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RESEARCH PAPER

Some exact solutions of the oscillatory motion of a generalized second grade fluid in an annular region of two cylinders

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Abstract The velocity field and the associated shear stress corresponding to the longitudinal oscillatory flow of a generalized second grade fluid, between two infinite coaxial circular cylinders, are determined by means of the Laplace and Hankel transforms. Initially, the fluid and cylinders are at rest and at $t = 0^+$ both cylinders suddenly begin to oscillate along their common axis with simple harmonic motions having angular frequencies Ω_1 and Ω_2. The solutions that have been obtained are presented under integral and series forms in terms of the generalized G and R functions and satisfy the governing differential equation and all imposed initial and boundary conditions. The respective solutions for the motion between the cylinders, when one of them is at rest, can be obtained from our general solutions. Furthermore, the corresponding solutions for the similar flow of ordinary second grade fluid and Newtonian fluid are also obtained as limiting cases of our general solutions. At the end, the effect of different parameters on the flow of ordinary second grade and generalized second grade fluid are investigated graphically by plotting velocity profiles.

Keywords Generalized second grade fluid · Velocity field · Shear stress · Longitudinal oscillatory flow · Laplace and Hankel transforms

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1 Introduction

In many fields, such as food industry, drilling operations, polymer chemical industry and bio-engineering, the fluids, either synthetic or natural, are mixtures of different stuffs such as water, particles, oils, red cells and other long chain molecules. The viscosity function varies non-linearly with the shear rate and the elasticity is felt through elongational effects and time-dependent effects. In these cases, the fluids have been treated as viscoelastic fluids. Because of the difficulty to suggest a single model, which exhibits all properties of viscoelastic fluids, they cannot be described as simply as Newtonian fluids. For this reason, many models or constitutive equations have been proposed and most of them are empirical or semi-empirical.

The second grade fluids are the common non-Newtonian viscoelastic fluids in industrial fields, such as polymer solutions. The ordinary linear constitutive model for a second grade fluid has the following form

$$\tau(t) = \mu\varepsilon(t) + E\frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t}, \tag{1}$$

where τ is the stress, ε is the strain, μ is the viscosity coefficient and E is the viscoelasticity constant. This mathematical model provides a reasonable qualitative description. However, it is not satisfactory from a quantitative view-point (see Ref. [1]).

Several authors [2–4] suggested that integer-order models for viscoelastic materials seem to be inadequate from both qualitative and quantitative point of view. At the same time, they proposed fractional-order laws of deformation for modeling the viscoelastic behavior of real materials. For instance, Caputo and Mainardi [1] formulated the following fractional-order model for a second grade fluid

$$\sigma(t) + a\frac{\mathrm{d}^{\alpha}\sigma}{\mathrm{d}t^{\alpha}} = m\varepsilon(t) + b\frac{\mathrm{d}^{\alpha}\varepsilon}{\mathrm{d}t^{\alpha}}, \quad 0 < \alpha \le 1, \tag{2}$$

where α, a, m and b are constants which depend on the nature of material. This model includes the classical law when $\alpha = 1$, $a = 0$, $m = \mu$ and $b = E$. Bagley and Torvik [5,6] and Rogers [7] have shown that law (2) is very useful for modeling of most viscoelastic materials. In addition to experimental findings, they proved that the four-parameter model (2) seems to be satisfactory for most real materials.

Bagley and Torvik [5,6], Koeller [8], Xu and Tan [9,10] proposed the fractional derivative approach to viscoelasticity in order to describe the properties of numerous viscoelastic materials. They suggest the general form of the model as

$$\sigma(t) = E_0\varepsilon(t) + E_1\mathrm{D}_t^{\beta}[\varepsilon(t)], \tag{3}$$

where $\mathrm{D}_t^{\beta}[\varepsilon(t)]$ is the Riemann–Liouville fractional differential operator of order β with respect to t defined by [11,12]

$$\mathrm{D}_t^{\beta}[\varepsilon(t)] = \frac{1}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\varepsilon(\tau)}{(t-\tau)^{\beta}}\mathrm{d}\tau, \quad 0 < \beta < 1, \tag{4}$$

where $\Gamma(\cdot)$ is Gamma function.

According to the molecular theory for dilute polymer solutions due to Rouse [13], the stress is

$$\sigma(t) = \mu_s\mathrm{D}_t^{1}[\varepsilon(t)] + \left[\frac{3}{2}(\mu_0 - \mu_s)nkT\right]^{1/2}\mathrm{D}_t^{1/2}[\varepsilon(t)], \tag{5}$$

where n is the number of molecules per unit volume of the polymer solution, k is the Boltzmann constant, T is the absolute temperature, μ_s is the steady-flow viscosity of the solvent in the solution and μ_0 is the steady-flow viscosity of the solution. Thus, the Rouse theory provides us presence of fractional derivative along with the first derivative of classical viscoelasticity in the relation between stress and strain for some polymers. Ferry et al. [14] modified the Rouse theory in concentrated polymer solutions and polymer solids with no cross-linking and obtained that

$$\sigma(t) = \left(\frac{3\mu\rho RT}{2M}\right)^{1/2}\mathrm{D}_t^{1/2}[\varepsilon(t)], \tag{6}$$

where M is the molecular weight, ρ is the density, μ is the viscosity and R is the universal gas constant.

Thus, the fractional calculus approach to viscoelasticity for the study of viscoelastic material properties is justified, at least for polymer solutions and for polymer solids without cross-linking.

Recently, the fractional calculus has encountered much success in the description of viscoelasticity. Specifically, rheological constitutive equations with fractional derivatives play an important role in the description of the properties of polymer solutions and melts. The starting point of the fractional derivative model of non-Newtonian fluids is usually a classical differential equation which is modified by replacing the time derivative of an integer order by so-called Riemann–Liouville fractional differential operator. This generalization allows us to define precisely non-integer order integrals or derivatives [12].

Flows in the neighborhood of spinning or oscillating bodies are of interest to both academic workers sand industry. Among them, the flows between oscillating cylinders are some of the most important and interesting problems of motion. As early as 1886, Stokes [15] established an exact solution for the rotational oscillations of an infinite rod immersed in a classical linearly viscous fluid. Casarella et al. [16] obtained an exact solution for the motion of the same fluid due to both longitudinal and torsional oscillations of the rod. Later, Rajagopal [17] found two simple but elegant solutions for the flow of a second grade fluid induced by the longitudinal and torsional oscillations of an infinite rod. These solutions have been already extended to Oldroyd-B fluids by Rajagopal et al. [18]. Other interesting results have been recently obtained by Khan et al. [19] and Fetecau et al. [20–25].

It is important to mention here that a number of research papers in Refs. [22–25] are devoted to the study of the flow of different viscoelastic fluids between two cylinders, when only one of them is oscillating and other is at rest. On the other hand, the exact solutions corresponding to the flow of these fluids between two cylinders, when both of them are oscillating along or around their common axis simultaneously, are very rare in literature. Recently, Mahmood et al. [21] have studied the flow of fractional Maxwell fluid between two cylinders, when both of them are oscillating around their common axis. In this paper, we are interested into the longitudinal oscillatory motion of a generalized second grade fluid between two infinite coaxial circular cylinders when both of them are oscillating along their common axis with given constant angular frequencies Ω_1 and Ω_2. Velocity field and associated tangential stress of the motion are determined by using Laplace and Hankel transforms and are presented under integral and series forms in terms of the generalized G and R functions. It is worthy to point out that the solutions that have been obtained satisfy the governing differential equation and all imposed initial and boundary conditions as well. The solutions corresponding to the ordinary second grade fluid and those for Newtonian fluid, performing same motion, are also determined as special cases of our general solutions. Furthermore, the respective solutions for the oscillatory motion between the cylinders, when one of them is at

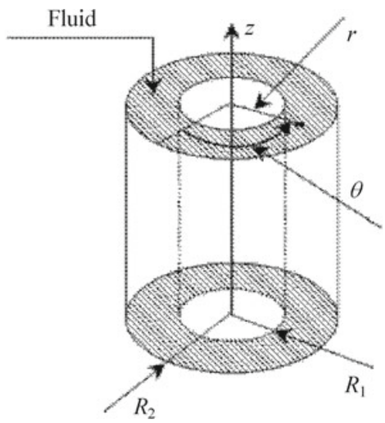

Fig. 1 Flow geometry

rest, can be obtained from our general solutions. Finally, the effects of different fractional and rheological parameters on the flow field are also examined by graphical illustrations.

2 Statement of the problem and governing equation

Consider an incompressible second grade or generalized second grade fluid at rest in the annular region between two infinite circular cylinders of radii R_1 and $R_2(> R_1)$, as shown in Fig. 1. At time $t = 0^+$, the cylinders suddenly begin to oscillate along their common axis ($r = 0$) with the velocities $V_1 \sin(\Omega_1 t)$ and $V_2 \sin(\Omega_2 t)$ of frequencies Ω_1 and Ω_2, respectively. Owing to the shear, the fluid between the cylinders is gradually moved, its velocity being of the form

$$\boldsymbol{v} = \boldsymbol{v}(r,t) = v(r,t)\boldsymbol{e}_z, \tag{7}$$

where $\boldsymbol{e}_z$ is the unit vector along z-axis of the cylindrical coordinates system r, θ and z. For such flows the constraint of incompressibility is automatically satisfied. The governing equation corresponding to such motions of second grade fluids, as it results from Ref. [17, Eq. (8.3)], is

$$\frac{\partial v(r,t)}{\partial t} = \left(\nu + \alpha\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)v(r,t),\quad r \in (R_1, R_2),\quad t > 0, \tag{8}$$

while the components of the extra-stress tensor are given by

$$\begin{aligned} S_{rr}(r,t) &= (2\alpha_1 + \alpha_2)\left[\frac{\partial v(r,t)}{\partial r}\right]^2,\\ S_{\theta\theta}(r,t) &= 0,\\ S_{zz}(r,t) &= \alpha_2\left[\frac{\partial v(r,t)}{\partial r}\right]^2,\\ \tau(r,t) = S_{rz}(r,t) &= \left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\frac{\partial v(r,t)}{\partial r},\\ S_{r\theta}(r,t) &= S_{\theta z}(r,t) = 0. \end{aligned} \tag{9}$$

In above equations μ is the viscosity, α_1 and α_2 are the normal stress moduli, $\nu = \mu/\rho$ is the kinematic viscosity, and $\alpha = \alpha_1/\rho$ (ρ being the constant density of the fluid). As regards the hydrostatic pressure p, it is obtained from the equalities [17, Eq. (8.1)]

$$\begin{aligned} \frac{\partial p}{\partial r} &= (2\alpha_1 + \alpha_2)\left[\frac{\partial v(r,t)}{\partial r}\frac{\partial^2 v(r,t)}{\partial r^2} + \frac{1}{r}\left(\frac{\partial v(r,t)}{\partial r}\right)^2\right]\\ &\quad + 2\alpha_1\frac{\partial v(r,t)}{\partial r}\frac{\partial^2 v(r,t)}{\partial r^2},\\ \frac{\partial p}{\partial \theta} &= \frac{\partial p}{\partial z} = 0, \end{aligned} \tag{10}$$

as soon as the velocity $v(r,t)$ is determined.

The governing equations corresponding to an incompressible generalized second grade fluid, performing the same motion, are (cf. [26, Eqs. (1) and (23)] with $\lambda = 0$)

$$\tau(r,t) = \left(\mu + \alpha_1 D_t^\beta\right)\frac{\partial v(r,t)}{\partial r}, \tag{11a}$$

$$\frac{\partial v(r,t)}{\partial t} = \left(\nu + \alpha D_t^\beta\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)v(r,t), \tag{11b}$$

where $r \in (R_1, R_2)$, $t > 0$.

Of course, the new material constants α_1 and α in Eq. (11) (although, for the sake of simplicity, we have kept the same notations) have the dimensions of μt^β and νt^β, respectively, and they reduce to the previous one for $\beta \to 1$. As regards Eq. (10), as well as other relations from Eq. (9), they remain unchanged in form both for ordinary and generalized second grade fluids, with the above understanding of dimensions. Consequently, the normal stress difference corresponding to the generalized second grade fluid is

$$S_{rr}(r,t) - S_{zz}(r,t) = 2\alpha_1\left[\frac{\partial v(r,t)}{\partial r}\right]^2,$$

where α_1 is the corresponding material constant. Thus α_1 and α have different significations for ordinary and generalized second grade fluids.

In the following, the fractional partial equation (11b) with the appropriate initial and boundary conditions

$$v(r,0) = 0,\quad r \in [R_1, R_2], \tag{12}$$

$$\begin{aligned} v(R_1,t) &= V_1\sin(\Omega_1 t),\quad \text{for } t \geq 0\\ v(R_2,t) &= V_2\sin(\Omega_2 t), \end{aligned} \tag{13}$$

will be solved by means of the Laplace and Hankel transforms. In order to avoid the burdensome calculations of residues and contour integrals, the discrete inverse Laplace transform method [26–29] will be used.

3 Calculation of the velocity field

Keeping in mind the initial condition (12), applying the Laplace transform to Eqs. (11b) and (13) and using the

Laplace transform formula for sequential fractional derivatives [12], we obtain the ordinary differential equation

$$\frac{\partial^2\bar{v}(r,q)}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{v}(r,q)}{\partial r}-\frac{q}{\alpha q^\beta+\nu}\bar{v}(r,q)=0,\quad r\in[R_1,R_2], \tag{14}$$

where the image function $\bar{v}(r,q)$ of $v(r,t)$ has to satisfy the conditions

$$\bar{v}(R_1,q)=\frac{V_1\Omega_1}{q^2+\Omega_1^2},\quad \bar{v}(R_2,q)=\frac{V_2\Omega_2}{q^2+\Omega_2^2}. \tag{15}$$

In the following, let us denote by

$$\bar{v}_n(q)=\int_{R_1}^{R_2} r\bar{v}(r,q)B_0(r\,r_n)\mathrm{d}r,\quad n=1,2,3,\ldots, \tag{16}$$

the finite Hankel transforms of $\bar{v}(r,q)$, where r_n are the positive roots of the transcendental equation $B_0(R_1\,r)=0$ and

$$B_0(r\,r_n)=\mathrm{J}_0(r\,r_n)\mathrm{Y}_0(R_2\,r_n)-\mathrm{J}_0(R_2\,r_n)\mathrm{Y}_0(r\,r_n). \tag{17}$$

In the above relation, $\mathrm{J}_0(\cdot)$ and $\mathrm{Y}_0(\cdot)$ are Bessel functions of order zero of the first and second kind, respectively. Applying the finite Hankel transform to Eq. (14) and taking into account the conditions (15), we find that [30]

$$\frac{2V_2\Omega_2}{\pi(q^2+\Omega_2^2)}-\frac{2V_1\Omega_1}{\pi(q^2+\Omega_1^2)}\frac{\mathrm{J}_0(R_2r_n)}{\mathrm{J}_0(R_1r_n)}-r_n^2\bar{v}_n(q)-\frac{q}{\alpha q^\beta+\nu}\bar{v}_n(q)=0, \tag{18}$$

or equivalently,

$$\bar{v}_n(q)=\frac{2V_2\Omega_2(\alpha q^\beta+\nu)}{\pi(q^2+\Omega_2^2)(\alpha r_n^2q^\beta+q+\nu r_n^2)}-\frac{2V_1\Omega_1(\alpha q^\beta+\nu)}{\pi(q^2+\Omega_1^2)(\alpha r_n^2q^\beta+q+\nu r_n^2)}\frac{\mathrm{J}_0(R_2r_n)}{\mathrm{J}_0(R_1r_n)}. \tag{19}$$

In order to determine $\bar{v}(r,q)$ from $\bar{v}_n(q)$, we apply inverse Hankel transform formula [30]

$$\bar{v}(r,q)=\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_n^2\mathrm{J}_0^2(R_1\,r_n)B_0(r\,r_n)}{\mathrm{J}_0^2(R_1\,r_n)-\mathrm{J}_0^2(R_2\,r_n)}\bar{v}_n(q) \tag{20}$$

to Eq. (19) and use Eq. (A1) from appendix. Furthermore, in order to avoid the burdensome calculations of residues and contour integrals, we apply the discrete inverse Laplace transform method. For this we write

$$\frac{1}{\alpha r_n^2q^\beta+q+\nu r_n^2}=\frac{1}{q^\beta[\nu r_n^2q^{-\beta}+(q^{1-\beta}+\alpha r_n^2)]}=\sum_{k=0}^{\infty}\frac{(-\nu r_n^2)^kq^{-\beta k-\beta}}{(q^{1-\beta}+\alpha r_n^2)^{k+1}}, \tag{21}$$

and use Eq. (A2), where [31]

$$G_{a,b,c}(d,t)=\sum_{j=0}^{\infty}\frac{(c)_jd^jt^{(j+c)a-b-1}}{j!\Gamma[(j+c)a-b]}, \tag{22}$$

is the generalized G function and $(c)_j$ is the Pochhammer polynomial [31].

Finally, Eqs. (19)–(22), Eq. (A5) and application of discrete inverse Laplace transform to $\bar{v}(r,q)$ give the velocity field

$$\begin{aligned}v(r,t)=&\frac{V_1\ln(R_2/r)\sin(\Omega_1t)+V_2\ln(r/R_1)\sin(\Omega_2t)}{\ln(R_2/R_1)}\\&-\pi\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}(-\nu r_n^2)^k\frac{\mathrm{J}_0(R_1r_n)B_0(rr_n)}{\mathrm{J}_0^2(R_1r_n)-\mathrm{J}_0^2(R_2r_n)}\\&\times\Bigg[V_2\Omega_2\mathrm{J}_0(R_1r_n)\int_0^t\cos\Omega_2(t-\tau)\\&\times G_{1-\beta,-\beta k-\beta,k+1}(-\alpha r_n^2,\tau)\mathrm{d}\tau\\&-V_1\Omega_1\mathrm{J}_0(R_2r_n)\int_0^t\cos\Omega_1(t-\tau)\\&\times G_{1-\beta,-\beta k-\beta,k+1}(-\alpha r_n^2,\tau)\mathrm{d}\tau\Bigg].\end{aligned} \tag{23}$$

4 Calculation of the shear stress

Applying the Laplace transform to Eq. (11a), we find that

$$\bar{\tau}(r,q)=(\mu+\alpha_1q^\beta)\frac{\partial\bar{v}(r,q)}{\partial r}, \tag{24}$$

where

$$\begin{aligned}\frac{\partial\bar{v}(r,q)}{\partial r}=&\frac{1}{r\ln(R_2/R_1)}\left(\frac{V_2\Omega_2}{q^2+\Omega_2^2}-\frac{V_1\Omega_1}{q^2+\Omega_1^2}\right)\\&+\pi\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}(-\nu r_n^2)^k\frac{r_n\mathrm{J}_0(R_1r_n)\widetilde{B}_0(rr_n)}{\mathrm{J}_0^2(R_1r_n)-\mathrm{J}_0^2(R_2r_n)}\\&\times\Bigg[V_2\Omega_2\mathrm{J}_0(R_1r_n)\frac{q}{q^2+\Omega_2^2}\frac{(k+1)_j(-\alpha r_n^2)^j}{j!\,q^{k+j(1-\beta)+1}}\\&-V_1\Omega_1\mathrm{J}_0(R_2r_n)\frac{q}{q^2+\Omega_1^2}\frac{(k+1)_j(-\alpha r_n^2)^j}{j!\,q^{k+j(1-\beta)+1}}\Bigg],\end{aligned} \tag{25}$$

has been obtained from Eq. (23) and Eq. (A8), where in the above relation

$$\widetilde{B}_0(rr_n)=\mathrm{J}_1(rr_n)\mathrm{Y}_0(R_2r_n)-\mathrm{J}_0(R_2r_n)\mathrm{Y}_1(rr_n).$$

Substituting Eq. (20) into Eq. (19), applying again the discrete inversion Laplace transform method to the obtained result and using Eq. (A3) and Eq. (A5), where

$$R_{a,b}(c,d,t)=\sum_{j=0}^{\infty}\frac{c^j(t-d)^{(j+1)a-b-1}}{\Gamma[(j+1)a-b]} \tag{26}$$

is R function [31], we find for the shear stress the following expression

$$\tau(r,t)=\frac{\mu\left[V_2\sin(\Omega_2 t)-V_1\sin(\Omega_1 t)\right]+\alpha_1\left[V_2\Omega_2R_{2,\beta}(-\Omega_2^2,0,t)-V_1\Omega_1R_{2,\beta}(-\Omega_1^2,0,t)\right]}{r\ln(R_2/R_1)}$$
$$+\pi\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}(-\nu r_n^2)^k\frac{r_n\mathrm{J}_0(R_1r_n)\widetilde{B}_0(rr_n)}{\mathrm{J}_0^2(R_1r_n)-\mathrm{J}_0^2(R_2r_n)}$$
$$\times\Bigg\{V_2\Omega_2\mathrm{J}_0(R_1r_n)\Bigg[\mu\int_0^t\cos\Omega_2(t-\tau)G_{1-\beta,-\beta k-\beta,k+1}(-\alpha r_n^2,\tau)\mathrm{d}\tau+\alpha_1G_{1-\beta,-\beta k-1,k+1}(-\alpha r_n^2,t)$$
$$-\alpha_1\Omega_2\int_0^t\sin\Omega_2(t-\tau)G_{1-\beta,-\beta k-1,k+1}(-\alpha r_n^2,\tau)\mathrm{d}\tau\Bigg]$$
$$-V_1\Omega_1\mathrm{J}_0(R_2r_n)\Bigg[\mu\int_0^t\cos\Omega_1(t-\tau)G_{1-\beta,-\beta k-\beta,k+1}(-\alpha r_n^2,\tau)\mathrm{d}\tau+\alpha_1G_{1-\beta,-\beta k-1,k+1}(-\alpha r_n^2,t)$$
$$-\alpha_1\Omega_1\int_0^t\sin\Omega_1(t-\tau)G_{1-\beta,-\beta k-1,k+1}(-\alpha r_n^2,\tau)\mathrm{d}\tau\Bigg]\Bigg\}. \tag{27}$$

5 Limiting cases

5.1 Solutions for ordinary second grade fluid ($\beta\to 1$)

Making the limit $\beta\to 1$ into Eqs. (23) and (27) and using Eqs. (A4) and (A7), we obtain the velocity field

$$v_{SG}(r,t)=\frac{V_1\ln(R_2/r)\sin(\Omega_1 t)+V_2\ln(r/R_1)\sin(\Omega_2 t)}{\ln(R_2/R_1)}$$
$$-\pi\sum_{n=1}^{\infty}\frac{\mathrm{J}_0(R_1r_n)B_0(rr_n)}{\mathrm{J}_0^2(R_1r_n)-\mathrm{J}_0^2(R_2r_n)}\Bigg\{\frac{V_2\Omega_2\mathrm{J}_0(R_1r_n)}{\nu^2r_n^4+\Omega_2^2(1+\alpha r_n^2)^2}$$
$$\times\left\{\nu r_n^2\left[\cos(\Omega_2 t)-\exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right)\right]\right.$$
$$\left.+\Omega_2(1+\alpha r_n^2)\sin(\Omega_2 t)\right\}-\frac{V_1\Omega_1\mathrm{J}_0(R_2r_n)}{\nu^2r_n^4+\Omega_1^2(1+\alpha r_n^2)^2}$$
$$\times\left\{\nu r_n^2\left[\cos(\Omega_1 t)-\exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right)\right]\right.$$
$$\left.+\Omega_1(1+\alpha r_n^2)\sin(\Omega_1 t)\right\}\Bigg\}, \tag{28}$$

and associated shear stress

$$\tau_{SG}(r,t)=\frac{\mu\left[V_2\sin(\Omega_2 t)-V_1\sin(\Omega_1 t)\right]+\alpha_1\left[V_2\Omega_2\cos(\Omega_2 t)-V_1\Omega_1\cos(\Omega_1 t)\right]}{r\ln(R_2/R_1)}$$
$$+\pi\sum_{n=1}^{\infty}\frac{r_n\mathrm{J}_0(R_1r_n)\widetilde{B}_0(rr_n)}{\mathrm{J}_0^2(R_1r_n)-\mathrm{J}_0^2(R_2r_n)}\left[\mathrm{J}_0(R_1r_n)g_2-\mathrm{J}_0(R_2r_n)g_1\right], \tag{29}$$

where, in above Eq. (29)

$$g_m=V_m\Omega_m\Bigg\{\frac{\mu}{\nu^2r_n^4+\Omega_m^2(1+\alpha r_n^2)^2}\left\{\nu r_n^2\left[\cos(\Omega_m t)\right.\right.$$
$$\left.\left.-\exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right)\right]+\Omega_m(1+\alpha r_n^2)\sin(\Omega_m t)\right\}$$
$$+\frac{\alpha_1}{1+\alpha r_n^2}\exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right)$$
$$-\frac{\alpha_1\Omega_m}{\nu^2r_n^4+\Omega_m^2(1+\alpha r_n^2)^2}\left\{\Omega_m(1+\alpha r_n^2)\left[\exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right)\right.\right.$$
$$\left.\left.-\cos(\Omega_m t)\right]+\nu r_n^2\sin(\Omega_m t)\right\}\Bigg\},\quad m=1,2,$$

corresponding to the ordinary or classical second grade fluid, performing the same motion.

5.2 Solutions for Newtonian fluid

Making $\alpha\to 0$ (equivalently $\alpha_1\to 0$) into Eqs. (28) and (29), velocity field and associated shear stress for

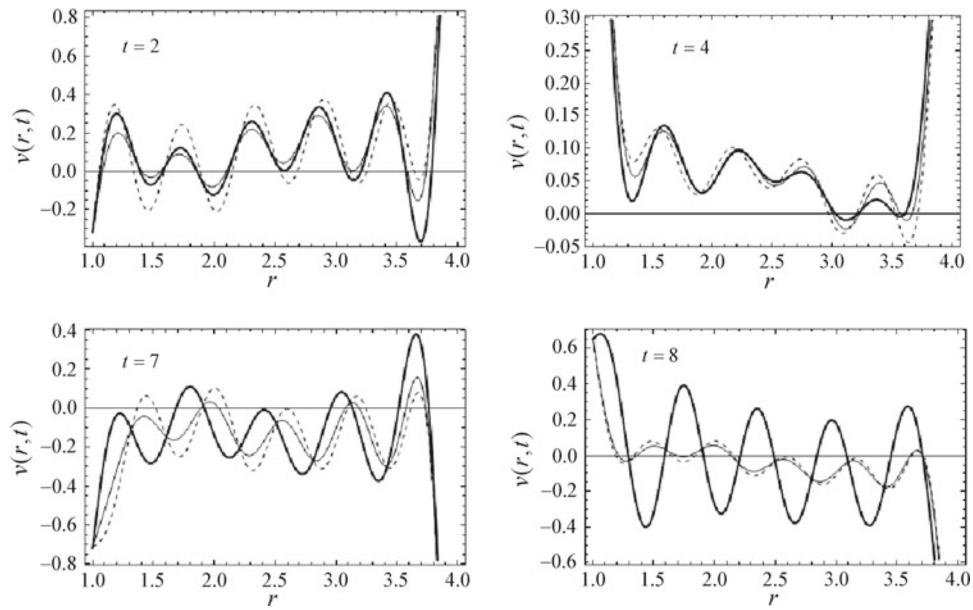

Fig. 2 Profiles of the velocity $v(r, t)$ versus r for different values of fractional parameter β (*bold solid curve* generalized second grade fluid ($\beta = 0.6$), *non-bold solid curve* generalized second grade fluid ($\beta = 0.9$) and *dashed curve* ordinary second grade fluid ($\beta = 1$)). The other parameters are chosen, respectively, as $R_1 = 1$, $R_2 = 4$, $V_1 = 1$, $V_2 = 4$, $\Omega_1 = 5$, $\Omega_2 = 7$, $\alpha = 0.009$ and $\nu = 1.1746 \times 10^{-3}$

Newtonian fluid, performing the same motion, can be obtained. For instance, the velocity field is

$$v_N(r,t) = \frac{V_1 \ln(R_2/r)\sin(\Omega_1 t) + V_2 \ln(r/R_1)\sin(\Omega_2 t)}{\ln(R_2/R_1)} - \pi \sum_{n=1}^{\infty} \frac{J_0(R_1 r_n) B_0(r r_n)}{J_0^2(R_1 r_n) - J_0^2(R_2 r_n)} \left\{ \frac{V_2 \Omega_2 J_0(R_1 r_n)}{\nu^2 r_n^4 + \Omega_2^2} \times \left[\nu r_n^2 \left[\cos(\Omega_2 t) - \exp(-\nu r_n^2 t)\right] + \Omega_2 \sin(\Omega_2 t)\right] - \frac{V_1 \Omega_1 J_0(R_2 r_n)}{\nu^2 r_n^4 + \Omega_1^2} \left[\nu r_n^2 \left(\cos(\Omega_1 t) - \exp(-\nu r_n^2 t)\right) + \Omega_1 \sin(\Omega_1 t)\right] \right\}, \tag{30}$$

6 Concluding remarks and numerical results

Our purpose in this paper was to establish exact solutions for the velocity field and shear stress corresponding to the flow of a generalized second grade fluid between infinite coaxial circular cylinders, by using Laplace and Hankel transforms. The motion of fluid was due to the simple harmonic sine oscillations of both cylinders along their common axis, with different angular frequencies Ω_1 and Ω_2 of their velocities. It is important to point out that the velocity field and the shear stress for the oscillatory motion between the cylinders, when one of them is at rest, can be obtained from our general solutions by making $V_1 = 0$, $V_2 = V$ and $\Omega_2 = \Omega$ (when inner cylinder is at rest) or $V_1 = V$, $V_2 = 0$ and $\Omega_1 = \Omega$ (when outer cylinder is at rest). For instance, the velocity field for the flow of generalized second grade fluid, when inner cylinder is at rest and outer cylinder is oscillating, is given by (from Eq. (23))

$$v(r,t) = \frac{V \ln(r/R_1)\sin(\Omega t)}{\ln(R_2/R_1)} - \pi V \Omega \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} (-\nu r_n^2)^k \frac{J_0^2(R_1 r_n) B_0(r r_n)}{J_0^2(R_1 r_n) - J_0^2(R_2 r_n)} \times \int_0^t \cos[\Omega(t-\tau)] G_{1-\beta,-\beta k-\beta,k+1}(-\alpha r_n^2, \tau) d\tau. \tag{31}$$

The solutions that have been obtained, presented under integral and series forms in terms of the generalized G and R functions, satisfy the governing equation and all imposed initial and boundary conditions and for $\beta \to 1$ reduce to the similar solutions for a second grade fluid. Furthermore, the solutions for the flow of Newtonian fluid for the similar flow between cylinders have also been recovered as special cases of our general solutions, when $\beta \to 1$ and $\alpha \to 0$.

Finally, we present the graphical results for the velocity field. We compare profiles of the velocity field for an ordinary or classical second grade fluid ($\beta \to 1$) and a generalized second grade fluid ($0 < \beta < 1$). We interpret these results with respect to the emerging parameters, especially the fractional parameter β. In making these graphical illustrations, we have used SI units for all the parameters and the roots r_n have been approximated by $n\pi/(R_2 - R_1)$ [32].

In Fig. 2, the profiles of the velocity corresponding to the unsteady oscillations of the ordinary second grade fluid and

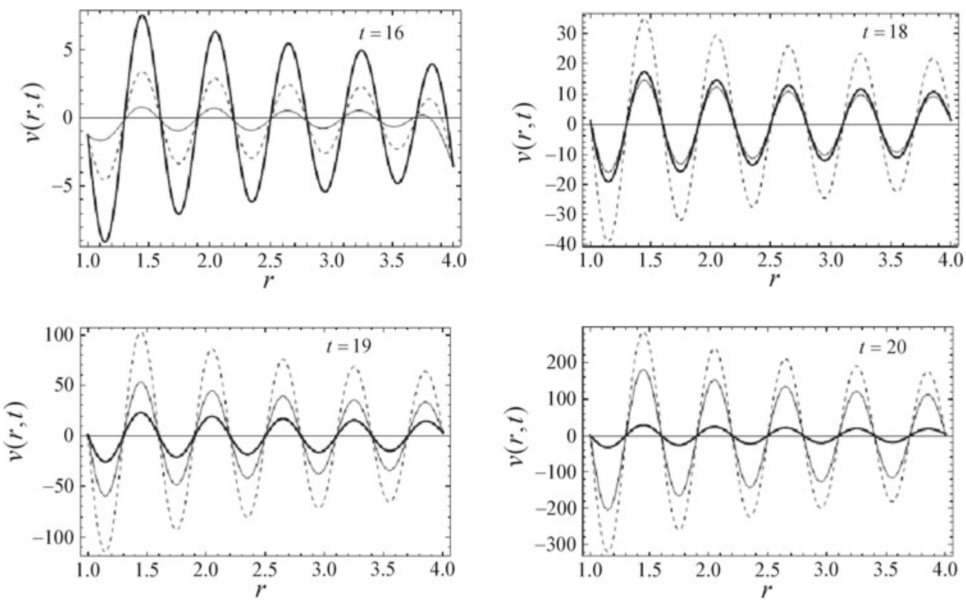

Fig. 3 Profiles of the velocity $v(r,t)$ versus r for relatively larger values of fractional parameter β (*bold solid curve* generalized second grade fluid ($\beta = 0.6$), *non-bold solid curve* generalized second grade fluid ($\beta = 0.9$) and *dashed curve* ordinary second grade fluid ($\beta = 1$)). The other parameters are chosen, respectively, as $R_1 = 1$, $R_2 = 4$, $V_1 = 1$, $V_2 = 4$, $\Omega_1 = 5$, $\Omega_2 = 7$, $\alpha = 0.009$ and $\nu = 1.1746 \times 10^{-3}$

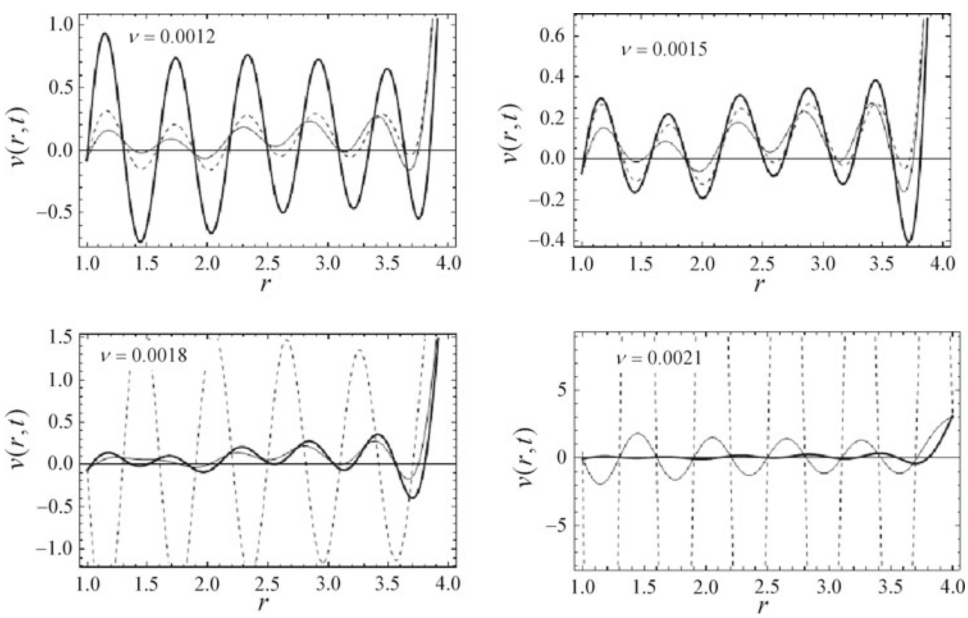

Fig. 4 Profiles of the velocity $v(r,t)$ versus r for different values of fractional parameter β and kinematic viscosity ν (*bold solid curve* generalized second grade fluid ($\beta = 0.6$), *non-bold solid curve* generalized second grade fluid ($\beta = 0.9$) and *dashed curve* ordinary second grade fluid ($\beta = 1$)). The other parameters are chosen, respectively, as $R_1 = 1$, $R_2 = 4$, $V_1 = 1$, $V_2 = 4$, $\Omega_1 = 5$, $\Omega_2 = 7$ and $\alpha = 0.009$

generalized second grade fluid are plotted at different values of time t. It is interesting to observe that the velocity profiles of ordinary second grade fluid and generalized second grade fluid (for differen values of β) show a nonlinear behavior with respect to the fractional parameter β in the sense that they do not remain in phase for smaller values of time. Furthermore, it is also observed from Fig. 2 that effect of fractional parameter β on the flow field is non-monotonous. More exactly, for some values of time, the velocity profiles corresponding to smaller values of β is smaller and for other values of time they are larger. A similar phenomenon is observed for larger values of β.

Figure 3 also shows the velocity profiles of the flow but for relatively larger values of time. It is noted that profiles corresponding to ordinary second grade fluid and generalized second grade fluid come into phase after some time and then remain in phase forever. But the interesting fact which has been observed that the flow field remains non-monotonous with respect to β for all values of time. This shows that the flow is strong function of the fractional parameter β.

In Fig. 4, the behavior of the velocity profiles is examined for different values of kinematic viscosity ν. We can see that a considerable increase in the fluctuation of velocity profiles is observed for ordinary second grade fluid when a small increase in the value of ν takes place. That is, velocity of ordinary second grade fluid increases with increase in the value of kinematic viscosity ν. On the other hand, the

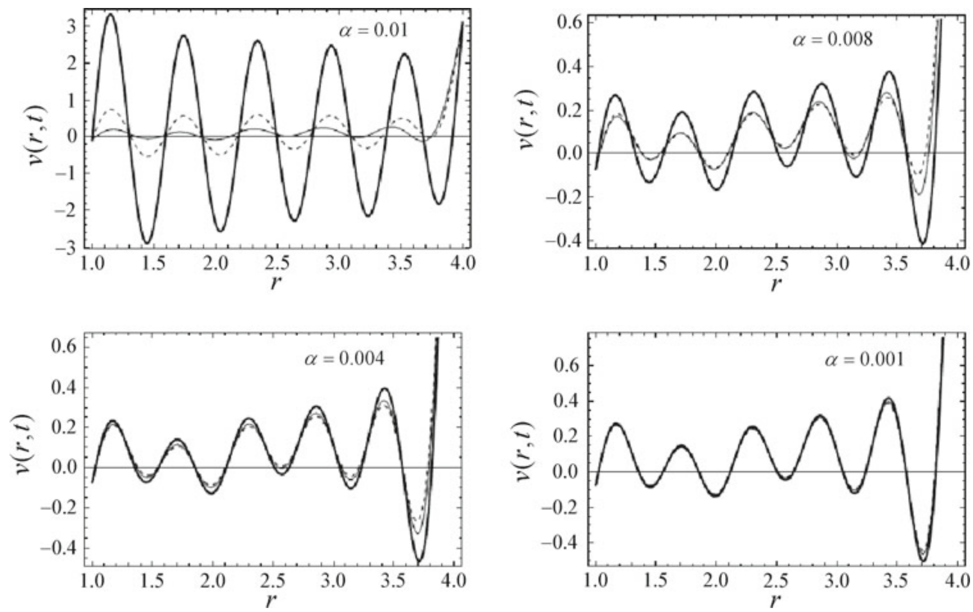

Fig. 5 Profiles of the velocity $v(r, t)$ versus r for different values of fractional parameter β and material constant α (*bold solid curve* generalized second grade fluid ($\beta = 0.6$), *non-bold solid curve* generalized second grade fluid ($\beta = 0.9$) and *dashed curve* ordinary second grade fluid ($\beta = 1$)). The other parameters are chosen, respectively, as $R_1 = 1$, $R_2 = 4$, $V_1 = 1$, $V_2 = 4$, $\Omega_1 = 5$, $\Omega_2 = 7$ and $\nu = 1.1746 \times 10^{-3}$

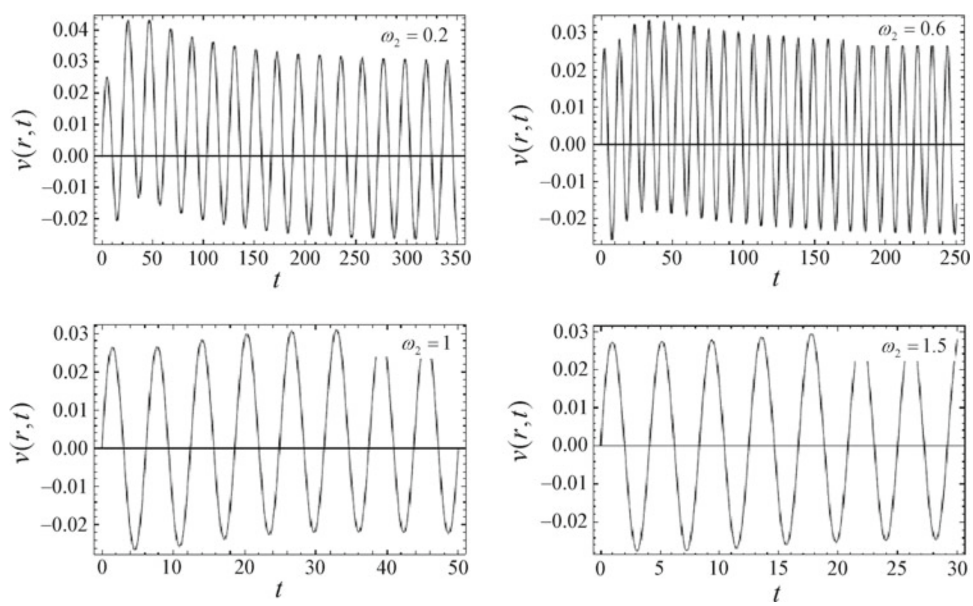

Fig. 6 Profiles of the velocity $v(r, t)$ versus t for different values of frequency of the inner cylinder ω_2. The other parameters are chosen, respectively, as $R_1 = 1$, $R_2 = 4$, $V_1 = 0$, $V_2 = 1$, $\beta = 1$, $\alpha = 0.009$ and $\nu = 1.1746 \times 10^{-3}$

velocity corresponding to the generalized second grade fluid decreases with the increase in ν.

Figure 5 shows the effect of the material constant α on velocity profiles. It is observed that by decreasing magnitude of α, the velocity becomes smaller and smaller. Furthermore, on decreasing α, the profiles of ordinary second grade fluid and generalized second grade fluid corresponding to the different values of fractional parameter β tend to superpose, which shows that for smaller values of material constant α, the flows of generalized and ordinary second grade fluids behave alike.

Figure 6 depicts the behavior of the time series of the velocity profiles for different values of frequency of inner cylinder ω_2. It is clearly observed that the time elapsed between transient motion and the steady-state motion decreases when the frequency of inner cylinder is increased and hence the steady-state regime is achieved earlier for larger values of frequency of inner cylinder. A similar behavior is observed for the frequency of outer cylinder, however figures are not included here for this case.

Appendix

Some results used in the text:

The finite Hankel transform of the function

$$a(r) = \frac{A \ln(R_2/r) + B \ln(r/R_1)}{\ln(R_2/R_1)}, \tag{A1}$$

satisfying $a(R_1) = A$ and $a(R_2) = B$ is

$$a_n = \int_{R_1}^{R_2} ra(r)\, B_0(rr_n)\mathrm{d}r = \frac{2B}{\pi r_n^2} - \frac{2A}{\pi r_n^2}\frac{\mathrm{J}_0(R_2 r_n)}{\mathrm{J}_0(R_1 r_n)}.$$

In order to prove Eq. (A1), we integrate by parts and use the next identities

$$\int \mathrm{J}_1(u)\mathrm{d}u = -\mathrm{J}_0(u),$$

$$\mathrm{J}_1(R_1 r_n)\mathrm{Y}_0(R_1 r_n) - \mathrm{J}_0(R_1 r_n)\mathrm{Y}_1(R_1 r_n) = \frac{2}{\pi R_1 r_n}$$

and

$$\mathrm{J}_1(R_2 r_n)\mathrm{Y}_0(R_2 r_n) - \mathrm{J}_0(R_2 r_n)\mathrm{Y}_1(R_2 r_n) = \frac{2}{\pi R_2 r_n},$$

if $B_0(R_1 r_n) = 0$.

$$L^{-1}\left[\frac{q^b}{(q^a - d)^c}\right] = G_{a,\,b,\,c}(d,t);$$

$$Re(ac-b) > 0, \qquad Re(q) > 0, \qquad \left|\frac{\mathrm{d}}{q^a}\right| < 1. \tag{A2}$$

$$L^{-1}\left(\frac{e^{-dq}q^b}{q^a - c}\right) = R_{a,\,b}(c,d,t) = \sum_{j=0}^{\infty}\frac{c^j(t-d)^{(j+1)a-b-1}}{\Gamma[(j+1)a-b]};$$

$$d \geq 0, \qquad Re[(j+1)a-b] > 0, \qquad Re(q) > 0. \tag{A3}$$

$$R_{2,\,1}(-a^2, 0, t) = \cos(at). \tag{A4}$$

If $u_1(t) = L^{-1}[\overline{u}_1(q)]$ and $u_2(t) = L^{-1}[\overline{u}_2(q)]$ then

$$L^{-1}[\overline{u}_1(q)\overline{u}_2(q)] = u_1 * u_2(t) = \int_0^t u_1(t-s)u_2(s)\mathrm{d}s = \int_0^t u_1(s)u_2(t-s)\mathrm{d}s, \tag{A5}$$

$$\frac{1}{z+a} = \sum_{k=1}^{\infty}(-1)^k\frac{z^k}{a^{k+1}}, \tag{A6}$$

$$\sum_{k=0}^{\infty}(-\nu r_n^2)^k G_{0,-k-1,k+1}(-\alpha r_n^2, t) = \sum_{k=0}^{\infty}(-\nu r_n^2)^k \sum_{j=0}^{\infty}\frac{(k+1)_j(-\alpha r_n^2)^j}{j\,!}\frac{t^k}{k\,!} = \sum_{k=0}^{\infty}(-\nu r_n^2)^k\frac{1}{(1+\alpha r_n^2)^{k+1}}\frac{t^k}{k\,!} = \frac{1}{1+\alpha r_n^2}\sum_{k=0}^{\infty}\frac{1}{k\,!}\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right)^k = \frac{1}{1+\alpha r_n^2}\exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right), \tag{A7}$$

$$\frac{\mathrm{d}}{\mathrm{d}u}[\mathrm{Y}_0(u)] = -\mathrm{Y}_1(u), \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}u}[\mathrm{J}_0(u)] = -\mathrm{J}_1(u). \tag{A8}$$

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References

1. Caputo, M., Mainardi, F.: A new dissipation model based on memory mechanism. Pure Appl. Geophys. **91**, 134–147 (1971)
2. Slonimsky, G.L.: On the law of deformation of highly elastic polymeric bodies. Dokl. Akad. Nauk BSSR **140**, 343–346 (1961)
3. Stiassnie, M.: On the application of fractional calculus for the formulation of viscoelastic models. Appl. Math. Modell. **3**, 300–302 (1979)
4. Mainardi, F.: Applications of fractional calculus in mechanics. In: Rusev, P., Dimovschi, I., Kiryakova, V. (eds.) Transform Methods and Special Functions, Varna'96, pp. 309–334. Bulgarian Academy of Sciences, Sofia (1998)
5. Bagley, R.L., Torvik, P.J.: A theoretical basis for the application of fractional calculus to viscoelastisity. J. Rheol. **27**, 201–210 (1983)
6. Bagley, R.L., Torvik, P.J.: On the fractional calculus model of viscoelastic behavior. J. Rheol. **30**, 133–155 (1986)
7. Rogers, L.: Operators and fractional derivatives for viscoelastic constitutive equations. J. Rheol. **27**, 351–372 (1983)
8. Koeller, R.C.: Applications of fractional calculus to the theory of viscoelasticity. Trans. ASME J. Appl. Mech. **51**, 299–307 (1984)
9. Xu, M., Tan, W.: Theoretical analysis of the velocity field, stress field and vortex sheet of generalized second order fluid with fractional anomalous diffusion. Sci. China Ser. A **44**, 1387–1399 (2001)
10. Xu, M., Tan, W.: The representation of the constitutive equation of viscoelastic materials by the generalized fractional element networks and its generalized solutions. Sci. China Ser. G **46**, 145–157 (2003)
11. Debnath, L.: Recent applications of fractional calculus to science and engineering. Int. J. Math. Math. Sci. **54**, 3413–3442 (2003)

12. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
13. Rouse, P.E.: The theory of the linear viscoelastic properties of dilute solutions of coiling polymers. J. Chem. Phys. **21**, 1272–1280 (1953)
14. Ferry, J.D., Landel, R.F., Williams, M.L.: Extensions of the Rouse theory of viscoelastic properties to undiluted linear polymers. J. Appl. Phys. **26**, 359–362 (1955)
15. Stokes, G.G.: On the Effect of the Rotation of Cylinders and Spheres About Their Own Axes in Increasing the Logarithmic Decrement of the Arc of Vibration. pp. 204–217. Cambridge University Press, Cambridge (1886)
16. Casarella, M.J., Laura, P.A.: Drag on oscillating rod with longitudinal and torsional motion. J. Hydronaut. **3**, 180–183 (1969)
17. Rajagopal, K.R.: Longitudinal and torsional oscillations of a rod in a non-Newtonian fluid. Acta Mech. **49**, 281–285 (1983)
18. Rajagopal, K.R., Bhatnagar, R.K.: Exact solutions for some simple flows of an Oldroyd-B fluid. Acta Mech. **113**, 233–239 (1995)
19. Khan, M., Asghar, S., Hayat, T.: Oscillating flow of a Burgers' fluid in a pipe. The Abdus Salam International Center for Theoretical Physics, IC/2005/071
20. Fetecau, C., Fetecau, C.: Starting solutions for the motion of a second grade fluid due to longitudinal and torsional oscillations of a circular cylinder. Int. J. Eng. Sci. **44**, 788–796 (2006)
21. Mahmood, A., Parveen, S., Ara, A. et al.: Exact analytic solutions for the unsteady flow of a non-Newtonian fluid between two cylinders with fractional derivative model. Commun. Nonlinear Sci. Numer. Simulat. **14**, 3309–3319 (2009)
22. Vieru, D., Akhtar, W., Fetecau, C. et al.: Starting solutions for the oscillating motion of a Maxwell fluid in cylindrical domains. Meccanica **42**, 573–583 (2007)
23. Fetecau, C., Hayat, T., Fetecau, C.: Starting solutions for oscillating motions of Oldroyd-B fluids in cylindrical domains. J. Non-Newtonian Fluid Mech. **153**, 191–201 (2008)
24. Massoudi, M., Phuoc, T.X.: On the motion of a second grade fluid due to longitudinal and torsional oscillations of a cylinder: a numerical study. Appl. Math. Comput. **203**(4), 471–481 (2008)
25. Khan, M., Ali, S.H., Qi, H.: Exact solutions of starting flows for a fractional Burgers' fluid between coaxial cylinders. Nonlinear Anal. Real World Appl. **10**(3), 1775–1783 (2009)
26. Tong, D., Wang, R., Yang, H.: Exact solutions for the flow of non-Newtonian fluid with fractional derivative in an annular pipe. Sci. China Ser. G **48**, 485–495 (2005)
27. Tan, W.C., Xu, M.Y.: The impulsive motion of flat plate in a generalized second grade fluid. Mech. Res. Commun. **29**, 3–9 (2002)
28. Shen, F., Tan, W.C., Zhao, Y. et al.: The Rayleigh–Stokes problem for a heated generalized second grade fluid with fractional derivative model. Non-Linear Anal. Real World Appl. **7**, 1072–1080 (2006)
29. Fetecau, C., Mahmood, A., Fetecau, C. et al.: Some exact solutions for the helical flow of a generalized Oldroyd-B fluid in a circular cylinder. Comput. Math. Appl. **56**, 3096–3108 (2008)
30. Sneddon, I.N.: Functional Analysis in: Encyclopedia of Physics, vol. II. Springer, Berlin (1955)
31. Lorenzo. C.F., Hartley, T.T.: Generalized functions for the fractional calculus. NASA/TP-1999-209424/Rev1 (1999)
32. Debnath, L., Bhatta, D.: Integral Transforms and Their Applications, 2nd edn. Chapman & Hall/CRC, London (2007)