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RESEARCH PAPER

Unsteady flow of viscoelastic fluid between two cylinders using fractional Maxwell model

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Abstract The unsteady flow of an incompressible fractional Maxwell fluid between two infinite coaxial cylinders is studied by means of integral transforms. The motion of the fluid is due to the inner cylinder that applies a time dependent torsional shear to the fluid. The exact solutions for velocity and shear stress are presented in series form in terms of some generalized functions. They can easily be particularized to give similar solutions for Maxwell and Newtonian fluids. Finally, the influence of pertinent parameters on the fluid motion, as well as a comparison between models, is highlighted by graphical illustrations.

Keywords Maxwell fluid · Fractional derivative · Exact solutions · Velocity field · Shear stress · Laplace and Hankel transforms

1 Introduction

The motion of a fluid in cylindrical domains has applications in the food industry, oil exploitation, chemistry and bio-engineering [1], and the first exact solutions corresponding to motions of non-Newtonian fluids in cylindrical domains seem to be those of Ting [2] for second grade fluids, Srivastava [3] for Maxwell fluids and Waters and King [4] for Oldroyd-B fluids. In the meantime a lot of papers regarding such motions have been published. However, most of them deal with motion problems in which the velocity field is given on the boundary. To the best of our knowledge, the first exact solutions for motions of non-Newtonian fluids due to a circular cylinder that applies a constant shear stress to the fluid are those of Bandelli and Rajagopal [5]. These solutions give the velocity field corresponding to the motion of a second grade fluid between two infinite circular cylinders, the inner one applies a constant longitudinal or rotational shear stress to the fluid. Other similar solutions have also been obtained in Refs. [6–13].

The aim of this note is to extend the results of Ref. [5, Sect. 5] to a class of rate type fluids. More exactly, our interest is to find the velocity field and the adequate shear stress corresponding to the motion of a Maxwell fluid between two infinite circular cylinders when the inner cylinder applies a rotational shear stress to the fluid. However, for generality, we shall develop exact solutions for a larger class of such fluids, namely Maxwell fluids with fractional derivative. Recently, the fractional calculus has witnessed much success in the description of viscoelasticity [14–22]. The interest for viscoelastic fluids with fractional derivatives came from practical problems. In order to predict the dynamic response of viscous dampers, Makris et al. [23] firstly used conventional models of viscoelasticity. It was not possible to achieve satisfactory fitting of experimental data over the entire range of frequencies. However, a very good fitting of the experimental data was archived when the fractionalized Maxwell model

$$\tau + \lambda \mathrm{D}^{\alpha}\tau = \mu \mathrm{D}^{\beta}\gamma \tag{1}$$

was used. Here τ and γ are the shear stress and strain, λ and μ are generalized material constants and D^{α} is a fractional derivative operator of order α with respect to time. This model collapses to the conventional Maxwell model with $\alpha = \beta = 1$, in which λ and μ become the relaxation time and dynamic viscosity, respectively. Based on the fact

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that at vanishingly small strain rates, the behavior of the viscoelastic fluid would be reduced to that of a Newtonian fluid, the parameter β was set equal to unity. The other three parameters were also determined and the predict mechanical properties are in excellent agreement with experimental results.

However, despite these successful results, it must be emphasized that a constitutive relation should be expressed in a three dimensional form such that it is also frame indifferent. The first objective law which characterizes an incompressible fractional Maxwell fluid seems to be that of Palade et al. [24, Eq. (16)]. Under linearization, this constitutive relation is reduced to the fractional integral Maxwell model [24, Eq. (8)] which is equivalent to Eq. (1). Consequently, if one wishes to study uni-dimensional behavior only, one could consider the present model as a qualified candidate. So, in the following we shall develop exact solutions for velocity and shear stress corresponding to the unsteady flows of an incompressible fractional Maxwell fluid due to an infinite circular cylinder that applies a time-dependent torsional shear stress to the fluid. These solutions satisfy all imposed initial and boundary conditions and can easily be reduced to the similar solutions for ordinary Maxwell and Newtonian fluids performing the same motion. Finally, the influence of the pertinent parameters on the fluid motion as well as comparison among models is graphically illustrated. It is found that the Newtonian fluid is the swiftest and fractional fluid is the slowest.

2 Statement of the problem

Consider an incompressible fractional Maxwell fluid (FMF) at rest in the annular region between two infinitely long co-axial cylinders. At time $t = 0^+$ let the inner cylinder of radius R_1 be set in rotation about its axis by a time-dependent torque per unit length $2\pi R_1 \tau(R_1, t)$ [5], where

$$\tau(R_1, t) = \frac{f}{\lambda} R_{\alpha,-1}\left(\frac{-1}{\lambda}, t\right), \quad 0 < \alpha < 1, \tag{2}$$

f is a constant and the generalized $R_{a,b}(c, t)$ functions are defined by [25]

$$R_{a,b}(c, t) = \sum_{n=0}^{\infty} \frac{c^n t^{(n+1)a-b-1}}{\Gamma[(n+1)a - b]}, \quad \mathrm{Re}(a - b) > 0. \tag{3}$$

Owing to the shear the fluid is gradually moved, its velocity being of the form of

$$\boldsymbol{V} = \boldsymbol{V}(r, t) = w(r, t)\boldsymbol{e}_\theta, \tag{4}$$

where $\boldsymbol{e}_\theta$ is the unit vector in the θ-direction of a cylindrical coordinate system r, θ and z. For such flows the constraint of incompressibility is automatically satisfied, while the governing equations (cf. [16, Eqs. (6) and (10)]) are

$$(1 + \lambda \mathrm{D}_t^\alpha)\frac{\partial w(r, t)}{\partial t} = \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)w(r, t), \quad r \in (R_1, R_2), \ t > 0, \tag{5}$$

$$(1 + \lambda \mathrm{D}_t^\alpha)\tau(r, t) = \mu\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)w(r, t), \quad r \in (R_1, R_2), \ t > 0, \tag{6}$$

where λ is a material constant having the dimension of t^α, μ is the dynamic viscosity, $\nu = \mu/\rho$ is the kinematic viscosity (ρ is the constant density of the fluid), $\tau(r, t) = S_{r\theta}(r, t)$ is the non-trivial shear stress, R_2 is the radius of the outer cylinder and D_t^α is the Caputo fractional derivative of order α defined by [26, 27],

$$\mathrm{D}_t^\alpha f(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{f'(\tau)}{(t - \tau)^\alpha}\, \mathrm{d}\,\tau, \quad 0 < \alpha < 1. \tag{7}$$

Of course, for $\alpha \to 1$ when $\mathrm{D}_t^\alpha f(t) \to \mathrm{d}f(t)/\mathrm{d}t$, Eqs. (5) and (6) are reduced to the governing equations for an ordinary Maxwell fluid, the constant λ becoming the relaxation time.

In the following the fractional partial differential equations (5) and (6), together with the appropriate initial and boundary conditions

$$w(r, 0) = \frac{\partial w(r, 0)}{\partial t} = 0\,, \ \tau(r, 0) = 0, \quad r \in (R_1, R_2], \tag{8}$$

and

$$(1 + \lambda \mathrm{D}_t^\alpha)\tau(r, t)\big|_{r=R_1} = \mu\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)w(r, t)\Big|_{r=R_1} = f, \quad w(R_2, t) = 0, \quad t > 0, \tag{9}$$

will be solved by means of Laplace and finite Hankel transforms. The expression of $\tau(R_1, t)$ given by Eq. (2) is just the solution of the fractional differential equation (9). For $\alpha \to 1$, it takes a simple form

$$\tau(R_1, t) = \frac{f}{\lambda} R_{1,-1}\left(\frac{-1}{\lambda}, t\right) = f[1 - \mathrm{e}^{-t/\lambda}], \tag{10}$$

corresponding to an ordinary Maxwell fluid. Furthermore, making $\lambda \to 0$ in Eq. (10) and having Eq. (9) in mind we obtain the boundary conditions of

$$\tau(R_1, t) = \mu\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)w(r, t)\Big|_{r=R_1} = f, \ w(R_2, t) = 0, \ t > 0, \tag{11}$$

corresponding to the motion of a Newtonian fluid due to a constant couple on the boundary (cf. [5, Eqs. (5.2) and (5.3)]).

3 The solution of the problem

In order to solve the fractional differential equations (5) and (6) with initial and boundary conditions (8) and (9) we shall follow the same line as adopted in Ref. [9]. Applying the Laplace transform to Eq. (5), and using the Laplace transform formula for sequential fractional derivatives [27], we find that

$$(q + \lambda q^{\alpha+1})\overline{w}(r,q) = \nu\Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\Big)\overline{w}(r,q), \quad r \in (R_1, R_2), \tag{12}$$

where the image function $\overline{w}(r,q) = \mathcal{L}\{w(r,t)\}$ has to satisfy the conditions

$$\Big(\frac{\partial}{\partial r} - \frac{1}{r}\Big)\overline{w}(R_1,q) = \frac{f}{\mu q} \quad \text{and} \quad \overline{w}(R_2,q) = 0. \tag{13}$$

In the following we denote by referring to [7, 28]

$$\overline{w}_H(r_n,q) = \int_{R_1}^{R_2} r\overline{w}(r,q)B(r,r_n)\,\mathrm{d}\,r, \quad n = 1,2,3,\cdots \tag{14}$$

the finite Hankel transform of $\overline{w}(r,q)$, where

$$B(r,r_n) = \mathrm{J}_1(rr_n)\mathrm{Y}_2(R_1r_n) - \mathrm{J}_2(R_1r_n)\mathrm{Y}_1(rr_n), \tag{15}$$

r_n are the positive roots of the transcendental equation $B(R_2,r) = 0$ while $\mathrm{J}_p(\bullet)$ and $\mathrm{Y}_p(\bullet)$ are Bessel functions of the first and second kind of order p.

Multiplying both sides of Eq. (12) by $rB(r,r_n)$, integrating with respect to r from R_1 to R_2 and taking into account conditions (13) and the identity of

$$\int_{R_1}^{R_2} r\left[\frac{\partial^2\overline{w}(r,q)}{\partial r^2} + \frac{1}{r}\frac{\partial\overline{w}(r,q)}{\partial r} - \frac{1}{r^2}\overline{w}(r,q)\right]B(r,r_n)\,\mathrm{d}\,r = \frac{2}{\pi r_n}\Big(\frac{\partial}{\partial r} - \frac{1}{r}\Big)\overline{w}(r,q)\Big|_{r=R_1} - r_n^2\overline{w}_H(r_n,q), \tag{16}$$

we find that

$$\overline{w}_H(r_n,q) = \frac{2f}{\rho\pi r_n}\frac{1}{q(q + \lambda q^{\alpha+1} + \nu r_n^2)}. \tag{17}$$

Now, for a suitable presentation of the final results, we rewrite Eq. (17) in the following equivalent form

$$\overline{w}_H(r_n,q) = \frac{2f}{\mu\pi r_n^3 q} - \frac{2f(1 + \lambda q^{\alpha})}{\mu\pi r_n^3(q + \lambda q^{\alpha+1} + \nu r_n^2)}, \tag{18}$$

apply the inverse finite Hankel transform formula [7, 28]

$$\overline{w}(r,q) = \frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_n^2\mathrm{J}_1^2(R_2r_n)B(r,r_n)}{\mathrm{J}_2^2(R_1r_n) - \mathrm{J}_1^2(R_2r_n)}\,\overline{w}_H(r_n,q), \tag{19}$$

and use the known result

$$\int_{R_1}^{R_2}(r^2 - R_2^2)B(r,r_n)\,\mathrm{d}\,r = \frac{4}{\pi r_n^3}\Big(\frac{R_2}{R_1}\Big)^2. \tag{20}$$

Finally, applying the discrete inverse Laplace transform method [27] and using the known result [25, Eq. (97)]

$$\mathcal{L}^{-1}\Big\{\frac{q^b}{(q^a - d)^c}\Big\} = G_{a,b,c}(d,t), \quad \mathrm{Re}\,(ac - b) > 0, \quad \mathrm{Re}(q) > 0, \quad \Big|\frac{d}{q^a}\Big| < 1, \tag{21}$$

where the generalized $G_{a,b,c}(d,t)$ function is defined by

$$G_{a,b,c}(d,t) = \sum_{j=0}^{\infty}\frac{d^j\Gamma(c+j)}{\Gamma(c)\Gamma(j+1)}\frac{t^{(c+j)a-b-1}}{\Gamma[(c+j)a-b]}, \tag{22}$$

we find the velocity field in the form of

$$\begin{aligned} w(r,t) = {} & \frac{f}{2\mu}\Big(\frac{R_1}{R_2}\Big)^2\Big(r - \frac{R_2^2}{r}\Big) - \frac{\pi f}{\mu\lambda}\sum_{n=1}^{\infty}\frac{\mathrm{J}_1^2(R_2r_n)B(r,r_n)}{r_n[\mathrm{J}_2^2(R_1r_n) - \mathrm{J}_1^2(R_2r_n)]} \\ & \times \sum_{k=0}^{\infty}\Big(\frac{-\nu r_n^2}{\lambda}\Big)^k\Big[G_{\alpha,-k-1,k+1}\Big(\frac{-1}{\lambda},t\Big) + \lambda G_{\alpha,\alpha-k-1,k+1}\Big(\frac{-1}{\lambda},t\Big)\Big]. \end{aligned} \tag{23}$$

The corresponding shear stress

$$\begin{aligned} \tau(r,t) = {} & \frac{f}{\lambda}\Big(\frac{R_1}{r}\Big)^2 R_{\alpha,-1}\Big(\frac{-1}{\lambda},t\Big) + \frac{\pi f}{\lambda}\sum_{n=1}^{\infty}\frac{\mathrm{J}_1^2(R_2r_n)\widetilde{B}(r,r_n)}{\mathrm{J}_2^2(R_1r_n) - \mathrm{J}_1^2(R_2r_n)} \\ & \times \sum_{k=0}^{\infty}\Big(\frac{-\nu r_n^2}{\lambda}\Big)^k G_{\alpha,-k-1,k+1}\Big(\frac{-1}{\lambda},t\Big), \end{aligned} \tag{24}$$

where $R_{a,b}(c,t)$ is defined by Eq. (3) and

$$\widetilde{B}(r,r_n) = \mathrm{J}_2(rr_n)\mathrm{Y}_2(R_1r_n) - \mathrm{J}_2(R_1r_n)\mathrm{Y}_2(rr_n), \tag{25}$$

is obtained in the same way as Eq. (6).

4 Special cases

4.1 Ordinary Maxwell fluid

Making $\alpha \to 1$ into Eqs. (23) and (24), we obtain the velocity field

$$\begin{aligned} w_M(r,t) = {} & \frac{f}{2\mu}\Big(\frac{R_1}{R_2}\Big)^2\Big(r - \frac{R_2^2}{r}\Big) - \frac{\pi f}{\mu\lambda}\sum_{n=1}^{\infty}\frac{\mathrm{J}_1^2(R_2r_n)B(r,r_n)}{r_n[\mathrm{J}_2^2(R_1r_n) - \mathrm{J}_1^2(R_2r_n)]} \\ & \times \sum_{k=0}^{\infty}\Big(\frac{-\nu r_n^2}{\lambda}\Big)^k\Big[G_{1,-k-1,k+1}\Big(\frac{-1}{\lambda},t\Big) + \lambda G_{1,-k,k+1}\Big(\frac{-1}{\lambda},t\Big)\Big], \end{aligned} \tag{26}$$

and the associated shear stress

$$\tau_M(r,t) = f\left(\frac{R_1}{r}\right)^2\left[1-\mathrm{e}^{-t/\lambda}\right] + \frac{\pi f}{\lambda}\sum_{n=1}^{\infty}\frac{\mathrm{J}_1^2(R_2r_n)\widetilde{B}(r,r_n)}{\mathrm{J}_2^2(R_1r_n)-\mathrm{J}_1^2(R_2r_n)} \times \sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{1,-k-1,k+1}\left(\frac{-1}{\lambda},t\right), \quad (27)$$

corresponding to an ordinary Maxwell fluid, performing the same motion. Direct computations show that $w_M(r,t)$ and $\tau_M(r,t)$ satisfy initial conditions (8), boundary conditions (10) and (9).

4.2 Newtonian fluid

Now letting $\lambda \to 0$ in Eqs. (26) and (27) and using the limit

$$\lim_{\lambda\to 0}\frac{1}{\lambda^\eta}G_{1,b,\eta}\left(\frac{-1}{\lambda},t\right) = \frac{t^{-b-1}}{\Gamma(-b)}, \quad b<0,$$

we obtain the corresponding solutions

$$w_N(r,t) = \frac{f}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(r-\frac{R_2^2}{r}\right) - \frac{\pi f}{\mu}\sum_{n=1}^{\infty}\frac{\mathrm{J}_1^2(R_2r_n)B(r,r_n)}{r_n[\mathrm{J}_2^2(R_1r_n)-\mathrm{J}_1^2(R_2r_n)]}\,\mathrm{e}^{-\nu r_n^2 t}, \quad (28)$$

$$\tau_N(r,t) = f\left(\frac{R_1}{r}\right)^2 + \pi f\sum_{n=1}^{\infty}\frac{\mathrm{J}_1^2(R_2r_n)\widetilde{B}(r,r_n)}{\mathrm{J}_2^2(R_1r_n)-\mathrm{J}_1^2(R_2r_n)}\,\mathrm{e}^{-\nu r_n^2 t}, \quad (29)$$

for a Newtonian fluid. Of course, as expected, Eq. (28), is identical to that coming from [5, Eq. (5.17)] for $\alpha_1 = 0$.

5 Conclusions

In this note, the velocity $w(r,t)$ and the shear stress $\tau(r,t)$ corresponding to the flow of an incompressible Maxwell fluid with fractional derivatives, in the annular region between two infinite coaxial circular cylinders, have been determined using the Laplace and finite Hankel transforms. The motion is produced by the inner cylinder that, after the initial moment, is subject to a time dependent couple. The solutions, that have been obtained, written in series form in terms of generalized G and R-functions, satisfy all imposed initial and boundary conditions. In order to verify the boundary condition (9), for instance, we use the known relation

$$\mathrm{D}_t^\alpha(t^a) = \frac{\Gamma(a+1)}{\Gamma(a-\alpha+1)}t^{a-\alpha}, \quad 0\le\alpha<1.$$

In the special cases, when $\alpha \to 1$ or $\alpha \to 1$ and $\lambda \to 0$, the corresponding solutions for the ordinary Maxwell and Newtonian fluids are obtained. Lengthy but straightforward computations show that $w_M(r,t)$ and $\tau_M(r,t)$ given by Eqs. (26) and (27) are equivalent to the limit solutions of (20) and (21) obtained in Ref. [29] by a different technique. Making $t \to \infty$ into Eqs. (26)–(29) and bearing in mind the previous equivalence, we obtain

$$w_M(r,\infty) = w_N(r,\infty) = \frac{f}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(r-\frac{R_2^2}{r}\right),$$

$$\tau_M(r,\infty) = \tau_N(r,\infty) = f\left(\frac{R_1}{r}\right)^2. \quad (30)$$

Consequently, the unsteady flow of a Newtonian fluid induced by a circular cylinder subject to a constant torque, as well as the flow of a Maxwell fluid induced by the same circular cylinder subject to a time-dependent torque of the form of (10), becomes steady. Moreover, the corresponding steady solutions are the same for both types of fluids. This is not a surprise, because for $t \to \infty$ the boundary condition (10), corresponding to Maxwell fluids, tends to that corresponding to Newtonian fluids. On the other hand, the flow of the fractional fluid is an unsteady flow and remains unsteady.

Now, in order to reveal some relevant physical aspects of the obtained results, the diagrams of velocity and the shear stress are drawn against r for different values of time t and pertinent parameters. Figure 1a shows that the velocity $w(r,t)$ is an increasing function of t in the neighborhood of the moving cylinder. The shear stress (in absolute value) is an increasing function of t on the whole domain. The influence of the material parameter λ and the fractional parameter α on the fluid motion is shown in Figs. 2 and 3. The two parameters have significant influences on the fluid motion, but their effects are opposite. The velocity of the fluid in the neighborhood of the moving cylinder and the shear stress on the whole flow domain are decreasing functions with respect to λ and increasing ones with regard to α. The influence of the material parameter λ on the fluid motion near the inner cylinder seems natural, λ being the relaxation time of the fluid.

Finally, for comparison, the profiles of the velocity and the shear stress corresponding to the motion of the three types of fluids (Newtonian, Maxwell and fractional Maxwell) are depicted together in Figs. 4 for the same values of t and material constants. The Newtonian fluid, as it result from Fig. 4a, is the swiftest, whereas the fractional Maxwell fluid is the slowest in the neighborhood of inner cylinder. The shear stress corresponding to Newtonian fluid is the highest on the whole flow domain. Of course, these results are in agreement with those resulting from Figs. 2 and 3. The units of the material constants are SI units in all pictures and the roots r_n have been approximated by $(2n-1)\pi/[2(R_2-R_1)]$.

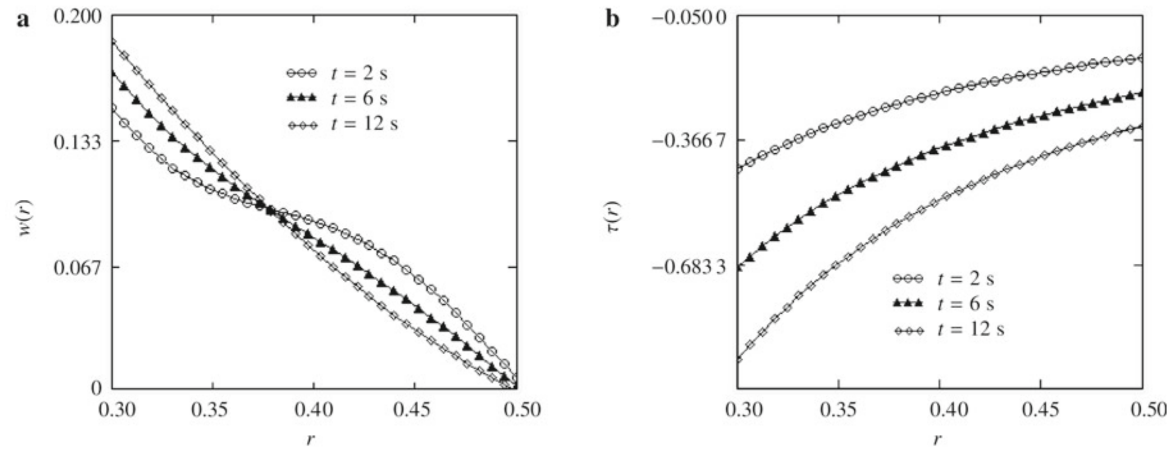

Fig. 1 Profiles of **a** The velocity $w(r,t)$ and **b** Shear stress $\tau(r,t)$ given by Eqs. (23) and (24) for $R_1 = 0.3$, $R_2 = 0.5$, $f = -2$, $\nu = 0.001\,188$, $\mu = 1.05$, $\alpha = 0.5$, $\lambda = 6$, and different values of t

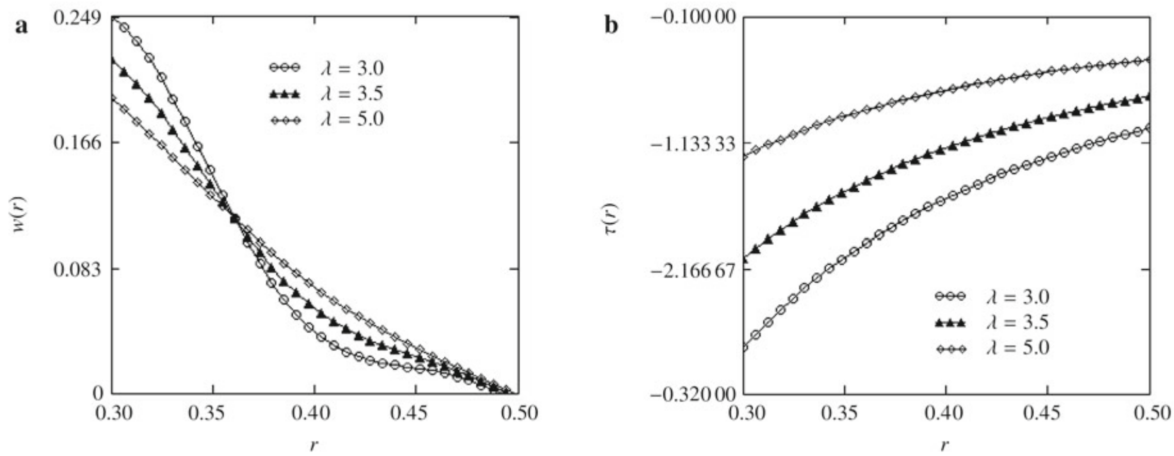

Fig. 2 Profiles of **a** The velocity $w(r,t)$ and **b** Shear stress $\tau(r,t)$ given by Eqs. (23) and (24) for $R_1 = 0.3$, $R_2 = 0.5$, $f = -2$, $\nu = 0.001188$, $\mu = 1.05$, $\alpha = 0.5$, $t = 15$ s, and different values of λ

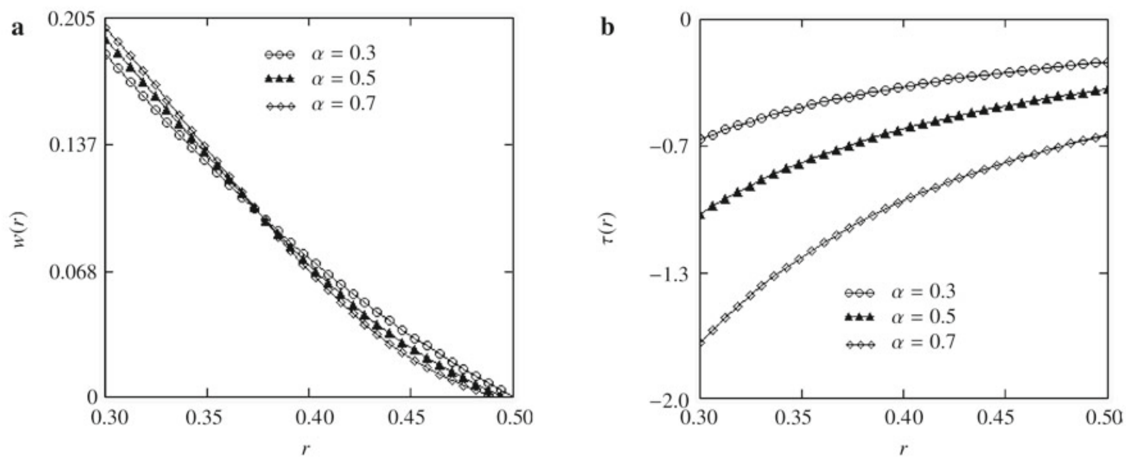

Fig. 3 Profiles of **a** The velocity $w(r,t)$ and **b** Shear stress $\tau(r,t)$ given by Eqs. (23) and (24) for $R_1 = 0.3$, $R_2 = 0.5$, $f = -2$, $\nu = 0.001\,188$, $\mu = 1.05$, $\lambda = 6$, $t = 15$ s and different values of α

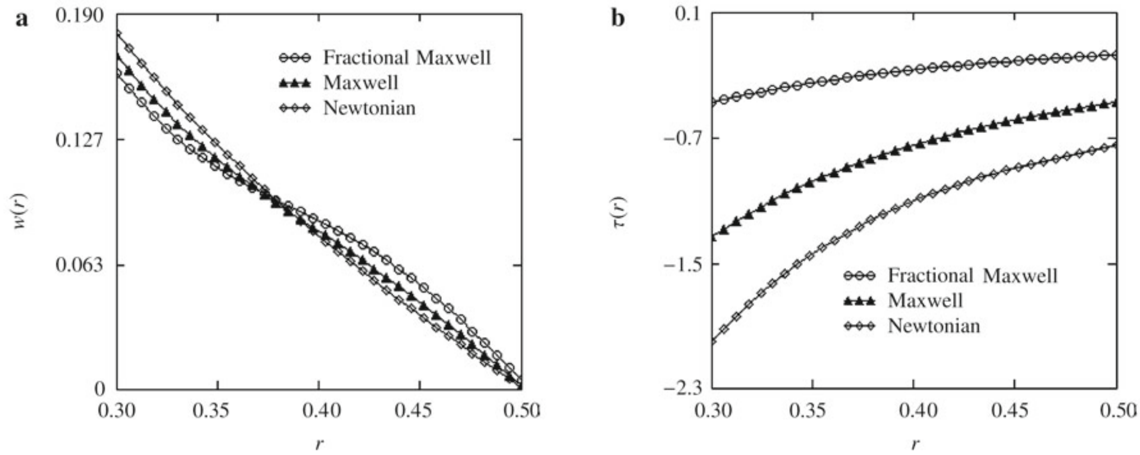

Fig. 4 Profiles of **a** The velocity $w(r,t)$ and **b** Shear stress $\tau(r,t)$ for fractional Maxwell, ordinary Maxwell and Newtonian fluids, for $R_1 = 0.3$, $R_2 = 0.5$, $f = -2$, $\nu = 0.001\,188$, $\mu = 1.05$, $\alpha = 0.2$, $\lambda = 5$ and $t = 5$ s

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