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RESEARCH PAPER

Stokes' first problem for a viscoelastic fluid with the generalized Oldroyd-B model

Haitao Qi · Mingyu Xu

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Abstract The flow near a wall suddenly set in motion for a viscoelastic fluid with the generalized Oldroyd-B model is studied. The fractional calculus approach is used in the constitutive relationship of fluid model. Exact analytical solutions of velocity and stress are obtained by using the discrete Laplace transform of the sequential fractional derivative and the Fox H-function. The obtained results indicate that some well known solutions for the Newtonian fluid, the generalized second grade fluid as well as the ordinary Oldroyd-B fluid, as limiting cases, are included in our solutions.

Keywords Generalized Oldroyd-B fluid · Stokes' first problem · Fractional calculus · Exact solution · Fox H-function

1 Introduction

Navier–Stokes equations are the most fundamental motion equations in fluid dynamics. However, there are only a few cases for which exact analytical solutions can be obtained. Exact solutions are very important not only because they are solutions of some fundamental flows, but also because they may serve as accuracy checks for experimental, numerical, and asymptotic methods.

The inadequacy of the classical Navier–Stokes theory to describe rheologically complex fluids such as polymer solutions, blood and heavy oils, has led to the development of theories of non-Newtonian fluids. In order to describe the non-linear relationship between the stress and the strain rate, numerous models or constitutive equations have been proposed. Models of differential type and rate type have received much attention [1]. In recent years, the Oldroyd-B fluid has obtained a special attention amongst many fluids of rate type, as it includes as special cases the classical Newtonian fluid and the Maxwell fluid. For some special flows, the model of second grade fluid is also included [2–6].

Recently, fractional calculus has seen some success in the description of the complex dynamics. In particular it proves to be a valuable tool to handle viscoelastic properties. The starting point of the fractional derivative model of viscoelastic fluid is usually a classical differential equation, which is modified by replacing the classical, time derivatives of an integer order by the fractional calculus operators. This generalization allows one to define precisely non-integer order integrals or derivatives [7]. Fractional calculus has been found to be quite flexible in describing viscoelastic behavior [8–13]. More recently, Huang et al. [14–22] discussed some unsteady flows of the generalized second grade fluid. The unidirectional flow of a viscoelastic fluid with the fractional Maxwell model was studied by Tan et al. [23–29]. The unsteady flow with a generalized Jeffreys model in an annular pipe was also studied by Tong et al. [30,31].

On the other hand, almost every student of fluid mechanics is familiar with Stokes' celebrated paper on pendulums (in 1851), in which he described a kind of classical problems

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H. Qi (✉)
Department of Applied Mathematics and Statistics,
Shandong University at Weihai, Weihai 264209, China
e-mail: htqi@sdu.edu.cn

M. Xu
School of Mathematics and System Science,
Shandong University, Jinan 250100, China
e-mail: xumingyu@sdu.edu.cn

of the impulsive and oscillatory motion of an infinite plate in its own plane. Nowadays, the flows over a plane wall which is initially at rest and is suddenly set into motion in its own plane with a constant velocity and with a harmonic vibration are termed Stokes' first (or Rayleigh-type) and second problems, respectively [32,33]. At present, one paid more and more attention to the Stokes problems due to their important theoretical and practical implications [4,6,17,23,34–37]. For example, Tan and Xu [17,23] gave the exact analytical solutions to Stokes' first problem not only for generalized second grade fluid but also for generalized Maxwell fluid. In another paper, Fetecau and Fetecau [4] established the sine transform-based solution to Stokes' first problem for ordinary Oldroyd-B fluid. Recently, Tan and Masuoka [6,37] considered Stokes' first problem for second grade fluid and Oldroyd-B fluid in porous half space.

The aim of this paper is to investigate the Stokes' first problem for a viscoelastic fluid with the generalized Oldroyd-B model. The fractional calculus approach is used in the constitutive relationship of fluid model. Exact analytical solutions of velocity and stress are obtained by the similar methods used in the above mentioned papers [12,17,21,23]. The obtained results indicate that some well known solutions for the Newtonian fluid [38], the generalized second grade fluid [17] as well as for the ordinary Oldroyd-B fluid, as limiting cases, are included in our solutions. A special case of Ref. [23] about the generalized Maxwell fluid is also included in our results. It is also found that the effect of the relaxation and retardation times and the fractional orders in the generalized Oldroyd-B model is significant, which is discussed in detail in Sect. 4.

2 Basic equations

The constitutive equation of an incompressible, ordinary Oldroyd-B fluid is of the form [1]

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{S}, \tag{1}$$

$$\boldsymbol{S} + \lambda \frac{\mathrm{D}\boldsymbol{S}}{\mathrm{D}t} = \mu\left[1 + \theta \frac{\mathrm{D}}{\mathrm{D}t}\right]\boldsymbol{A}, \tag{2}$$

where λ and θ are the relaxation and retardation times, μ is the dynamic viscosity of fluid, $\boldsymbol{T}$ is the Cauchy stress tensor, $-p\boldsymbol{I}$ denotes the indeterminate spherical stress, $\boldsymbol{A} = \boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}}$ with $\boldsymbol{L}$ being the velocity gradient and $\mathrm{D}/\mathrm{D}t$ (upper convected time derivative) an operator on any tensor $\boldsymbol{B}$, as is defined by

$$\frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t} = \frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{V} \cdot \nabla \boldsymbol{B} - \boldsymbol{L} \cdot \boldsymbol{B} - \boldsymbol{B} \cdot \boldsymbol{L}^{\mathrm{T}}, \tag{3}$$

where $\boldsymbol{V}$ is the velocity vector, ∇ is the gradient operator, the superscript T denotes a transpose operation.

Generally, the constitutive relationship of the generalized Oldroyd-B fluid also takes the form (1) and (2), but $\mathrm{D}\boldsymbol{S}/\mathrm{D}t$ and $\mathrm{D}\boldsymbol{A}/\mathrm{D}t$ are defined as follows [11]

$$\frac{\mathrm{D}\boldsymbol{S}}{\mathrm{D}t} = \mathrm{D}_t^{\alpha}\boldsymbol{S} + \boldsymbol{V} \cdot \nabla \boldsymbol{S} - \boldsymbol{L} \cdot \boldsymbol{S} - \boldsymbol{S} \cdot \boldsymbol{L}^{\mathrm{T}}, \tag{4}$$

$$\frac{\mathrm{D}\boldsymbol{A}}{\mathrm{D}t} = \mathrm{D}_t^{\beta}\boldsymbol{A} + \boldsymbol{V} \cdot \nabla \boldsymbol{A} - \boldsymbol{L} \cdot \boldsymbol{A} - \boldsymbol{A} \cdot \boldsymbol{L}^{\mathrm{T}}, \tag{5}$$

where D_t^{α} and D_t^{β} are the fractional differentiation operators of order α and β with respect to t, respectively, and may be defined as [7]

$$\mathrm{D}_t^p f(t) = \frac{1}{\Gamma(1-p)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(\tau)}{(t-\tau)^p} \mathrm{d}\tau, \quad 0 \le p \le 1, \tag{6}$$

where $\Gamma(\cdot)$ is the Gamma function. It should be noted that this model can be reduced to the ordinary Oldroyd-B model when $\alpha = \beta = 1$, the generalized second grade Fluid [14–21] when $\alpha = 0$, $\lambda \to 0$ and the fractional Maxwell fluid when $\beta = 0$, $\theta \to 0$ [9,28,29].

In the following, we will determine the velocity field and the associated stress field corresponding to the first problem of Stokes for a viscoelastic fluid with the generalized Oldroyd-B model lying over an infinitely extended flat plate. Initially, the fluid is at rest, and at time $t = 0^+$ the plate is impulsively set in motion with constant velocity U. By the influence of shear, the fluid above the plate is gradually set in motion. For the problem under consideration, we seek a velocity field of the form

$$\boldsymbol{V} = u(y,t)\boldsymbol{i}, \tag{7}$$

where u is the velocity in the x-coordinate direction, $\boldsymbol{i}$ the unit vector in the x-direction, x the coordinate along the plate and y the coordinate perpendicular to the plate.

Substituting Eq. (7) into the above formula and taking account of the initial condition

$$\boldsymbol{S}(y,0) = 0, \tag{8}$$

we get

$$(1 + \lambda \mathrm{D}_t^{\alpha}) S_{xy} = \mu(1 + \theta \mathrm{D}_t^{\beta}) \partial_y u(y,t), \tag{9}$$

and $S_{yy} = S_{zz} = S_{xz} = S_{yz} = 0$, where $S_{xy} = S_{yx}$. In the absence of a pressure gradient in the x-direction and body forces the equation of motion leads to

$$\rho \partial_t u = \partial_y S_{xy}, \tag{10}$$

where ρ is the constant density of the fluid. Eliminating S_{xy} with Eqs. (9) and (10), we arrive at the following fractional differential equation

$$(1 + \lambda \mathrm{D}_t^{\alpha}) \frac{\partial u}{\partial t} = \nu(1 + \theta \mathrm{D}_t^{\beta}) \frac{\partial^2 u}{\partial y^2}, \tag{11}$$

where $\nu = \mu/\rho$. The corresponding initial and boundary conditions are as follows

$$u(y,0) = \partial_t u(y,0) = 0, \quad y > 0, \tag{12}$$

$$u(0,t) = U, \quad t > 0. \tag{13}$$

And the natural conditions are

$$u(y,t), \partial_y u(y,t) \to 0 \quad \text{as} \quad y \to \infty. \tag{14}$$

3 Exact solutions to the Stokes' first problem

Let us introduce the following dimensionless variables

$$u^* = \frac{u}{U}, \quad y^* = \frac{yU\rho}{\mu}, \quad t^* = \frac{tU^2\rho}{\mu},$$
$$\lambda^* = \lambda\Big(\frac{U^2\rho}{\mu}\Big)^{\alpha}, \quad \text{and} \quad \theta^* = \theta\Big(\frac{U^2\rho}{\mu}\Big)^{\beta}, \tag{15}$$

in which U and $\mu/U^2\rho$ denote the characteristic velocity and time, respectively. The dimensionless governing equation (11) and its initial and boundary conditions (12)–(14) can be written as (for simplicity, the superscript * is omitted)

$$(1+\lambda D_t^{\alpha})\frac{\partial u}{\partial t} = (1+\theta D_t^{\beta})\frac{\partial^2 u}{\partial y^2}, \tag{16}$$

$$u(y,0) = \partial_t u(y,0) = 0, \quad y > 0, \tag{17}$$

$$u(0,t) = 1, \quad t > 0, \tag{18}$$

$$u(y,t), \partial_y u(y,t) \to 0 \quad \text{as} \quad y \to \infty. \tag{19}$$

Let us suppose that

$$\tilde{u}(y,s) = L\{u(y,t),s\} = \int_0^{\infty} e^{-st}u(y,t)dt \tag{20}$$

is the image function of $u(y,t)$, where s is a transform parameter. Because the fractional order equation (16) has the property of the integer order initial conditions, the fractional derivative must be interpreted as a properly chosen sequential fractional derivative [7]. Using the Laplace transform formula for sequential fractional derivatives, we can obtain the equation of the image function as:

$$\frac{d^2\tilde{u}}{dy^2} = \frac{s(1+\lambda s^{\alpha})}{1+\theta s^{\beta}}\tilde{u}, \tag{21}$$

subject to the boundary conditions

$$\tilde{u}(0,s) = \frac{1}{s}, \tag{22}$$

$$\tilde{u}(y,s), \partial_y \tilde{u}(y,s) \to 0 \quad \text{as} \quad y \to \infty. \tag{23}$$

Solving Eqs. (21)–(23) yields

$$\tilde{u}(y,s) = \frac{1}{s}\cdot e^{-y\left(\frac{s(1+\lambda s^{\alpha})}{1+\theta s^{\beta}}\right)^{1/2}}. \tag{24}$$

Because the fluid is moved by the action of stress at the plate, the stress field may be calculated. From Eq. (11) the dimensionless stress can be represented by

$$(1+\lambda D_t^{\alpha})F(y,t) = (1+\theta D_t^{\beta})\partial_y u(y,t), \tag{25}$$

where $F(y,t) = S_{xy}/\rho U^2$. Using Eqs. (8) and (24), the Laplace transform of Eq. (25) is

$$\tilde{F}(y,s) = -\Big(\frac{s(1+\lambda s^{\alpha})}{1+\theta s^{\beta}}\Big)^{-1/2} e^{-y\left(\frac{s(1+\lambda s^{\alpha})}{1+\theta s^{\beta}}\right)^{1/2}}. \tag{26}$$

In order to obtain an analytical solution for this problem and to avoid lengthy calculations of residues and contour integrals, we will apply the discrete inverse Laplace transform method to obtain the velocity and stress distribution. First, we rewrite (24) and (26) in series forms

$$\tilde{u}(y,s) = \frac{1}{s} + \sum_{k=1}^{\infty}\frac{(-y\sqrt{\lambda/\theta})^k}{k!}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\lambda^m} \times \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma\left(m-\frac{k}{2}\right)\Gamma\left(\frac{k}{2}+n\right)}{n!\theta^n\Gamma\left(\frac{k}{2}\right)\Gamma\left(-\frac{k}{2}\right)s^{\frac{k}{2}(\beta-\alpha-1)+m\alpha+n\beta+1}}, \tag{27}$$

$$\tilde{F}(y,s) = -\sum_{k=0}^{\infty}\frac{(-y)^k(\lambda/\theta)^{\frac{k-1}{2}}}{k!}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\lambda^m} \times \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma\left(m+\frac{1-k}{2}\right)\Gamma\left(\frac{k-1}{2}+n\right)}{n!\theta^n\Gamma\left(\frac{1-k}{2}\right)\Gamma\left(\frac{k-1}{2}\right)s^{\frac{k-1}{2}(\beta-\alpha-1)+m\alpha+n\beta}}. \tag{28}$$

Applying the discrete inverse Laplace transform to Eqs. (27) and (28), we obtain

$$\begin{aligned} u(y,t) &= 1 + \sum_{k=1}^{\infty}\frac{(-y\sqrt{\lambda/\theta})^k}{k!}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\lambda^m}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\theta^n} \\ &\times\frac{\Gamma\left(\frac{k}{2}+n\right)\Gamma\left(m-\frac{k}{2}\right)t^{\frac{k}{2}(\beta-\alpha-1)+m\alpha+n\beta}}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(-\frac{k}{2}\right)\Gamma\left(\frac{k}{2}(\beta-\alpha-1)+m\alpha+n\beta+1\right)} \\ &= 1 + \sum_{k=1}^{\infty}\frac{(-y\sqrt{\lambda/\theta})^k}{k!}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\lambda^m}t^{\frac{k}{2}(\beta-\alpha-1)+m\alpha} \\ &\times H_{2,4}^{1,2}\left[\frac{t^{\beta}}{\theta}\Big|_{(0,1),(1-\frac{k}{2},0),(1+\frac{k}{2},0),(\frac{k}{2}(1+\alpha-\beta)-m\alpha,\beta)}^{(1-\frac{k}{2},1),(1-m+\frac{k}{2},0)}\right], \end{aligned} \tag{29}$$

$$F(y,t) = -\sum_{k=0}^{\infty}\frac{(-y)^k(\lambda/\theta)^{\frac{k-1}{2}}}{k!}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\lambda^m}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\theta^n} \times\frac{\Gamma\left(\frac{k-1}{2}+n\right)\Gamma\left(m+\frac{1-k}{2}\right)t^{\frac{k-1}{2}(\beta-\alpha-1)+m\alpha+n\beta-1}}{\Gamma\left(\frac{k-1}{2}\right)\Gamma\left(\frac{1-k}{2}\right)\Gamma\left(\frac{k-1}{2}(\beta-\alpha-1)+m\alpha+n\beta\right)}$$
$$= -\sum_{k=0}^{\infty}\frac{(-y)^k(\lambda/\theta)^{\frac{k-1}{2}}}{k!}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\lambda^m}t^{\frac{k-1}{2}(\beta-\alpha-1)+m\alpha-1} \times H^{1,2}_{2,4}\left[\frac{t^\beta}{\theta}\middle|^{(\frac{k+1}{2}-m,0),(\frac{3-k}{2},1)}_{(0,1),(\frac{3-k}{2},0),(\frac{k+1}{2},0),(\frac{k-1}{2}(1+\alpha-\beta)-m\alpha+1,\beta)}\right], \tag{30}$$

in which $H^{m,n}_{p,q}(z)$ is the Fox H-function [11]. In Eqs. (29) and (30), the following property of the Fox H-function is used:

$$\sum_{n=0}^{\infty}\frac{(-z)^n\prod_{j=1}^{p}\Gamma(a_j+A_jn)}{n!\prod_{j=1}^{q}\Gamma(b_j+B_jn)} = H^{1,p}_{p,q+1}\left[z\middle|^{(1-a_1,A_1),\ldots,(1-a_p,A_p)}_{(0,1),(1-b_1,B_1),\ldots,(1-b_q,B_q)}\right]. \tag{31}$$

Particularly, taking $y=0$ in Eq. (30), we get the following formula to calculate the shear stress on the plate

$$F_p(t) = -\left(\frac{\theta}{\lambda}\right)^{\frac{1}{2}}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\lambda^m}t^{m\alpha+\frac{\alpha-\beta-1}{2}} \times H^{1,2}_{2,4}\left[\frac{t^\beta}{\theta}\middle|^{(\frac{1}{2}-m,0),(\frac{3}{2},1)}_{(0,1),(\frac{3}{2},0),(\frac{1}{2},0),(\frac{\beta-\alpha+1}{2}-m\alpha,\beta)}\right]. \tag{32}$$

4 Limiting cases and numerical results

Now let us apply formulas (29) and (30) in special cases.

(1) If $\alpha=0, \lambda\to 0, \theta=\eta$, the medium is the generalized second grade fluid, and Eqs. (29) and (30) reduce to the same form as obtained in Refs. [17,19]

$$u(y,t) = 1+\sum_{k=1}^{\infty}\frac{(-yt^{\frac{\beta-1}{2}})^k}{\eta^{\frac{k}{2}}k!} \times H^{1,1}_{1,3}\left[\frac{t^\beta}{\eta}\middle|^{(1-\frac{k}{2},1)}_{(0,1),(1-\frac{k}{2},0),(\frac{k}{2}(1-\beta),\beta)}\right], \tag{33}$$

$$F(y,t) = -\sum_{k=0}^{\infty}\frac{(-y)^k\eta^{\frac{1-k}{2}}}{k!}t^{\frac{k-1}{2}(\beta-1)-1} \times H^{1,1}_{1,3}\left[\frac{t^\beta}{\eta}\middle|^{(\frac{3-k}{2},1)}_{(0,1),(\frac{3-k}{2},0),(\frac{k-1}{2}(1-\beta)+1,\beta)}\right]. \tag{34}$$

(2) If $\beta=0, \theta\to 0$, then Eqs. (29) and (30) are simplified into

$$u(y,t) = 1+\sum_{k=1}^{\infty}\frac{(-y\sqrt{\lambda})^k}{k!}t^{-\frac{(\alpha+1)k}{2}} \times H^{1,1}_{1,3}\left[\frac{t^\alpha}{\lambda}\middle|^{(1+\frac{k}{2},1)}_{(0,1),(1+\frac{k}{2},0),(\frac{k}{2}(\alpha+1),\alpha)}\right], \tag{35}$$

$$F(y,t) = -\sum_{k=0}^{\infty}\frac{(-y)^k\lambda^{\frac{k-1}{2}}}{k!}t^{-1-\frac{(\alpha+1)(k-1)}{2}} \times H^{1,1}_{1,3}\left[\frac{t^\alpha}{\lambda}\middle|^{(\frac{k+1}{2},1)}_{(0,1),(\frac{k+1}{2},0),(\frac{k-1}{2}(\alpha+1)+1,\alpha)}\right]. \tag{36}$$

They are the solutions of velocity and stress for the fractional Maxwell fluid, in which the derivatives of the stress and strain are fractional and first order, respectively [9,23,28,29].

(3) Setting $\beta=0, \theta\to 0$ in Eqs. (33) and (34) or setting $\alpha=0, \lambda\to 0$ in Eqs. (35) and (36), we obtain the classical Rayleigh's similarity solutions of Newtonian fluid [38]

$$u(y,t) = 1-\text{erf}\left(\frac{y}{2\sqrt{t}}\right), \tag{37}$$

$$F(y,t) = -\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{y^2}{4t}\right). \tag{38}$$

(4) When $\alpha=\beta=1$, Eqs. (29) and (30) can be simplified into

$$u(y,t) = 1+\sum_{k=1}^{\infty}\frac{(-y\sqrt{\lambda/\theta})^k}{k!}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\lambda^m}t^{m-\frac{k}{2}} \times H^{1,2}_{2,4}\left[\frac{t}{\theta}\middle|^{(1-\frac{k}{2},1),(1-m+\frac{k}{2},0)}_{(0,1),(1-\frac{k}{2},0),(1+\frac{k}{2},0),(\frac{k}{2}-m,1)}\right], \tag{39}$$

$$F(y,t) = -\sum_{k=0}^{\infty}\frac{(-y)^k(\lambda/\theta)^{\frac{k-1}{2}}}{k!}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\lambda^m}t^{m-\frac{k+1}{2}} \times H^{1,2}_{2,4}\left[\frac{t}{\theta}\middle|^{(\frac{k+1}{2}-m,0),(\frac{3-k}{2},1)}_{(0,1),(\frac{3-k}{2},0),(\frac{k+1}{2},0),(\frac{k+1}{2}-m,1)}\right], \tag{40}$$

which are the solutions for ordinary Oldroyd-B fluid. It is worth pointing out that the results (39) and (40) are obviously different from the sine transform-based solution given in Ref. [4].

(5) From Eqs. (24) and (26), we also have

$$\tilde{F}(y,s) = -\left(\frac{s(1+\theta s^\beta)}{1+\lambda s^\alpha}\right)^{1/2}\tilde{u}(y,s). \tag{41}$$

Using the same method as used in Sect. 3 and the definition of fractional calculus, we have

$$F(y,t) = -\sqrt{\frac{\theta}{\lambda}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\theta^n}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma\left(\frac{1}{2}+m\right)\Gamma\left(n-\frac{1}{2}\right)}{m!\lambda^m\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)} \times \mathrm{D}_t^{-m\alpha-n\beta+\frac{1+\beta-\alpha}{2}}u(y,t). \tag{42}$$

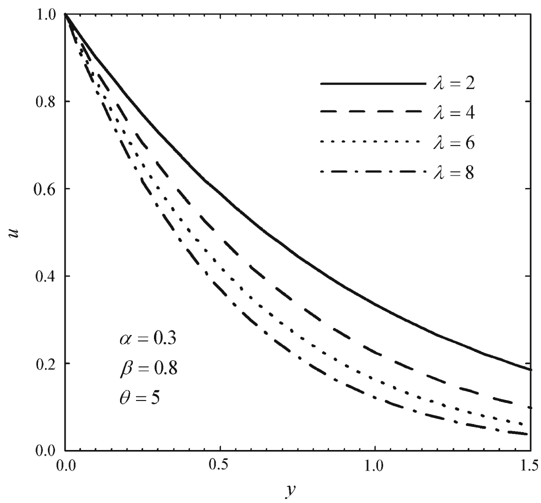

Fig. 1 Velocity $u(y, t)$ versus y for various values of λ at a fixed time $t = 0.1$

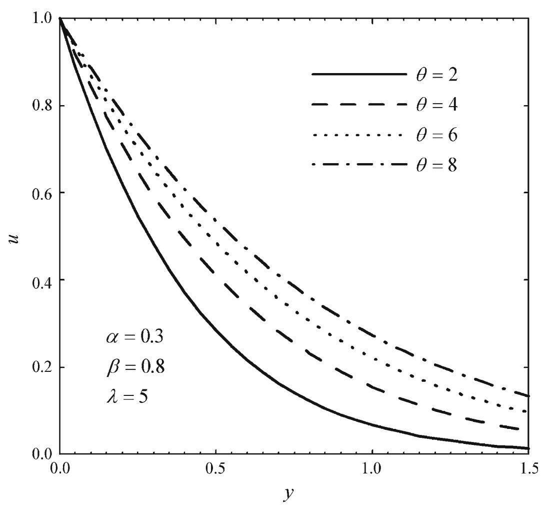

Fig. 2 Velocity $u(y, t)$ versus y for various values of θ at a fixed time $t = 0.1$

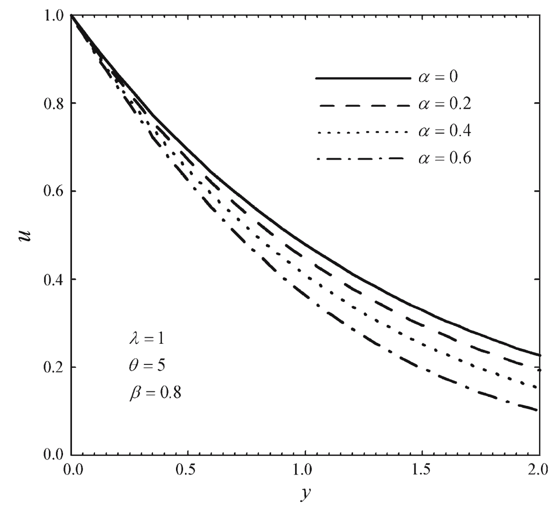

Fig. 3 Velocity $u(y, t)$ versus y for various values of α at a fixed time $t = 0.1$

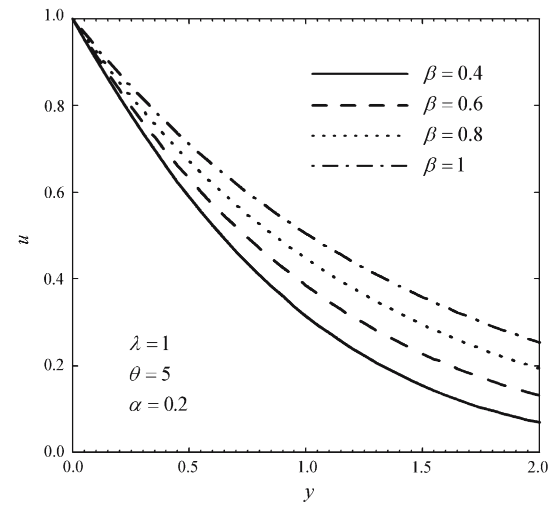

Fig. 4 Velocity $u(y, t)$ versus y for various values of β at a fixed time $t = 0.1$

If one sets $\alpha = 0$, $\lambda \to 0$, $\theta = \eta$, the formula (42) can be simplified to the same form as obtained in Ref. [17]:

$$F(y, t) = -\eta^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma\left(n - \frac{1}{2}\right)}{n! \eta^n \Gamma\left(-\frac{1}{2}\right)} \times \mathrm{D}_t^{-n\beta+(\beta+1)/2} u(y, t). \quad (43)$$

The physical meaning of Eq. (42) is that the stress at a given point at any time is dependent on the time history of the velocity profile at that point, and the history can be obtained by the fractional calculus [10].

(6) The variations of $u(y, t)$ with y for various values of λ, θ, α and β at a fixed time ($t = 0.1$) are illustrated in Figs. 1, 2, 3 and 4, which are calculated by formula (29). It is clearly seen that the smaller the λ (or α), the more slowly the velocity decays, and the larger the effect region becomes. But one sees an opposite trend for the values of θ (or β). It is obvious that the relaxation and retardation times and the orders of the time fractional derivative have effect on the velocity field. Figures 5 and 6 demonstrate the velocity changes with time at a given point ($y = 1$). It is clearly seen that the smaller the α, the more rapidly the velocity changes in the initial time and the more like a solid the generalized Oldroyd-B model behaves. The effect of β on the velocity is contrary to that of α. However, it seems that their effect on the velocity changes is reversed at a critical time point.

5 Conclusion

Fractional calculus approach is introduced to the constitutive relationship model of generalized Oldroyd-B fluid in

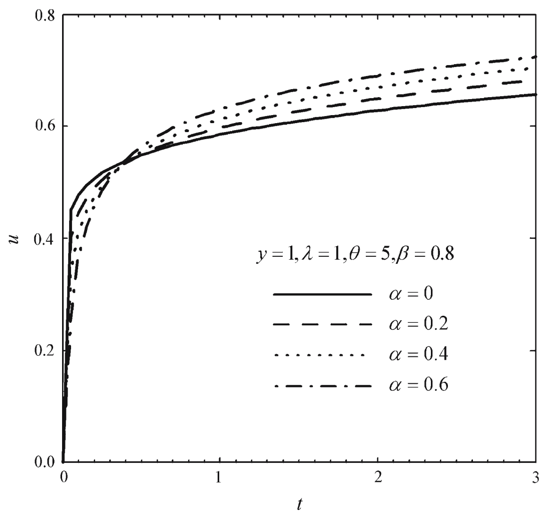

Fig. 5 Velocity $u(y, t)$ versus t for various values of α

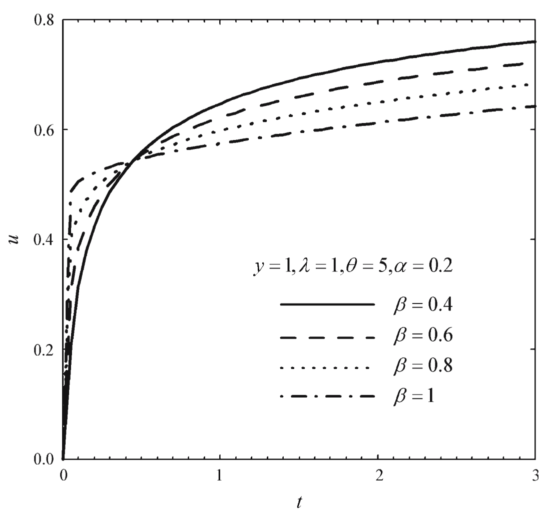

Fig. 6 Velocity $u(y, t)$ versus t for various values of β

this paper. The flow model is more useful compared with the traditional model. Exact analytical solutions of velocity and stress for the Stokes' first problem are obtained by using the discrete Laplace transform of the sequential fractional derivatives and the Fox H-function. Our results can provide new models and analytical solutions for studying the complicated fluid in rheology.

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