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RESEARCH PAPER

Magnetohydrodynamic peristaltic flow of a hyperbolic tangent fluid in a vertical asymmetric channel with heat transfer

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Abstract In the present paper we discuss the magnetohydrodynamic (MHD) peristaltic flow of a hyperbolic tangent fluid model in a vertical asymmetric channel under a zero Reynolds number and long wavelength approximation. Exact solution of the temperature equation in the absence of dissipation term has been computed and the analytical expression for stream function and axial pressure gradient are established. The flow is analyzed in a wave frame of reference moving with the velocity of wave. The expression for pressure rise has been computed numerically. The physical features of pertinent parameters are analyzed by plotting graphs and discussed in detail.

Keywords MHD flow · Hyperbolic tangent fluid · Vertical asymmetric channel · Heat transfer

1 Introduction

The study of magnetohydrodynamic (MHD) flows is quite interesting and useful because it is used in magnetic wound or cancer tumor treatment causing hypothermia, bleeding reduction during surgeries, targeted transport of drugs using magnetic particles as drug carries and MRI (magnetic resonance imaging) to diagnose the disease. Some significant studies involving MHD flows are discussed in Refs. [1,2]. The effects of the magnetic field on the peristaltic mechanism are also significant in connection with certain problems of the movement of the conductive physiological fluids, e.g., the blood and blood pump machines. Extensive analytical and numerical studies have been undertaken which involve such fluids. Significant studies concerning the topic include the work in Refs. [3–6]. The study of heat transfer and peristaltic flow has also received some attention, as it might be relevant in processes such as hemodialysis and oxygenation, obtaining information about the properties of tissues, hypothermia treatment, sanitary fluid transport, blood pump in heart-lung machines and transport of corrosive fluids. Limited attention has been focused on this study. A few of them are cited in Refs. [7–12] Recently, Nadeem and Akbar [13] have discussed the peristaltic flow of a viscous fluid in a vertical uniform channel. More recently, Srinivas and Gayathri [14] have discussed the peristaltic transport of a Newtonian fluid in a vertical asymmetric channel with heat transfer and porous media. Motivated by the above analysis, we discuss the peristaltic flow of a tangent hyperbolic fluid in a vertical asymmetric channel with heat transfer. To the best of the authors' knowledge, no attempt has been made to study the peristaltic flow of a non-Newtonian fluid in a vertical channel. The governing equations of tangent hyperbolic fluid model are simplified under the assumptions of long wavelength and low Reynolds number. The simplified equations are then solved analytically by regular perturbation method. The expression for pressure rise has been computed

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numerically. Finally, the physical features of pertinent parameters of interest are discussed through graphs.

2 Mathematical formulation

We consider MHD flow of an electrically conducting hyperbolic tangent fluid in a vertical asymmetric channel. The right side of the wall channel is maintained at temperature T_0, while the left wall has temperature T_1. We assume that the fluid is subject to a constant transverse magnetic field $\boldsymbol{B}_0$. A very small magnetic Reynolds number is assumed and hence the induced magnetic field can be neglected. When the fluid moves into the magnetic field, two major physical effects arise. The first one is that an electric field $\boldsymbol{E}$ is induced in the flow. We assume that the flow is 2D, and $\boldsymbol{B}_0 \cdot (\nabla \times \boldsymbol{V}) = 0$. The current density $\boldsymbol{J}$ is given by Ohm's law as $\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B})$. From the charge conservation law, we have $\nabla \cdot \boldsymbol{J} = \sigma\nabla\cdot\boldsymbol{E} + \sigma\nabla\cdot(\boldsymbol{V}\times\boldsymbol{B}_0) = \sigma\nabla\cdot\boldsymbol{E} + \sigma\boldsymbol{B}_0\cdot(\nabla\times\boldsymbol{V}) = 0$, and $\nabla \cdot \boldsymbol{E} = 0$ is satisfied. Neglecting the induced magnetic field also implies $\nabla \times \boldsymbol{E} = 0$. Because both of the divergence and the curl of the electric field are zero, the electric field is zero, and the current density $\boldsymbol{J}$ is given by Ohm's law as $\boldsymbol{J} = \sigma(\boldsymbol{V} \times \boldsymbol{B}_0)$. The second effect is dynamic in nature, i.e., a Lorentz force $(\boldsymbol{J} \times \boldsymbol{B})$, which acts on the fluid and modifies its motion. This results in the transfer of energy from the electromagnetic field to the fluid.

As the vertical asymmetric channel is considered, the channel flow is produced due to different amplitudes and phases of the peristaltic waves on the channel.

The geometry of the wall surface is defined as

$$Y = H_1 = d_1 + a_1 \cos\left[\frac{2\pi}{\lambda}(X - ct)\right],$$
$$Y = H_2 = -d_2 - b_1 \cos\left[\frac{2\pi}{\lambda}(X - ct) + \phi\right], \tag{1}$$

where a_1 and b_1 are the amplitudes of the waves, λ is the wavelength, $d_1 + d_2$ is the width of the channel, c is the velocity of propagation, t is the time, and X is the direction of wave propagation. The phase difference ϕ varies in the range $0 \le \phi \le \pi$, in which $\phi = 0$ corresponds to the symmetric channel with waves out of phase and $\phi = \pi$ corresponds to that with waves in phase. a_1, b_1, d_1, d_2 and ϕ further satisfy the condition

$$a_1^2 + b_1^2 + 2a_1b_1 \cos\phi \le (d_1 + d_2)^2.$$

For the two-dimensional incompressible flow, the governing equations of motion and energy for the vertical channel are

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \tag{2}$$

$$\rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y}\right) = -\frac{\partial P}{\partial X} - \frac{\partial \tau_{XX}}{\partial X} - \frac{\partial \tau_{YY}}{\partial Y} - \sigma B_0^2 U + \rho g\alpha(T - T_0), \tag{3}$$

$$\rho\left(\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y}\right) = -\frac{\partial P}{\partial Y} - \frac{\partial \tau_{XY}}{\partial X} - \frac{\partial \tau_{YY}}{\partial Y}, \tag{4}$$

$$C'\rho\left(\frac{\partial T}{\partial t} + U\frac{\partial T}{\partial X} + \bar{V}\frac{\partial T}{\partial Y}\right) = K'\nabla^2 T + Q_0, \tag{5}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2},$$

U and V are the velocities in the X and Y directions in the fixed frame, ρ is the constant density, P is the pressure, σ is the electrical conductivity, K' is the thermal conductivity, C' is the specific heat, T is the temperature, and Q_0 is the heat absorption.

The constitutive equation for hyperbolic tangent fluid is given by [15]

$$\tau = -[\eta_\infty + (\eta_0 + \eta_\infty)\tanh(\Gamma\dot{\gamma})^n]\dot{\gamma}, \tag{6}$$

where η_∞ is the infinite shear rate viscosity, η_0 is the zero shear rate viscosity, Γ is the time constant, n is the power law index, and $\dot{\gamma}$ is defined as

$$\dot{\gamma} = \sqrt{\frac{1}{2}\sum_i\sum_j \dot{\gamma}_{ij}\dot{\gamma}_{ji}} = \sqrt{\frac{1}{2}\Pi}, \tag{7}$$

where Π is the second invariant strain tensor. We consider the constitutive Eq. (6) for the case in which $\eta_\infty = 0$ and $\Gamma\dot{\gamma} < 1$. Therefore, the component of extra stress tensor can be written as

$$\begin{aligned}\tau &= -\eta_0(\Gamma\dot{\gamma})^n\dot{\gamma}\\ &= -\eta_0(1 + \Gamma\dot{\gamma} - 1)^n\dot{\gamma}\\ &= -\eta_0[1 + n(\Gamma\dot{\gamma} - 1)]\dot{\gamma}.\end{aligned} \tag{8}$$

Introduce a wave frame (x, y) moving with velocity c away from the fixed frame (X, Y) by the transformation

$$x = X - ct, \quad y = Y, \quad u = U - c, \quad v = V, \quad p(x) = P(X, t). \tag{9}$$

Define

$$\bar{x} = \frac{x}{\lambda}, \quad \bar{y} = \frac{y}{d_1}, \quad \bar{u} = \frac{u}{c}, \quad \bar{V} = \frac{V}{c}, \quad \bar{t} = \frac{c}{\lambda}t, \quad h_1 = \frac{H_1}{d_1}, \quad h_2 = \frac{H_2}{d_1}, \quad \bar{\tau}_{\bar{x}\bar{x}} = \frac{\lambda}{\eta_0 c}\tau_{xx},$$

$$\bar{\tau}_{\bar{x}\bar{y}} = \frac{\bar{d}_1}{\eta_0 c}\tau_{xy}, \qquad \bar{\tau}_{\bar{y}\bar{y}} = \frac{\bar{d}_1}{\eta_0 c}\tau_{yy},$$
$$\delta = \frac{d_1}{\lambda}, \qquad Re = \frac{\rho c d_1}{\eta_0},$$
$$We = \frac{\Gamma c}{d_1}, \qquad P = \frac{d_1^2}{c\lambda\eta_0}p,$$
$$\bar{\dot{\gamma}} = \frac{\dot{\gamma} d_1}{c}, \qquad \theta = \frac{T - T_0}{T_1 - T_0},$$
$$\beta = \frac{Q_0 d_1^2}{K'(T_1 - T_0)}, \qquad Pr = \frac{\rho\nu C'}{K'},$$
$$M = \sqrt{\frac{\sigma\rho}{\mu}}B_0 d, \qquad Gr = \frac{\alpha g(T_1 - T_0)d_1^3}{\nu^2}. \tag{10}$$

Using the above non-dimensional quantities and the resulting equations in terms of stream function $\Psi(u = \partial\Psi/\partial y$, $v = -\delta\partial\Psi/\partial x$, and dropping bars), Eqs. (3)–(5) and (8) can be written as

$$-\delta Re\left[\left(\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial\Psi}{\partial y}\right] = -\frac{\partial P}{\partial x} - \delta^2\frac{\partial\tau_{xx}}{\partial x}$$
$$-\frac{\partial\tau_{xy}}{\partial y} - M^2\left(\frac{\partial\Psi}{\partial y} + 1\right) + Gr\theta, \tag{11}$$

$$-\delta^3 Re\left[\left(\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial\Psi}{\partial x}\right]$$
$$= -\frac{\partial P}{\partial y} - \delta^2\frac{\partial\tau_{xy}}{\partial x} - \delta\frac{\partial\tau_{yy}}{\partial y}, \tag{12}$$

$$Pr\,Re\delta\left(\frac{\partial\Psi}{\partial y}\frac{\partial\theta}{\partial x} - \frac{\partial\Psi}{\partial x}\frac{\partial\theta}{\partial y}\right) = \delta^2\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \beta, \tag{13}$$

where

$$\tau_{xx} = -2[1 + n(We\dot{\gamma} - 1)]\frac{\partial^2\Psi}{\partial x\partial y},$$
$$\tau_{xy} = -[1 + n(We\dot{\gamma} - 1)]\left(\frac{\partial^2\Psi}{\partial y^2} - \delta^2\frac{\partial^2\Psi}{\partial x^2}\right),$$
$$\tau_{yy} = 2\delta[1 + n(We\dot{\gamma} - 1)]\frac{\partial^2\Psi}{\partial x\partial y}, \tag{14}$$
$$\dot{\gamma} = \left[2\delta^2\left(\frac{\partial^2\Psi}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2\Psi}{\partial y^2} - \delta^2\frac{\partial^2\Psi}{\partial x^2}\right)^2 + 2\delta^2\left(\frac{\partial^2\Psi}{\partial x\partial y}\right)^2\right]^{1/2},$$

in which δ, Re and We represent the wave, Reynolds and Weissenberg numbers, respectively. Under the assumptions of long wavelength $\delta \ll 1$ and low Reynolds number, neglecting the terms of order δ and higher, Eqs. (11)–(13) take the form

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial y}\left[(1 - n)\frac{\partial^2\Psi}{\partial y^2} + nWe\left(\frac{\partial^2\Psi}{\partial y^2}\right)^2\right]$$
$$-M^2\left(\frac{\partial\Psi}{\partial y} + 1\right) + Gr\theta, \tag{15}$$

$$\frac{\partial P}{\partial y} = 0, \tag{16}$$

$$\frac{\partial^2\theta}{\partial y^2} + \beta = 0. \tag{17}$$

Elimination of the pressure from Eqs. (15) and (16) yields

$$\frac{\partial^2}{\partial y^2}\left[(1 - n)\frac{\partial^2\Psi}{\partial y^2} + nWe\left(\frac{\partial^2\Psi}{\partial y^2}\right)^2\right]$$
$$-M^2\frac{\partial^2\Psi}{\partial y^2} + Gr\frac{\partial\theta}{\partial y} = 0. \tag{18}$$

The dimensionless mean flow Q is defined as

$$Q = F + 1 + d, \tag{19}$$

where

$$F = \int_{h_2(x)}^{h_1(x)}\frac{\partial\Psi}{\partial y}\mathrm{d}y = \Psi(h_1(x)) - \Psi(h_2(x)), \tag{20}$$

$$h_1(x) = 1 + a\cos 2\pi x, \qquad h_2(x) = -d - b\cos(2\pi x + \phi). \tag{21}$$

The corresponding dimensionless boundary conditions are defined as follows

$$\Psi = \frac{F}{2}, \quad \frac{\partial\Psi}{\partial y} = -1, \quad \text{for } y = h_1(x), \qquad \Psi = -\frac{F}{2}, \quad \frac{\partial\Psi}{\partial y} = -1, \quad \text{for } y = h_2(x), \tag{22}$$

$$\theta = 0, \quad \text{for } y = h_1(x), \qquad \theta = 1, \quad \text{for } y = h_2(x), \tag{23}$$

where a, b, ϕ and d satisfy the following relation

$$a^2 + b^2 + 2ab\cos\phi \le (1 + d)^2.$$

3 Solution of the problem

The solution of Eq. (17) satisfying the boundary conditions (23) can be written as follows

$$\theta = \frac{1}{2(h_2 - h_1)}\left\{\beta h_1 h_2(h_1 - h_2) - 2h_1 + \left[2 + \beta\left(h_2^2 - h_1^2\right)\right]y - \beta(h_2 - h_1)y^2\right\}. \tag{24}$$

As Eq. (18) is a highly non-linear equation, its close form solution is very difficult. Therefore, we are interested to find the analytical solution with the help of perturbation method. For perturbation solution, we expand Ψ, F, and P as follows

$$\Psi = \Psi_0 + We\Psi_1 + O(We^2), \tag{25}$$

$$F = F_0 + WeF_1 + O(We^2), \tag{26}$$

$$P = P_0 + WeP_1 + O(We^2). \tag{27}$$

Using expressions (24)–(27) in Eqs. (15), (18) and (22), collecting the powers of We, we obtain the following systems.

3.1 System of order We^0

$$\frac{\partial^4\Psi_0}{\partial y^4} = \frac{M^2}{1-n}\left(\frac{\partial^2\Psi_0}{\partial y^2}\right) + \frac{Gr}{(n-1)}\left[\frac{1}{h_2-h_1} + \frac{\beta}{2}(h_2+h_1) - \beta y\right], \tag{28}$$

$$\frac{\partial P_0}{\partial x} = (1-n)\left(\frac{\partial^3\Psi_0}{\partial y^3}\right) - M^2\left(\frac{\partial^2\Psi_0}{\partial y^2}+1\right) + \frac{1}{2(h_2-h_1)}\Big\{\beta h_1h_2(h_1-h_2) - 2h_1 + \left[2+\beta\left(h_2^2-h_1^2\right)\right]y - \beta(h_2-h_1)y^2\Big\}, \tag{29}$$

$$\Psi_0 = \frac{F_0}{2}, \quad \frac{\partial\Psi_0}{\partial y} = -1, \quad \text{at } y = h_1(x),$$
$$\Psi_0 = -\frac{F_0}{2}, \quad \frac{\partial\Psi_0}{\partial y} = -1, \quad \text{at } y = h_2(x). \tag{30}$$

3.2 System of order We^1

$$\frac{\partial^4\Psi_1}{\partial y^4} - \left(\frac{M^2}{1-n}\right)\frac{\partial^2\Psi_1}{\partial y^2} = \frac{n}{n-1}\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\Psi_0}{\partial y^2}\right)^2, \tag{31}$$

$$\frac{\partial P_1}{\partial x} = (1-n)\frac{\partial^3\Psi_1}{\partial y^3} - M^2\left(\frac{\partial\Psi_1}{\partial y}\right) + n\frac{\partial}{\partial y}\left(\frac{\partial^2\Psi_0}{\partial y^2}\right)^2, \tag{32}$$

$$\Psi_1 = \frac{F_1}{2}, \quad \frac{\partial\Psi_1}{\partial y} = 0, \quad \text{at } y = h_1(x),$$
$$\Psi_1 = -\frac{F_1}{2}, \quad \frac{\partial\Psi_1}{\partial y} = 0, \quad \text{at } y = h_2(x). \tag{33}$$

3.3 Solution for system of order We^0

Solution of Eq. (28) satisfying the boundary conditions (30) can be written as

$$\Psi_0 = A + By + A_1\cosh(m_1y) + B_1\sinh(m_1y) - \frac{C_1}{2m_1^2}y^2 + \frac{C_2}{6m_1^2}y^3. \tag{34}$$

The axial pressure gradient at this order is

$$\frac{dP_0}{dx} = m_1\left[(1-n)m_1^2 - M^2\right]\left[A_1\sinh(m_1y) + B_1\cosh(m_1y)\right] + \frac{C_2}{m_1^2}\left(1-n-\frac{M^2}{2}y^2\right) + \frac{C_1}{m_1^2}yM^2 - M^2(B+1) + \frac{1}{2(h_2-h_1)}\Big\{\beta h_1h_2(h_1-h_2) - 2h_1 + \left[2+\beta\left(h_2^2-h_1^2\right)\right]y - \beta(h_2-h_1)y^2\Big\}. \tag{35}$$

For one wavelength the integration of Eq. (35) yields

$$\Delta P_0 = \int_0^1 \frac{dP_0}{dx}dx. \tag{36}$$

3.4 Solution for system of order We^1

Solution of Eq. (31) satisfying the boundary conditions (33) can be written as

$$\Psi_1 = \{e^{-2m_1(h_1+h_2+y)}[L_1(x,y) + L_2(x,y) + L_3(x,y) + L_4(x,y) + L_5(x,y)]\}\Big/\Big\{12(e^{h_1m_1} - e^{h_2m_1})m_1^7 \times[e^{h_1m_1}(-2+h_1m_1-h_2m_1) + e^{h_2m_1}(2+h_1m_1-h_2m_1)]\Big\}. \tag{37}$$

The axial pressure gradient at this order is

$$\frac{dP_1}{dx} = e^{-2m_1(h_1+h_2+y)}\Big\{\frac{1-n}{a_{06}(x,y)}[a_{12}(x,y) + a_{13}(x,y) + a_{14}(x,y) + a_{15}(x,y) + a_{16}(x,y)] - \frac{M^2}{m_1^2a_{06}(x,y)}[a_{07}(x,y) + a_{08}(x,y) + a_{09}(x,y) + a_{10}(x,y) + a_{11}(x,y)]\Big\}2n\left[-\frac{C_1}{m_1^2} + \frac{C_2y}{m_1^2} + A_1m_1^2\cosh(m_1y) + B_1m_1^2\sinh(m_1y)\right] \times\left[\frac{C_2}{m_1^2} + B_1m_1^3\cosh(m_1y) + A_1m_1^3\sinh(m_1y)\right]. \tag{38}$$

For one wavelength the integration of Eq. (38) yields

$$\Delta P_1 = \int_0^1 \frac{dP_1}{dx}dx. \tag{39}$$

Summarizing the perturbation results for the small parameter We, we have

$$\Psi = \Psi_0 + \delta\Psi_1, \tag{40}$$

$$\frac{dP}{dx} = \frac{dP_0}{dx} + \delta\frac{dP_1}{dx}, \tag{41}$$

$$\Delta P = \Delta P_0 + \delta\Delta P_1, \tag{42}$$

where Ψ_0, Ψ_1, dP_0/dx, dP_1/dx, ΔP_0 and ΔP_1 are defined in Eqs. (34)–(39).
Defining

$$F = F_0 + \delta F_1, \tag{43}$$

using $F_0 = F - \delta F_1$ and then neglecting the terms greater than $O(\delta)$, the results given by Eqs. (47)–(50) are expressed up to δ.

4 Results and discussion

In this section graphical results are displayed to show the effects of various physical parameters on pressure rise, pressure gradient and velocity profile. The expressions for pressure rise and pressure gradient are calculated using a mathematic software Mathematica. To study the effects of pressure rise on various physical parameters, Figs. 1–7 are plotted. Figure 1 shows that in the pumping region ($\Delta P > 0$), the pumping rate increases with the increase in

Weissenberg number We, while in free ($\Delta P = 0$) and copumping ($\Delta P < 0$) regions, the pumping rate decreases with the increase in Weissenberg number We. To study the effect of Grashof number Gr, Fig. 2 is plotted. Figure 2 shows that the pumping rate increases with the increase of Grashof number Gr in pumping ($\Delta P > 0$) and free pumping ($\Delta P = 0$) regions, while the behavior is quite opposite in the copumping ($\Delta P < 0$) region. Figure 3 shows that the pumping rate decreases throughout all the regions with the increase in heat source/ sink parameter. Figure 4 is plotted to study the effects of phase difference ϕ on pressure rise. Figure 4 shows that in the pumping ($\Delta P > 0$) and free pumping ($\Delta P = 0$) regions the pumping rate decreases with the increase of phase difference ϕ. Figure 5 shows that the pumping rate increases throughout all the regions with the increase in power law index n. Figure 6 shows that the pumping rate decreases with the increase in the width of the channel d in pumping ($\Delta P > 0$) and free pumping ($\Delta P = 0$) regions, while in the copumping region ($\Delta P < 0$) the pumping rate increases with the increase in the width of the channel d. Figure 7 shows

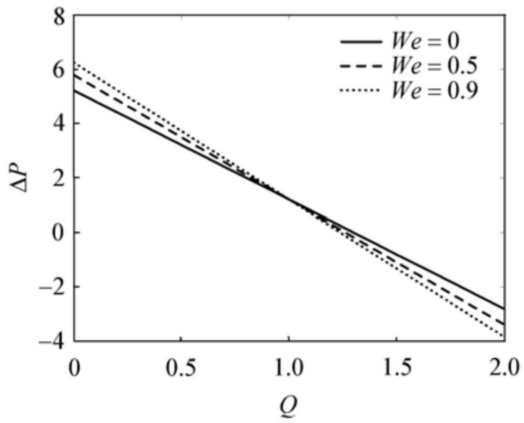

Fig. 1 Variation of ΔP with Q for different values of We. The other parameters are $a = 0.9$, $b = 0.5$, $d = 1.3$, $n = 0.2$, $\beta = 0.02$, $M = 1$, $\phi = \pi/6$, $Gr = 0.9$

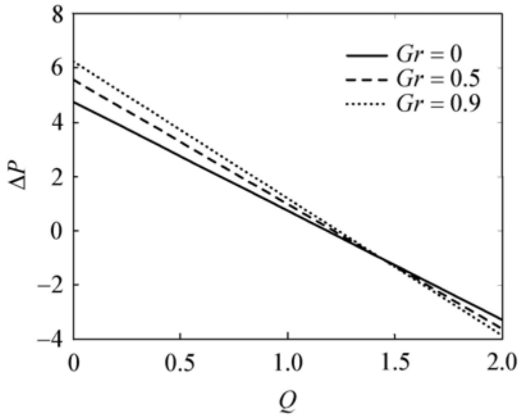

Fig. 2 Variation of ΔP with Q for different values of Gr. The other parameters are $a = 0.9$, $b = 0.5$, $d = 1$, $\beta = 0.02$, $M = 1$, $\phi = \pi/6$, $n = 0.06$, $We = 0.04$

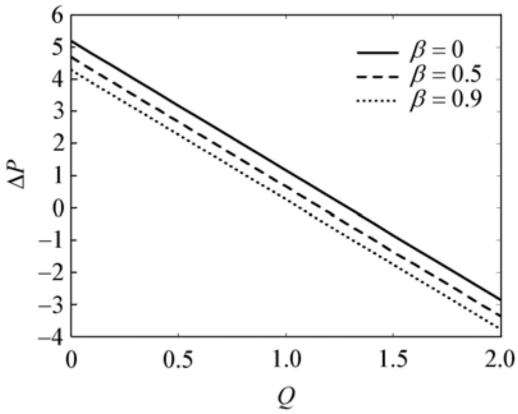

Fig. 3 Variation of ΔP with Q for different values of β. The other parameters are $a = 0.9$, $b = 0.5$, $d = 1.3$, $n = 0.02$, $We = 0.09$, $M = 1$, $\phi = \pi/6$, $Gr = 0.9$

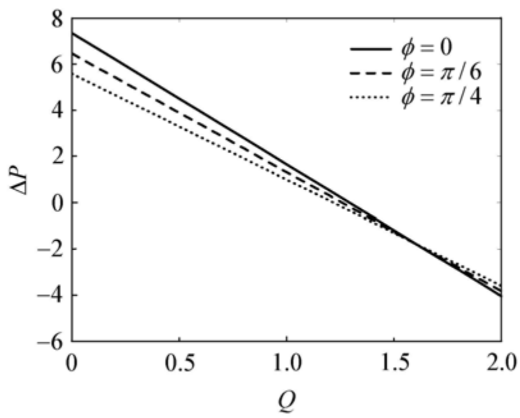

Fig. 4 Variation of ΔP with Q for different values of ϕ. The other parameters are $a = 0.9$, $b = 0.5$, $d = 1.2$, $n = 0.06$, $We = 0.04$, $M = 1$, $\beta = 0.02$, $Gr = 0.9$

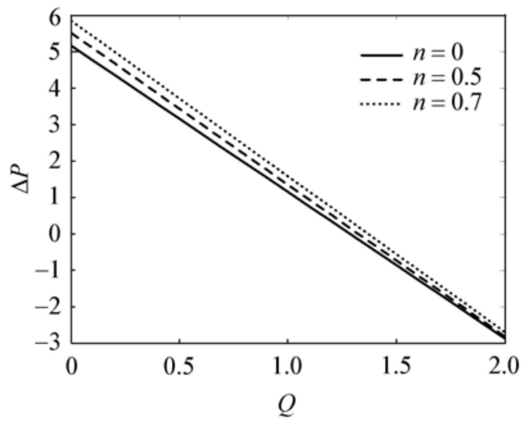

Fig. 5 Variation of ΔP with Q for different values of n. The other parameters are $a = 0.9$, $b = 0.5$, $d = 1.3$, $\beta = 0.02$, $M = 1$, $\phi = \pi/6$, $n = 0.06$, $Gr = 0.9$, $We = 0.04$

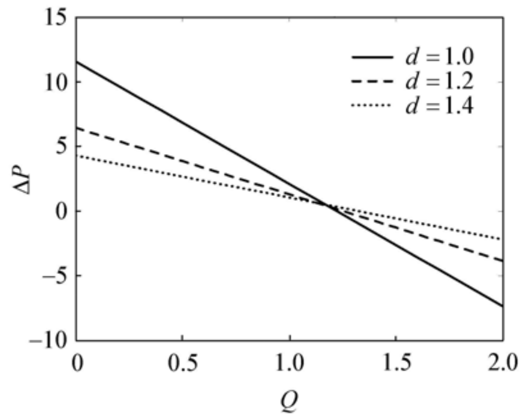

Fig. 6 Variation of ΔP with Q for different values of d. The other parameters are $a = 0.9$, $b = 0.5$, $Gr = 0.9$, $\beta = 0.02$, $M = 1$, $\phi = \pi/6$, $n = 0.06$, $We = 0.04$

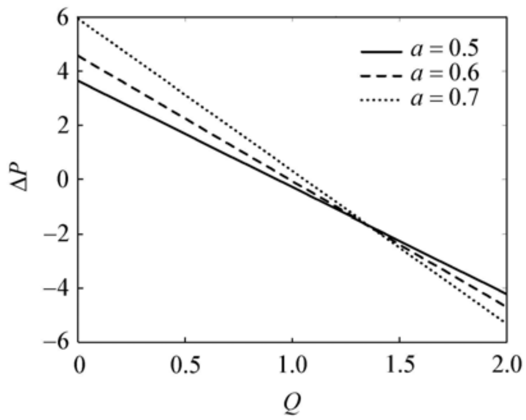

Fig. 7 Variation of ΔP with Q for different values of a. The other parameters are $Gr = 0.9$, $b = 0.5$, $d = 1$, $\beta = 0.02$, $M = 1$, $\phi = \pi/6$, $n = 0.06$, $We = 0.04$

that the pumping rate increases in pumping ($\Delta P > 0$) and free pumping ($\Delta P = 0$) regions with the increase in the amplitude of wave a, while in the copumping region ($\Delta P < 0$) the behavior is quite opposite. Figures 8–11 illustrate the effects of β, Gr, n and M on $\mathrm{d}P/\mathrm{d}x$. Figures 8 and 10 show that for $x \in [0, 0.2]$ and $[0.6, 1]$, the pressure gradient is small, i.e., the flow can easily pass without imposition of large pressure gradient, while in the narrow part of the channel $x \in [0.2, 0.6]$, to retain the same flux, large pressure gradient is required. Moreover, in the narrow part of the channel, the pressure gradient increases with the increase in Gr and n. Figure 11 also shows that the pressure gradient increases with the increase in Hartmann number M. The velocity profiles for different values of emerging parameters are plotted in Figs. 12–15. Figures 12 and 13 show that with the increase in Grashof number Gr and power law index n, the velocity profile decreases in the right side of the wall channel, while it increases in the left wall of the channel, but the velocity profile is maximum at the center of the channel. Figure 14 shows that the velocity profile increases in the interval $y \in [-2, -1]$ and $[0.4, 1.7]$ with the increase in heat source/sink parameter β, while the behavior is quite opposite in the interval $[-1, 0.5]$. Figure 15 shows that the velocity profile decreases with the increase in volume flow rate Q.

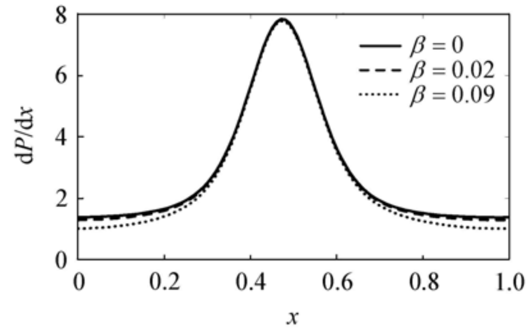

Fig. 8 Variation of $\mathrm{d}P/\mathrm{d}x$ with x for different values of β. The other parameters are $a = 0.7$, $b = 0.5$, $d = 1.2$, $Q = 1$, $M = 2$, $\phi = \pi/8$, $n = 0.06$, $We = 0.04$

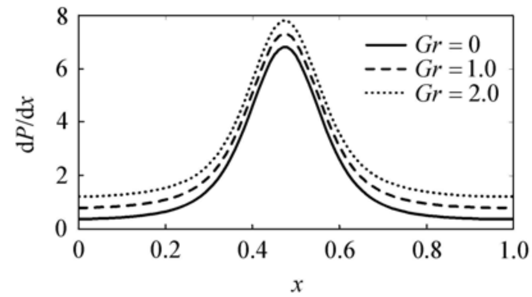

Fig. 9 Variation of $\mathrm{d}P/\mathrm{d}x$ with x for different values of Gr. The other parameters are $a = 0.7$, $b = 0.5$, $d = 1.2$, $Q = 1$, $M = 2$, $\beta = 0.04$, $\phi = \pi/8$, $n = 0.06$, $We = 0.04$

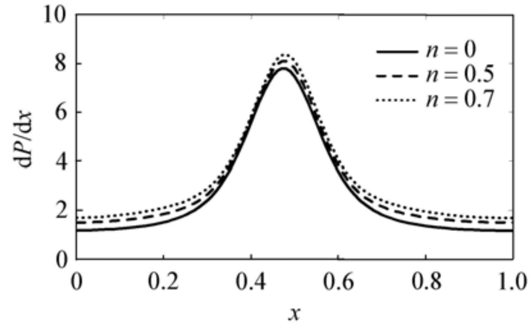

Fig. 10 Variation of $\mathrm{d}P/\mathrm{d}x$ with x for different values of n. The other parameters are $a = 0.7$, $b = 0.5$, $d = 1.2$, $Q = 1$, $M = 2$, $\beta = 0.04$, $\phi = \pi/8$, $We = 0.04$

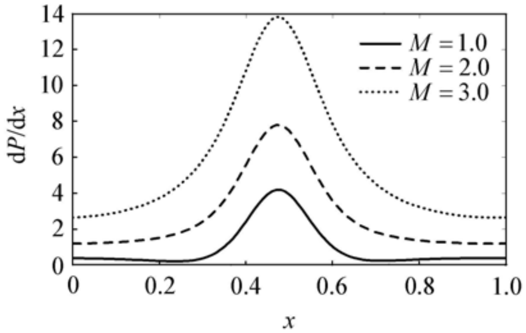

Fig. 11 Variation of dP/dx with x for different values of M. The other parameters are $a = 0.7$, $b = 0.5$, $d = 1.2$, $Q = 1$, $\beta = 0.04$, $\phi = \pi/8$, $n = 0.06$, $We = 0.04$

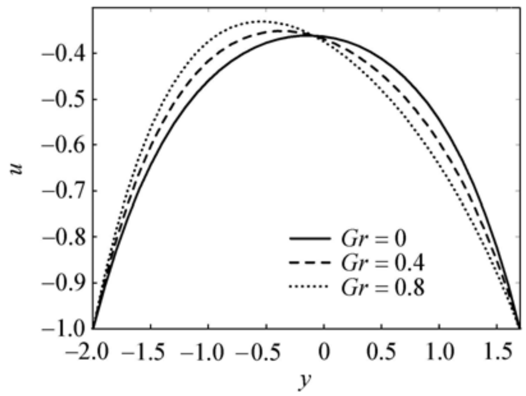

Fig. 12 Velocity profile for different values of Gr. The other parameters are $a = 0.7$, $b = 1.2$, $d = 2$, $Q = 1$, $M = 1$, $\beta = 0.06$, $\phi = \pi/2$, $n = 0.4$, $We = 0.06$

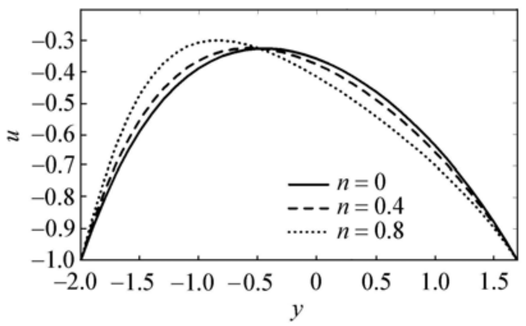

Fig. 13 Velocity profile for different values of n. The other parameters are $a = 0.7$, $b = 1.2$, $d = 2$, $Q = 1$, $M = 1$, $\beta = 0.06$, $\phi = \pi/2$, $Gr = 0.9$, $We = 0.06$

Trapping phenomena

Another interesting phenomenon in peristaltic motion is trapping. It is basically the formation of an internally circulating bolus of fluid by closed stream lines. This trapped bolus pushes a head along peristaltic waves. The trapping phenomena for different values of M, Gr and Q are shown in Figs. 16–18. Figure 16 shows that the size of the trapping bolus decreases on both sides of the channel wall with the increase in Hartmann number M. Figures 17 and 18 show that with the increase in Grashof number Gr and volume flow rate Q, the size of the trapping bolus increases on both sides of the channel wall. Table 1 shows the comparison with the already existing literature.

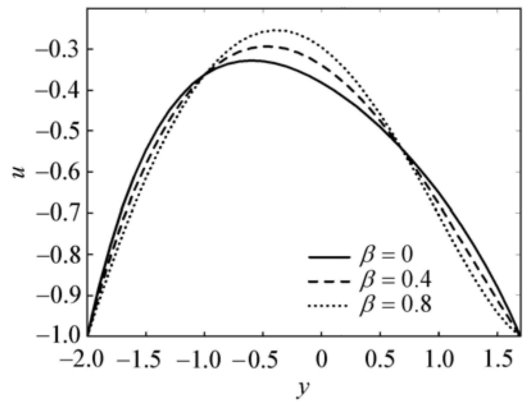

Fig. 14 Velocity profile for different values of β. The other parameters are $a = 0.7$, $b = 1.2$, $d = 2$, $Q = 1$, $M = 1$, $Gr = 0.9$, $\phi = \pi/2$, $n = 0.4$, $We = 0.06$

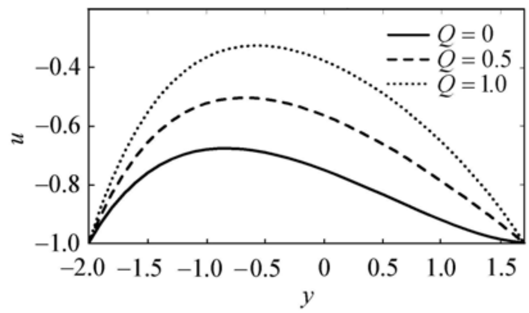

Fig. 15 Velocity profile for different values of Q. The other parameters are $a = 0.7$, $b = 1.2$, $d = 2$, $Gr = 0.9$, $M = 1$, $\beta = 0.06$, $\phi = \pi/2$, $n = 0.4$, $We = 0.06$

5 Conclusion

In the present paper we have discussed the magnetohydrodynamic (MHD) peristaltic flow of a hyperbolic tangent fluid model in a vertical asymmetric channel under a zero Reynolds number and long wavelength approximation. Exact solution of the temperature equation in the absence of dissipation term has been computed and the analytical expression for stream function and axial pressure gradient are established. The expression for pressure rise has been computed numerically. The main findings are summarized as follows:

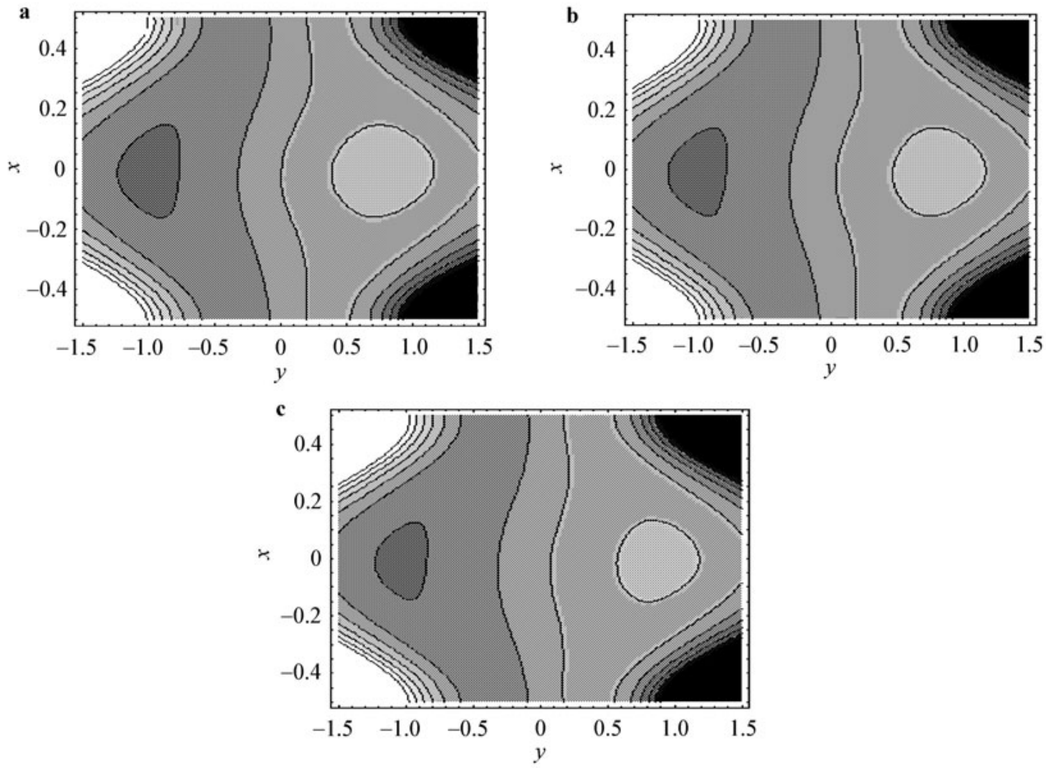

Fig. 16 Stream lines for different values of M. **a** $M = 0.5$; **b** $M = 1$; **c** $M = 1.5$. The other parameters are $a = 0.5$, $b = 0.5$, $d = 1$, $Gr = 2$, $\beta = 2$, $\phi = 0.1$, $Q = 2.5$, $We = 0.03$, $n = 0.398$

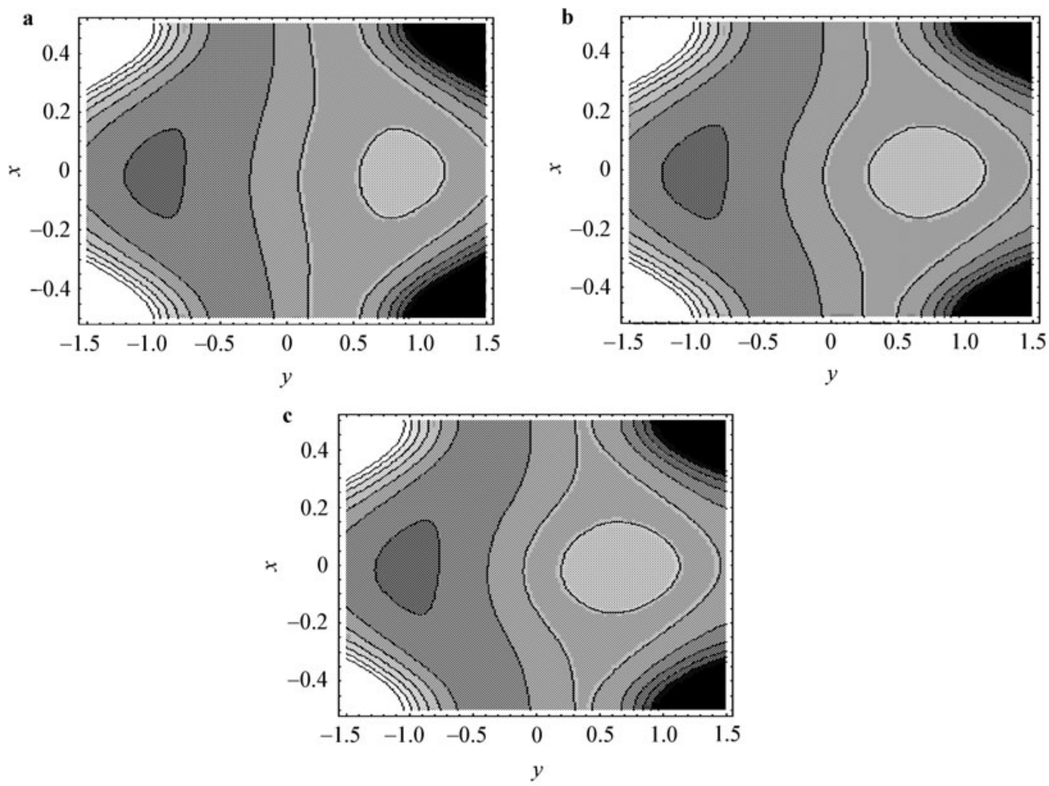

Fig. 17 Stream lines for different values of Gr. **a** $Gr = 1$; **b** $Gr = 3$; **c** $Gr = 4$. The other parameters are $a = 0.5$, $b = 0.5$, $d = 1$, $M = 0.5$, $\beta = 2$, $\phi = 0.1$, $Q = 2.5$, $We = 0.03$, $n = 0.398$

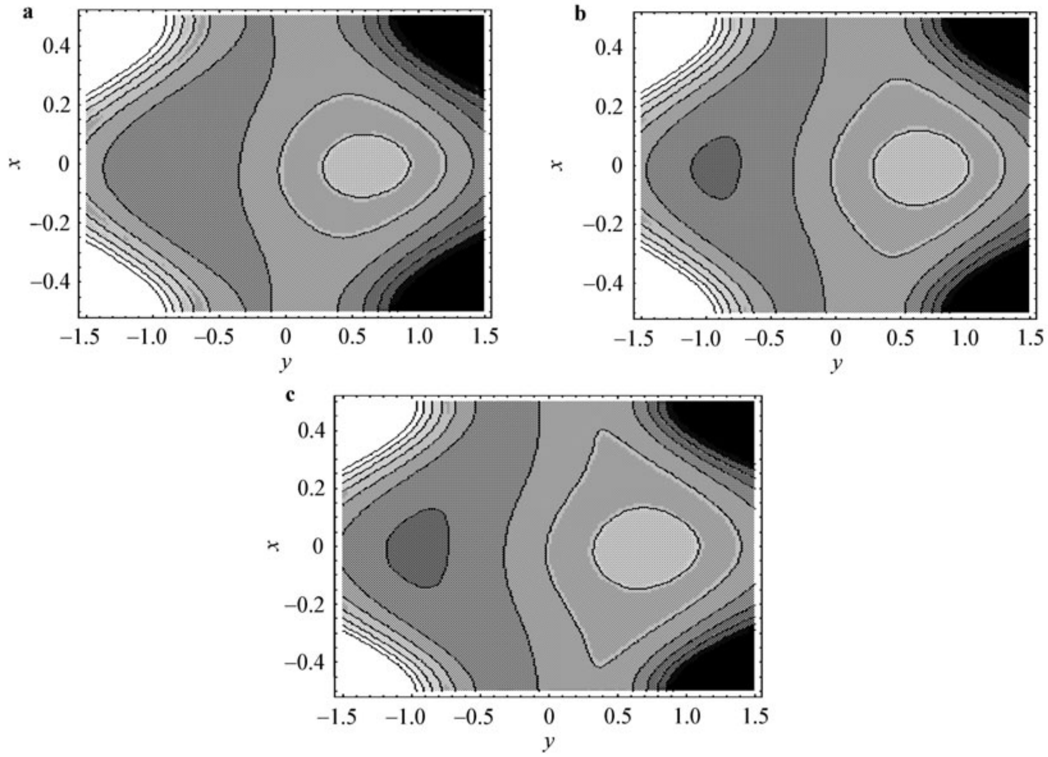

Fig. 18 Stream lines for different values of Q. **a** $Q = 1.8$; **b** $Q = 2.0$; **c** $Q = 2.2$. The other parameters are $a = 0.5$, $b = 0.5$, $d = 1$, $M = 0.5$, $\beta = 2$, $\phi = 0.1$, $Gr = 2.0$, $We = 0.03$, $n = 0.398$

Table 1 Comparison of velocity for fixed $a = 0.7$, $b = 1.2$, $d = 2$, $Q = 1$, $x = 0$, $\phi = \pi/2$

y	$u(x, y)/(\text{m}\cdot\text{s}^{-1})$ for viscous fluid when $\beta = 0$, $Gr = 0$, $We = 0$, $M = 0$	$u(x, y)/(\text{m}\cdot\text{s}^{-1})$ for Nadeem et al. [16] when $\beta = 0$, $Gr = 0$, $M = 0$ $We = 0.04$, $n = 0.7$	$u(x, y)(\text{m}\cdot\text{s}^{-1})$ for Present work when $\beta = 0.06$, $Gr = 0.9$, $M = 1$, $We = 0.04$, $n = 0.7$
−2.0	−1	−1	−1
−1.6	−0.734 191	−0.754 057	−0.556 590
−1.2	−0.532 821	−0.556 718	−0.363 179
−0.8	−0.395 890	−0.418 809	−0.311 454
−0.4	−0.323 396	−0.342 266	−0.336 445
0	−0.315 342	−0.329 027	−0.404 571
0.4	−0.371 725	−0.381 030	−0.500 267
0.8	−0.492 547	−0.500 212	−0.618 497
1.2	−0.677 808	−0.688 513	−0.762 316
1.6	−0.927 507	−0.947 868	−0.945 070
1.7	−1	−1	−1

(1) It is observed that the pumping rate increases with an increase in Weissenberg number, Grashof number and amplitude of wave α in the pumping and free pumping regions, while in the copumping region the behavior is quite opposite.

(2) The pumping rate decreases throughout all the regions with an increase in heat source/ sink parameter, while the pumping rate increases throughout all the regions with an increase in power law index n.

(3) The pumping rate decreases with an increase of phase difference ϕ and width of the channel d in the pumping and free pumping regions.

(4) The pressure gradient increases with an increase in Gr and n in the narrow part of the channel.

(5) The pressure gradient increases with an increase in Hartmann number M.

(6) The velocity profile decreases in the right side of the channel wall with an increase in Gr and n, while it increases in the left side of the channel wall, but the velocity profile is maximum at the center of the channel.
(7) The velocity profile increases with an increase in heat source/ sink parameter in the interval $y \in [-2, -1]$ and [0.4, 1.7], while the behavior is opposite in the interval [−1, 0.5].
(8) The velocity profile decreases with an increase in volume flow rate Q.
(9) The size of the trapping bolus decreases on both sides of the channel wall with an increase in Hartmann number M, while it increases with an increase in Grashof number Gr and volume flow rate Q.

Appendix

$$m_1^2 = \frac{M^2}{(1-n)}, \qquad C_1 = \frac{Gr}{(n-1)}\left[\frac{1}{h_2 - h_1} + \frac{\beta(h_2+h_1)}{2}\right],$$

$$C_2 = \frac{Gr\beta}{(n-1)}, \qquad A = \frac{a_{02}}{2m_1 a_{01}},$$

$$B = \frac{a_{03}}{a_{01}}, \qquad A_1 = \frac{a_{04}}{2m_1 a_{01}}, \qquad B_1 = \frac{a_{05}}{2m_1 a_{01}},$$

$$\begin{aligned} a_{01} = {} & 6m_1^2\left\{(h_1-h_2)m_1\cosh\left[\frac{1}{2}(h_1-h_2)m_1\right] \right. \\ & \left. -2\sinh\left[\frac{1}{2}(h_1-h_2)m_1\right]\right\}, \end{aligned}$$

$$\begin{aligned} a_{02} = {} & -\Big\{2\Big\{3(h_1-h_2)[-2C_1+C_2(h_1+h_2)] \\ & +h_1h_2(-h_1+h_2)[-3C_1+C_2(h_1+h_2)]m_1^2 \\ & +3F_0(h_1+h_2)m_1^4\Big\}\cosh\left[\frac{1}{2}(h_1-h_2)m_1\right] \\ & +m_1\Big\{-3(h_1-h_2)^2[-2C_1+C_2(h_1+h_2)] \\ & \times\mathrm{csch}\left[\frac{1}{2}(h_1-h_2)m_1\right] + 2\Big\{\left(h_1^2-4h_1h_2+h_2^2\right) \\ & \times[3C_1-C_2(h_1+h_2)] + 6(h_1+h_2)m_1^2\Big\} \\ & \times\sinh\left[\frac{1}{2}(h_1-h_2)m_1\right]\Big\}\Big\}, \end{aligned}$$

$$\begin{aligned} a_{03} = {} & \Big\{m_1\Big[3C_1\left(h_1^2-h_2^2\right)+C_2\left(-h_1^3+h_2^3\right)+6F_0m_1^2\Big] \\ & \times\cosh\left[\frac{1}{2}(h_1-h_2)m_1\right] + 3\Big[-2C_1(h_1+h_2) \\ & +C_2\left(h_1^2+h_2^2\right)+4m_1^2\Big]\sinh\left[\frac{1}{2}(h_1-h_2)m_1\right]\Big\}, \end{aligned}$$

$$\begin{aligned} a_{04} = {} & \mathrm{csch}\left[\frac{1}{2}(h_1-h_2)m_1\right]\Big\{m_1\Big\{-(h_1-h_2)^2 \\ & \times[-3C_1+C_2(h_1+2h_2)] \\ & +6(F_0+h_1-h_2)m_1^2\Big\}\cosh(h_1m_1) \\ & -m_1\Big\{(h_1-h_2)^2[-3C_1+C_2(2h_1+h_2)] \\ & +6(F_0+h_1-h_2)m_1^2\Big\}\cosh(h_2m_1) \\ & +3(h_1-h_2)[-2C_1+C_2(h_1+h_2)] \\ & \times[\sinh(h_1m_1)-\sinh(h_2m_1)]\Big\}, \end{aligned}$$

$$\begin{aligned} a_{05} = {} & \mathrm{csch}\left[\frac{1}{2}(h_1-h_2)m_1\right] \\ & \times\Big\{-3(h_1-h_2)[-2C_1+C_2(h_1+h_2)]\cosh(h_1m_1) \\ & +3(h_1-h_2)[-2C_1+C_2(h_1+h_2)]\cosh(h_2m_1) \\ & +m_1\Big\{\Big\{(h_1-h_2)^2[-3C_1+C_2(h_1+2h_2)] \\ & +6(F_0+h_1-h_2)m_1^2\Big\}\sinh(h_1m_1) \\ & +\Big\{(h_1-h_2)^2[-3C_1+C_2(2h_1+h_2)] \\ & +6(F_0+h_1-h_2)m_1^2\Big\}\sinh(h_2m_1)\Big\}\Big\} \end{aligned}$$

$$\begin{aligned} L_1(x,y) = {} & (A_1-B_1)^2m_1^9m_2\Big\{4e^{3(h_1+h_2)m_1}+e^{3m_1(h_1+y)} \\ & +e^{3m_1(h_2+y)}-3e^{m_1(3h_1+2h_2+y)}-3e^{m_1(2h_1+3h_2+y)} \\ & -e^{m_1(4h_1+h_2+y)}(-3+2h_1m_1-2h_2m_1) \\ & +e^{m_1(h_1+2h_2+3y)}(-1+2h_1m_1-2h_2m_1) \\ & -e^{m_1(2h_1+h_2+3y)}(1+2h_1m_1-2h_2m_1) \\ & +e^{m_1(h_1+4h_2+y)}(3+2h_1m_1-2h_2m_1) \\ & +e^{2(2h_1+h_2)}(-2+h_1m_1-h_2m_1) \\ & -e^{2(h_1+2h_2)}(2+h_1m_1-h_2m_1) \\ & +3e^{2m_1(h_1+h_2+y)}[2+m_1(h_1+h_2-2y)] \\ & -2e^{m_1(h_1+3h_2+2y)}(1+2h_1m_1-2m_1y) \\ & -2e^{m_1(3h_1+h_2+2y)}(1+2h_2m_1-2m_1y) \\ & +e^{2m_1(2h_1+y)}(-1+h_1m_1-m_1y) \\ & +e^{2m_1(2h_2+y)}(-1+h_2m_1-m_1y)\Big\}, \end{aligned}$$

$$\begin{aligned} L_2(x,y) = {} & (A_1+B_1)^2m_1^9m_2(e^{m_1(6h_1+3h_2+y)} \\ & +e^{m_1(3h_1+6h_2+y)}-3e^{m_1(3h_1+4h_2+3y)} \\ & -3e^{m_1[4h_1+3(h_2+y)]}+4e^{3(h_1+h_2)m_1+4m_1y} \\ & -e^{m_1(5h_1+2h_2+3y)}(-3+2h_1m_1-2h_2m_1) \\ & +\Big\{e^{m_1(4h_1+5h_2+y)}(-1+2h_1m_1-2h_2m_1) \\ & -e^{m_1(5h_1+4h_2+y)}(1+2h_1m_1-2h_2m_1) \\ & +e^{m_1(2h_1+5h_2+3y)}(3+2h_1m_1-2h_2m_1) \\ & +e^{2m_1(2h_1+h_2+2y)}(-2+h_1m_1-h_2m_1) \\ & -e^{2m_1[h_1+2(h_2+y)]}(2+h_1m_1-h_2m_1) \\ & -3e^{2m_1[2(h_1+h_2)+y]}[-2+m_1(h_1+h_2-2y)] \\ & +2e^{m_1(5h_1+3h_2+2y)}(-1+2h_1m_1-2m_1y) \\ & +2e^{m_1(3h_1+5h_2+2y)}(-1+2h_2m_1-2m_1y) \\ & -e^{2m_1(h_1+3h_2+y)}(1+h_1m_1-m_1y) \\ & -e^{2m_1(3h_1+h_2+y)}(1+h_2m_1-m_1y)\Big\}, \end{aligned}$$

$$L_3(x,y) = (A_1+B_1)m_1^4 m_2\Big\{(C_2+2C_1m_1-2C_2h_1m_1)$$
$$\times\big(-3e^{m_1(5h_1+3h_2+y)}-3e^{m_1(3h_1+5h_2+y)}\big)$$
$$-3e^{m_1[4(h_1+h_2)+y]}\Big\{-2C_2-4C_1m_1+2C_2(h_1+h_2)m_1$$
$$+C_2(h_1-h_2)^2m_1^2+(h_1-h_2)^2$$
$$\times[-2C_1+C_2(h_1+h_2)]m_1^3\Big\}$$
$$+3e^{m_1(2h_1+5h_2+2y)}(C_2+2C_1m_1-2C_2h_2m_1)$$
$$\times(1+h_1m_1-m_1y)+3e^{m_1[5h_1+2(h_2+y)]}$$
$$\times(1+h_2m_1-m_1y)(C_2+2C_1m_1-2C_2h_1m_1)$$
$$-6e^{3m_1(h_1+h_2+y)}\Big\{-2C_1m_1[1+m_1(h_1+h_2-2y)]$$
$$+C_2\Big\{-1+m_1\Big[2y+m_1\Big(h_1^2+h_2^2-2y^2\Big)\Big]\Big\}\Big\}$$
$$+3e^{m_1(4h_1+3h_2+2y)}\Big\{C_2\Big\{-1+m_1\Big[h_1+2h_2^2m_1$$
$$+2h_1\Big(h_1^2-h_2^2\Big)m_1^2+y-2m_1\Big(h_1+h_1^2m_1-h_2^2m_1\Big)y\Big]\Big\}$$
$$+2C_1m_1\{-1+m_1\{-2h_1^2m_1+y-2(h_2+h_2m_1y)$$
$$+h_1[1+2m_1(h_2+y)]]\}\}+3e^{m_1(3h_1+4h_2+2y)}$$
$$\times\{C_2\{-1+m_1[h_2+2h_1^2m_1+2h_2m_1^2(-h_1^2+h_2^2)$$
$$+y-2m_1(h_2-h_1^2m_1+h_2^2m_1)y]\}$$
$$+2C_1m_1\{-1+m_1\{h_2+y+2[h_2m_1(-h_2+y)$$
$$+h_1(-1+h_2m_1-m_1y)]\}\}\}+3e^{m_1(2h_1+4h_2+3y)}$$
$$\times\{-2C_1m_1\{1+m_1[h_2-2y+h_2m_1(-h_2+y)$$
$$+h_1(1+h_2m_1-m_1y)]\}+C_2\{-1+m_1\{h_2+2y$$
$$-m_1(h_2+y)[2y+h_2(-1+h_2m_1-m_1y)]$$
$$+h_1(-1+m_1(h_2+y)(1+h_2m_1-m_1y))\}\}\}$$
$$-3e^{m_1(4h_1+2h_2+3y)}\Big\{2C_1m_1\Big\{1+m_1\Big[h_1+h_2-h_1^2m_1$$
$$+h_1h_2m_1+(-2+h_1m_1-h_2m_1)y\Big]\Big\}$$
$$+C_2\Big\{1+m_1\Big\{h_2+h_1^3m_1^2-h_1^2m_1(1+h_2m_1)$$
$$-2y+m_1y[2y+h_2(-1+m_1y)]$$
$$-h_1\{1+m_1[h_2+y(-1+m_1y)]\}\Big\}\Big\}\Big\}\Big\},$$

$$L_4(x,y) = (A_1-B_1)m_1^4m_2\Big\{(C_2-2C_1m_1+2C_2h_1m_1)$$
$$\times\big(3e^{m_1[h_1+3(h_2+y)]}+3e^{m_1(3h_1+h_2+3y)}\big)$$
$$-3e^{2(h_1+h_2)m_1+3m_1y}\Big\{2C_2-4C_1m_1+2C_2(h_1+h_2)m_1$$
$$-C_2m_1^2(h_1-h_2)^2+(h_1-h_2)^2$$
$$\times[-2C_1+C_2(h_1+h_2)]m_1^3\Big\}+3e^{m_1(4h_1+h_2+3y)}$$
$$\times(C_2-2C_1m_1+2C_2h_2m_1)(-1+h_1m_1-m_1y)$$
$$+3e^{m_1(h_1+4h_2+2y)}(C_2-2C_1m_1+2C_2h_1m_1)$$
$$\times(-1+h_2m_1-m_1y)-3e^{m_1(2h_1+4h_2+y)}$$
$$\times\Big\{-2C_1m_1\Big\{-1+m_1\Big[h_1+h_2+h_1^2m_1-h_1h_2m_1$$
$$+(-2-h_1m_1+h_2m_1)y\Big]\Big\}+C_2\Big\{-1$$
$$+m_1\Big\{h_2+h_1\Big\{-1+m_1[h_1+h_2+h_1(h_1-h_2)m_1]\Big\}$$
$$-2y+(-h_1+h_2)m_1y+m_1(-2-h_1m_1+h_2m_1)y^2\Big\}\Big\}\Big\}$$
$$+6e^{m_1[3(h_1+h_2)+y]}\Big\{-2C_1m_1[-1+m_1(h_1+h_2-2y)]$$
$$+C_2\Big\{-1+m_1\Big[m_1\Big(h_1^2+h_2^2\Big)-2y(1+m_1y)\Big]\Big\}\Big\}$$
$$+3e^{m_1(2h_1+3h_2+2y)}\Big\{-2C_1m_1\Big\{1+m_1\Big\{2h_1^2m_1+y$$
$$+2h_2(-1+m_1y)+h_1[1-2m_1(h_2+y)]\Big\}\Big\}$$
$$+C_2\Big\{1+m_1\Big\{2h_1^3m_1^2+y-2h_1^2m_1^2y$$
$$+2h_2^2m_1(-1+m_1y)+h_1\Big[1+2m_1\Big(-h_2^2m_1+y\Big)\Big]\Big\}\Big\}\Big\}$$
$$+3e^{m_1(4h_1+2h_2+y)}\Big\{2C_1m_1\Big\{-1+m_1\Big[h_1+h_2-h_1h_2m_1$$
$$+h_2^2m_1+(-2+h_1m_1-h_2m_1)y\Big]\Big\}$$
$$+C_2\Big\{1+m_1\Big\{h_2+2y-m_1(h_2+y)$$
$$\times\Big(h_2+h_2^2m_1-2y-h_2m_1y\Big)$$
$$+h_1[-1+m_1(h_2+y)(-1+h_2m_1-m_1y)]\Big\}\Big\}$$
$$+3e^{m_1[3h_1+2(h_2+y)]}\Big\{C_2\Big\{1+m_1\Big\{2h_2^3m_1^2+y$$
$$-2h_2^2m_1^2y+2h_1^2m_1(-1+m_1y)$$
$$+h_2\Big[1+2m_1\Big(-h_1^2m_1+y\Big)\Big]\Big\}\Big\}$$
$$+2C_1m_1\{-1-m_1\{h_2+y-2[h_2m_1(-h_2+y)$$
$$+h_1(1+h_2m_1-m_1y)]\}\}\},$$

$$L_5(x,y) = -12e^{m_1(3h_1+2h_2+3y)}\Big[F_1m_1^7-C_2^2(h_1-h_2)$$
$$\times(-2+h_1m_1-h_2m_1)m_2\Big]+12e^{m_1(4h_1+3h_2+y)}$$
$$\times\Big[F_1m_1^7+C_2^2(h_1-h_2)(-2+h_1m_1-h_2m_1)m_2\Big]$$
$$-12e^{m_1(3h_1+4h_2+y)}\Big[F_1m_1^7-C_2^2(h_1-h_2)$$
$$\times(2+h_1m_1-h_2m_1)m_2\Big]+12e^{m_1[2h_1+3(h_2+y)]}$$
$$\times\Big[F_1m_1^7+C_2^2(h_1-h_2)(2+h_1m_1-h_2m_1)m_2\Big]$$
$$-6e^{2m_1(2h_1+h_2+y)}\Big\{F_1m_1^8(h_1+h_2-2y)$$
$$+2C_2^2(-2+h_1m_1-h_2m_1)m_2h_1-h_2+h_1h_2m_1$$
$$-(h_1+h_2)m_1y+m_1y^2\Big\}+6e^{2m_1(h_1+2h_2+y)}$$
$$\times\Big\{F_1m_1^8(h_1+h_2-2y)+2C_2^2(2+h_1m_1-h_2m_1)$$
$$\times m_2\Big[-h_1+h_2+h_1h_2m_1-(h_1+h_2)m_1y+m_1y^2\Big]\Big\},$$

$$a_{06}(x,y) = 12(e^{h_1m_1}-e^{h_2m_1})m_1^4\Big[e^{h_1m_1}(-2+h_1m_1-h_2m_1)$$
$$+e^{h_2m_1}(2+h_1m_1-h_2m_1)\Big]$$

$$a_{07}(x,y) = (A_1-B_1)^2m_1^9m_2\Big[-8e^{3(h_1+h_2)m_1}+e^{3m_1(h_1+y)}$$
$$-e^{2m_1(2h_1+y)}+e^{3m_1(h_2+y)}-e^{2m_1(2h_2+y)}$$
$$-6e^{2m_1(h_1+h_2+y)}+3e^{m_1(3h_1+2h_2+y)}+3e^{m_1(2h_1+3h_2+y)}$$

$$
\begin{aligned}
&+4e^{m_1(3h_1+h_2+2y)}+4e^{m_1(h_1+3h_2+2y)}\\
&+e^{m_1(4h_1+h_2+y)}(-3+2h_1m_1-2h_2m_1)\\
&+e^{m_1(h_1+2h_2+3y)}(-1+2h_1m_1-2h_2m_1)\\
&-e^{m_1(2h_1+h_2+3y)}(1+2h_1m_1-2h_2m_1)\\
&-e^{m_1(h_1+4h_2+y)}(3+2h_1m_1-2h_2m_1)\\
&-2e^{2m_1(2h_1+h_2)}(-2+h_1m_1-h_2m_1)\\
&+2e^{2m_1(h_1+2h_2)}(2+h_1m_1-h_2m_1)\Big]
\end{aligned}
$$

$$
\begin{aligned}
a_{08}(x,y)=&(A_1+B_1)^2m_1^9m_2\Big[e^{2m_1(3h_1+h_2+y)}+e^{2m_1(h_1+3h_2+y)}\\
&-e^{m_1(6h_1+3h_2+y)}-e^{m_1(3h_1+6h_2+y)}+6e^{2m_1[2(h_1+h_2)+y]}\\
&-4e^{m_1(5h_1+3h_2+2y)}-4e^{m_1(3h_1+5h_2+2y)}-3e^{m_1(3h_1+4h_2+3y)}\\
&+8e^{3(h_1+h_2)m_1+4m_1y}-3e^{m_1[4h_1+3(h_2+y)]}\\
&-e^{m_1(5h_1+2h_2+3y)}(-3+2h_1m_1-2h_2m_1)\\
&-e^{m_1(4h_1+5h_2+y)}(-1+2h_1m_1-2h_2m_1)\\
&+e^{m_1(5h_1+4h_2+y)}(1+2h_1m_1-2h_2m_1)\\
&+e^{m_1(2h_1+5h_2+3y)}(3+2h_1m_1-2h_2m_1)\\
&+2e^{2m_1(2h_1+h_2+2y)}(-2+h_1m_1-h_2m_1)\\
&-2e^{2m_1[h_1+2(h_2+y)]}(2+h_1m_1-h_2m_1)\Big],
\end{aligned}
$$

$$
\begin{aligned}
a_{09}(x,y)=&(A_1+B_1)m_1^4m_2\Big\{(C_2+2C_1m_1-2C_2h_1m_1)\\
&\times(3e^{m_1(5h_1+3h_2+y)}-3e^{m_1[5h_1+2(h_2+y)]})\\
&+(C_2+2C_1m_1-2C_2h_2m_1)(3e^{m_1(3h_1+5h_2+y)}\\
&-3e^{m_1(2h_1+5h_2+2y)})-3e^{m_1(4h_1+3h_2+2y)}\\
&\times\Big\{-C_2-2C_1m_1+2C_2h_1m_1+2(h_1-h_2)\\
&\times[-2C_1+C_2(h_1+h_2)]m_1^2\Big\}+3e^{m_1(3h_1+4h_2+2y)}\\
&\times\Big\{C_2+2(C_1-C_2h_2)m_1+2(h_1-h_2)\\
&\times[-2C_1+C_2(h_1+h_2)]m_1^2\Big\}+3e^{m_1[4(h_1+h_2)+y)}\\
&\times\Big\{-2C_2-4C_1m_1+2C_2(h_1+h_2)m_1\\
&+C_2(h_1-h_2)^2m_1^2+(h_1-h_2)^2\\
&\times[-2C_1+C_2(h_1+h_2)]m_1^3\Big\}\\
&-6e^{3m_1(h_1+h_2+y)}\Big\{-2C_1m_1[-1+m_1(h_1+h_2-2y)]\\
&+C_2\Big\{1+m_1\Big[-2y+m_1\Big(h_1^2+h_2^2-2y^2\Big)\Big]\Big\}\Big\}\\
&-3e^{m_1(4h_1+2h_2+3y)}\Big\{-2C_1m_1\\
&\times[1+m_1(-2+h_1m_1-h_2m_1)\\
&\times(h_1-y)]+C_2\Big\{-1+m_1\Big[h_1^3m_1^2-h_1^2m_1(1+h_2m_1)\\
&+(2+h_2m_1)y(1+m_1y)-h_1m_1\Big(h_2+y+m_1y^2\Big)\Big]\Big\}\Big\}\\
&-3e^{m_1(2h_1+4h_2+3y)}\{-2C_1m_1\\
&\times[1+m_1(-2-h_1m_1+h_2m_1)\\
&\times(h_2-y)]+C_2\{-1+m_1\{2y+m_1(h_2-y)\\
&\times\{-2y+h_2[-1+m_1(h_2+y)]\\
&-h_1[1+m_1(h_2+y)]\}\}\}\}\},
\end{aligned}
$$

$$
\begin{aligned}
a_{10}(x,y)=&(A_1-B_1)m_1^4m_2\Big\{(C_2-2C_1m_1+2C_2h_1m_1)\\
&\times(-3e^{m_1(h_1+4h_2+2y)}+3e^{m_1[h_1+3(h_2+y)]})\\
&\times(C_2-2C_1m_1+2C_2h_2m_1)(-3e^{m_1(4h_1+h_2+2y)}\\
&+3e^{m_1(3h_1+h_2+3y)})-3e^{m_1(2h_1+3h_2+2y)}\\
&\times\Big\{-C_2+2(C_1-C_2h_1)m_1+2(h_1-h_2)\\
&\times[-2C_1+C_2(h_1+h_2)]m_1^2\Big\}+3e^{m_1[3h_1+2(h_2+y)]}\\
&\times\Big\{C_2-2C_1m_1+2C_2h_2m_1+2(h_1-h_2)\\
&\times[-2C_1+C_2(h_1+h_2)]m_1^2\Big\}-3e^{2m_1(h_1+h_2)+3m_1y}\\
&\times\Big\{2C_2-4C_1m_1+2C_2(h_1+h_2)m_1-C_2(h_1-h_2)^2m_1^2\\
&+(h_1-h_2)^2[-2C_1+C_2(h_1+h_2)]m_1^3\Big\}\\
&+3e^{m_1(2h_1+4h_2+y)}\Big\{2C_1m_1[-1+m_1(2+h_1m_1-h_2m_1)\\
&\times(-h_1+y)]+C_2\Big\{1+m_1\Big\{2y+m_1(h_1-y)\\
&\times[h_1+h_2+h_1(h_1-h_2)m_1\\
&+2y+(h_1-h_2)m_1y]\Big\}\Big\}\Big\}-6e^{m_1[3(h_1+h_2)+y]}\\
&\times\Big\{-2C_1m_1[1+m_1(h_1+h_2-2y)]+C_2\Big\{1+m_1\Big[2y\\
&+m_1\Big(h_1^2+h_2^2-2y^2\Big)\Big]\Big\}\Big\}+3e^{m_1(4h_1+2h_2+y)}\\
&\times\{2C_1m_1[-1+m_1(2-h_1m_1+h_2m_1)(-h_2+y)]\\
&+C_2\{1+m_1\{2y+m_1(h_2-y)\{h_2+2y+h_2m_1(h_2+y)\\
&-h_1[-1+m_1(h_2+y)]\}\}\}\}\},
\end{aligned}
$$

$$
\begin{aligned}
a_{11}(x,y)=&-12e^{m_1(3h_1+2h_2+3y)}\Big[F_1m_1^7-C_2^2(h_1-h_2)\\
&\times(-2+h_1m_1-h_2m_1)m_2\Big]-12e^{m_1(4h_1+3h_2+y)}\\
&\times\Big[F_1m_1^7+C_2^2(h_1-h_2)(-2+h_1m_1-h_2m_1)m_2\Big]\\
&+12e^{m_1(3h_1+4h_2+y)}\Big[F_1m_1^7-C_2^2(h_1-h_2)\\
&\times(2+h_1m_1-h_2m_1)m_2\Big]+12e^{m_1[2h_1+3(h_2+y)]}\\
&\times\Big[F_1m_1^7+C_2^2(h_1-h_2)(2+h_1m_1-h_2m_1)m_2\Big]\\
&+12e^{2m_1(2h_1+h_2+y)}\Big[F_1m_1^7+C_2^2(-2+h_1m_1-h_2m_1)\\
&\times m_2(h_1-h_2-2y)\Big]-12e^{2m_1(h_1+2h_2+y)}\\
&\times\Big[F_1m_1^7+C_2^2(2+h_1m_1-h_2m_1)m_2(h_1+h_2-2y)\Big]\\
&+48m_2C_2^2e^{3m_1(h_1+h_2)+2m_1y}(h_1+h_2-2y),
\end{aligned}
$$

$$
\begin{aligned}
a_{12}(x,y)=&(A_1-B_1)^2m_1^9m_2\Big[-32e^{3(h_1+h_2)m_1}+e^{3(h_1+y)m_1}\\
&+e^{3(h_2+y)m_1}+3e^{(3h_1+2h_2+y)m_1}+3e^{(2h_1+3h_2+y)m_1}\\
&+(-3+2h_1m_1-2h_2m_1)e^{(4h_1+h_2+y)m_1}\\
&+e^{(h_1+2h_2+3y)m_1}(-1+2h_1m_1+2h_2m_1)
\end{aligned}
$$

$$
\begin{aligned}
&-(1+2h_1m_1-2h_2m_1)e^{(2h_1+h_2+3y)m_1}\\
&-(3+2h_1m_1-2h_2m_1)e^{(h_1+4h_2+y)m_1}\\
&-8e^{(2h_1+h_2)m_1}(-2+h_1m_1-h_2m_1)\\
&+8(2+h_1m_1-h_2m_1)e^{2(h_1+2h_2)m_1}\Big],
\end{aligned}
$$

$$
\begin{aligned}
a_{13}(x,y)=\;&(A_1+B_1)m_1^4m_2\Big\{(C_2+2C_1m_1-2C_2h_1m_1)\\
&\times 3e^{(5h_1+3h_2+y)m_1}+3e^{(3h_1+5h_2+y)m_1}\\
&\times(C_2+2C_1m_1-2C_2h_1m_1)+3e^{[4(h_1+h_2)+y]m_1}\\
&\quad\times\Big\{-2C_2-4C_1m_1+2C_2(h_1+h_2)m_1\\
&+C_2m_1^2(h_1-h_2)^2+(h_1-h_2)^2\\
&\times[-2C_1+C_2(h_1+h_2)]m_1^3\Big\}\\
&-6e^{3(h_1+h_2+y)m_1}\Big\{-2C_1m_1[-5+m_1(h_1+h_2-2y)]\\
&+C_2\Big\{-7+m_1\Big[h_1^2m_1+h_2^2m_1-2y(5+m_1y)\Big]\Big\}\Big\}\\
&+3e^{(2h_1+4h_2+3y)m_1}\Big\{2C_1m_1\Big\{5+m_1\Big[2y\\
&+h_2(-4+h_2m_1-m_1y)+h_1(2-h_2m_1+m_1y)\Big]\Big\}\\
&+C_2\Big\{-7+m_1\Big\{h_2\Big[4+h_2m_1(1-h_2m_1)\Big]\\
&+5(-2+h_2m_1)y+m_1(-2+h_2m_1)y^2\\
&+h_1\Big\{-4+m_1\Big[h_2+h_2^2m_1-y(5+m_1y)\Big]\Big\}\Big\}\Big\}\Big\}\\
&-3e^{(4h_1+2h_2+3y)m_1}\Big\{-2C_1m_1\Big\{5+m_1\Big\{h_1^2m_1+h_2m_1y\\
&+2(h_2+y)-h_1\Big[4+m_1(h_2+y)\Big]\Big\}\Big\}\\
&+C_2\Big\{7+m_1\Big\{h_1^3m_1^2-h_1^2m_1(1+h_2m_1)\\
&+h_2(1+m_1y)(4+m_1y)+2y(5+m_1y)\\
&-h_1\{4+m_1[h_2+y(5+m_1y)]\}\}\}\}\},
\end{aligned}
$$

$$
\begin{aligned}
a_{14}(x,y)=\;&(A_1-B_1)m_1^4m_2\Big\{3e^{[h_1+3(h_2+y)]m_1}\\
&\times(C_2-2C_1m_1+2C_2h_1m_1)\\
&+3e^{(3h_1+h_2+3y)m_1}(C_2-2C_1m_1+2C_2h_2m_1)\\
&-3e^{2(h_1+h_2)m_1+3m_1y}\Big\{2C_2-4C_1m_1\\
&+2C_2(h_1+h_2)m_1-C_2(h_1-h_2)^2m_1^2\\
&+(h_1-h_2)^2[-2C_1+C_2(h_1+h_2)\Big]m_1^3\}\\
&-6e^{[3(h_1+h_2)+y]m_1}\Big\{-2C_1m_1[5+m_1(h_1+h_2-2y)]\\
&+C_2\Big\{-7+m_1\Big[10y+m_1\Big(h_1^2+h_2^2-2y^2\Big)\Big]\Big\}\Big\}\\
&+3e^{m_1(2h_1+4h_2+y)}\{C_2\{-7+m_1\{4h_2+h_1\{-4\\
&+m_1[h_1+h_2+h_1(h_1-h_2)m_1]\}+10y\\
&+5(h_1-h_2)m_1y-m_1(2+h_1m_1-h_2m_1)y^2\}\}\\
&+2C_1m_1\{-5+m_1\{-h_1^2m_1-h_2m_1y+2(h_2+y)\\
&+h_1[-4+m_1(h_2+y)]\}\}\}\\
&-3e^{m_1(4h_1+2h_2+y)}\{2C_1m_1[5+h_2m_1(4+h_2m_1)\\
&-m_1(2+h_2m_1)y+h_1m_1(-2-h_2m_1+m_1y)]\\
&+C_2\{7+m_1\{-h_2[-4+h_2m_1(1+h_2m_1)]\\
&-5(2+h_2m_1)y+m_1(2+h_2m_1)y^2\\
&+h_1\{-4+m_1[h_2(-1+h_2m_1)+5y-m_1y^2]\}\}\}\}\},
\end{aligned}
$$

$$
\begin{aligned}
a_{15}(x,y)=\;&(A_1+B_1)^2m_1^9m_2[-e^{(6h_1+3h_2+y)m_1}\\
&-e^{(3h_1+6h_2+y)m_1}-3e^{(3h_1+4h_2+3y)m_1}\\
&-3e^{[4h_1+3(h_2+y)]m_1}+32e^{3(h_1+h_2)m_1+4m_1y}\\
&-e^{(5h_1+2h_2+3y)m_1}(-3+2h_1m_1-2h_2m_1)\\
&-e^{(4h_1+5h_2+y)m_1}(-1+2h_1m_1-2h_2m_1)\\
&+e^{(5h_1+4h_2+y)m_1}(1+2h_1m_1-2h_2m_1)\\
&+e^{(2h_1+5h_2+3y)m_1}(3+2h_1m_1-2h_2m_1)\\
&+8e^{2[h_1+2(h_2+y)]m_1}(2+h_1m_1-h_2m_1)\\
&+8e^{2(2h_1+h_2+2y)m_1}(-2+h_1m_1-h_2m_1)\Big],
\end{aligned}
$$

$$
\begin{aligned}
a_{16}(x,y)=\;&-12e^{m_1(3h_1+2h_2+3y)}\Big[F_1m_1^7-C_2^2(h_1-h_2)\\
&\times(-2+h_1m_1-h_2m_1)m_2\Big]-12e^{m_1(4h_1+3h_2+y)}\\
&\times\Big[F_1m_1^7+C_2^2(h_1-h_2)(-2+h_1m_1-h_2m_1)m_2\Big]\\
&+12e^{m_1(3h_1+4h_2+y)}\Big[F_1m_1^7-C_2^2(h_1-h_2)\\
&\times(2+h_1m_1-h_2m_1)m_2\Big]+12e^{m_1[2h_1+3(h_2+y)]}\\
&\times\Big[F_1m_1^7+C_2^2(h_1-h_2)(2+h_1m_1-h_2m_1)m_2\Big].
\end{aligned}
$$

References

1 Srivastava, L.M., Agrawal, R.P.: Oscillating flow of a conducting fluid with suspension of spherical particles. J. Appl. Mech. **47**, 196–199 (1980)

2 Agrawal, H.L., Anwaruddin, B.: Peristaltic flow of blood in a branch. Ranchi Uni. Math. J. **15**, 111 (1984)

3 Nadeem, S., Akram, S.: Slip effects on the peristaltic flow of a Jeffrey fluid in an asymmetric channel under the effect of induced magnetic field. Int. J. Numer. Meth. Fluids. Doi: 10.1002/fld.2081

4 Hayat, T., Ali, N.: Peristaltically induced motion of a MHD third grade fluid in a deformable tube. Phys. A. **370**, 225–239 (2006)

5 Mekheimer, Kh.S.: Non linear peristaltic transport of magnetohydrodynamic flow in an inclined planar channel. Arab J. Sci. Eng. **28**, 183–201 (2003)

6 Ealshahed, M., Haroun, M.H.: Peristaltic transport of Johnson-Segalman fluid under effect of a magnetic field. Math. Prob. Eng. **6**, 663–677 (2005)

7 Radhakrishnamacharya, G., Srinivasulu, Ch.: Influence of wall properties on peristaltic transport with heat transfer. C. R. Mecanique. **335**, 369–373 (2007)

8 Radhakrishnamacharya, G., Radhakrishna Murthy, V.: Heat transfer to peristaltic transport in a non-uniform channel. Def. Sci. J. **43**, 275–280 (1993)

9 Srinivas, S., Kothandapani, M.: Peristaltic transport in an asymmetric channel with heat transfer. Int. Commun. Heat Mass Transf. **35**, 514–522 (2008)

10 Mekheimer, Kh. S., Abdelmaboud, Y.: Influence of heat transfer and magnetic field on peristaltic transport of a Newtonian fluid in a vertical annulus. Application of an endoscope. Phys. Lett. A. **372**, 1657–1665 (2008)

11 Nadeem, S., Akram, S.: Heat transfer in a peristaltic flow of MHD fluid with partial slip. Commun. Non Linear. Sci. Numer. Simulat. **15**, 312–321 (2010)

12 Nadeem, S., Akbar, N.S.: Influence of heat transfer on a peristaltic transport of Herschel-Bulkley fluid in a non-uniform inclined tube. Commun. Non Linear. Sci. Numer. Simulat. **14**, 4100–4113 (2009)

13 Nadeem, S., Akbar, N.S.: Effects of heat transfer on the peristaltic transport of MHD Newtonian fluid with variable viscosity. Application of adomian decomposition method. Commun. Non Linear. Sci. Numer. Simulat. **14**, 3844–3855 (2009)

14 Srinivas, S., Gayathri, R.: Peristaltic transport of a Newtonian fluid in a vertical asymmetric channel with heat transfer and porous medium. Appl. Math. Comput. **215**, 185–196 (2009)

15 Pop, I., Ingham, D.B.: Convective Heat Transfer: Mathematical and Computational Modelling of Viscous Fluids and Porous Media. Pergamon, Amsterdam, NewYork. 2001

16 Nadeem, S., Akram, S.: Peristaltic transport of a hyperbolic tangent fluid model in an asymmetric channel. ZNA, **64a**, 559–567 (2009)