eq. (3) into Eq. (48) makes

$$E\{y(t)x(t-\tau)\} = \sum_{z=1}^{Q} \int_{R^z} h_z(\tau_1, \tau_2, \cdots, \tau_z) E \times \left\{ x(t-\tau) \prod_{i=1}^{z} x(t-\tau_i) \right\} \mathrm{d}\tau_{1\to z}. \tag{49}$$

From the Isserlis' theorem described as Eqs. (20) and (21), it can be known that when z is even number, $E\left\{ x(t-\tau) \prod_{i=1}^{z} x(t-\tau_i) \right\} = 0$, then the cross-covariance of the output and the input for Volterra nonlinear systems can be calculated as

$$\begin{aligned} E\{y(t)x(t-\tau)\} &= \sum_{z=1}^{[(Q+1)/2]} E\{y_{(2z-1)}(t)x(t-\tau)\} \\ &= \sum_{z=1}^{[(Q+1)/2]} \int_{R^{2z-1}} h_{2z-1}(\tau_1, \tau_2, \cdots, \tau_{2z-1}) E \\ &\quad \times \left\{ x(t-\tau) \prod_{i=1}^{2z-1} x(t-\tau_i) \right\} \mathrm{d}\tau_{1\to 2z-1}. \end{aligned} \tag{50}$$

The cross-PSD of the input and output is defined as the Fourier spectrum of the cross-covariance. Using Eq. (50) and following a similar procedure used to calculate the PSD of the system output in the above section, we can easily derive the cross-PSD of the input and output as

$$S_{xy}(\omega) = \sum_{z=1}^{[(Q+1)/2]} S_{xy_{(2z-1)}}(\omega), \tag{51}$$

with

$$\begin{aligned} S_{xy_{(2z-1)}}(\omega) &= \frac{(2z-1)! S_{xx}(\omega)}{2!(z-1)!(2\pi)^{z-1}} \int_{R^{z-1}} H_{(2z-1)} \\ &\quad \times (-\omega_1, \omega_1, \cdots, -\omega_{(z-1)}, \omega_{(z-1)}, \omega) \\ &\quad \times \prod_{i=1}^{z-1} S_{xx}(\omega_i) \mathrm{d}\omega_{1\to(z-1)}. \end{aligned} \tag{52}$$

Equation (51) indicates that, for nonlinear systems, only the nonlinearities with odd number order could make contribution to the cross-PSD, and so the cross-PSD does not possess any information about the even number order nonlinearity.

Applying Eqs. (51) and (52) to the nonlinear oscillator (12), it can be derived that the cross-PSD could be approximately calculated as

$$\begin{aligned} S_{xy}(\omega) \approx S_{xx}(\omega) \Bigg(& H_1(\omega) + \frac{3}{2\pi} \int_{R^1} H_3(-\omega_1, \omega_1, \omega) \\ & \times S_{xx}(\omega_1) \mathrm{d}\omega_1 + \frac{15}{4\pi^2} \int_{R^2} H_5(-\omega_1, \omega_1, -\omega_2, \omega_2, \omega) \\ & \times \prod_{i=1}^{2} S_{xx}(\omega_i) \mathrm{d}\omega_{1\to 2} + \cdots \Bigg). \end{aligned} \tag{53}$$

3.3 Parametric characteristic analysis of the PSD and the cross-PSD

In the above two sections, two expressions have been derived to calculate respectively the PSD and the cross-PSD for nonlinear systems. However, these expressions are very complicated so that it is very difficult to be implemented in practices. An immediate difficulty is that the results generated by the expressions can not be readily used to explicitly reveal how the nonlinear characteristic parameters in the system (4) and the input intensity affect the PSD. To solve this problem, two different new formulations are devised for the expressions of the PSDs, through which analytical explicit relationships between the PSD and the input intensity and between the PSD and the nonlinear characteristic parameters of system (4) can be established. Particularly, in this study the input auto-spectrum is considered to be constant over all frequencies for the Gaussian white input, i.e., $S_{xx}(\omega) = P$.

Proposition 1: When the nonlinear systems are subjected to a Gaussian white noise input with constant spectrum, i.e., $S_{xx}(\omega) = P$, the PSD of the system outputs can be expressed as a polynomial function of the input intensity such that

$$S_{yy}(\omega) = \sum_{i=1}^{Q} P^i \Lambda_{(i)}(\omega), \tag{54}$$

where $\Lambda_{(i)}(\omega)$ $(i = 1, 2, \cdots, Q)$ is a function that is related to the system parameters only, and is independent of the system input [45].

The proof of the proposition is quite straightforward and the details are omitted here. Obviously, by setting

$$\Lambda_{(1)}(\omega) = H_1(\omega) H_1(-\omega),$$

$$\Lambda_{(2)}(\omega) = \frac{3}{\pi} \mathrm{Re}\left(\int_{\substack{R^2 \\ \omega_1=\omega}} H_1(-\omega_1) H_3(\omega_1, \omega_2, -\omega_2) \mathrm{d}\omega_{1\to 2} \right),$$

$$\begin{aligned} \Lambda_{(3)}(\omega) = & \frac{30}{\pi^2} \mathrm{Re}\left(\int_{\substack{R^3 \\ \omega_1=\omega}} H_1(-\omega_1) H_5 \times (\omega_1, \omega_2, -\omega_2, \omega_2, -\omega_2) \mathrm{d}\omega_{1\to 3} \right) \\ & + \frac{9}{4\pi^2} \mathrm{Re}\left(\int_{\substack{R^3 \\ \omega_2=\omega}} H_3(-\omega_1, \omega_1, -\omega_2) H_3 \times (\omega_2, \omega_3, -\omega_3) \mathrm{d}\omega_{1\to 3} \right) \\ & + \frac{1}{4\pi^2} \mathrm{Re}\left(\int_{\substack{R^3 \\ \omega_1+\omega_2+\omega_3=\omega}} H_3(-\omega_1, -\omega_2, -\omega_3) H_3 \times (\omega_1, \omega_2, \omega_3) \mathrm{d}\omega_{1\to 3} \right), \end{aligned}$$

$$\Lambda_4(\omega) = \frac{45}{2\pi^3}\mathrm{Re}\Bigg(\int_{\substack{R^4\\ \omega_2=\omega}} H_3(-\omega_1,\omega_1,-\omega_2) \times H_5(\omega_2,\omega_3,-\omega_3,\omega_4,-\omega_4)\mathrm{d}\omega_{1\to4}\Bigg) + \frac{15}{\pi^3}\mathrm{Re}\Bigg(\int_{\substack{R^4\\ \omega_1+\omega_2+\omega_3=\omega}} H_3(-\omega_1,-\omega_2,-\omega_3) \times H_5(\omega_1,\omega_2,\omega_3,\omega_4,-\omega_4)\mathrm{d}\omega_{1\to4}\Bigg),$$

then Eq. (47) can be reformulated as a polynomial form described by Eq. (54).

Expression (54) reveals a distinct difference between the nonlinear and the linear systems, that is, the output intensity of the nonlinear systems is a polynomial function of the input intensity while the output intensity is a linear function of the input intensity, i.e., $S_{yy}(\omega) = P\Lambda_{(1)}(\omega)$ for linear systems. In addition, expression (54) indicates that, if $\Lambda_{(i)}(\omega)$ ($i = 1, 2, \cdots, Q$) can be known a priori, then given a specific input intensity, the PSD of the nonlinear system output can be determined. Then the remaining problem becomes how to evaluate$\Lambda_{(i)}(\omega)$. Given the nonlinear differential equation and the values of all system parameters, obviously the values of $\Lambda_{(i)}(\omega)$ can be straightforwardly calculated from their definitions. However, the highly complex forms of $\Lambda_{(i)}(\omega)$ and GFRFs imply that it would be not easy (though straightforward) to implement the calculation procedure in practice. As in linear system analysis people usually prefer to estimating the FRFs from the system input and output responses, an algorithm is proposed here to directly estimate the values of $\Lambda_{(i)}(\omega)$ using the input and output responses of the nonlinear systems.

From Eq. (54), the calculation of the PSD can be rewritten in the following form

$$S_{yy}(\omega) = \begin{pmatrix} P^1 & P^2 & \cdots & P^Q \end{pmatrix} \begin{pmatrix} \Lambda_{(1)}(\omega) \\ \Lambda_{(2)}(\omega) \\ \vdots \\ \Lambda_{(Q)}(\omega) \end{pmatrix}. \tag{55}$$

Excite the system under study U ($U \geqslant Q$) times by the Gaussian white noise inputs with different intensities, i.e., $P_{(1)}$, $P_{(2)}$, $\cdots$, $P_{(U)}$, to generate the output responses, $y_{(1)}$, $y_{(2)}$, $\cdots$, $y_{(U)}$. Without loss of generality, it is assumed that $P_{(1)} > P_{(2)} > \cdots > P_{(U)}$, then the PSDs of the output responses can be related to $\Lambda_{(i)}(\omega)$ ($i = 1, 2, \cdots, Q$) as below

$$\begin{pmatrix} S_{y_{(1)}y_{(1)}}(\omega) \\ \vdots \\ S_{y_{(U)}y_{(U)}}(\omega) \end{pmatrix} = \begin{pmatrix} P_{(1)} & \cdots & P_{(1)}^Q \\ \vdots & \ddots & \vdots \\ P_{(U)} & \cdots & P_{(U)}^Q \end{pmatrix} \begin{pmatrix} \Lambda_{(1)}(\omega) \\ \vdots \\ \Lambda_{(Q)}(\omega) \end{pmatrix}. \tag{56}$$

Consequently, the values of $\Lambda_{(i)}(\omega)$ ($i = 1, 2, \cdots, Q$) can be determined using a least square based approach as

$$\begin{pmatrix} \Lambda_{(1)}(\omega) \\ \vdots \\ \Lambda_{(Q)}(\omega) \end{pmatrix} = \left[\boldsymbol{P}'_{(U\times Q)}\boldsymbol{P}_{(U\times Q)}\right]^{-1} \boldsymbol{P}'_{(U\times Q)} \begin{pmatrix} S_{y_{(1)}y_{(1)}}(\omega) \\ \vdots \\ S_{y_{(U)}y_{(U)}}(\omega) \end{pmatrix}, \tag{57}$$

where

$$\boldsymbol{P}_{(U\times Q)} = \begin{pmatrix} P_{(1)} & \cdots & P_{(1)}^Q \\ \vdots & \ddots & \vdots \\ P_{(U)} & \cdots & P_{(U)}^Q \end{pmatrix}.$$

It can be seen that, to determine $\Lambda_{(i)}(\omega)$ ($i = 1, 2, \cdots, Q$), this algorithm requires experimental or simulation results for the nonlinear system under U different Gaussian white noise signal excitations.

To achieve an explicit expression for the relationship between the PSD and the nonlinear characteristic parameters, another formulation is proposed here, described as proposition 2.

Proposition 2: When the nonlinear systems are subjected to a Gaussian white noise input with constant spectrum, i.e., $S_{xx}(\omega) = P$, the PSD of the system outputs can be expressed as a polynomial function of the nonlinear characteristic parameters in nonlinear systems such that

$$S_{yy}(\omega) = \sum_{n=1}^{Q} P^n \sum_{(j_1,j_2,\cdots,j_{s_n})\in J_n} \Phi^{(2n:j_1\ j_2\ \cdots\ j_{s_n})}_{\lambda_1\ \lambda_2\ \cdots\ \lambda_{s_n}}(\omega) \times \lambda_1^{j_1}\ \lambda_2^{j_2}\ \cdots\ \lambda_{s_n}^{j_{s_n}}, \tag{58}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_{s_n}$ are the nonlinear characteristic parameters of the nonlinear system, $\Phi^{(2n:j_1\ j_2\ \cdots\ j_{s_n})}_{\lambda_1\ \lambda_2\ \cdots\ \lambda_{s_n}}(\omega)$ is a function of ω and $H_1(\cdot)$ and depends only on the linear characteristic parameters of the system. J_n is a set of s_n dimensional integer vectors, which contains the exponents of all the monomials $\lambda_1^{j_1}\ \lambda_2^{j_2}\ \cdots\ \lambda_{s_n}^{j_{s_n}}$ in the polynomial representation (58).

Proof of the proposition is very straightforward and needs an important result about the GFRFs derived by Lang et al. [25], which states that the GFRFs of nonlinear systems can be expressed as a polynomial function of the nonlinearity characteristic parameters in a form similar to expression (58). The details of the proof are omitted here. To demonstrate Proposition 2, the nonlinear oscillator (12) is considered again. It is easy to known that there are two nonlinear characteristic parameters in the nonlinear oscillator, c_2 and k_2. By substituting Eqs. (14)–(17) into Eq. (47), the output PSD for the Volterra nonlinear systems can be expressed as

$$S_{yy}(\omega) = P\Phi^{(2:0,0)}_{c_2,k_2}(\omega) + c_2P^2\Phi^{(4:1,0)}_{c_2,k_2}(\omega) + k_2P^2\Phi^{(4:0,1)}_{c_2,k_2}(\omega) + c_2k_2P^3\Phi^{(6:1,1)}_{c_2,k_2}(\omega) + c_2^2P^3\Phi^{(6:2,0)}_{c_2,k_2}(\omega) + k_2^2P^3\Phi^{(6:0,2)}_{c_2,k_2}(\omega) + c_2^3P^4\Phi^{(8:3,0)}_{c_2,k_2}(\omega)$$

$$+k_2^3P^4\Phi_{c_2,k_2}^{(8:0,3)}(\omega)+c_2^2P^4k_2\Phi_{c_2,k_2}^{(8:2,1)}(\omega)$$
$$+c_2k_2^2P^4\Phi_{c_2,k_2}^{(8:1,2)}(\omega)+\cdots, \tag{59}$$

where the expressions of $\Phi_{c_2,k_2}^{(2:0,0)}(\omega)$, $\Phi_{c_2,k_2}^{(4:1,0)}(\omega)$, $\Phi_{c_2,k_2}^{(4:0,1)}(\omega)$, $\Phi_{c_2,k_2}^{(6:1,1)}(\omega)$, $\Phi_{c_2,k_2}^{(6:2,0)}(\omega)$, $\Phi_{c_2,k_2}^{(6:0,2)}(\omega)$, $\Phi_{c_2,k_2}^{(8:3,0)}(\omega)$, $\Phi_{c_2,k_2}^{(8:0,3)}(\omega)$, $\Phi_{c_2,k_2}^{(8:2,1)}(\omega)$ and $\Phi_{c_2,k_2}^{(8:1,2)}(\omega)$ are given in Appendix A as Eqs. (A1)–(A10) respectively.

The dependence of $\Phi_{\lambda_1\ \lambda_2\ \cdots\ \lambda_{s_n}}^{(2n:j_1\ j_2\ \cdots\ j_{s_n})}(\mathrm{j}\omega)$ in Eq. (58) on the parameters in $H_1(\cdot)$ implies that the system linear characteristic parameters play an important role in the system output behavior. Moreover, when the linear characteristic parameters are fixed, the system output PSD can be determined through the polynomial function of the nonlinear characteristic parameters, which is an explicit analytical relationship between the nonlinear characteristic parameters and the system output PSD. Clearly, to know how the nonlinear characteristic parameters affect the system output PSD, $\Phi_{\lambda_1\ \lambda_2\ \cdots\ \lambda_{s_n}}^{(2n:j_1\ j_2\ \cdots\ j_{s_n})}(\mathrm{j}\omega)$ must be determined a priori, for example the values of $\Phi_{c_2,k_2}^{(2:0,0)}(\omega)$, $\Phi_{c_2,k_2}^{(4:1,0)}(\omega)$, $\Phi_{c_2,k_2}^{(4:0,1)}(\omega)$, $\Phi_{c_2,k_2}^{(6:1,1)}(\omega)$, $\Phi_{c_2,k_2}^{(6:2,0)}(\omega)$, $\Phi_{c_2,k_2}^{(6:0,2)}(\omega)$, $\Phi_{c_2,k_2}^{(8:3,0)}(\omega)$, $\Phi_{c_2,k_2}^{(8:0,3)}(\omega)$, $\Phi_{c_2,k_2}^{(8:2,1)}(\omega)$ and $\Phi_{c_2,k_2}^{(8:1,2)}(\omega)$ for the nonlinear oscillator (12). Of course, these functions can be determined directly using numerical integrations from their definitions, i.e., from Eqs. (A1)–(A10). However, as indicated by their expressions, such a numerical method would be very complex and so may not be applicable in some practices. To resolve this problem and ensure that the new expression of the system output PSD can be used practically to perform nonlinear system analysis, an algorithm is proposed to evaluate the values of $\Phi_{\lambda_1\ \lambda_2\ \cdots\ \lambda_{s_n}}^{(2n:j_1\ j_2\ \cdots\ j_{s_n})}(\mathrm{j}\omega)$ directly from the output response data. Here, the algorithm is introduced using the nonlinear oscillator (12) as an example.

Denote $\boldsymbol{\Phi}_{10\times1}(\omega)=\big(\Phi_{c_2,k_2}^{(2:0,0)}(\omega), \Phi_{c_2,k_2}^{(4:1,0)}(\omega), \Phi_{c_2,k_2}^{(4:0,1)}(\omega), \Phi_{c_2,k_2}^{(6:1,1)}(\omega), \Phi_{c_2,k_2}^{(6:2,0)}(\omega), \Phi_{c_2,k_2}^{(6:0,2)}(\omega), \Phi_{c_2,k_2}^{(8:3,0)}(\omega), \Phi_{c_2,k_2}^{(8:0,3)}(\omega), \Phi_{c_2,k_2}^{(8:2,1)}(\omega), \Phi_{c_2,k_2}^{(8:1,2)}(\omega)\big)'$. Equation (59) can be written as

$$S_{yy}(\omega)=\Big(P, c_2P^2, k_2P^2, c_2k_2P^3, c_2^2P^3, k_2^2P^3, c_2^3P^4, k_2^3P^4, c_2^2k_2P^4, c_2k_2^2P^4\Big)\Phi_{10\times1}(\omega). \tag{60}$$

Equation (60) indicates that the values of $\Phi_{10\times1}(\omega)$ can be evaluated and, to achieve this objective, at least 10 simulation studies or experimental tests are needed, and the nonlinear characteristic parameters c_2 and k_2 should take different values in the simulations or experimental tests. Assume $U \geqslant 10$ testes are conducted by taking $(c_2, k_2) = (c_{2(1)}, k_{2(1)}), \cdots, (c_{2(U)}, k_{2(U)})$ respectively, then the system responses of the U tests can be written as

$$\begin{pmatrix} S_{y_{(1)}y_{(1)}}(\omega) \\ \vdots \\ S_{y_{(U)}y_{(U)}}(\omega) \end{pmatrix} = \boldsymbol{\Theta}_{U\times10}\boldsymbol{\Phi}_{10\times1}(\omega), \tag{61}$$

where

$$\boldsymbol{\Theta}_{U\times10}=\begin{pmatrix} P, c_{2(1)}P^2, k_{2(1)}P^2, c_{2(1)}k_{2(1)}P^3, c_{2(1)}^2P^3, k_{2(1)}^2P^3, c_{2(1)}^3P^4, k_{2(1)}^3P^4, c_{2(1)}^2k_{2(1)}P^4, c_{2(1)}k_{2(1)}^2P^4 \\ \vdots \\ P, c_{2(U)}P^2, k_{2(U)}P^2, c_{2(U)}k_{2(U)}P^3, c_{2(U)}^2P^3, k_{2(U)}^2P^3, c_{2(U)}^3P^4, k_{2(U)}^3P^4, c_{2(U)}^2k_{2(U)}P^4, c_{2(U)}k_{2(U)}^2P^4 \end{pmatrix}, \tag{62}$$

then the values of $\boldsymbol{\Phi}_{10\times1}(\omega)$ can be determined from Eq. (62) using the least square (LS) based approach as

$$\boldsymbol{\Phi}_{10\times1}(\omega)=\left[\boldsymbol{\Theta}'_{(U\times10)}\boldsymbol{\Theta}_{(U\times10)}\right]^{-1}\boldsymbol{\Theta}'_{(U\times10)}\begin{pmatrix} S_{y_{(1)}y_{(1)}}(\omega) \\ \vdots \\ S_{y_{(U)}y_{(U)}}(\omega) \end{pmatrix}. \tag{63}$$

Obviously, to use the method (63) to estimate the values of $\Phi_{\lambda_1\ \lambda_2\ \cdots\ \lambda_{s_n}}^{(2n:j_1\ j_2\ \cdots\ j_{s_n})}(\mathrm{j}\omega)$, the information about what and how many monomials, $\lambda_1^{j_1}\ \lambda_2^{j_2}\ \cdots\ \lambda_{s_n}^{j_{s_n}}$, are included in Eq. (58) should be known a prior. To achieve this, a recursive algorithm is derived from Eqs. (9)–(11) and Eqs. (45) and (46) to determine what and how many monomials involved in Eq. (58), and it is given as Appendix B.

The results of Proposition 1 and Proposition 2 provide us two methods to analyze the nonlinear systems from different perspectives, so as to achieve a comprehensive observation on the effects of the input intensity and the nonlinear characteristic parameters on the random vibration of the nonlinear systems. However, as stated above, to utilize the two propositions in practice, one may need to resort to numerical simulation method and LS-based regression technique, but it can be seen that by using the two propositions one can depict a comprehensive pattern about the effectiveness of the nonlinear characteristic parameter or the input intensity on the output PSD through much less set of numerical simulation signals.

It is worth noting here that although the two propositions are concerned only with the output PSD, similar conclusions also hold for the cross-PSD.

4 Numerical simulations

In this section, numerical simulation studies are provided to demonstrate the theoretical results above. The nonlinear oscillator (12) is considered, taking the system linear characteristic parameters m, c and k to be 1 kg, 31.582 7 s·N/m and

15 791 N/m, respectively.

4.1 Case 1: The effect of the input intensity P

It is assumed that the PSD of the system output can be approximated as

$$S_{yy}(\omega) = \sum_{i=1}^{4} P^i \Lambda_{(i)}(\omega) + \mathrm{O}(P^5), \tag{64}$$

where $\mathrm{O}(P^5)$ represents the ignorable contribution of other higher order terms.

To estimate $\Lambda_{(i)}(\omega)$, $(i = 1, 2, 3, 4)$, four sets of responses of the oscillator (12) were calculated using the fourth-order Runge–Kutta method under $c_2 = 656.31\,\mathrm{Ms{\cdot}N/m^3}$ and $k_2 = 15.751\,\mathrm{GN/m^3}$ with the intensity of the Gaussian white noise input, P, taken as 0.319 1, 0.459 5, 0.625 4 and 0.816 9, respectively. The sampling frequency is considered to be 200 Hz.

From the numerical simulation responses, the output PSDs are first estimated via the Welch's method which is executed with the pwelch routine in the mathematic software Matlab 14.0, and then the least square based method described as Eq. (57) is used to estimate the values of $\Lambda_{(i)}(\omega)$, $(i = 1, 2, 3, 4)$. The estimated results are shown in Fig. 2. With the estimated $\Lambda_{(i)}(\omega)$, $(i = 1, 2, 3, 4)$, the PSDs are predicted using Eq. (64) for $P = 0.003\,2$–$2.871\,9$. The predicted results are shown in Fig. 3a. For comparison, the system responses are also numerically calculated by using the Runge–Kutta method for $P = 0.003\,2$–$2.871\,9$, and the PSDs which are estimated directly using the simulation responses are given in Fig. 3b; the difference between the two PSDs are presented in Fig. 3c. It can be seen that there is an excellent agreement between the two output PSDs. This agreement verifies that the output PSDs of nonlinear systems subjected to a Gaussian white noise input can be expressed as a polynomial function of the input intensity, i.e., Eq. (54). In addition, Fig. 3c indicates that the differences between the predicted PSDs and the estimated PSDs increase with the increase of the input intensity P. It is mainly because that, when the input intensity increases, the contribution, $\mathrm{O}(P^5)$, of the ignored higher order terms to the output PSDs will increase accordingly.

From the expression of $\Lambda_{(1)}(\omega)$, i.e., $\Lambda_{(1)}(\omega) = |H_1(\omega)|^2$, it is known that $\Lambda_{(1)}(\omega)$ is actually the power of the linear systems' FRF, which can be directly estimated from the output PSD and input PSD, i.e.

$$|H_1(\omega)|^2 = \frac{S_{yy}(\omega)}{S_{xx}(\omega)}. \tag{65}$$

Therefore, $\Lambda_{(1)}(\omega)$ of a nonlinear system would be equal to the power of the FRF of its associated linear system, which is generated by keeping the linear characteristic parameters unchanged and setting all the nonlinear characteristic parameters to be zero. To validate this, the power of the FRF of the associated linear system of the oscillator (12) under consideration is estimated from the numerical simulation response generated using the fourth-order Runge–Kutta method. The estimated $\Lambda_{(1)}(\omega)$ of the oscillator (12) and the estimated $|H_1(\omega)|^2$ of its associated linear system are shown in Fig. 4. In addition, the ratio between the output PSD and the input PSD of the oscillator (12) is also presented in Fig. 4

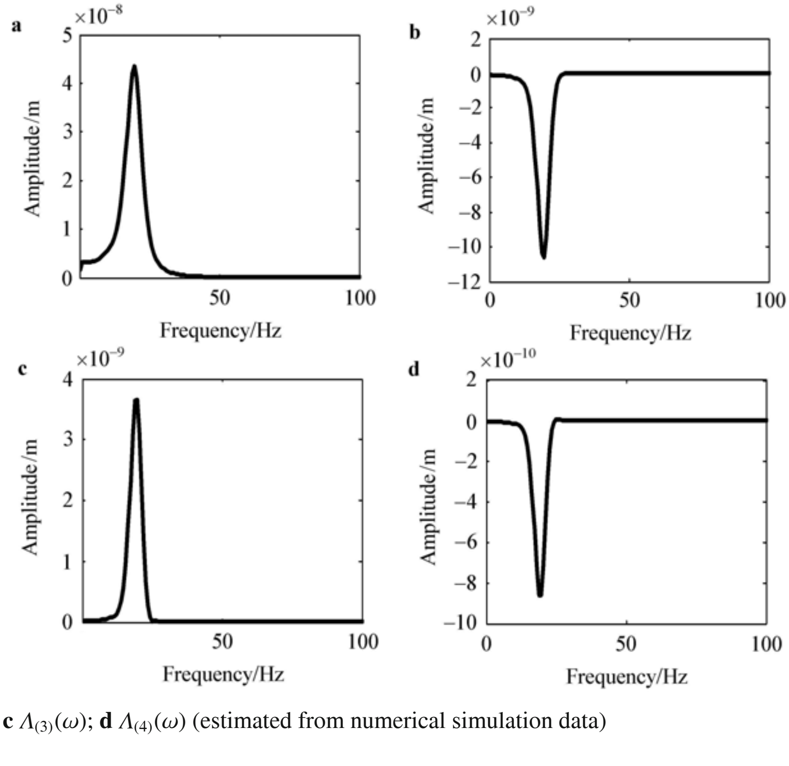

Fig. 2 **a** $\Lambda_{(1)}(\omega)$; **b** $\Lambda_{(2)}(\omega)$; **c** $\Lambda_{(3)}(\omega)$; **d** $\Lambda_{(4)}(\omega)$ (estimated from numerical simulation data)

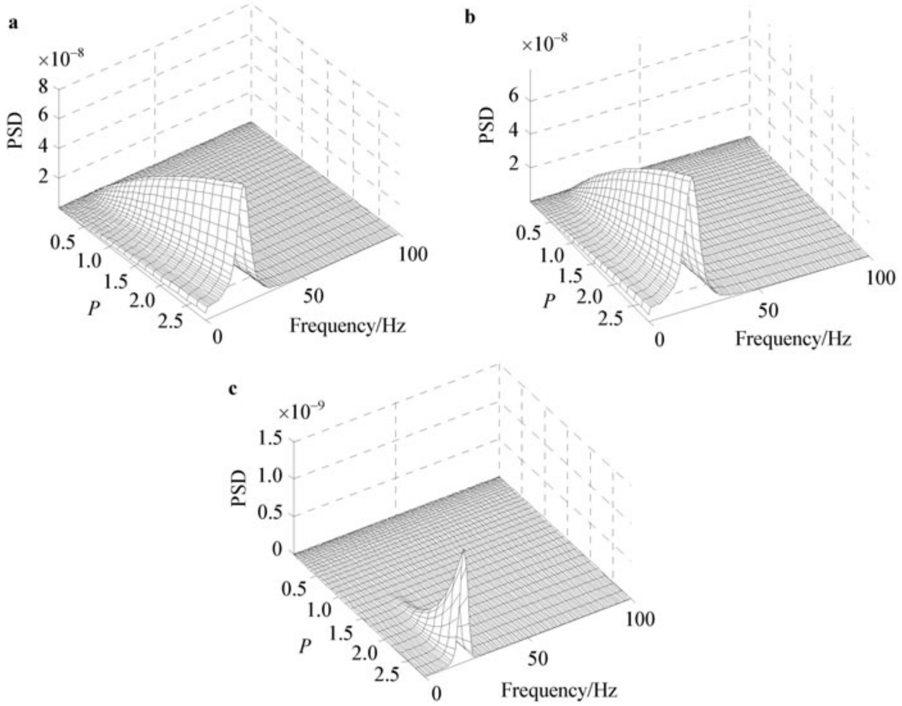

Fig. 3 The effect of P on PSD. **a** The predicted PSDs; **b** The estimated PSDs; **c** The difference between **a** and **b**

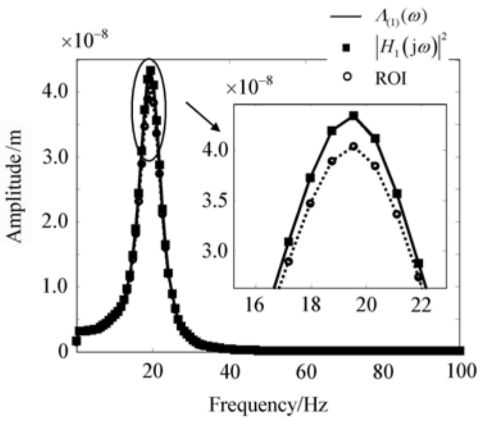

Fig. 4 Comparison among the estimated $\Lambda_{(1)}(\omega)$, $\left|H_1(\mathrm{j}\omega)\right|^2$ and the ROI of system (13)

as the ROI curve. It can be seen that the estimated $\Lambda_{(1)}(\omega)$ matches the estimated $|H_1(\omega)|^2$ very well, however, significant difference can be observed between the estimated $|H_1(\omega)|^2$ and the ROI curve around the resonance region, and the difference comes from the nonlinear terms in Eq. (54). To a certain extent, this confirms that, as stated by Proposition 1, there is a polynomial relationship between the output PSD of nonlinear systems and the input intensity.

4.2 Case 2: The effect of the nonlinear characteristic parameters

In this case, it is assumed that the output PSD can be approximated as

$$\begin{aligned} S_{yy}(\omega) &= P\Phi^{(2:0,0)}_{c_2,k_2}(\omega) + c_2P^2\Phi^{(4:1,0)}_{c_2,k_2}(\omega) + k_2P^2\Phi^{(4:0,1)}_{c_2,k_2}(\omega) \\ &+c_2k_2P^3\Phi^{(6:1,1)}_{c_2,k_2}(\omega) + c_2^2P^3\Phi^{(6:2,0)}_{c_2,k_2}(\omega) \\ &+k_2^2P^3\Phi^{(6:0,2)}_{c_2,k_2}(\omega) + \mathrm{O}(P^4). \end{aligned} \tag{66}$$

To estimate $\Phi^{(2:0,0)}_{c_2,k_2}(\omega)$, $\Phi^{(4:1,0)}_{c_2,k_2}(\omega)$, $\Phi^{(4:0,1)}_{c_2,k_2}(\omega)$, $\Phi^{(6:1,1)}_{c_2,k_2}(\omega)$, $\Phi^{(6:2,0)}_{c_2,k_2}(\omega)$, $\Phi^{(6:0,2)}_{c_2,k_2}(\omega)$, nine sets of simulation responses are generated using the fourth-order Runge–Kutta method with the nonlinear characteristic parameters c_2 and k_2 taking the values listed in Table 1.

Table 1 Values of the nonlinear characteristic parameters

c_2/(Gs·N·m^{-3})	6.826	6.826	6.826	9.188	9.188	9.188	10.763	10.763	10.763
k_2/(GN·m^{-3})	216.58	287.46	334.72	216.58	287.46	334.72	216.58	287.46	334.72

Following the same procedure used in the above case study, the values of $\Phi^{(2:0,0)}_{c_2,k_2}(\omega)$, $\Phi^{(4:1,0)}_{c_2,k_2}(\omega)$, $\Phi^{(4:0,1)}_{c_2,k_2}(\omega)$, $\Phi^{(6:1,1)}_{c_2,k_2}(\omega)$, $\Phi^{(6:2,0)}_{c_2,k_2}(\omega)$, $\Phi^{(6:0,2)}_{c_2,k_2}(\omega)$ are estimated by using the algorithm described as Eq. (63); the estimated results are shown in Figs. 5a–5f respectively. Comparing Fig. 5a with Fig. 2a, we can see that the two estimated results match each other very well; it is no surprising since $\Lambda_{(1)}(\omega)$ and

$\Phi^{(2:0,0)}_{c_2,k_2}(\omega)$ actually have the same expression, i.e., $|H_1(\omega)|^2$. In addition, from Eq. (A2) it can be expected that, theoretically, the estimated $\Phi^{(4:1,0)}_{c_2,k_2}(\omega)$ and $\mathrm{Re}\,(\mathrm{j}\omega H_1(-\omega))$ should have similar waveforms, as well as the estimated $\Phi^{(4:0,1)}_{c_2,k_2}(\omega)$ and $\mathrm{Re}\,(-H_1(-\omega))$. The values of $\mathrm{Re}\,(\mathrm{j}\omega H_1(-\omega))$ and $\mathrm{Re}\,(-H_1(-\omega))$ can be readily calculated directly from their analytic expressions, and the results are given in Figs. 6a and 6b, respectively. Obviously, just as expected, the estimated $\Phi^{(4:1,0)}_{c_2,k_2}(\omega)$ and $\mathrm{Re}\,(\mathrm{j}\omega H_1(-\omega))$, as well as $\Phi^{(4:0,1)}_{c_2,k_2}(\omega)$ and $\mathrm{Re}\,(-H_1(-\omega))$, have similar waveforms. To a certain extent, the consistencies between the estimated $\Lambda_{(1)}(\omega)$ and $\Phi^{(2:0,0)}_{c_2,k_2}(\omega)$ and between the waveforms validate the theoretical result expressed as Proposition 2.

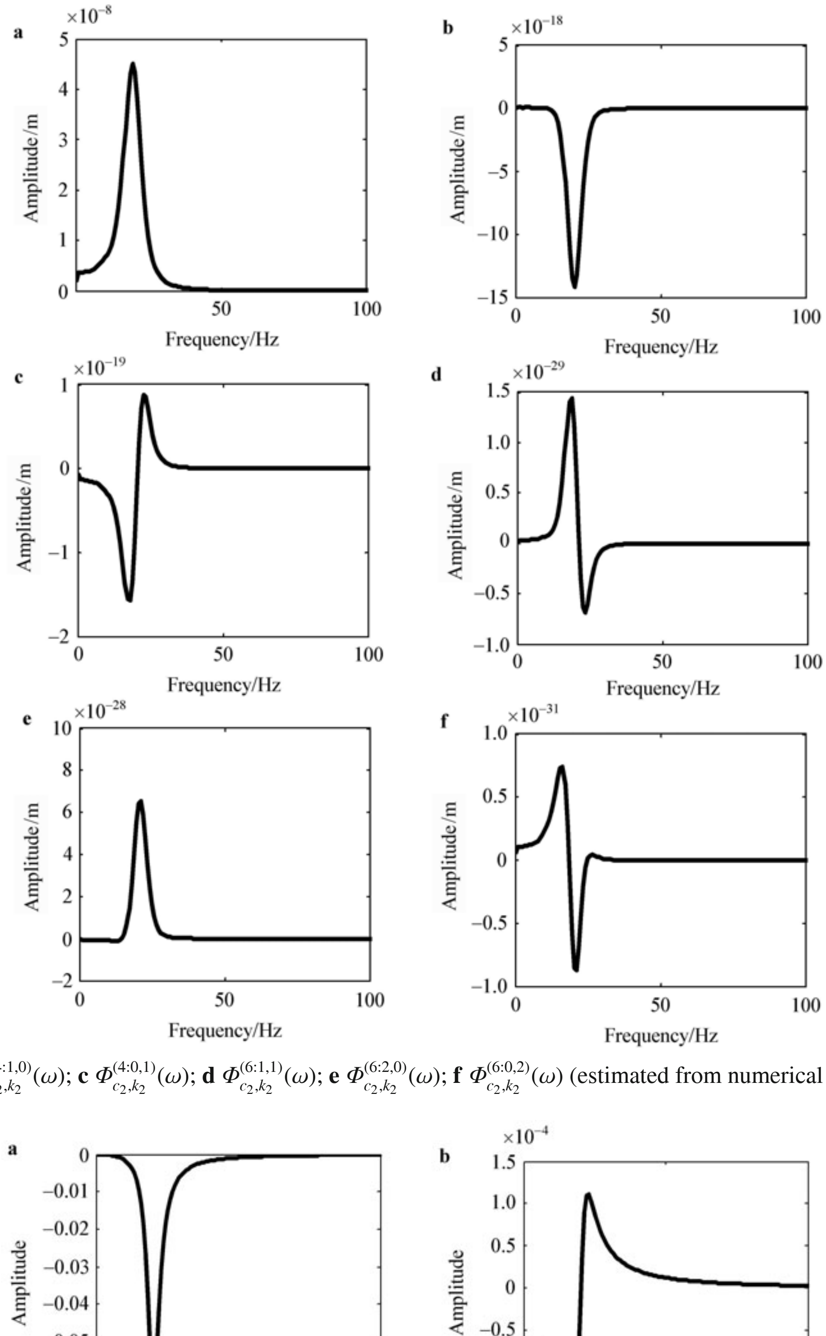

Fig. 5 **a** $\Phi^{(2:0,0)}_{c_2,k_2}(\omega)$; **b** $\Phi^{(4:1,0)}_{c_2,k_2}(\omega)$; **c** $\Phi^{(4:0,1)}_{c_2,k_2}(\omega)$; **d** $\Phi^{(6:1,1)}_{c_2,k_2}(\omega)$; **e** $\Phi^{(6:2,0)}_{c_2,k_2}(\omega)$; **f** $\Phi^{(6:0,2)}_{c_2,k_2}(\omega)$ (estimated from numerical simulation data)

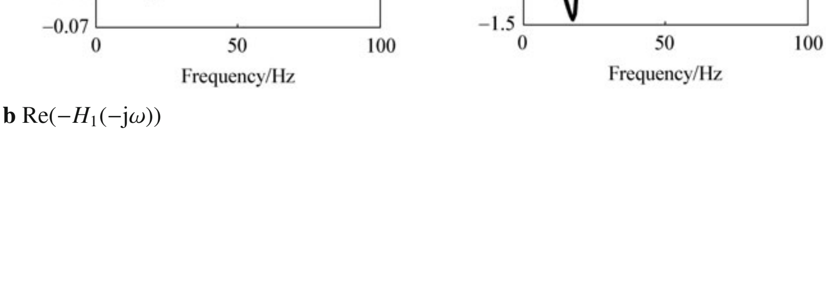

Fig. 6 **a** $\mathrm{Re}(\mathrm{j}\omega H_1(-\mathrm{j}\omega))$; **b** $\mathrm{Re}(-H_1(-\mathrm{j}\omega))$

With the estimated $\Phi^{(2:0,0)}_{c_2,k_2}(\omega)$, $\Phi^{(4:1,0)}_{c_2,k_2}(\omega)$, $\Phi^{(4:0,1)}_{c_2,k_2}(\omega)$, $\Phi^{(6:1,1)}_{c_2,k_2}(\omega)$, $\Phi^{(6:2,0)}_{c_2,k_2}(\omega)$, $\Phi^{(6:0,2)}_{c_2,k_2}(\omega)$, the effects of the nonlinear characteristic parameters c_2 and k_2 on the output PSD are investigated. The PSD shown in Fig. 7a is predicted using Eq. (66) with c_2 taken as 2.887 8 Gs·N/m^3 and k_2 varying from 3.937 9 to 476.48 GN/m^3. For comparison, the PSDs, which are estimated using the system responses numerically calculated with the Runge–Kutta method, are given as Fig. 7b. The difference between the two PSDs is presented in Fig. 7c. From both the predicted and estimated PSDs, it can be seen that around the resonance region the increase of the nonlinear stiffness characteristic parameter k_2 will reduce the output PSD of the Volterra nonlinear system; but in the non-resonance region the effect of k_2 on the output PSD is very weak and ignorable. The same procedure is used to investigate the effect of the nonlinear damping characteristic parameter c_2 on the output PSD. The value of k_2 is taken as 98.446 GN/m^3; the value of c_2 varies from 0.131 26 to 15.883 Gs·N/m^3. The results are shown in Fig. 8. Similarly, only around the resonance region is the effect of c_2 significant; increasing c_2 will effectively reduce the output PSD. In addition, from both Figs. 7 and 8 it can be seen that there is an excellent agreement between the output PSD predicted using Eq. (66) and the PSD estimated from the numerical simulation responses. This confirms that, as stated by Proposition 2, the PSDs of nonlinear systems subjected to a Gaussian white noise input can be expressed as a polynomial function of the nonlinear characteristic parameters, i.e., Eq. (58).

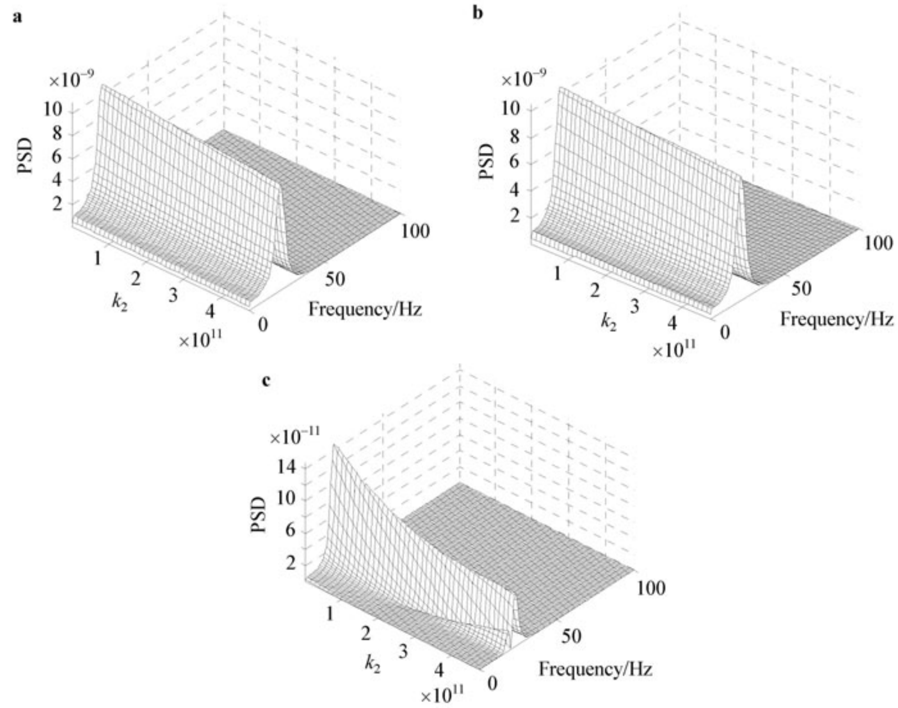

Fig. 7 The effects of k_2 on the PSD. **a** The predicted PSDs; **b** The estimated PSDs; **c** The difference between **a** and **b**

Overall, the results in Case 1 and Case 2 verify the theoretical analysis in the previous sections and demonstrate the effectiveness of the analytical descriptions Eqs. (54) and (58) for the PSD of the nonlinear systems. It is worthy noting here that similar numerical experiments could be designed to verify that the input-output cross-PSD of the nonlinear systems subjected to Gaussian white noise signal could also be expressed as a polynomial function of the nonlinear characteristic parameters or a polynomial function of the input intensity.

5 Conclusions

Analytical expressions for the calculation of output PSD and input-output cross-PSD of nonlinear systems subjected to a Gaussian white noise excitation have been derived using the Volterra series. Based on these expressions, the relationship between the output PSD and the nonlinear characteristic parameters and the relationship between the PSD and the input intensity have been investigated. The results show that the output PSD as well as the input-output cross-PSD of the nonlinear systems can be expressed as a polynomial function of the input intensity or a polynomial function of the nonlinear characteristic parameters. Results from simulation studies have been used to verify the theoretical analysis and to demonstrate the effectiveness of the derived relationship. As demonstrated in the present study, this analytical relationship is of significance to the analysis and design of a wide range of nonlinear engineering systems and structures which can be well represented by a Volterra series model.

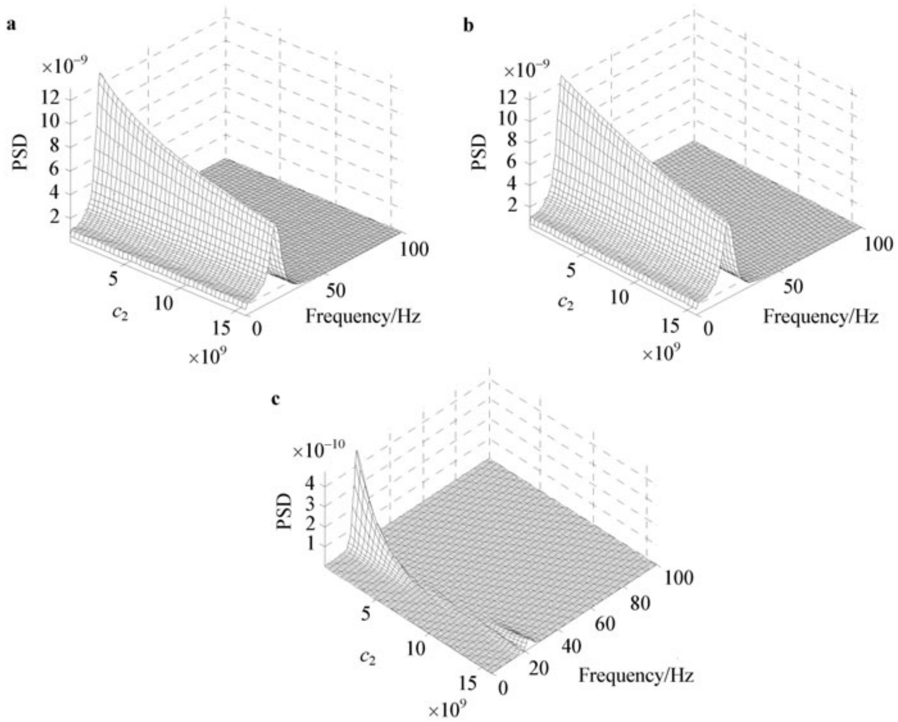

Fig. 8 The effects of c_2 on the PSD. **a** The predicted PSDs; **b** The estimated PSDs; **c** The difference between **a** and **b**

Appendix I

$$\Phi^{(2:0,0)}_{c_2,k_2}(\omega) = \left|H_1(\mathrm{j}\omega)\right|^2, \tag{A1}$$

$$\Phi^{(4:1,0)}_{c_2,k_2}(\omega) = \mathrm{Re}\left(\frac{\mathrm{j}2\omega}{\pi} H_1(-\mathrm{j}\omega)\left|H_1(\mathrm{j}\omega)\right|^2 \int \left|H_1(\mathrm{j}\omega_2)\right|^2 \mathrm{d}\omega_2\right), \tag{A2}$$

$$\Phi^{(4:0,1)}_{c_2,k_2}(\omega) = \mathrm{Re}\left(-\frac{6}{\pi} H_1(-\mathrm{j}\omega)\left|H_1(\mathrm{j}\omega)\right|^2 \int \left|H_1(\mathrm{j}\omega_2)\right|^2 \mathrm{d}\omega_2\right), \tag{A3}$$

$$\begin{aligned}\Phi^{(6:1,1)}_{c_2,k_2}(\omega) = {} & \mathrm{Re}\left\{\frac{6\mathrm{j}}{\pi^2}\int_{\substack{R^3\\ \omega_1=\omega}} H_1(\mathrm{j}\omega_1)\prod_{i=1}^{3}\left|H_1(\mathrm{j}\omega_i)\right|^2\right.\\ & \times\left\{\sum_{\substack{(i1,i2,\cdots,i3)\text{ is all per-}\\ \text{mutations}(1,2,-2,3,-3)}}\left(\omega_1+\sum_{l=1}^{3}\omega_{il}\right)\right.\\ & \left.\left.\times H_1\left[\mathrm{j}(\omega_{i1}+\omega_{i2}+\omega_{i3})\right]\right\}\mathrm{d}\omega_{1\to 3}\right\}\\ & +\mathrm{Re}\left(\frac{3j}{2\pi^2}\int_{\substack{R^3\\ \omega_2=\omega}}\omega_2\left|H_1(\mathrm{j}\omega_2)\right|^2\prod_{i=1}^{3}\left|H_1(\mathrm{j}\omega_i)\right|^2\mathrm{d}\omega_{1\to 3}\right),\end{aligned} \tag{A4}$$

$$\begin{aligned}\Phi^{(6:2,0)}_{c_2,k_2}(\omega) = {} & 2\mathrm{Re}\left\{-\frac{1}{\pi^2}\int_{\substack{R^3\\ \omega_1=-\omega}} H_1(\mathrm{j}\omega_1)\prod_{i=1}^{3}\left|H_1(\mathrm{j}\omega_i)\right|^2\right.\\ & \times\left\{\sum_{\substack{(i1,i2,\cdots,i3)\text{ is all per-}\\ \text{mutations}(1,2,-2,3,-3)}}\omega_1\left(\sum_{l=1}^{3}\omega_{il}\right)\right.\\ & \left.\left.\times H_1\left[\mathrm{j}(\omega_{i1}+\omega_{i2}+\omega_{i3})\right]\right\}\mathrm{d}\omega_{1\to 3}\right\}\\ & -\frac{\omega^2\left|H_1(\mathrm{j}\omega)\right|^2}{4\pi^2}\left(\int_{\substack{R^3\\ \omega_2=\omega}}\prod_{i=1}^{3}\left|H_1(\mathrm{j}\omega_i)\right|^2\mathrm{d}\omega_{1\to 3}\right.\\ & \left.-\frac{1}{9}\int_{\substack{R^3\\ \omega_1+\omega_2+\omega_3=\omega}}\prod_{i=1}^{3}\left|H_1(\mathrm{j}\omega_i)\right|^2\mathrm{d}\omega_{1\to 3}\right),\end{aligned} \tag{A5}$$

$$\begin{aligned}\Phi^{(6:0,2)}_{c_2,k_2}(\omega) = {} & 2\mathrm{Re}\left\{\frac{9H_1(-\mathrm{j}\omega)}{\pi^2}\int_{\substack{R^3\\ \omega_1=\omega}}\prod_{i=1}^{3}\left|H_1(\mathrm{j}\omega_i)\right|^2\right.\\ & \left.\times\left\{\sum_{\substack{(i1,i2,\cdots,i3)\text{ is all per-}\\ \text{mutations}(1,2,-2,3,-3)}} H_1\left[\mathrm{j}(\omega_{i1}+\omega_{i2}+\omega_{i3})\right]\right\}\mathrm{d}\omega_{1\to 3}\right\}\\ & +\frac{\left|H_1(\mathrm{j}\omega)\right|^2}{4\pi^2}\left(9\int_{\substack{R^3\\ \omega_2=\omega}}\prod_{i=1}^{3}\left|H_1(\mathrm{j}\omega_i)\right|^2\mathrm{d}\omega_{1\to 3}\right.\\ & \left.+\int_{\substack{R^3\\ \omega_1+\omega_2+\omega_3=\omega}}\prod_{i=1}^{3}\left|H_1(\mathrm{j}\omega_i)\right|^2\mathrm{d}\omega_{1\to 3}\right),\end{aligned} \tag{A6}$$

$$\begin{aligned}\Phi^{(8:3,0)}_{c_2,k_2}(\omega) = {} & \mathrm{Re}\left\{\frac{\mathrm{j}\omega^2}{2\pi^3}\int_{\substack{R^4\\ \omega_2=-\omega}}\left|H_1(\mathrm{j}\omega)\right|^2\prod_{i=1}^{4}\left|H_1(\mathrm{j}\omega_i)\right|^2\right.\\ & \times\sum_{\substack{(i1,i2,\cdots,i5)\text{ is all per-}\\ \text{mutations}(2,3,-3,4,-4)}}\left(\sum_{l=3}^{5}\omega_{il}\right)H_1\\ & \left.\times\left[\mathrm{j}(\omega_{i3}+\omega_{i4}+\omega_{i5})\right]\mathrm{d}\omega_{1\to 4}\right\}\\ & +\mathrm{Re}\left\{\frac{\mathrm{j}\omega^2}{3\pi^3}\int_{\substack{R^4\\ \omega_1+\omega_2+\omega_3=-\omega}}\left|H_1(\mathrm{j}\omega)\right|^2\prod_{i=1}^{4}\left|H_1(\mathrm{j}\omega_i)\right|^2\right.\end{aligned}$$

$$\times \sum_{\substack{(i1,i2,\cdots,i5)\text{ is all per-}\\ \text{mutations}(2,3,-3,4,-4)}} \left(\sum_{l=3}^{5} \omega_{il}\right) H_1$$

$$\times \left[\mathrm{j}(\omega_{i3}+\omega_{i4}+\omega_{i5})\right] \mathrm{d}\omega_{1\to 4}\Bigg\}, \quad \text{(A7)}$$

$$\Phi^{(8:0,3)}_{c_2,k_2}(\omega) = \mathrm{Re}\left\{ -\frac{27\left|H_1(\mathrm{j}\omega)\right|^2}{2\pi^3} \int_{\substack{R^4\\ \omega_2=\omega}} \prod_{i=1}^{4} \left|H_1(\mathrm{j}\omega_i)\right|^2 \right.$$

$$\left. \times \sum_{\substack{(i1,i2,\cdots,i5)\text{ is all per-}\\ \text{mutations}(2,3,-3,4,-4)}} H_1\left[\mathrm{j}(\omega_{i3}+\omega_{i4}+\omega_{i5})\right] \mathrm{d}\omega_{1\to 4} \right\}$$

$$-\mathrm{Re}\left\{ \frac{9\left|H_1(\mathrm{j}\omega)\right|^2}{\pi^3} \int_{\substack{R^4\\ \omega_1+\omega_2+\omega_3=\omega}} \prod_{i=1}^{4} \left|H_1(\mathrm{j}\omega_i)\right|^2 \right.$$

$$\left. \times \sum_{\substack{(i1,i2,\cdots,i5)\text{ is all per-}\\ \text{mutations}(2,3,-3,4,-4)}} H_1\left[\mathrm{j}(\omega_{i3}+\omega_{i4}+\omega_{i5})\right] \mathrm{d}\omega_{1\to 4} \right\}, \quad \text{(A8)}$$

$$\Phi^{(8:2,1)}_{c_2,k_2}(\omega) = \mathrm{Re}\left\{ \frac{3\omega^2\left|H_1(\mathrm{j}\omega)\right|^2}{2\pi^3} \int_{\substack{R^4\\ \omega_2=\omega}} \prod_{i=1}^{4} \left|H_1(\mathrm{j}\omega_i)\right|^2 \right.$$

$$\left. \times \sum_{\substack{(i1,i2,\cdots,i5)\text{ is all per-}\\ \text{mutations}(2,3,-3,4,-4)}} H_1\left[\mathrm{j}(\omega_{i3}+\omega_{i4}+\omega_{i5})\right] \mathrm{d}\omega_{1\to 4} \right\}$$

$$+\mathrm{Re}\left\{ \frac{9\omega^2\left|H_1(\mathrm{j}\omega)\right|^2}{8\pi^3} \int_{\substack{R^4\\ \omega_1+\omega_2+\omega_3=\omega}} \prod_{i=1}^{4} \left|H_1(\mathrm{j}\omega_i)\right|^2 \right.$$

$$\left. \times \sum_{\substack{(i1,i2,\cdots,i5)\text{ is all per-}\\ \text{mutations}(2,3,-3,4,-4)}} H_1\left[\mathrm{j}(\omega_{i3}+\omega_{i4}+\omega_{i5})\right] \mathrm{d}\omega_{1\to 4} \right\}, \quad \text{(A9)}$$

$$\Phi^{(8:1,2)}_{c_2,k_2}(\omega) = \mathrm{Re}\left\{ \frac{9\mathrm{j}\omega^2}{2\pi^3} \int_{\substack{R^4\\ \omega_2=\omega}} \left|H_1(\mathrm{j}\omega)\right|^2 \prod_{i=1}^{4} \left|H_1(\mathrm{j}\omega_i)\right|^2 \right.$$

$$\left. \times \sum_{\substack{(i1,i2,\cdots,i5)\text{ is all per-}\\ \text{mutations}(2,3,-3,4,-4)}} \left(\sum_{l=3}^{5}\omega_{il}\right) H_1\left[\mathrm{j}(\omega_{i3}+\omega_{i4}+\omega_{i5})\right] \mathrm{d}\omega_{1\to 4} \right\}$$

$$+\mathrm{Re}\left\{ \frac{27\mathrm{j}\omega^2}{8\pi^3} \int_{\substack{R^4\\ \omega_1+\omega_2+\omega_3=\omega}} \left|H_1(\mathrm{j}\omega)\right|^2 \prod_{i=1}^{4} \left|H_1(\mathrm{j}\omega_i)\right|^2 \right.$$

$$\times \sum_{\substack{(i1,i2,\cdots,i5)\text{ is all per-}\\ \text{mutations}(2,3,-3,4,-4)}} \left(\sum_{l=3}^{5}\omega_{il}\right) H_1$$

$$\left. \times \left[\mathrm{j}(\omega_{i3}+\omega_{i4}+\omega_{i5})\right] \mathrm{d}\omega_{1\to 4} \right\}. \quad \text{(A10)}$$

Appendix II

Denote the set of all monomials involved in Eq. (58) as Γ, then Γ can be determined as follows

$$\Gamma = \left(\bigcup_{z=1}^{[(Q+1)/2]} \bigcup_{d=1}^{[(Q+1)/2]} \Gamma_{(2z-1)} \otimes \Gamma_{(2d-1)} \right)$$
$$\bigcup \left(\bigcup_{z=1}^{[Q/2]} \bigcup_{d=1}^{[Q/2]} \Gamma_{2z} \otimes \Gamma_{2d} \right), \quad \text{(A11)}$$

where $\otimes$ is the Kronecker product, $\Gamma_1 = \{1\}$ and Γ_n is determined as

$$\Gamma_n = \left[\bigcup_{l_1,l_2,\cdots,l_n=0}^{Z} \left[c_{0n}(l_1,l_2,\cdots,l_n)\right] \right]$$
$$\bigcup \left[\bigcup_{q=1}^{n-1} \bigcup_{p=1}^{n-q} \bigcup_{l_1,l_2,\cdots,l_n=0}^{Z} \left(\left[c_{pq}(l_1,l_2,\cdots,l_n)\right] \otimes \Gamma_{n-q,p} \right) \right]$$
$$\bigcup \left[\bigcup_{p=2}^{n} \bigcup_{l_1,l_2,\cdots,l_p=0}^{Z} \left(\left[c_{p0}\left(l_1,l_2,\cdots,l_p\right)\right] \otimes \Gamma_{np} \right) \right], \quad \text{(A12)}$$

with

$$\Gamma_{np} = \bigcup_{i=1}^{n-p+1} \left(\Gamma_i \otimes \Gamma_{n-i,p-1} \right), \qquad \Gamma_{n1} = \Gamma_n. \quad \text{(A13)}$$

To demonstrate the algorithm, it is applied to the oscillator (12) up to the 5th order, and the results are given below

$\Gamma_1 = \{1\}$

$\Gamma_2 = \text{Null}$

$\Gamma_3 = \{\{c_2\} \otimes \{1\}\} \cup \{\{k_2\} \otimes \{1\}\} = \{c_2,\ k_2\}$

$\Gamma_4 = \text{Null}$

$\Gamma_5 = \{\{c_2\} \otimes \Gamma_3\} \cup \{\{k_2\} \otimes \Gamma_3\} = \left\{ c_2^2\ \ c_2k_2\ \ k^2 \right\}$

and

$$\Gamma = (\Gamma_1 \otimes \Gamma_1) \bigcup (\Gamma_1 \otimes \Gamma_3) \bigcup (\Gamma_1 \otimes \Gamma_5)$$
$$\bigcup (\Gamma_3 \otimes \Gamma_3) \bigcup (\Gamma_3 \otimes \Gamma_5) \bigcup (\Gamma_5 \otimes \Gamma_5)$$
$$= \{1,\ c_2,\ k_2,\ c_2^2,\ c_2k_2,\ k_2^2,\ c_2^3,\ c_2^2k_2,\ c_2k_2^2,\ k_2^3,\ c_2^4,$$
$$c_2^3k_2,\ c_2^2k_2^2,\ c_2k_2^3,\ k_2^4\}.$$

Obviously, the first 10 elements in Γ are those elements of the vector used in Eq. (59). The algorithm can be readily implemented with symbolic operation method, providing a convenient way to determine the monomials involved in Eq. (58). This simplifies the procedure of implementing the result expressed as Proposition 2 to analyze the Volterra nonlinear system subjected to a Gaussian white noise input.

References

1 Li, W.M., Hong, J.Z.: Research on the iterative method for model updating based on the frequency response function. Acta Mechanica Sinica **28**, 450–457 (2012)

2 Cao, D.X., Zhang, W., Yao, M.H.: Analytical and experimental studies on nonlinear characteristics of an L-shape beam structure. Acta Mechanica Sinica **26**, 967–976 (2010)

3 Ogunfunmi, T.: Adaptive Nonlinear System Identification: The Volterra and Wiener Model Approaches. Springer, Berlin Heidelberg, New York (2008)

4 Boyd,S., Chua, L.O.: Fading memory and the problem of approximating nonlinear operators with Volterra series. IEEE Transactions on Circuits and Systems **32**, 1150–1161 (1985)

5 Franz, M.O., Schölkopf, B.: A unifying view of Wiener and Volterra theory and polynomial kernel regression. Neural Computation **18**, 3097–3118 (2006)

6 Jeffreys,H., Jeffreys, B.S.: Methods of Mathematical Physics, 3rd edn. Cambridge University Press, Cambridge (1988)

7 Friston, K. J.: Imaging neuroscience: Principles or maps? Proceedings of the National Academy of Sciences **95**, 796–802 (1998)

8 Korenberg, M.J., Hunter, I.W.: The identification of nonlinear biological systems: Volterra kernel approaches. Annals of Biomedical Engineering **24**, 250–268 (1996)

9 Chen, G., Li, Y. M.: Advances and prospects of the reduced order model for unsteady flow and its application. Advances in Mechanics **41**, 686–701 (2011)

10 Zhang, Z.J., Xu, M., Chen, S.L.: Investigation of model reduction for aeroservoelasticity based on unsteady aerodynamics estimating. Acta Mechanica Sinica **41**, 641–650 (2009)

11 Naess, A., Gaidai, O.: The asymptotic behaviour of second-order stochastic Volterra series models of slow drift response. Probabilistic Engineering Mechanics **22**, 343–352 (2007)

12 Worden, K., Manson, G.: Nonlinearity in experimental modal analysis, Philosophical Transaction of Royal Society of London, Series A **359**, 113–130 (2001)

13 Chatterjee, A., Vyas, N.S.: Nonlinear parameter estimation in rotor-bearing system using volterra series and method of harmonic probing. Journal of Vibration and Acoustics **125**, 299–306 (2003)

14 Worden, K., Manson, G., Tomlinson, G. R.: Pseudo-Nonlinearities in Engineering Structures. Proceedings of the Royal Society of London A **445**, 193–220 (1994)

15 Ruotolo, R., Surace, C., Crespo, P., et al.: Harmonic analysis of the vibration of a cantilevered beam with closing crack. Computers and Structures **61**, 1057–1074 (1996)

16 George, D. A.: Continuous Nonlinear Systems. MIT Research Laboratory of Electrics. Cambridge, Massachusetts (1959)

17 Billings, S.A., Peyton Jones, J.C.: Mapping nonlinear integro-differential equations into the frequency domain. International Journal of Control **52**, 863–879 (1990)

18 Peyton Jones, J.C., Billings, S.A.: A recursive algorithm for the computing the frequency response of a class of nonlinear difference equation models. International Journal of Control **50**, 1925–1940 (1989)

19 Zhang, H., Billings, S.A.: Analysing non-linear systems in the frequency domain, I: The transfer function. Mechanical Systems and Signal Processing **7**, 531–550 (1993)

20 Zhang, H., Billings, S.A.: Analysing nonlinear systems in the frequency domain, II: The phase response. Mechanical Systems and Signal Processing **8**, 45–62 (1994)

21 Vazquez Feijoo, J.A., Worden, K., Stanway, R.: Associated linear equations for Volterra operators. Mechanical Systems and Signal Processing **19**, 57–69 (2005)

22 Vazquez Feijoo, J.A., Worden, K., Stanway, R.: System identification using associated linear equations. Mechanical Systems and Signal Processing **18**, 431–455 (2004)

23 Vazquez Feijoo, J.A., Worden, K., Stanway, R.: Analysis of time-invariant systems in the time and frequency domain by associated linear equations (ALEs). Mechanical Systems and Signal Processing **20**, 896–919 (2006)

24 Lang, Z.Q., Billings, S.A.: Energy transfer properties of nonlinear systems in the frequency domain. International Journal of Control **78**, 354–362 (2005)

25 Lang, Z.Q., Billings, S.A., Yue, R., et al.: Output frequency response functions of nonlinear Volterra systems. Automatica **143**, 805–816 (2007)

26 Tomlinson, G.R., Manson, G., Lee, G.M.: A simple criterion for establishing an upper limit to the harmonic excitation level of the Duffing oscillator using the Volterra series. Journal of Sound and Vibration **190**, 751–762 (1996)

27 Chatterjee, A., Vyas, N.S.: Convergence analysis of Volterra series response of nonlinear systems subjected to harmonic excitation. Journal of Sound and Vibration **236**, 339–358 (2006)

28 Peng, Z.K., Lang, Z.Q.: On the convergence of the Volterra-series representation of the Duffing's oscillators subjected to harmonic excitations. Journal of Sound and Vibration **305**, 322–332 (2007)

29 Jing, X.J., Lang, Z.Q., Billings, S.A.: New bound characteristics of NARX model in the frequency domain. International Journal of Control **80**, 140–149 (2007)

30 Billings, S.A., Lang, Z.Q.: A bound for the magnitude characteristics of nonlinear output frequency response functions. 1. Analysis and computation. International Journal of Control **65**, 309–328 (1996)

31 Sandberg, I.W.: Bounds for discrete-time volterra series representations. IEEE Transactions on Circuits and Systems—I: Fundamental Theory and Applications **46**, 135–139 (1999)

32 Thouverez, F.: A new convergence criteria of Volterra series for harmonic inputs. Proceedings of the 16th International Modal Analysis Conference **3243**, 723–727 (1998)

33 Worden, K., Manson, G.: A Volterra series approximation to the coherence of the Duffing oscillator. Journal of Sound and Vibration **286**, 529–547 (2005)

34 Chatterjee, A.: Identification and parameter estimation of a bilinear oscillator using Volterra series with harmonic probing. International Journal of Non-Linear Mechanics **45**, 12–20 (2010)

35 Lang, Z.Q., Billings, S.A.: Output frequency characteristics of nonlinear system. International Journal of Control **64**, 1049–1067 (1996)

36 Bedrosian, E., Rice, S.O.: The output properties of Volterra systems (nonlinear systems with memory) driven by harmonic and Gaussian inputs. Proeedings of the IEEE **59**, 1688–1708 (1971)

37 Bussgang, J.J., Ehrman, L., Graham, J.W.: Analysis of nonlinear systems with multiple inputs, Proceedings of the IEEE **62**, 1088–1119 (1974)

38 Victor, J.D., Knight, B.W.: Nonlinear analysis with an arbitrary stimulus ensemble. Quarterly of Applied Mathematics **37**, 113–136 (1979)

39 Billings, S.A., Tsang, K.M.: Spectral analysis for nonlinear systems, part 1: parametric nonlinear spectral analysis. Mechanical Systems and Signal Processing **3**, 319–339 (1989)

40 Swain, A.K., Billings, S.A.: Generalized frequency response function matrix for MIMO nonlinear systems. International Journal of Control **74**, 829–844 (2001)

41 Barrett, J.F.: Formula for output autocorrelation and spectrum of a Volterra system with stationary Gaussian input. IEE Proceedings-Control Theory and Applications **127**, 286–289 (1980)

42 Worden, K., Manson, G.: Random vibrations of a Duffing oscillator using the Volterra series. Journal of Sound and Vibration **217**, 781–789 (1998)

43 Marzocca, P., Nichols, J.M., Milanese, A., et al.: Second-order spectra for quadratic nonlinear systems by Volterra functional series: Analytical description and numerical simulation. Mechanical Systems and Signal Processing **22**, 1882–1895 (2008)

44 Michalowicz, J.V., Nichols, J.M., Bucholtz, F., et al.: An Isserlis' Theorem for mixed gaussian variables: Application to the auto-bispectral density. Journal of Statistical Physics **136**, 89–102 (2009)

45 Cheng, C.M., Peng, Z.K., Meng, G.: Random vibration frequency response analysis of nonlinear systems based on Volterra series. Acta Mechanica Sinica **43**, 905–913 (2011)