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TECHNICAL NOTE

Singular analysis of bifurcation systems with two parameters

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Abstract Bifurcation properties of dynamical systems with two parameters are investigated in this paper. The definition of transition set is proposed, and the approach developed is used to investigate the dynamic characteristic of the nonlinear forced Duffing system with nonlinear feedback controller. The whole parametric plane is divided into several persistent regions by the transition set, and then the bifurcation diagrams in different persistent regions are obtained.

Keywords Two parameters · Bifurcation · Singular analysis

1 Introduction

Since the powerful tool of singularity theory was introduced to the study of bifurcation problems by Golubitsky and Schaeffer [1] in 1980s, the singularity theory has been greatly developed. In 1980, Golubitsky and Keyfitz [2] studied the steady state solutions for a continuous flow stirred tank chemical reactor by the singularity theory and obtained the qualitative bifurcation diagrams. In 1981, Schaeffer and Golubitsky [3] applied the singularity theory to investigate the bifurcation phenomena of a model chemical reaction. In 1988, Chen and Colleagues [4,5] firstly proposed a new method, C–L method, to study the bifurcation of the periodic solution of nonlinear dynamical equation undergoing parameter excitation by combining the L–S method and the singularity theory. It was proved in Ref. [4] that in different parametric regions there were six kinds of bifurcation pattern of response and successfully unified the inconsistent results in academic world. In 1992, Jin and Matsuzaki [6] investigated the local behavior of the double pendulum with a follower force. In 1996, Bi and Chen [7] studied the universal unfolding of a nonlinear vibration mill and obtained all kinds of dynamical phenomena of the system. In 2003, Jin and Zou [8] applied singularity theory to a restrained pipe conveying fluid and obtained the dynamical behavior in different persistent regions. In the last few years, the singularity theory has been employed to study the dynamic behavior in rotary machine [9,10]. Up to now the singularity theory has mainly been applied to the dynamical systems with one bifurcation parameter. In actual systems, however, such as chemical system and power system, there are many physical parameters and some parameters are of the same importance, i.e. the change of each important parameter may cause the change of the dynamical behavior—bifurcation. Therefore these parameters may be considered as bifurcation parameters. This demands further study of bifurcation properties of the multiparameter system. In 1993, Lari-Lavassani and Lu [11] gave a full and accessible account of unfolding, finite determinacy and stability theorems of multiparameter bifurcation problem using the singularity theory. In 2003, Gao [12] studied the unfolding of multiparameter equivalent bifurcation problems with respect to left-right equivalence and obtained the $A(\Gamma)$-universal unfolding theorem. In 2005, Guo and Li [13] discussed the unfolding of a multiparameter equivariant bifurcation problem with two groups of state variables with respect to left-right equivalence and obtained the necessary and sufficient condition for an universal unfolding. In this paper, the definition of transition set is proposed and the bifurcation properties of the system with two parameters are studied.

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2 Singular analysis of bifurcation systems with two bifurcation parameters

The universal unfolding theory of multiparameter bifurcation system with multidimension has been developed by Li [14]. Here, it is assumed that the universal unfolding of the bifurcation system is

$$G = G(x, \lambda_1, \lambda_2, \alpha) = 0, \tag{1}$$

where $x \in \mathbf{R}$ is the state variable, $\lambda_1 \in \mathbf{R}$ and $\lambda_2 \in \mathbf{R}$ are bifurcation parameters, and $\alpha \in \mathbf{R}^k$ is unfolding parameter. The conditions satisfying the bifurcation, hysteresis and double limit point set are discussed respectively as follows.

2.1 Bifurcation set

The bifurcation set B consists of those α for which the curve (1) contains a singular point. It follows from the implicit function theorem that the bifurcation set satisfies

$$G_x = G_{\lambda_1} = G_{\lambda_2} = 0, \tag{2}$$

Therefore, the definition of the bifurcation set is given as

$$B = \left\{\alpha \in \mathbf{R}^k : \exists(x, \lambda_1, \lambda_2)\text{s.t.} G = G_x = G_{\lambda_1} = G_{\lambda_2} = 0\right\}. \tag{3}$$

2.2 Double limit point set

The definition of double limit set is that there are two different limit points x_1 and x_2 for the same λ. Thus the double limit point set satisfies

$$G(x_i, \lambda_1, \lambda_2) = G_{x_i}(x_i, \lambda_1, \lambda_2) = 0, \quad i = 1, 2, \tag{4}$$

where $x_1 \neq x_2$.

Additionally, from Fig. 1 we can see that the normal vectors at the two limit points are parallel. Thus the double limit point set is defined as

$$D = \left\{\alpha \in \mathbf{R}^k : \exists(x, \lambda_1, \lambda_2)\text{s.t.} x_1 \neq x_2, G = G_x = 0, \frac{G_{\lambda_1}(x_1, \lambda_1, \lambda_2)}{G_{\lambda_1}(x_2, \lambda_1, \lambda_2)} = \frac{G_{\lambda_2}(x_1, \lambda_1, \lambda_2)}{G_{\lambda_2}(x_2, \lambda_1, \lambda_2)}\right\}. \tag{5}$$

2.3 Hysteresis set

The hysteresis set H consists of those α for which the curve (1) makes at least quadratic contact with the vertical plane $\{\lambda = \text{const}\}$. According to the definition of quadratic contact, a portion of curve (1) is parameterized as

$$\phi(t) = (x(t), \lambda_1(t), \lambda_2(t)). \tag{6}$$

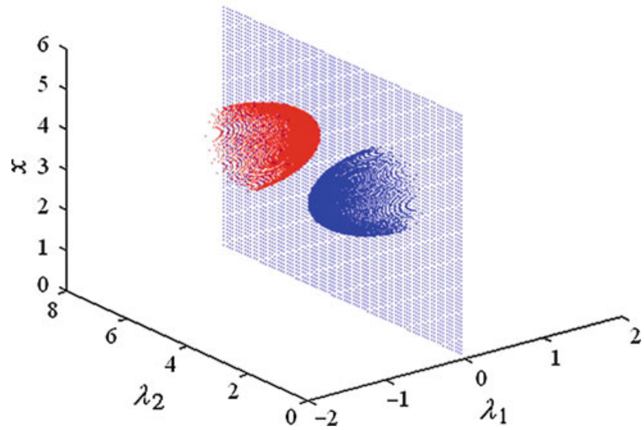

Fig. 1 Double limit point set

It is said that ϕ makes quadratic contact with the vertical plane $\{\lambda = \text{const}\}$ at $t = 0$ if the second and third components in Eq. (6) satisfy

$$\lambda_i'(0) = 0, \quad \lambda_i''(0) = 0, \quad i = 1, 2. \tag{7}$$

(To avoid trivialities, $\phi'(0) \neq 0$ is imposed.) Necessary conditions for quadratic contact are derived as follows. Differentiating the identity $G = 0$ with respect to t and evaluating at $t = 0$, the following equation is obtained

$$G_x = G_{xx} = 0. \tag{8}$$

Now the different cases of hysteresis set are discussed, respectively.

Case A: when λ_2 is fixed and λ_1 is changed, α can be denoted as a function of λ_2 from Eq. (8). Thus the hysteresis set H_1 satisfies

$$H_1 = \left\{\alpha \in \mathbf{R}^k : \exists(x, \lambda_1, \lambda_2)\text{s.t.} G = G_x = G_{xx} = 0, \text{ and } \alpha \text{ not change with } \lambda_2\right\}. \tag{9}$$

If $H_1 = \emptyset$, α will change with λ_2. Then the hysteresis set is not yet a parametric curve, but a parametric plane. If α is

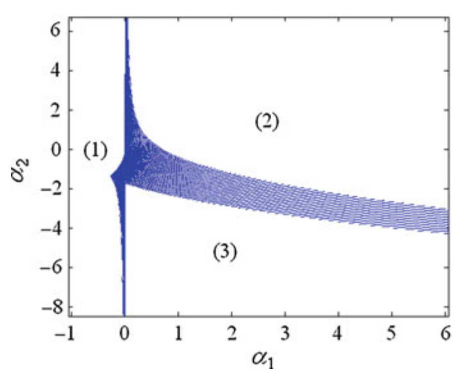

Fig. 2 Hysteresis set of system (18)

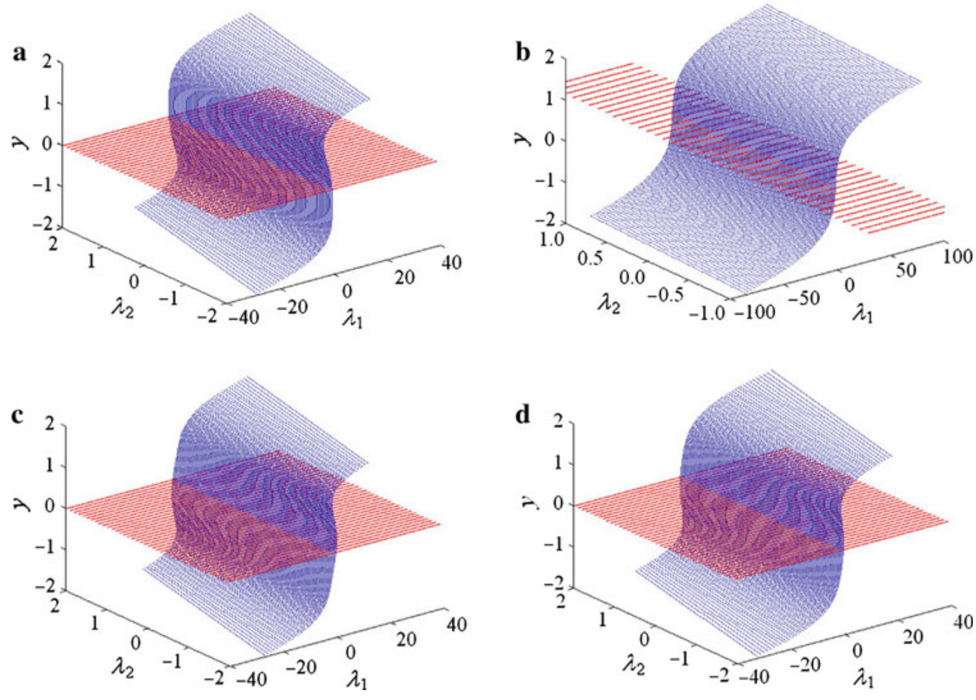

Fig. 3 Bifurcation diagrams in each region. **a** Region (1), **b** region (2), **c** region (3), **d** hysteresis set

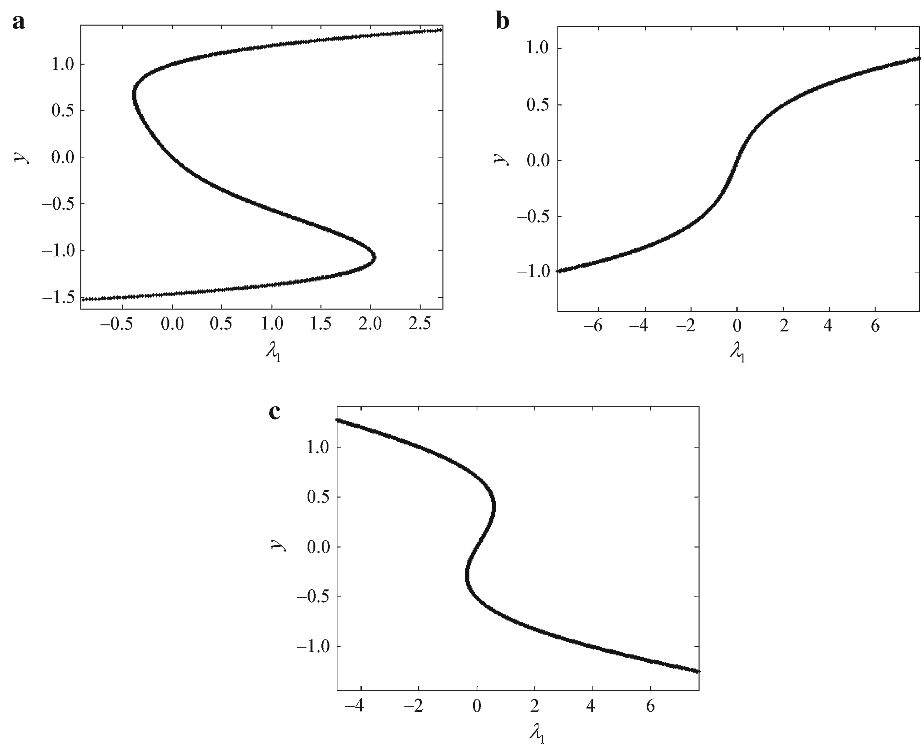

Fig. 4 $x - \lambda_1$ sections of 3D-diagrams in regions (1)–(3)

chosen in the plane, the corresponding λ_2 can be calculated. It is said that critical hysteresis can only occur with λ_1 at some fixed λ_2.

Case B: The same as case A, when λ_1 is fixed and λ_2 is changed, α can be denoted as a function of λ_1 from Eq. (8). Thus the hysteresis set H_2 satisfies

$$H_2 = \Big\{\alpha \in \mathbf{R}^k : \exists(x, \lambda_1, \lambda_2)\text{s.t.}G = G_x = G_{xx} = 0, \text{ and } \alpha \text{ not change with } \lambda_1\Big\}. \quad (10)$$

If $H_2 = \emptyset$, α will change with λ_1. Then the hysteresis set is also not a parametric curve, but a parametric plane. If α is

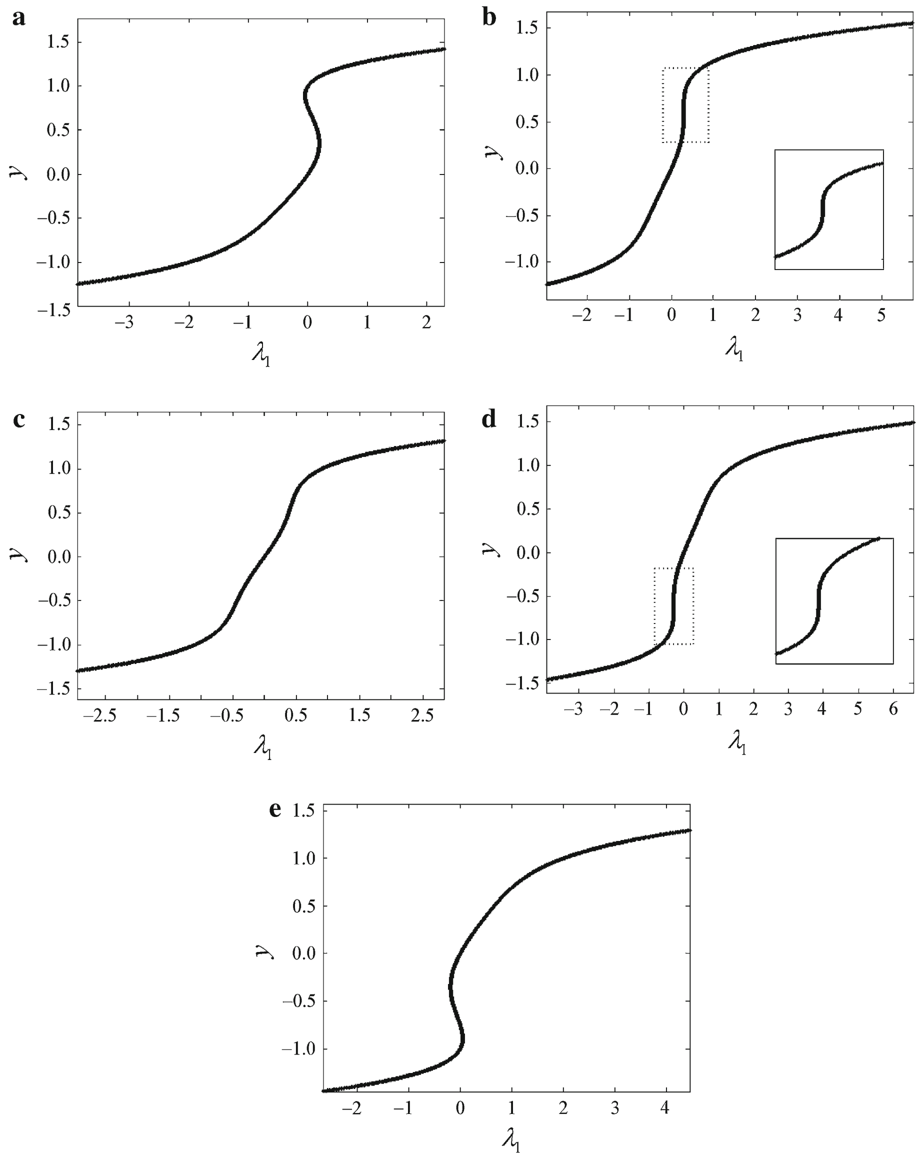

Fig. 5 $x - \lambda_1$ sections of 3D-diagrams in hysteresis set for different λ_2. **a** $\lambda_2 = -1$, **b** $\lambda_2 = -0.4720$, **c** $\lambda_2 = -0.1$, **d** $\lambda_2 = 0.4720$, **e** $\lambda_2 = 1$

chosen in the plane, the corresponding λ_1 can be calculated. It is said that critical hysteresis can only occur with λ_2 at some fixed λ_1.

Therefore, the definition of hysteresis set is given as

$$H = H_1 \cap H_2. \tag{11}$$

As in one parameter, transition set $\sum$ is a union of the three subsets.

$$\sum = B \cup H \cup D. \tag{12}$$

3 A typical example

Now, as a typical example, consider the nonlinear forced Duffing system with nonlinear feedback controller [15]

$$\ddot{x} + \omega_0^2 x + \varepsilon(2\mu\dot{x} + \alpha x^3) = \varepsilon f \cos\Omega t + 2\varepsilon k x^2 \cos\Omega t, \tag{13}$$

where ω_0 is the natural frequency, ε is a small parameter, μ is the damping coefficient, f and Ω are the amplitude and frequency of external excitation, respectively, and k is the control coefficient.

For main resonance analysis, the frequency of external excitation can be written as

$$\Omega = \omega_0 + \varepsilon\sigma, \tag{14}$$

where σ is detuning parameter. Using multiple scale method, the bifurcation equation of system (13) is obtained as [15]

$$\frac{\mu^2\omega_0^2 a^2}{(f/2 + 1/4ka^2)^2} + \frac{(\omega_0\sigma a - 3/8\alpha a^3)^2}{(f/2 + 3/4ka^2)^2} = 1. \tag{15}$$

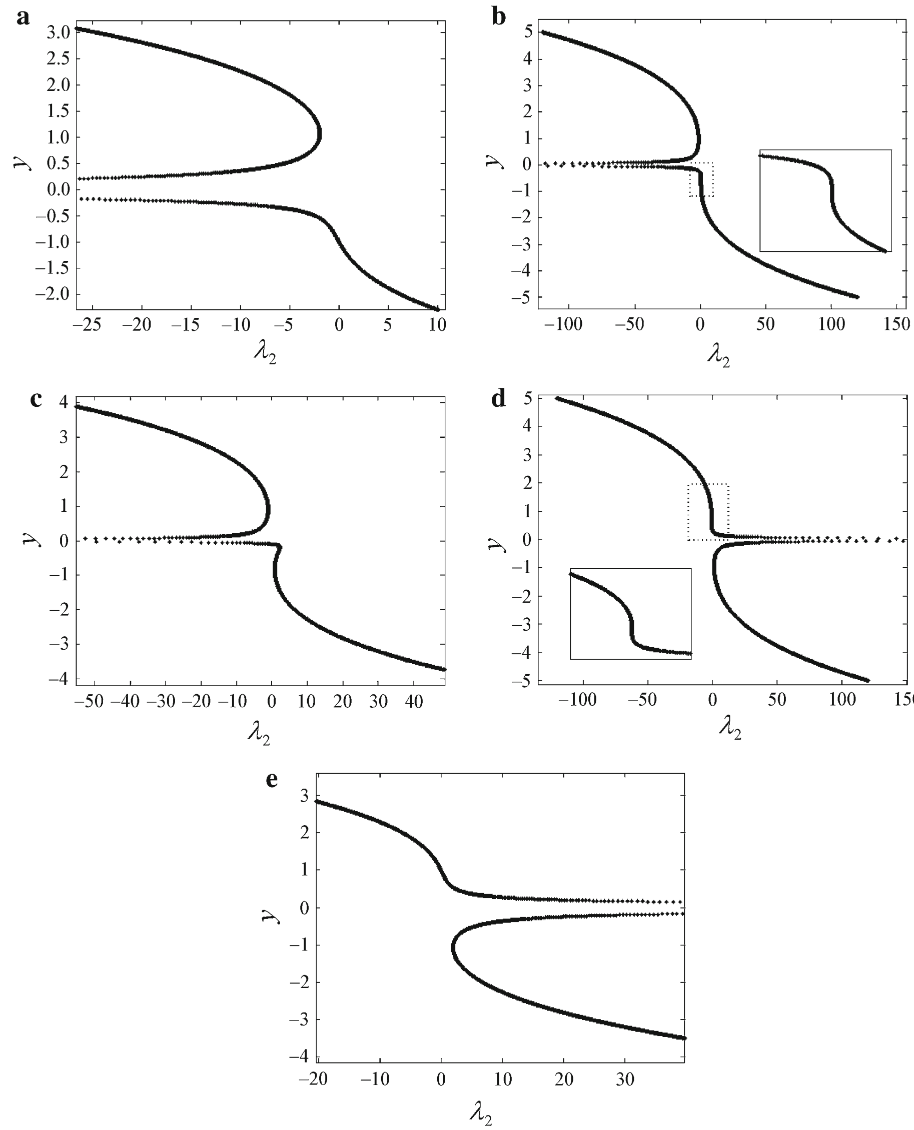

Fig. 6 $x - \lambda_2$ sections of 3D-diagrams in hysteresis set for different λ_1. **a** $\lambda_2 = -1$, **b** $\lambda_2 = -0.2918$, **c** $\lambda_2 = -0.1$, **d** $\lambda_2 = 0.2918$, **e** $\lambda_2 = 1$

Letting

$$r = a^2, \quad b_0 = \frac{9k^2\alpha^2}{1024},$$

$$b_1 = \frac{9fk\alpha^2}{256\varepsilon} - \frac{3k^2\alpha\sigma\omega_0}{64} - \frac{9k^4}{256},$$

$$b_2 = \frac{9f^2\alpha^2}{256} - \frac{3kf\alpha\sigma\omega_0}{16} + \frac{9k^2\mu^2\omega_0^2}{16} + \frac{k^2\sigma^2\omega_0^2}{16} - \frac{3fk^3}{16},$$

$$b_3 = -\frac{3f^2\alpha\sigma\omega_0}{16} + \frac{3kf\mu^2\omega_0^2}{4} + \frac{fk\sigma^2\omega_0^2}{4} - \frac{11f^2k^2}{32},$$

$$b_4 = \frac{f^2\mu^2\omega_0^2}{4} + \frac{f^2\sigma^2\omega_0^2}{4} - \frac{f^3k}{4}, \; b_5 = -\frac{f^4}{16},$$

Equation (15) is transformed into

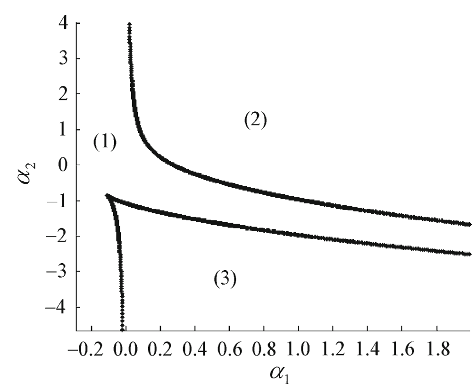

Fig. 7 Transition set of system (18) for fixed λ_2

$$b_0r^5 + b_1r^4 + b_2r^3 + b_3r^2 + b_4r + b_5 = 0. \qquad (16)$$

After taking the linear transformation $r = y - \frac{b_1}{5b_0}$ and letting $h_i = \frac{b_i}{b_0}$, $(i = 1, 2, \ldots, 5)$, Eq. (16) becomes

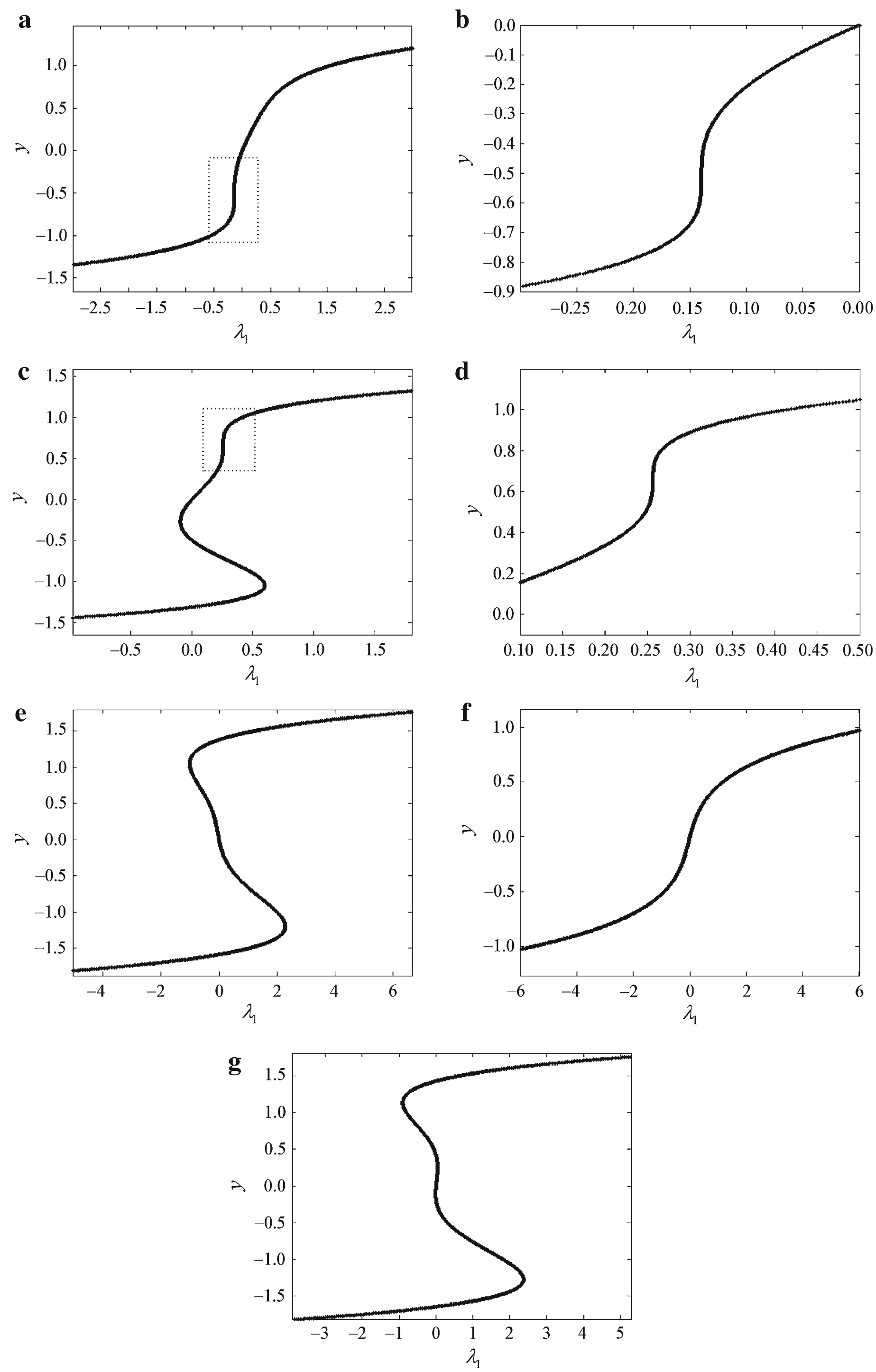

Fig. 8 Bifurcation diagrams in each region of system (18) for fixed λ_2. **a** Up branch of hysteresis set, **b** enlarged view of **a**, **c** low branch of hysteresis set, **d** enlarged view of **c**, **e** region (1), **f** region (2), **g** region (3)

$$y^5 - \lambda_1 + \lambda_2 y^2 + \alpha_1 y + \alpha_2 y^3 = 0, \tag{17}$$

where

$$\lambda_1 = -\frac{4h_1^5}{3125} + \frac{h_1^3 h_2}{125} - \frac{h_1^2 h_3}{25} + \frac{h_1 h_4}{5} - h_5,$$

$$\lambda_2 = \frac{4h_1^3}{25} - \frac{3h_1 h_2}{5} + h_3,$$

$$\alpha_1 = -\frac{2h_1^2}{5} + h_2,$$

$$\alpha_2 = -\frac{3h_1^4}{125} + \frac{3h_1^2 h_2}{25} - \frac{2h_1 h_3}{5} + h_4$$

G is denoted as

$$G = y^5 - \lambda_1 + \lambda_2 y^2 + \alpha_1 y + \alpha_2 y^3. \tag{18}$$

Then G can be considered as an unfolding of $g = y^5 - \lambda_1 + \lambda_2 y^2$.

It follows from the definition of transition set that the bifurcation set and hysteresis set are both empty. Thus the hysteresis set will change with λ_1 and λ_2. And for a set of (α_1, α_2), x will appear critical hysteresis only at some fixed λ_1 or λ_2. The hysteresis set which changes with λ_1 and λ_2 is shown in Fig. 2.

From Fig. 2, we can see that the whole parametric plane is divided by the hysteresis set into three different persistent regions. The bifurcation diagrams in different regions are shown in Fig. 3.

Additionally, the $x-\lambda_1$ sections of 3D-diagrams in regions (1)–(3) are shown in Fig. 4.

From Fig. 4 we can see that if the unfolding parameters are chosen in region (2), the bifurcation, jump and hysteresis will not occur. This provides a basis for bifurcation control.

Furthermore, the bifurcation of the system whose unfolding parameters are chosen in hysteresis set is discussed for different cases. A point $(\alpha_1 = 1, \alpha_2 = -1)$ is chosen from the hysteresis set. After calculating, it is found that the critical hysteresis can only occur at several sections for $\lambda_2 = \{-0.4720, 0.4720\}$ or $\lambda_1 = \{-0.2918, 0.2918\}$. The bifurcation diagrams in different regions are shown in Figs. 5 and 6.

Figures 5 and 6 show that for fixed λ_1, if λ_2 is chosen between -0.4720 and 0.4720 hysteresis will not occur, and for fixed λ_2, if λ_1 is chosen between $-\infty$ and -1 or 1 and $+\infty$ hysteresis will not occur as well.

The above bifurcation analysis is based on singularity theory for systems with two parameters. For comparing the change of bifurcation properties, now only one parameter is considered as the bifurcation parameter.

(a) If λ_1 is fixed and λ_2 is considered as bifurcation parameter, bifurcation condition is not satisfied. Therefore, there is no bifurcation in the small neighborhood of $(0, 0)$.
(b) If λ_2 is fixed and λ_1 is considered as bifurcation parameter, the hysteresis set of G is obtained as Fig. 7.

From Fig. 7, we can see that the whole parametric plane is divided by the hysteresis set into three different persistent regions. The bifurcation diagrams in different persistent regions are shown in Fig. 8.

From Fig. 8 we can see that critical hysteresis will occur if the unfolding parameters are chosen on the up or low branch of the hysteresis set. If the unfolding parameters are chosen in other regions, hysteresis will occur.

4 Conclusions

The definition of transition set of bifurcation systems with two parameters is proposed in this paper, and the approach developed is used to investigate the dynamic characteristic of the nonlinear forced Duffing system with nonlinear feedback controller. It is found that the whole parametric plane is divided by the transition set into three different persistent regions, and the bifurcation diagrams in different persistent regions are then obtained. For the two bifurcation parameters case, the hysteresis set is not yet a parametric curve, but a parametric plane. It will change with λ_1 or λ_2. It is said that critical hysteresis can only occur with λ_1 at some fixed λ_2 or occur with λ_2 at some fixed λ_1. Additionally, the system with one bifurcation parameter is also studied. From comparison it is found that λ_2 will produce effects on the bifurcation properties of the system. Therefore, in the parameter control of engineering processes, both parameters λ_1 and λ_2 need to be considered.

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