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RESEARCH PAPER

Homotopy analysis solutions for the asymmetric laminar flow in a porous channel with expanding or contracting walls

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Abstract In this paper, the asymmetric laminar flow in a porous channel with expanding or contracting walls is investigated. The governing equations are reduced to ordinary ones by using suitable similar transformations. Homotopy analysis method (HAM) is employed to obtain the expressions for velocity fields. Graphs are sketched for values of parameters and associated dynamic characteristics, especially the expansion ratio, are analyzed in detail.

Keywords Homotopy analysis method · Expanding or contracting wall · Asymmetric laminar flow · Porous channel · Expansion ratio

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1 Introduction

The flow in channels and in circular pipes with permeable walls has received considerable attention in the past few years. The earliest work of steady flow across permeable and stationary walls can be traced back to Berman [1], who showed that the governing equations can be reduced to a single fourth-order nonlinear ordinary differential equation which includes permeation Reynolds number R, and associated solution can be obtained. Since that a large number of theoretical investigations dealing with steady symmetric flow and stationary walls were made [2–6]. Terrill and Shrestha [7] investigated the laminar flow through parallel and uniformly porous walls of different permeability.

The flow of fluid in a porous channel with deformable walls has also gained importance because of its applications in the modeling of pulsating diaphragms, filtration, and the grain regression during solid propellant combustion. Especially the flow of a particular fluid in biological organisms is often treated as to be in a porous expanding or contracting vessel. For example, a valved vessel exhibits deformable boundaries and alternating wall contractions produce the effect of a physiological pump. The flow in the lymphatics also exhibits a similar behavior character [8]. Filtration also plays an important role between blood and tissue in many parts of the body. While in biological organisms filtration is often studied in long circular tubes, and extracorporeal duct flows take place more commonly between parallel flat membranes.

Uchida and Aoki [9] examined the viscous flow inside an impermeable tube of contracting cross section. Ohki [10] investigated unsteady flow in a semi-infinite tube with porous, elastic wall whose length varied with time, but its cross section does not vary. In order to simulate the laminar flow field in cylindrical solid rocket motors, Goto et al. [11] analyzed the laminar incompressible flow in a semi-infinite porous pipe whose radius varied with time. Bujurke et al. [12] obtained a series solution for unsteady flow in a contracting or expanding pipe. Ma et al. [13, 14] and Barron et al. [15] first achieved experimentally the time-dependent motion in a long rectangular channel with porous walls. They all used the sublimation process of carbon dioxide to simulate the injection process at the walls. As a result, the walls of their channel expanded during the sublimation process. Majdalani et al. [16, 17] and Dauenhauer et al. [18] obtained both numerically and asymptotically solutions for different Reynolds numbers. In very recent, Asghar et al. [19] discussed the flow in a slowly deforming channel with weak permeability using AMD. Si et al. [20] obtained an asymptotic solution for large suction using the singular perturbation method.

In membrane devices Villarroel et al. [21] studied gas exchange to blood flowing in semipermeable tubes under steady and pulsatile flow conditions. Wang [22] and Bhatnager [23] analyzed the unsteady flow for the case where on one plate it is injected with a constant wall velocity and at the same time it is sucked off at the opposite plates with the same velocity. Muhammad et al. [24] also studied asymmetric micropolar fluids in a porous channel numerically. In biological organisms, when the degree of ischemia is different, the permeability may be different.

Motivated by the above mentioned work, in this paper we extend previous investigations by presenting theoretical solutions for different permeabilities in a porous channel with expanding or contracting walls. A powerful technique recently developed by Liao [25, 26] named the homotopy analysis method (HAM) is adopted to solve the problems for velocity fields. Hayat et al. [27–29], Abbas et al. [30] and Sajid et al. [31] have successfully applied the HAM to several nonlinear problems. Using the HAM, the velocity of the fluid influenced by the expansion ratio and Reynolds number are taken into consideration.

2 Statement of the problem and governing equations

In our study, a channel with a rectangular cross section is considered. The distance between the porous walls is much smaller than the other two. Both walls have different permeabilities and expand or contract uniformly at a time-dependent rate $\dot{a}(t)$. The channel is assumed to be semi-infinite length. One end is closed by a compliant membrane.

As shown in Fig. 1, a coordinate system may be chosen with the origin at the center of the channel. The axial and normal velocity components are defined as u and v, which are parallel to the axial and normal coordinates x and y, respectively.

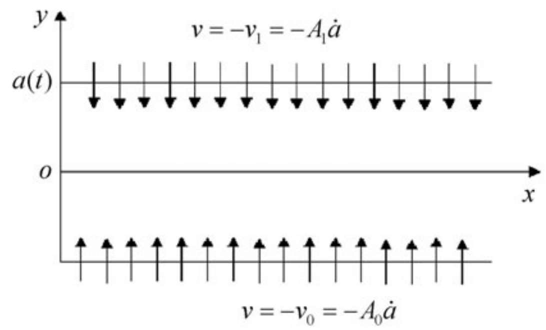

Fig. 1 The model of channel flow with expanding or contracting walls

Under these assumptions, the Navier-Stokes equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{3}$$

where ρ, p and ν are density, pressure and kinematic viscosity, respectively.

The boundary conditions are

$$u = 0, \quad v = -v_0 = -A_0\dot{a}, \quad \text{at } y = -a(t), \tag{4}$$

$$u = 0, \quad v = -v_1 = -A_1\dot{a}, \quad \text{at } y = a(t), \tag{5}$$

where $A_0 = v_0/\dot{a}$ and $A_1 = v_1/\dot{a}$ are the measure of wall permeabilities.

Introducing the stream function

$$\psi = \frac{\nu x}{a}F(\eta, t), \tag{6}$$

where $\eta = y/a$.

Substituting ψ into Eqs. (1)–(3) and eliminating the pressure term from the momentum equation, one can obtain the following expression

$$F_{\eta\eta\eta\eta} + \alpha(\eta F_{\eta\eta\eta} + 3F_{\eta\eta}) + FF_{\eta\eta\eta} - F_\eta F_{\eta\eta} - a^2\nu^{-1}F_{\eta\eta t} = 0, \tag{7}$$

where α is the wall expansion ratio defined by

$$\alpha = \frac{a\dot{a}}{\nu}. \tag{8}$$

Note that the expansion ratio will be positive for expansion and negative for contraction.

The boundary conditions are

$$F(-1) = R_0, \quad F_\eta(-1) = 0, \quad F(1) = R, \quad F_\eta(1) = 0, \tag{9}$$

where R_0 and R are the permeation Reynolds numbers defined by $R_0 = av_0/\nu$ and $R = av_1/\nu$, respectively. In physical meaning, R_0 and R are positive for injection and negative for suction.

Let

$$f=\frac{F}{R},\quad l=\frac{x}{a}. \tag{10}$$

A similar solution with respect to both space and time was developed by Uchida et al. [9], α is constant and $f=f(\eta)$, which leads to $f_{\eta\eta t}=0$.

Under these assumption, Eq. (7) becomes

$$f''''+\alpha(\eta f'''+3f'')+R(ff'''-f'f'')=0, \tag{11}$$

with boundary conditions

$$f(-1)=A,\quad f'(-1)=0,\quad f(1)=1,\quad f'(1)=0, \tag{12}$$

where $A=v_0/v_1$.

3 The HAM solution of the problem

For HAM solutions of the governing equation, we choose the initial approximation of $f(\eta)$ as follows

$$f_0(\eta)=\frac{A-1}{4}\eta^3+\frac{3-3A}{4}\eta+\frac{A+1}{2}, \tag{13}$$

and the auxiliary linear operator is

$$\ell_1(f)=f''''. \tag{14}$$

This auxiliary linear operator satisfies

$$\ell_1(C_1\eta^3+C_2\eta^2+C_3\eta+C_4)=0, \tag{15}$$

where $C_i(i=1,2,3,4)$ are constants.

$$(1-p)\ell_1(f(\eta,p)-f_0(\eta))=ph\mathrm{N}(f(\eta,p)), \tag{16}$$

$$\begin{aligned} &f(-1,p)=A, && f'(-1,p)=0,\\ &f(1,p)=1, && f'(1,p)=0, \end{aligned} \tag{17}$$

where N is nonlinear operator, which is defined as

$$\begin{aligned}\mathrm{N}(f(\eta,p))&=\frac{\partial^4 f(\eta,p)}{\partial\eta^4}+\alpha\left(\eta\frac{\partial^3 f(\eta,p)}{\partial\eta^3}+3\frac{\partial^2 f(\eta,p)}{\partial\eta^2}\right)\\ &\quad+R\left(f(\eta,p)\frac{\partial^3 f(\eta,p)}{\partial\eta^3}-\frac{\partial f(\eta,p)}{\partial\eta}\frac{\partial^2 f(\eta,p)}{\partial\eta^2}\right).\end{aligned} \tag{18}$$

As p increases from 0 to 1, $f(\eta,p)$ varies from $f_0(\eta)$ to $f(\eta)$. Using Taylor's theorem, we can write

$$\begin{aligned} f(\eta,p)&=f_0(\eta)+\sum_{n=1}^{\infty}f_n(\eta)p^n,\\ f_n(\eta)&=\frac{1}{n!}\frac{\partial^n f(\eta,p)}{\partial p^n}\bigg|_{p=0}. \end{aligned} \tag{19}$$

The convergence of the series is strongly dependent upon h. Assume that h is chosen so that the series (19) are convergent at $p=1$. From Eq. (19) we have

$$f=f_0(\eta)+\sum_{n=1}^{\infty}f_n(\eta). \tag{20}$$

Differentiating Eq. (16) n times with respect to p, then setting $p=0$, and finally dividing them by $n!$, we obtain the following n-th-order deformation problems

$$\ell_1(f_n(\eta)-\chi_n f_{n-1}(\eta))=h\aleph_n^f(\eta), \tag{21}$$

$$f_n(-1)=0,\quad f_{n'}(-1)=0,\quad f_n(1)=0,\quad f_{n'}(1)=0, \tag{22}$$

$$\begin{aligned}\aleph_n^f(\eta)&=f''''_{n-1}+\alpha(\eta f'''_{n-1}+3f''_{n-1})\\ &\quad+\sum_{k=0}^{n-1}R(f_{n-k-1}f'''_k-f'_{n-k-1}f''_k),\end{aligned} \tag{23}$$

where

$$\chi_n=\begin{cases}0, & n\le 1,\\ 1, & n>1.\end{cases} \tag{24}$$

In this way it is easy to solve the linear non-homogeneous equation (21) by using Maple one after the other in the order $n=1,2,\cdots$. The first deformation of the solution is presented below

$$\begin{aligned} f_1(\eta)&=\frac{hRA^2(-\eta^7+35\eta^4-70\eta^2+3\eta^3-2\eta+35)}{1120}\\ &\quad+\frac{hR(-35\eta^4-\eta^7-2\eta-35+3\eta^3+70\eta^2)}{1120}\\ &\quad+\frac{h\alpha(112\eta^3-56\eta-56\eta^5)}{1120}\\ &\quad+\frac{h\alpha A(-112\eta^3+56\eta+56\eta^5)}{1120}\\ &\quad+\frac{hRA(2\eta^7-6\eta^3+4\eta)}{1120}. \end{aligned} \tag{25}$$

4 Results and discussion

As pointed out by Liao [25, 26] the convergence of the series (16) depends upon h. The value of h determines the convergence region and the rate of approximation for HAM. For this purpose the h-curves are plotted in Fig. 2 for different approximations. The range for the admissible values of h for different values of A is different. The range is $-1\le h\le -0.2$ as $A=-0.2$ and $-1.2\le h\le -0.2$ as $A=-0.6$. In order to see the effects of parameters R, α on the velocity fields f, f', the proper value of $h=-0.3$ is chosen. As a result, we can get the convergent HAM series solution on the 20-th approximation.

In order to investigate the effect of dimensionless constant expansion ratio α and Reynolds number R on the flow fields, the graphical presentation of the influence of R on $f(\eta)$ and $f'(\eta)$ is given in Figs. 3 and 4 for fixed A and α. In Fig. 3 when the non-zero suction velocity is imposed on the upper wall (i.e. $R<0$), for $A=-0.2$ which corresponds to the suction velocity at the lower wall to be 20% of that at the upper wall and $\alpha = 0.5$ which represents that the wall is expanding, the profile becomes asymmetric pushed towards the upper wall. As the magnitude of R is increased, the profile becomes more asymmetric. As the non-zero injection velocity is imposed on the upper wall (i.e. $R > 0$), the point of maximum velocity is shifted towards the lower wall and the profile becomes asymmetric pushed towards the lower wall. Figure 4 shows the influence of R on flow fields as $\alpha = -2$, the similar trends of velocity can be found compared with Fig. 3. One interesting conclusion can be made that the effect of varying negative R tends to become insignificant for large magnitude of R.

In Figs. 5 and 6, streamwise and normal velocities for different expansion ratios α and fixed A and R are shown. When $\alpha = 0$, the wall is stationary. When $\alpha \neq 0$, the maximum of streamwise velocity becomes bigger when the wall is expanding (i.e. $\alpha > 0$) and the maximum of the streamwise velocity becomes smaller when the wall is contracting (i.e. $\alpha < 0$) whether R is positive or not.

The effect of A on the velocity is illustrated in Figs. 7 and 8 for fixed values of positive R and α. The streamwise velocity increases by the increasing magnitude of A and the profile tends to become symmetric as A is decreased from 0 to -1. The maximum value of streamwise velocity increases by decreasing A.

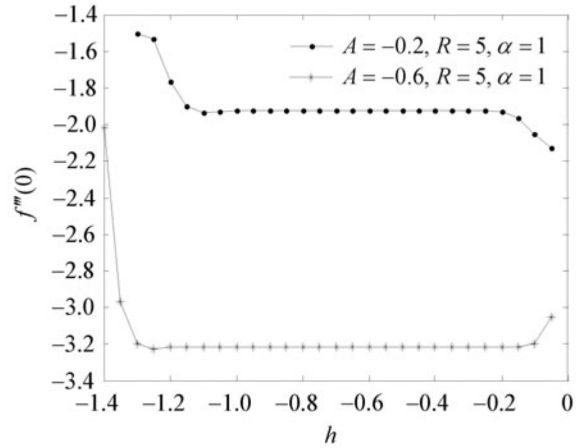

Fig. 2 h-curve on 20th-order approximation for different A values

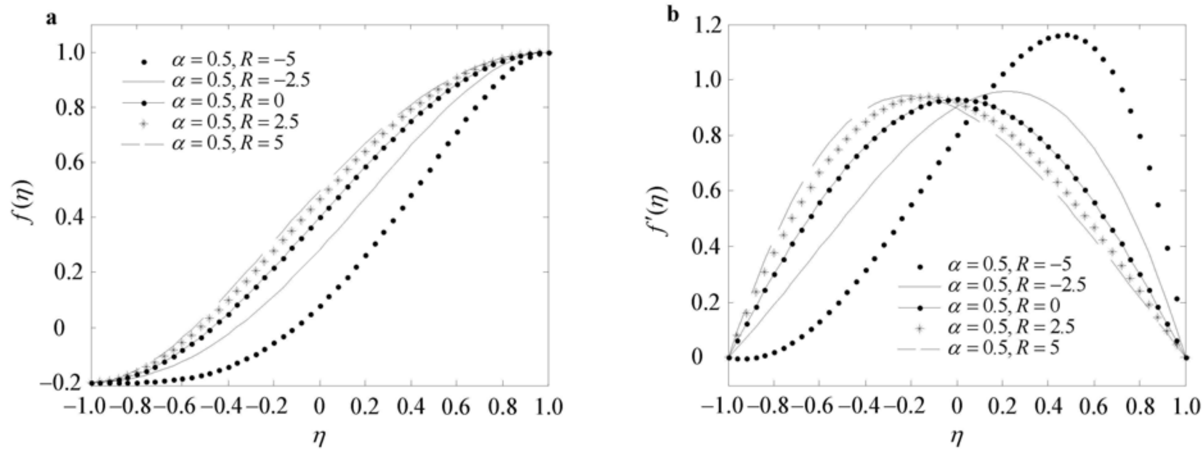

Fig. 3 Variation of f', f for different R values as $\alpha = 0.5$ and $A = -0.2$

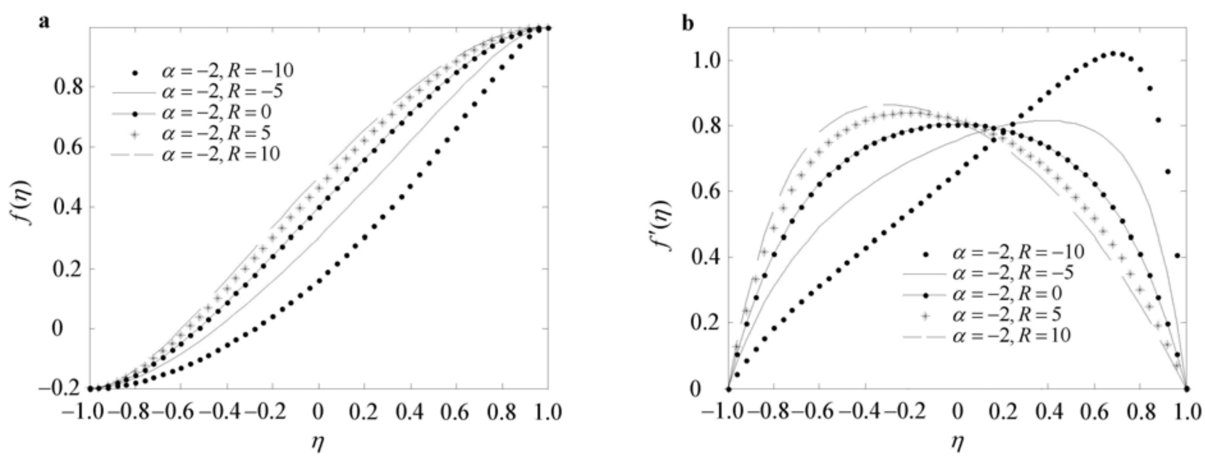

Fig. 4 Variation of f', f for different R values as $\alpha = -2$ and $A = -0.2$

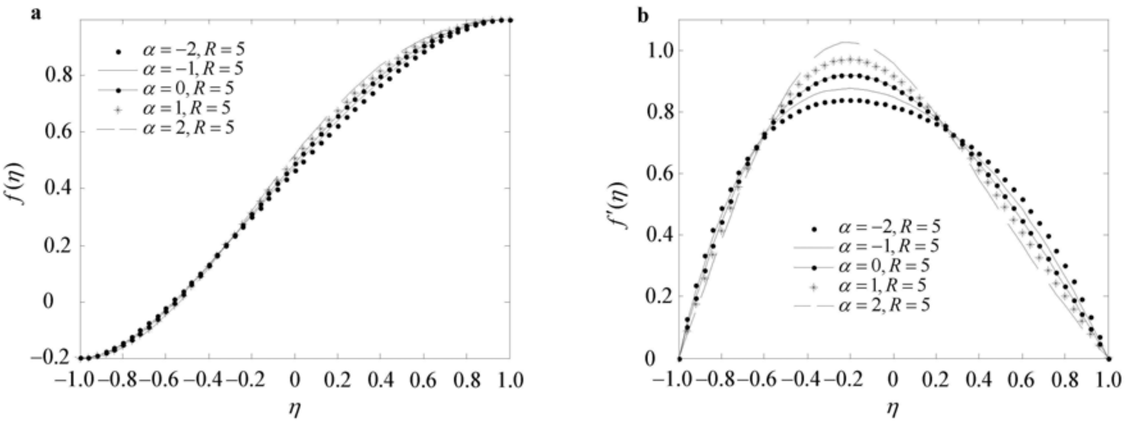

Fig. 5 Variation of f', f for different α values as $R = 5$ and $A = -0.2$

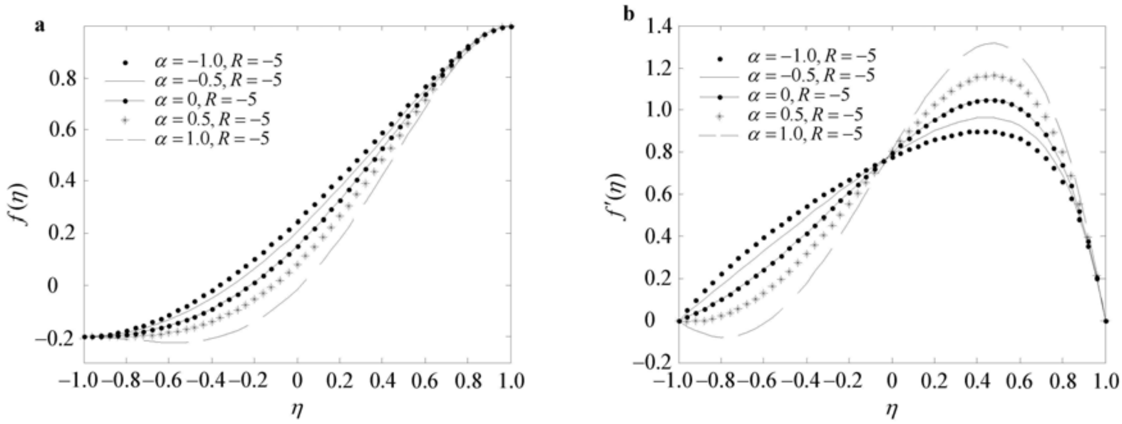

Fig. 6 Variation of f', f for different α values as $R = -5$ and $A = -0.2$

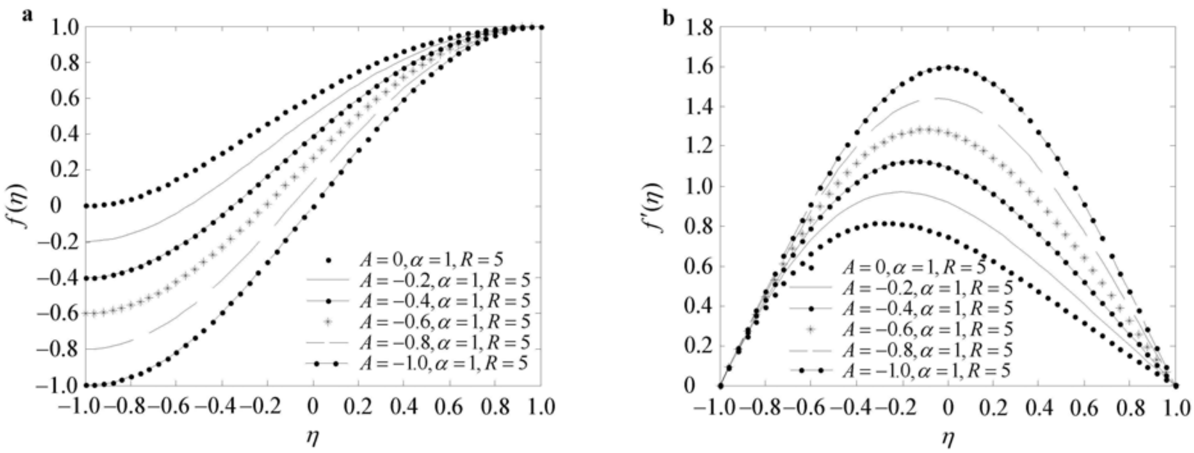

Fig. 7 Variation of f', f for different A values as $R = 5$ and $\alpha = 1$

The position of viscous layer is a point where $f(\eta) = 0$ which is developed due to the injection at the two walls. We observe that the position of viscous layer moves towards the centre of the channel by decreasing A. A comparison of Figs. 7 and 8 shows that the profile of $f'(\eta)$ becomes more symmetric when the wall is contracting.

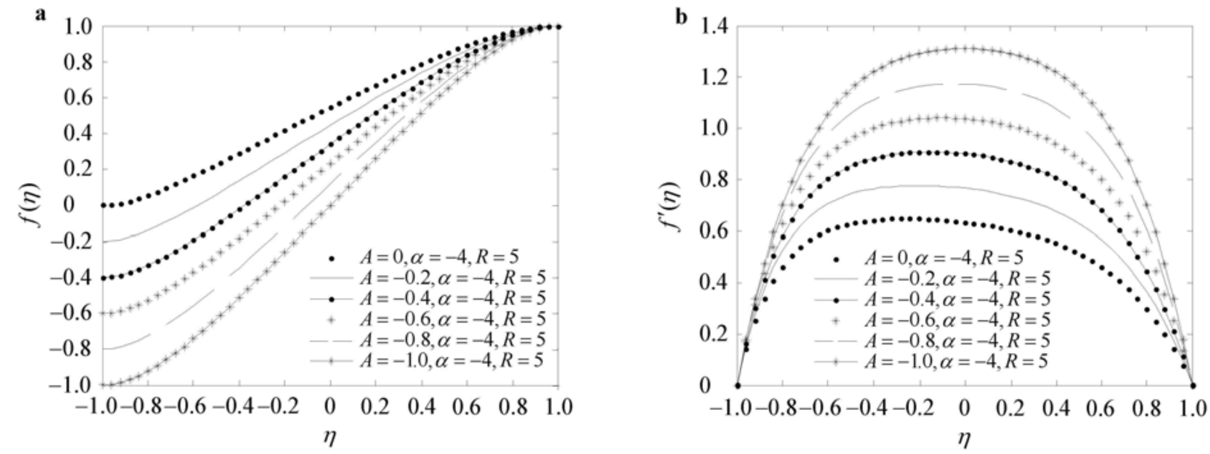

Fig. 8 Variation of f', f for different A values as $R = 5$ and $\alpha = -4$

It is well-known that there are multiple solutions for some R values and $\alpha = 0$ when $A = -1$. In fact, for $\alpha \neq 0$, there should exist the similar results. Using the shooting method and fourth-order Runge–Kutta integration, Fig. 9 indicates the multiple solution for different small expansion ratios and the symmetrical flow (i.e. $A = -1$). We shall seek the multiple solutions for the case of asymmetrical flow in future work.

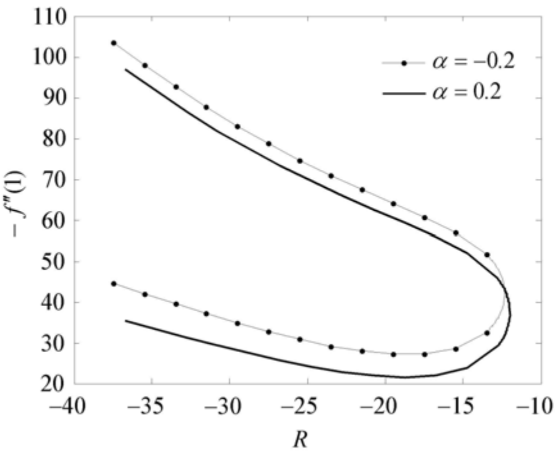

Fig. 9 Variation of $-f''(1)$ against R for $\alpha = -0.2$ and $\alpha = 0.2$ as $A = -1$

5 Conclusion

In the paper, the series solution of two-dimensional unsteady, laminar and incompressible flow in a channel with deformable porous walls of different permeabilities is obtained. The problem of the symmetric channel flow with both permeable walls occurs as a special case of the present problem corresponding to the parametric values $A = -1$. The objective of the present study is to investigate the effects of the permeation Reynolds number R, A and expansion ratio α on the flow variables. The results show that they have a strong influence on the velocity profiles. The maximum velocity of the streamwise velocity is shifted towards the wall with the increase of the magnitude of Reynolds number. The velocity profiles change from the most asymmetric shape to the symmetric shape as the permeability parameter A is decreased from 0 to -1. The streamwise velocity becomes more symmetric for different A values as the wall is contracting for positive Reynolds number R. The maximum of the streamwise velocity becomes bigger as the wall is expanding and it becomes smaller as the wall is contracting.

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