Acta Mech. Sin. (2012) 28(3):848–853
DOI 10.1007/s10409-012-0085-8

RESEARCH PAPER

Optimal design of vibration absorber using minimax criterion with simplified constraints

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Received: 16 January 2012 / Revised: 15 March 2012 / Accepted: 31 March 2012

Abstract In this paper, a minimax design of damped dynamic vibration absorber for a damped primary system is investigated to minimize the vibration magnitude peaks. Moreover, to reduce the sensitivity of the primary system response to variations of the forcing frequency for a two-degree-of-freedom system, the primary system should have two equal resonance magnitude peaks. To meet this requirement, a set of simplified constraint equations including distribution characteristics of the resonant frequencies of the primary system is established for the minimax objective function. The modified constraint equations have less unknown variables than those by other authors, which not only simplifies the computation but also improves the accuracy of the optimal values. The advantage of the proposed method is illustrated through numerical simulations.

Keywords Minimax · Optimization · Dynamic vibration absorber (DVA) · Sensitivity

1 Introduction

The dynamic vibration absorber (DVA) is a widely used passive vibration device. Classically, an undamped dynamic vibration absorber consists of a mass and a spring attached to a vibrating body which vibrates harmonically or in a narrow-band frequency range. A damped vibration absorber consists of a damper and a spring which are added to the primary mass on one side and attached to the absorber mass on the other side. Since Watts and Frahm [1, 2] reported on the first use of the dynamic vibration absorber, amount of works have been completed on the optimization design [3–7] and tuning of the vibration absorber [8–12].

For a two-degree-of-freedom (2-DOF) system that consists of a primary mass and an absorber mass, as it introduces two resonant frequencies in the neighborhood of the suppressed frequency and becomes sensitive to variations of the forcing frequency, a type of optimal design is brought forward. The usual goals pursued in this type of design are the minimization of vibration amplitudes of both primary and absorber masses and the reduction in sensitivity of the primary system response to the forcing frequency change. Den Hartog [13] did early analytical work with no damping in the primary system. It is proposed that the most favorable response curve of the primary mass should have two equal resonance magnitude peaks which make the primary mass less sensitive to variations of the forcing frequency.

To obtain two equal resonance magnitude peaks for a damped primary system, Pennestrì developed a minimax design of a damped dynamic vibration absorber [14]. He established the minimax objective function subject to six constraint equations with seven unknown variables. However, on the one hand, the constraint equations are very complicated so both the computation time and accuracy are affected to a certain extent. On the other hand, some of the variables have no obvious physical meaning, the initials of which are hard to be chosen, such as L and Δ (see also Ref. [6], etc.). Therefore, in this investigation, a modified minimax design of damped dynamic vibration absorber for a damped primary system is presented.

This paper is organized as follows: Section 2 presented the motion equations of a vibrating system; Section 3 presented the minimax optimization formulation, where a set of simplified constraint equations containing three equalities and one inequality was given; Section 4 presented numerical examples and comparisons to illustrate effectiveness of the proposed design.

The project was supported by the National Natural Science Foundation of China (11072014 and 11172018).

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2 Motion equations of the vibrating system

For practical systems, there exists damping inevitably. In most cases, the damping is viscous. Figure 1 shows a damped primary system with a damped DVA, which has been studied by Randall, Pennestrì and many others [4, 8, 14, 15]. The motion equations of this 2-DOF system are given by

$$m_1\ddot{x}_1 = b_2(\dot{x}_2-\dot{x}_1)+k_2(x_2-x_1)-b_1\dot{x}_1-k_1x_1+F\sin(\omega t), \quad (1)$$

$$m_2\ddot{x}_2 = b_2(\dot{x}_1-\dot{x}_2)+k_2(x_1-x_2), \quad (2)$$

where m_1, k_1, b_1, x_1 and m_2, k_2, b_2, x_2 are the mass, stiffness, damping, displacement of the primary mass and the absorber mass, respectively, F and ω are the amplitude and frequency of the excitement force.

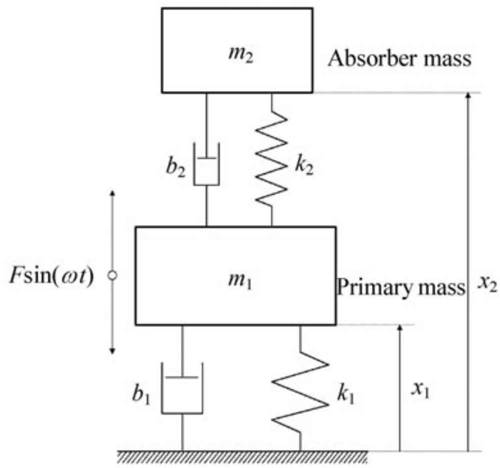

Fig. 1 Damped vibration absorber system model

When the transient responses have vanished, the harmonic responses of the two masses are given by the following expressions

$$x_1 = X_1\sin(\omega t+\Psi_1), \quad (3)$$

$$x_2 = X_2\sin(\omega t+\Psi_2). \quad (4)$$

Substituting Eqs. (3) and (4) into Eqs. (1) and (2), and solving for X_1, X_2 yields

$$X_1 = \frac{F\sqrt{(k_2-m_2\omega^2)^2+(b_2\omega)^2}}{H}, \quad (5)$$

$$X_2 = \frac{F\sqrt{k_2^2+(b_2\omega)^2}}{H}, \quad (6)$$

where

$$H^2 = [k_1k_2-\omega^2(m_1k_2+m_2k_1+b_1b_2+k_2m_2)+m_1m_2\omega^4]^2 + [(m_1b_2+b_1m_2+m_2b_2)\omega^3-(k_1b_2+k_2b_1)\omega]^2. \quad (7)$$

Taking the following normalized variables

$$\omega_i = \sqrt{\frac{k_i}{m_i}}, \quad \beta = \frac{\omega}{\omega_1}, \quad T = \frac{\omega_2}{\omega_1}, \quad \mu = \frac{m_2}{m_1}, \quad \xi_i = \frac{b_i}{2\sqrt{k_im_i}}, \quad i=1,2, \quad (8)$$

and substituting these normalized variables into Eqs. (5) and (6), we obtain the frequency responses for the system as

$$\alpha = \frac{k_1X_1}{F} = \frac{\sqrt{(1-\beta^2/T^2)^2+4(\xi_2\beta/T)^2}}{Z}, \quad (9)$$

$$\gamma = \frac{k_1X_r}{F} = \frac{\beta^2/T^2}{Z}, \quad (10)$$

where α is the normalized maximum displacement of the primary mass, X_r and γ are the maximum displacement and normalized maximum displacement of the absorber mass relative to the primary one, respectively, and Z^2 is defined as

$$Z^2 = \left[\frac{\beta^4}{T^2}-\frac{\beta^2}{T^2}-\beta^2(1+\mu)-\frac{4\xi_1\xi_2\beta^2}{T}+1\right]^2 + 4\left[\frac{\xi_1\beta^3}{T^2}+\frac{\xi_2\beta^3(1+\mu)-\xi_2\beta}{T}-\xi_1\beta\right]^2. \quad (11)$$

3 Minimax optimization formulation

The focus of this section is to formulate the constraint equations for the optimal design of vibration absorber. A minimax objective function subject to three equalities and one inequality is given for the spring-mass-damper system to solve for optimal design parameters.

For a 2-DOF system, the frequency response curve is characterized by two peak points which correspond to two maxima α_{pl}, α_{pr} as shown in Fig. 2. It can be seen that the difference between the two resonance magnitude peaks is large and the amplitude varies quickly near the suppressed frequency, namely, the amplitude is sensitive to variations of the forcing frequency, which should be avoided in practical applications. Therefore, to minimize the maximum vibration magnitude of the primary system and reduce meanwhile the sensitivity of the primary system response to variations of the forcing frequency near the suppressed frequency, some constraint conditions must be added to the optimization. According to Den Hartog [13], the most favorable response curve of the primary system should have the same maximum amplitudes. Therefore, a constraint equation of $\alpha_{pl} = \alpha_{pr}$ should be included to constrain the magnitude peaks of the primary system.

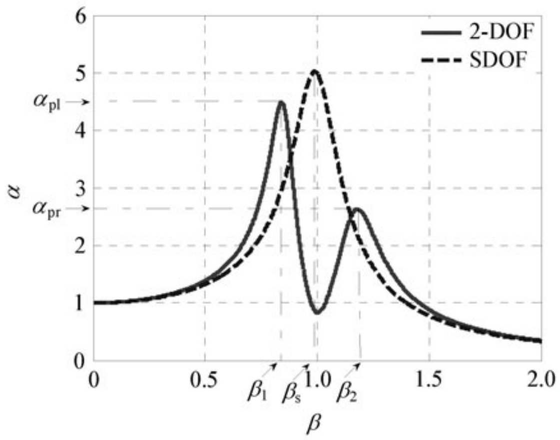

Fig. 2 Damped system frequency response

In fact, three extreme values including two maximum values and one minimum value can be found on the α vs. β curve as shown in Fig. 2. As a result, firstly, we must find out the two maximum values from the three extreme values, and then judge whether they have the same value. In order to effectively seek the two peaks, many additional constraint conditions have been reported. In Pennestrì's paper [14], he believed that the optimal curve α had two equal peak values with minimum distance from a straight line $\alpha = L$, where L is initially unknown. So he proposed the following minimax design for the optimal absorber parameters

$$\min_{\xi_2,T} \max_{\beta} \alpha, \tag{12}$$

$$\begin{aligned} \text{subject to } & \left.\frac{\mathrm{d}\alpha}{\mathrm{d}\beta}\right|_{\beta=\beta_1} = \left.\frac{\mathrm{d}\alpha}{\mathrm{d}\beta}\right|_{\beta=\beta_2} = \left.\frac{\mathrm{d}\alpha}{\mathrm{d}\beta}\right|_{\beta=\beta_3} = 0, \\ & -\alpha(\beta_1) + L + \Delta = 0, \\ & -\alpha(\beta_2) + L - \Delta = 0, \\ & -\alpha(\beta_3) + L + \Delta = 0, \end{aligned} \tag{13}$$

where Δ is the maximum deviation of the response curve from the value $\alpha = L$ and β_1, β_2 and β_3 are the frequency ratios where such a curve attains a maximum or a minimum.

In this paper, the distribution characteristic of the resonant frequencies of the primary system is utilized to establish an additional constraint equation for seeking the two peaks. It is well known that the two resonance magnitude peaks of the 2-DOF system must bracket the solo amplitude peak of single-degree-of-freedom (SDOF) system (see Fig. 2), so the forcing frequency range can be divided into two sections, with one amplitude peak on each side of the frequency of the solo amplitude peak

$$\beta_{\rm s} = \sqrt{1 - 2\xi_1^2}, \tag{14}$$

which is the resonant frequency of the non-dimensional SDOF system without an absorber mass. Then two resonance magnitude peaks can be obtained, with one amplitude peak on each side of $\beta_{\rm s}$. And $\alpha_{\max}$ is set equal to the larger of the two peak amplitudes.

According to the reasoning of Randall et al. [4], it has been assumed that ξ_1 and μ are independent parameters, varying respectively. Thus, the remaining parameters to be optimized are ξ_2 and T. Making use of the conditions required by the arguments above, the optimization problem proposed in this investigation can be posed as the following minimax problem

$$\min_{\xi_2,T} \max_{\beta} \alpha, \tag{15}$$

$$\begin{aligned} \text{subject to } & \left.\frac{\mathrm{d}\alpha}{\mathrm{d}\beta}\right|_{\beta=\beta_i,\, i=1,2} = 0, \\ & \alpha(\beta_1) = \alpha(\beta_2), \qquad \beta_1 < \beta_{\rm s} < \beta_2, \end{aligned} \tag{16}$$

where β_1 and β_2 represent the two resonant frequencies corresponding to two resonance magnitude peaks, respectively.

It should be noted that the constraint conditions are composed of four equations with four unknown variables (i.e., ξ_2, T, β_1, β_2). The solution of Eq. (15) will be the set of ξ_2 and T which will minimize the maximum $\alpha_{\max}$ with constraint Eq. (16).

4 Numerical examples and comparison

For the classic system in which $\xi_1 = 0$, a closed form analytical solution was given by Den Hartog [13]. He reported the following optimal choice of parameters that assured the curve α vs. β had two minimum equal peak values

$$\xi_{\rm 2opt} = \sqrt{\frac{3\mu}{8(1+\mu)}}, \qquad T_{\rm opt} = \frac{1}{1+\mu}. \tag{17}$$

Now for a damped primary system, we use numerical simulation to find the optimal absorber parameters and illustrate the formulations proposed in Sect. 3. In the simulation procedure, the Optimization package in Matlab is used. Consider a linear damped primary system with the following characteristics: $m_1 = 100\,\text{kg}$, $\omega_1 = 100\,\text{rad/s}$ [8]. It is assumed that ξ_1 and μ are in the range of $0 \le \xi_1 \le 0.4$ and $0 \le \mu \le 0.4$ [4].

First, as a comparison and a reference, the unconstrained $\alpha_{\max}$ surface is presented in Fig. 3 with varying ξ_2 and T at $\xi_1 = 0.1$, $\mu = 0.1$, from which it can be seen that the optimal parameters are those lead to the bottom on the curved surface. Then, the function $\max \alpha$ is solved under constraint Eq. (16), and a series of solution sets of ξ_2 and T that form a shaded area in Fig. 4 are obtained.

As can be seen in Fig. 4, the optimal choice of parameters ξ_2 and T, which minimizes the maximum amplitude of the primary system and meanwhile reduces sensitivity of the primary system response to the forcing frequency change near the suppressed frequency, is the solution set with the minimum of $\alpha_{\max}$. In Fig. 4, the optimal set of $\xi_{\rm 2opt}$, $T_{\rm opt}$ is marked as

$$\xi_{\rm 2opt} = 0.197\,3, \quad T_{\rm opt} = 0.862\,0, \quad \alpha_{\max} = 2.622\,5.$$

It should be noticed that the optimal set of $\xi_{2\mathrm{opt}}$, T_{opt} in Fig. 3 is

$$\xi_{2\mathrm{opt}} = 0.21, \quad T_{\mathrm{opt}} = 0.86, \quad \alpha_{\max} = 2.63.$$

The difference between the two optimal sets of $\xi_{2\mathrm{opt}}$, T_{opt} in Figs. 3 and 4 is due to the computation accuracy. In Fig. 3, the steps of ξ_2, T are both 0.05 , whereas in Fig. 4, the steps of ξ_2, T are both 10^{-4} to obtain the accurate result. As a matter of fact, if the steps of ξ_2, T in Fig. 3 are 10^{-4}, the minimum of $\alpha_{\max}$ will be equal to that given in Fig. 4, which indicates that, there is no loss in the reduction of the response amplitude when the condition of the optimal design is met.

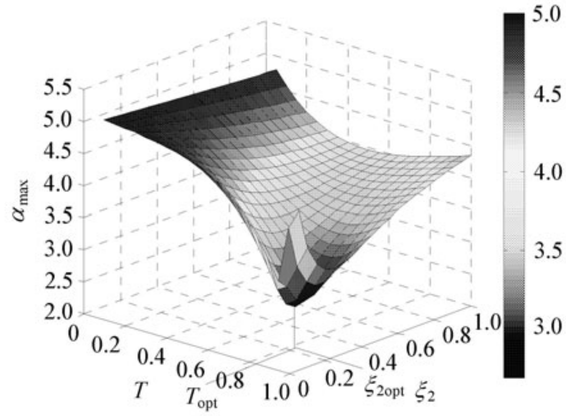

Fig. 3 Unconstrained $\alpha_{\max}$ surface with varying ξ_2 and T for $\xi_1 = 0.1, \mu = 0.1$

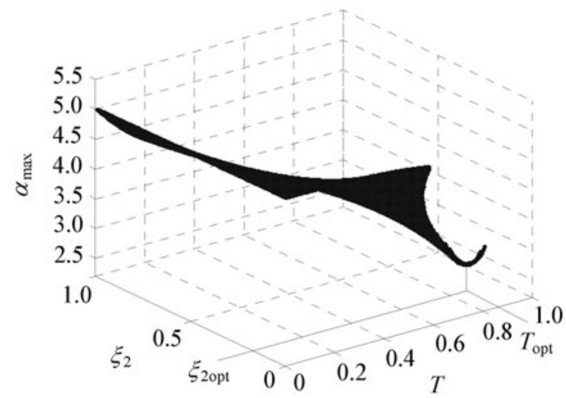

Fig. 4 Constrained $\alpha_{\max}$ surface with varying ξ_2 and T for $\xi_1 = 0.1$, $\mu = 0.1$

Similarly, by changing ξ_1 and μ, the curves of $\xi_{2\mathrm{opt}}$, T_{opt} are obtained as shown in Figs. 5 and 6, respectively. From Figs. 5 and 6 it can be concluded that, T_{opt} decreases monotonically with increasing ξ_1 and μ, whereas $\xi_{2\mathrm{opt}}$ increases monotonically with increasing ξ_1 and μ. When ξ_1 is small, $\alpha_{\max}$ and $\gamma_{\max}$ are both strongly dependent on μ, but when ξ_1 grows, they both become less sensitive to μ as shown in Figs. 7 and 8. As a matter of fact, when $\xi_1 \geq 0.4$, they vary little whatever μ is. This phenomenon is of great importance to the design of the vibration absorber.

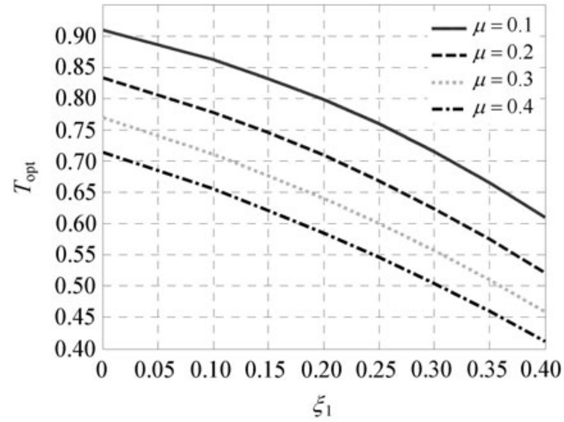

Fig. 5 Optimal T for prescribed ξ_1 and μ

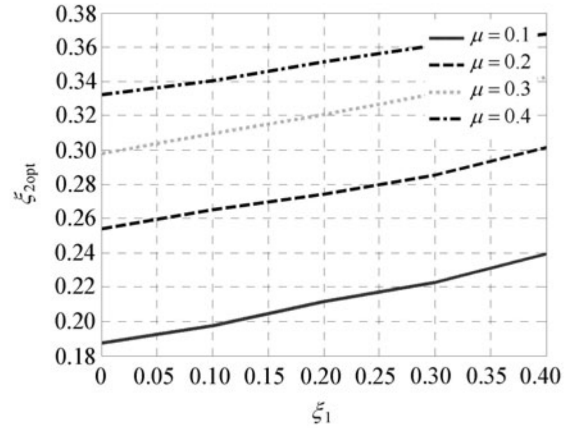

Fig. 6 Optimal ξ_2 for prescribed ξ_1 and μ

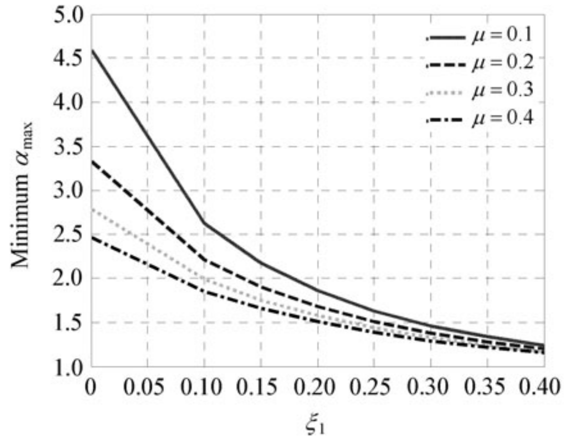

Fig. 7 Optimal $\alpha_{\max}$ for prescribed ξ_1 and μ

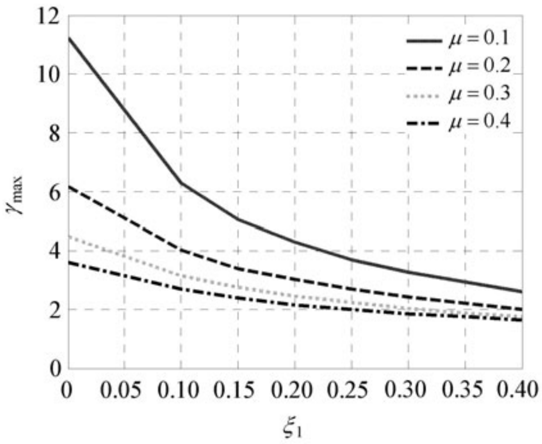

Fig. 8 Optimal γ_{max} for prescribed ξ_1 and μ

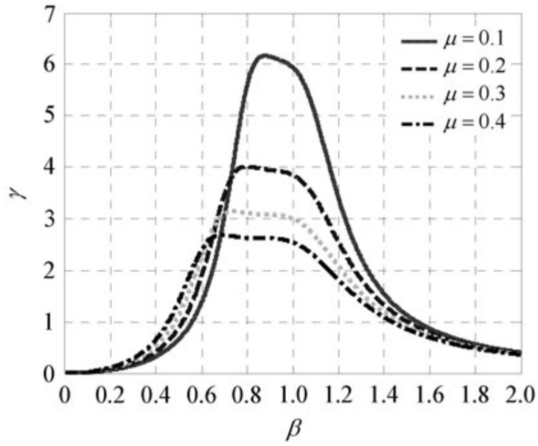

Fig. 10 γ vs. β at ξ_{2opt}, T_{opt} for $\xi_1 = 0.1$

The curves of α vs. β and γ vs. β at ξ_{2opt}, T_{opt} for $\xi_1 = 0.1$ are shown in Figs. 9 and 10. From Fig. 9 it can be seen that the primary system response curves have two equal resonance magnitude peaks, which satisfies the requirement of the system's low sensitivity to the forcing frequency change. Also it can be confirmed that the two peaks distribute on both sides of β_s, with one peak on each side of β_s. Moreover it can be seen clearly from Figs. 9 and 10 that both α_{max} and γ_{max} decrease with increasing mass ratio μ.

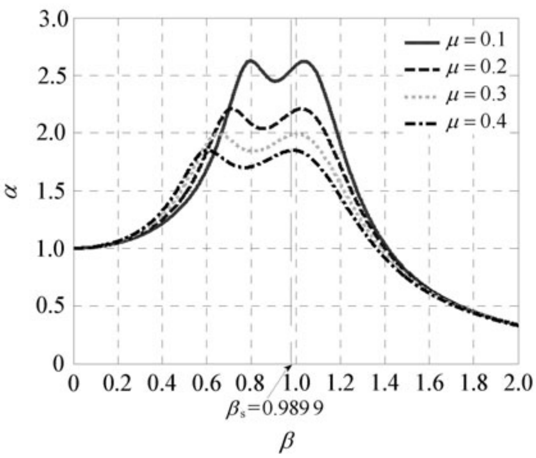

Fig. 9 α vs. β at ξ_{2opt}, T_{opt} for $\xi_1 = 0.1$

A comparison of the optimal design parameters obtained with the proposed formulations and those from other researchers is reported in Table 1. Notice that the reported α_{max}, γ_{max} do not exactly match those presented in related papers as a result of different computation accuracy. In Brown's paper [15], he believed that the working frequency range in real life is actually a narrow band, and limited the forcing frequency range to achieve a relatively accurate solution. In this investigation, the forcing frequency range is not limited and a more accurate solution is obtained through our formulations which prove to be well-performed.

Table 1 Optimal results given by different authors ($\xi_1 = 0.1, \mu = 0.1$)

Authors	T_{opt}	ξ_{2opt}	α_{max}	γ_{max}
Den Hartog [13]	0.909	0.185	2.899 4	6.624 1
Randall [4]	0.861	0.204	2.627 1	6.157 9
Pennestrì [14]	0.861	0.202	2.627 2	6.188 3
Brown [15]	0.862	0.199	2.622 7	6.248 8
This paper	0.862	0.197	2.622 5	6.272 7

To further advance the understanding of our simplified formulations, the main indexes in two different methods are listed in Table 2. Without loss of generality, ξ_2, T are defined in the range of $0.19 \le \xi_2 \le 0.21$ and $0.86 \le T \le 0.87$, and the steps size of ξ_2, T are both 10^{-4}. It is shown that the numerical simulation by our method saves CPU time compared to Pennestrì's method (Intel Core 2 CPU, 2.13G Hz).

Table 2 Comparison of main indexes in two different methods ($\xi_1 = 0.1, \mu = 0.1$)

	Authors	Number of constraint equations	Number of unknown variables	Optimal value	CPU time
$\xi_2 \in [0.19, 0.21]$ $T \in [0.86,\ 0.87]$ step size : 10^{-4}	Pennestrì [14]	6	7	$T_{opt} = 0.861$ $\xi_2 = 0.202$	568.7 s
	This paper	4	4	$T_{opt} = 0.862$ $\xi_2 = 0.197$	409.5 s

Moreover, for $\xi_1 = 0.1$, $\mu = 0.1$, the curves of α vs. β and γ vs. β at optimal parameters obtained in three different investigations are illustrated in Fig. 11. From Fig. 11 it can be seen that the curves match well. For other parameters such as $\xi_1 = 0$, $\mu = 0.2$, a little superiority can be observed from the results by our formulations as shown in Fig. 12.

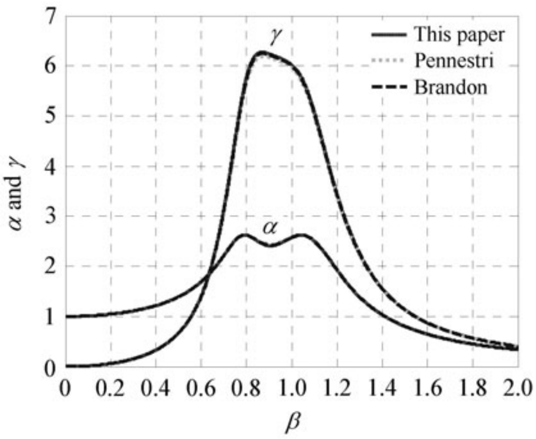

Fig. 11 Optimal $\alpha_{\max}$, $\gamma_{\max}$ for $\xi_1 = 0.1$, $\mu = 0.1$

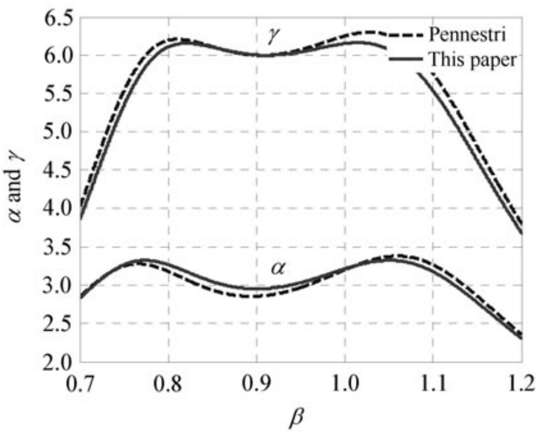

Fig. 12 Optimal $\alpha_{\max}$, $\gamma_{\max}$ for $\xi_1 = 0$, $\mu = 0.2$

5 Conclusions

In this investigation, a minimax problem formulation is presented for the design of vibration absorber. A set of simplified constraint equations is given. The distribution characteristic of the resonant frequencies of the primary system is utilized to establish the constraint equations for the minimax objective function, which has obvious physical meaning and helps to simplify the constraint equations. Most importantly, the comparisons with the results by other authors show good agreement, and the results by our formulations show a little superiority for some parameters. These comparisons show that the minimax formulation and the simplified constraint equations work effectively.

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