Acta Mechanica Sinica (2006) 22:487–501
DOI 10.1007/s10409-006-0018-5

RESEARCH PAPER

Transversal vibrations of double-plate systems

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Received: 27 July 2005 / Accepted: 6 January 2006 / Published online: 21 July 2006

Abstract This paper presents an analytical and numerical analysis of free and forced transversal vibrations of an elastically connected double-plate system. Analytical solutions of a system of coupled partial differential equations, which describe corresponding dynamical free and forced processes, are obtained using Bernoulli's particular integral and Lagrange's method of variation constants. It is shown that one-mode vibrations correspond to two-frequency regime for free vibrations induced by initial conditions and to three-frequency regime for forced vibrations induced by one-frequency external excitation and corresponding initial conditions. The analytical solutions show that the elastic connection between plates leads to the appearance of two-frequency regime of time function, which corresponds to one eigenamplitude function of one mode, and also that the time functions of different vibration modes are uncoupled, for each shape of vibrations. It has been proven that for both elastically connected plates, for every pair of *m* and *n*, two possibilities for appearance of the resonance dynamical states, as well as for appearance of the dynamical absorption, are present. Using the MathCad program, the corresponding visualizations of the characteristic forms of the plate middle surfaces through time are presented.

Keywords Double plate system · Elastic connection · Vibration · Multi-frequency · Visualization

The English text was polished by Keren Wang.

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1 Introduction

Plates have been extensively used as structural elements in many industrial applications. The investigation of the vibrations of plates dates back to the 19th century. There had been a great amount of research and literature over the last century. The problem of free vibration of a circular plate was first investigated by Poisson [1]. Rayleigh (see Ref. [2] reprint 1945) presented a well-known general method of solution to determine the resonant frequencies of vibrating systems. The method was improved by Ritz and this approach is one of the most popular approximate methods for the analysis of vibrations. Reviews of these problems may be found in Refs. [3,4].

With the availability of inexpensive and high performance computers, the theoretical analysis is frequently employed to optimize problems of the plate vibrations in practical engineering designs [5,6].

The study of transversal vibrations of an elastically connected double-plate system is important for both theoretical and practical reasons. Many important structures can be modeled as composite structures. This elastically connected double-plate system can be used for the acoustic and vibration isolation as a wall or a ground.

Current research in the theory of discrete and continuous dynamical system oscillations is directed to nonlinear phenomena as well as to nonstationary processes [7], and also to stochastic and chaotic processes in pure deterministic dynamical systems and conditions. In the theory of oscillations of continuous systems nonlinear phenomena [6,8,9] as well as damage and fracture are the topics of some leading journals and international scientific meetings while pure linear elastic systems are not in the focus of researchers.

In Refs. [10–12], the partial fractional differential equations of creeping and vibrations of plate as well as of the beam were derived by Hedrih. A fractional-differential operator with the creep material parameters was introduced. Plate material creeping and constitutive relations were expressed by fractional order derivatives. An equation of deformed middle surface of the plate was derived for the case of plate free oscillations.

The one- and two-frequencies stationary and non-stationary regimes of the nonlinear transversal free and forced vibrations of the beams, plates and shells have been studied in Hedrih's articles (see Ref. [13]). Transversal vibrating beam on the elastic Winkler's foundation exposed to the multi-frequency forces with frequencies from first frequency resonant range of the beam have also been studied, and some results of the investigation of multi-frequency vibrations in the single-frequency regime in nonlinear systems with many degrees of the freedom and with slow changing parameters were presented in the article by Stevanović and Rašković from 1974 [14]. Application of the Krilov–Bogolyulybov–Mitrolpolskiy asymptotic method (see Ref. [7] for the study of nonlinear oscillations of elastic bodies and energetic analysis of the elastic bodies oscillatory motions gave new results in the theses by Stevanović in 1975 [9]. In Ref. [15] a mesh free approach, called displacement boundary method, for anisotropic Kirchhoff plate dynamic analysis was presented.

An accurate laminate model developed by Bruno et al. [16] by using multi-layered shear deformable plate modeling and interface elements, based on fracture mechanics and contact mechanics, was proposed to analyse mixed mode delamination in composite laminates.

There were few papers on the bending and buckling of functionally graded structures in contract with the extensive investigations on isotropic and composite plates and shells. By Ma and Wang [17], the third-order shear deformation plate theory (TPT) was employed to solve the axisymmetric bending and buckling problems of functionally graded circular plates.

The first-known exact solutions for buckling and vibration of stepped rectangular Mindlin plates with two opposite edges being simply supported and the remaining two edges being either free, simply supported or clamped, are presented in Ref. [18]. The general Levy type solution method and a domain decomposition technique are employed to develop an analytical approach to deal with the stepped rectangular Mindlin plates. The paper by Shukla et al. [6] presents an analytical approach to examine the nonlinear dynamic responses of a laminated composite plate composed of spatially oriented short fibers in each layer of the composite.

Many researchers have studied plate mechanics using experimental methods, but most of the papers on vibration analysis of the plates published in the literature are analytical and numerical, and very few experimental results are available. In Ref. [19] the experimental whole-field interferometry for investigation of the transverse vibration of plates is used.

I think that it is very important to make some new classical examples of the plate system vibrations with corresponding analytical solutions useful for teaching process in theory of vibrations of plate systems, as well as for comparison with numerical solutions obtained by numerical methods using powerful computer possibilities.

New computer tools like MKE and BEM as well as MathCad, Mathematica, MathLab offer powerful possibilities for the visualization of the oscillatory processes in dynamic systems applied in engineering practice. At the same time they are very useful for the university teaching of the theory of oscillations as tools for the analytical method and pure mathematical explanations.

2 Theoretical problem formulation and governing equations

Let us consider two isotropic, elastic, thin plates, with width h_i, modulus of elasticity E_i, Poisson's ratio μ_i and shear modulus G_i, mass density ρ_i, $i = 1, 2$ in this paper. The plates have constant thickness in the z-direction (see Fig. 1). The contours of the plates are parallel. The plates are interconnected by a linear elastic Winkler-type layer with constant surface stiffness c. This elastically connected double-plate system represents a type of a composite structure, or sandwich or layered plates.

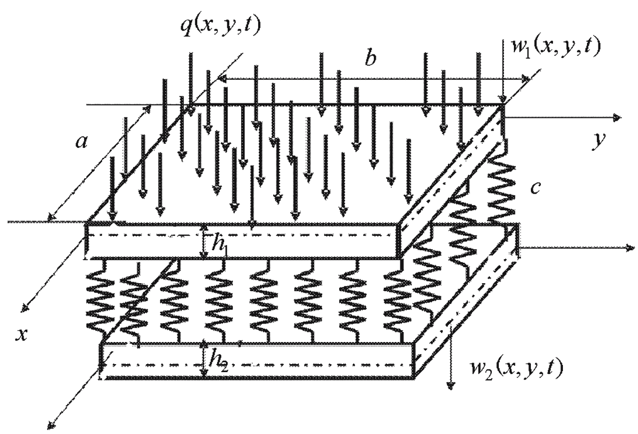

Fig. 1 A elastically connected double plate system

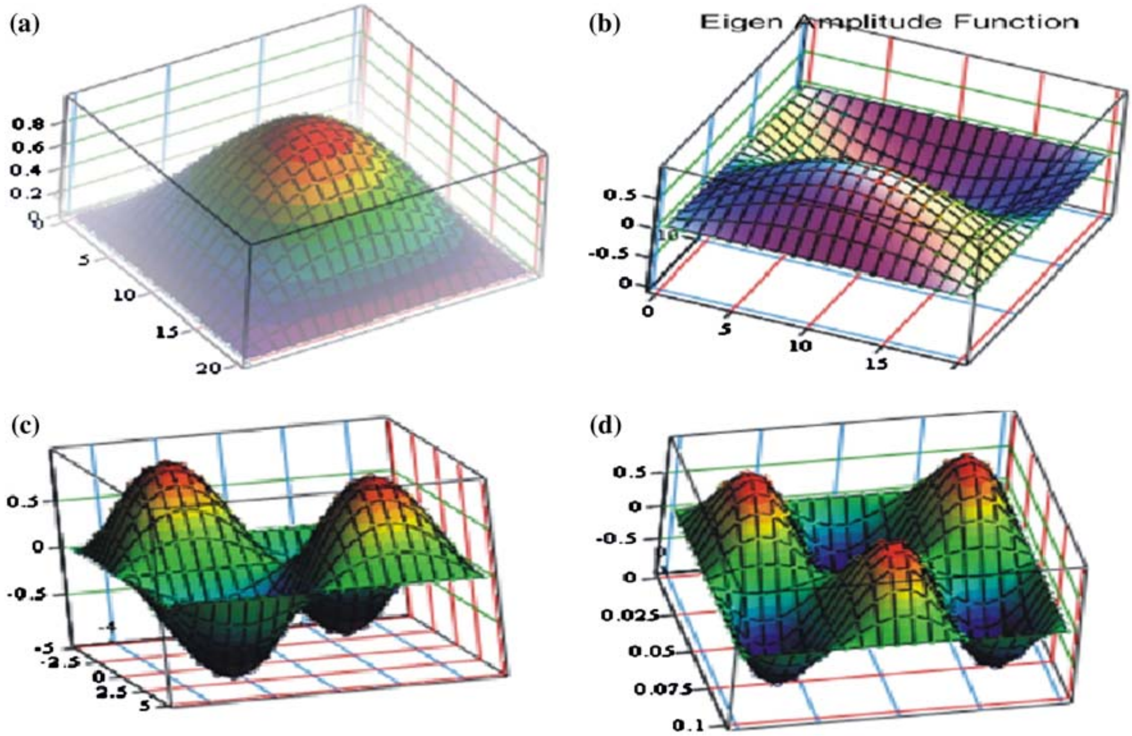

Fig. 2 The eigenamplitude functions for first four shapes

The origins of the two coordinate systems are at the corresponding centers in the nondeformed plates middle surfaces as shown in Fig. 1. and with parallel corresponding axes. Both plates may be subjected to transversal distributed external loads $q_i(x, y, t)$, along the corresponding external surfaces of the plates. The problem at hand is to determine solutions and the eigenfrequencies for such a double-plate system elastically connected by spring layer distributed along the contour surfaces of the plates.

The use of Love–Kirchhoff approximation makes the classical plate theory essentially a two dimensional phenomenon, in which the normal and transverse forces and bending and twisting moments on plate cross sections can be found in term of the displacements $w_i(x, y, t)$ of the middle surface points [20]. The plates are assumed to be with the same contour forms and boundary conditions.

Let us suppose that the plate middle surfaces are planes for the undeformed system. The plate transverse deflections $w_i(x, y, t)$ are small as compared to the thickness of the plates, h_i, and the vibrations occur only in the vertical direction.

Let us denote with $D_i = E_i h^3/12(1 - \mu^2)$, the corresponding bending cylindrical rigidity of plates. Differential operator is

$$\Delta\Delta = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}.$$

We suppose that the plate displacements $u_i(x, y, z, t)$, and $v_i(x, y, z, t)$, of the corresponding plate point $N_i(x, y, z)$ in the direction of the coordinate axes x and y, can be expressed as functions of its distance z from the corresponding plate middle surface and its transversal displacement $w_i(x, y, t)$ in the direction of the z-axis, and also the displacements of the corresponding point $N_{i0}(x, y, 0)$ in the corresponding plate middle surface.

The governing equations are formulated in terms of two unknowns: the transversal displacement $w_1(x, y, t)$ and $w_2(x, y, t)$. The two coupled partial differential equations are derived using d'Alembert's or variational principle [20]. These partial differential equations of the elastically connected double-plate system are:

$$\begin{aligned}
&\rho_1 h_1 \frac{\partial^2 w_1(x, y, t)}{\partial t^2} + D_1 \Delta\Delta w_1(x, y, t) \\
&\quad - c[w_2(x, y, t) - w_1(x, y, t)] = q_1(x, y, t), \\
&\rho_2 h_2 \frac{\partial^2 w_2(x, y, t)}{\partial t^2} + D_2 \Delta\Delta w_2(x, y, t) \\
&\quad + c[w_2(x, y, t) - w_1(x, y, t)] = -q_2(x, y, t).
\end{aligned} \tag{1}$$

Let us introduce the following notations: $a_{(i)}^2 = c/\rho_i h_i$, and $c_{(i)}^4 = D_i/\rho_i h_i$. Decoupling Eq. (1), we obtain two

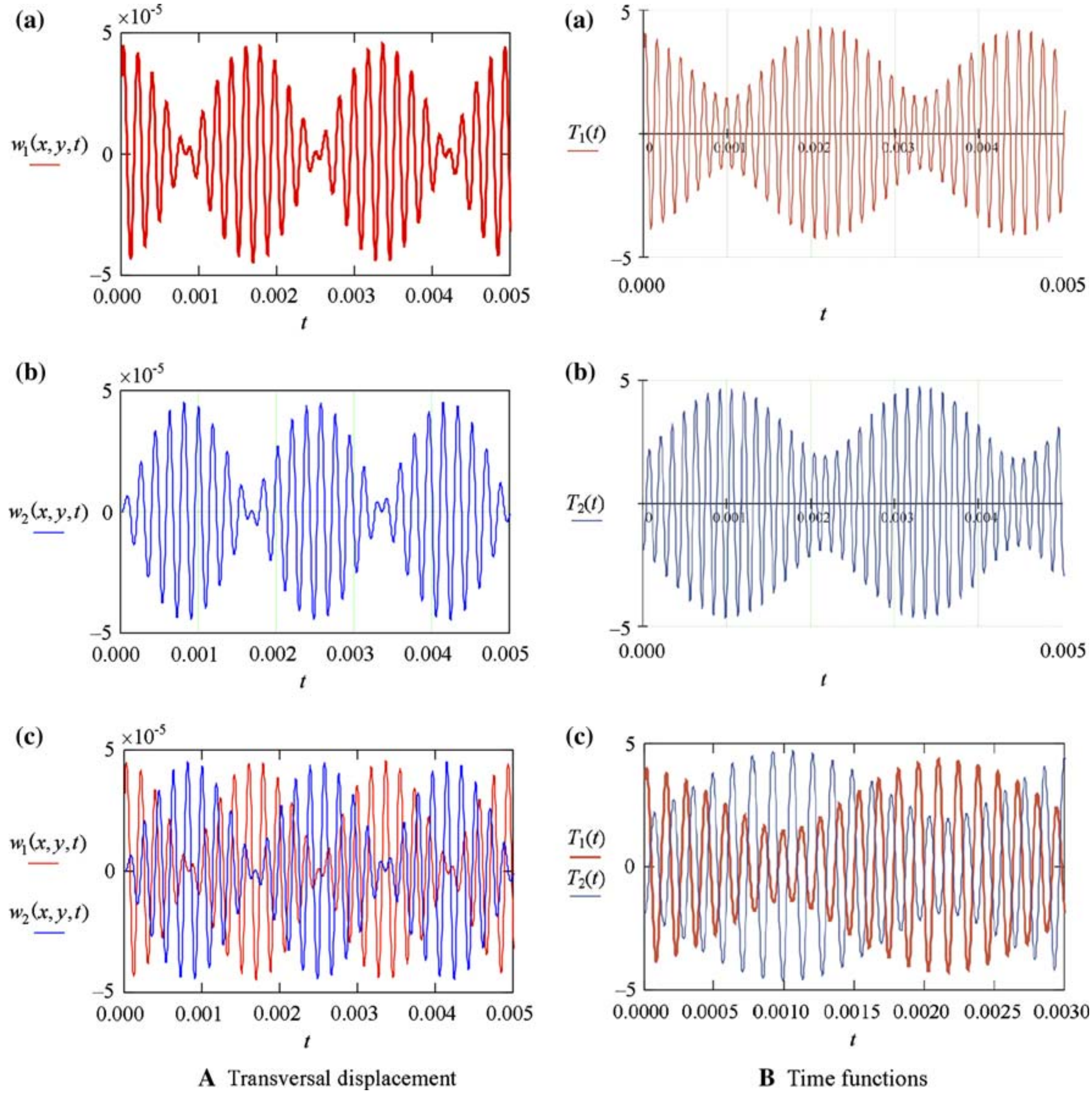

Fig. 3 The time history diagrams of the plate middle surface points corresponding to the time functions $T_{(1)}(t)$ and $T_{(2)}(t)$

associated corresponding partial differential equations which describe two partial plates founded on the elastic foundation of the Winkler type. These partial differential equations are:

$$\frac{\partial^2 w_1(x,y,t)}{\partial t^2} + c_{(1)}^4 \Delta\Delta w_1(x,y,t) + a_{(1)}^2 w_1(x,y,t) = 0,$$
$$\frac{\partial^2 w_2(x,y,t)}{\partial t^2} + c_{(2)}^4 \Delta\Delta w_2(x,y,t) + a_{(2)}^2 w_2(x,y,t) = 0. \tag{2}$$

3 Particular solutions of governing basic decoupled equations

Solutions of the previous partial-differential equations are found using Bernoulli's method of particular integrals in the form of product of two corresponding functions $W_{(i)}(x,y)$ and $T_{(i)}(t)$:

$$w_i(x,y,t) = W_{(i)}(x,y)T_{(i)}(t). \tag{3}$$

The assumed solution (3) is introduced into Eqs. (1) and (2) and after transformation and introducing the notation of the characteristic constants: $\omega_{(i)}^2$, $k_{(i)}^4$, k_T and k_W, we have:

(a) two second order ordinary differential equations for the unknown time-functions $T_{(i)}(t)$

$$\ddot{T}_{(i)}(t) + \omega_{(i)}^2 T_{(i)}(t) = 0, \tag{4}$$

(b) two four order partial differential equations of the unknown amplitude eigenfunctions $W_{(i)}(x,y)$ in the form:

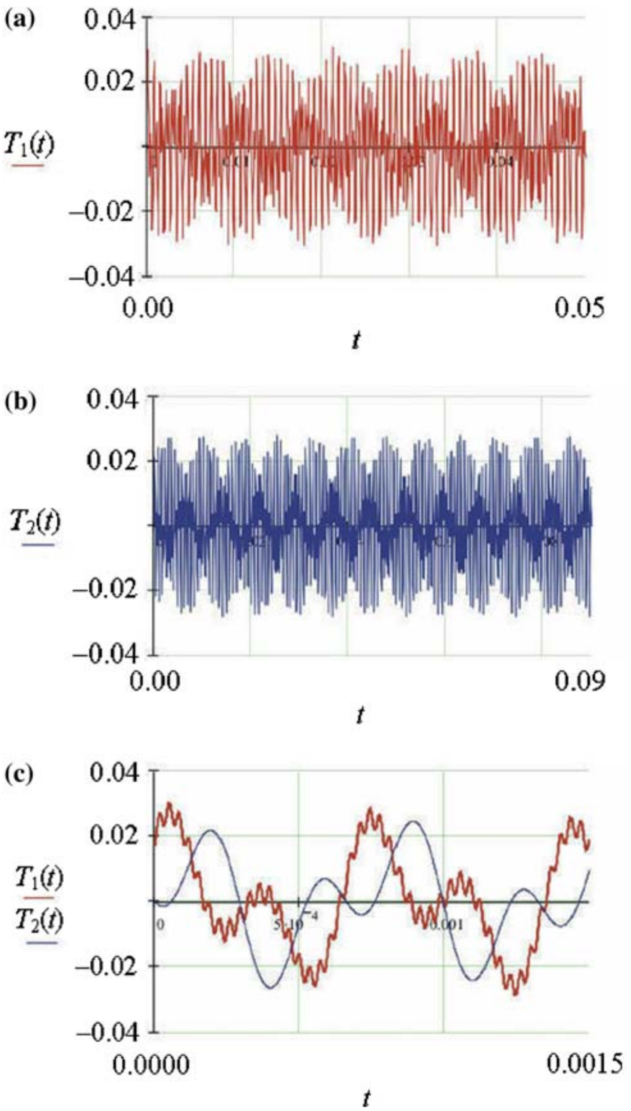

Fig. 4 The time history diagrams of the plate middle surface points in the forced regime

$$c_{(1)}^4 \frac{\Delta\Delta W_{(1)}(x,y)}{W_{(1)}(x,y)} + a_{(1)}^2 - \omega_{(1)}^2 = a_{(1)}^2 k_T^2 k_W^2,$$
$$c_{(2)}^4 \frac{\Delta\Delta W_{(2)}(x,y)}{W_{(2)}(x,y)} + a_{(2)}^2 - \omega_{(2)}^2 = a_{(2)}^2 \frac{1}{k_T^2 k_W^2}, \tag{5}$$

where

$$k_W^2 = \frac{W_{(2)}(x,y)}{W_{(1)}(x,y)}, \quad k_T^2 = \frac{T_{(2)}(t)}{T_{(1)}(t)},$$

and corresponding basic system,

$$c_{(i)}^4 \frac{\Delta\Delta W_{(i)}(x,y)}{W_{(i)}(x,y)} + a_{(i)}^2 - \omega_{(i)}^2 = 0. \tag{6}$$

From Eqs. (4), (5) and (6), we see that

(a) if plates are rectangular, we can use Descartes' coordinates

$$\ddot{T}_{(i)}(t) + \omega_{(i)}^2 T_{(i)}(t) = 0,$$
$$\Delta\Delta W_{(i)}(x,y) - k_{(i)}^4 W_{(i)}(x,y) = 0, \tag{7}$$

$$\omega_{(i)}^2 = k_{(i)}^4 c_{(i)}^4 + a_{(i)}^2. \tag{8}$$

(b) if plates are circular, it is suitable to use the polar-cylindrical coordinate system, and we have:

$$\Delta\Delta W_{(i)}(r,\varphi) - k_{(i)}^4 W_{(i)}(r,\varphi) = 0. \tag{9}$$

Circular eigenfrequencies are

$$\omega_{(i)}^2 = k_{(i)}^4 c_{(i)}^4 + a_{(i)}^2 = k_{(i)}^4 \frac{D_{0(i)}}{\rho_i h_i} + \frac{c}{\rho_i h_i}. \tag{10}$$

Substituting $D_{0(i)} = E_{0(i)} h_i^2 / 12(1-\mu^2)$ into Eq. (10), we obtain

$$\omega_{(i)}^2 = k_{(i)}^4 \frac{E_{0(i)} h_i^2}{12\rho_i(1-\mu^2)} + \frac{c}{\rho_i h_i}. \tag{11}$$

It is easy to find the following time functions $T_{(i)}(t)$ in the following forms

$$T_{(i)}(t) = A_{(i)} \cos \omega_{(i)} t + B_{(i)} \sin \omega_{(i)} t. \tag{12}$$

General solutions for the transversal middle surface point displacement for the classical case are in the following forms:

(a) if the plate is rectangular, we can express them in Descartes' coordinates

$$w_i(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_{(i)nm}(x,y) T_{(i)nm}(t), \tag{13}$$

(b) if the plate is circular, it is suitable to use the cylindrical coordinate system:

$$w_{(i)}(r,\varphi,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} [\mathrm{J}_n(k_{(i)nm} r) + K_{(i)nm} \mathrm{I}_n(k_{(i)nm} r)]$$
$$\times \sin(n\varphi + \varphi_{(i)0n}) T_{(i)mn}(t), \tag{14}$$

or

$$w_i(r,\varphi,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_{(i)nm}(r,\varphi) T_{(i)nm}(t), \tag{15}$$

where amplitude eigenfunctions $W_{(i)}(x,y)$, or $W_{(i)}(r,\varphi)$ satisfy the corresponding boundary conditions. For more explanations see Appendices A and B.

4 Particular solutions of the coupled partial differential equations for free oscillations

To solve the corresponding coupled partial differential Eq. (1) for free double plate oscillations, we use following equations to describe their time evolution:

$$w_i(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{(i)nm}(x,y)T_{(i)nm}(t), \tag{16}$$

where the eigenamplitude functions $W_{(i)nm}(x,y)$ are the same, for both plates in the system, as in the case for decoupled plate problems. Then introducing Eq. (16) into the following coupled partial differential equations for free double plate oscillations:

$$\begin{aligned}&\frac{\partial^2 w_1(x,y,t)}{\partial t^2}+c_{(1)}^4\Delta\Delta w_1(x,y,t)\\&\quad-a_{(1)}^2[w_2(x,y,t)-w_1(x,y,t)]=0,\\&\frac{\partial^2 w_2(x,y,t)}{\partial t^2}+c_{(2)}^4\Delta\Delta w_2(x,y,t)\\&\quad+a_{(2)}^2[w_2(x,y,t)-w_1(x,y,t)]=0,\end{aligned} \tag{17}$$

multiplying first and second equation with $W_{(i)sr}(x,y)\mathrm{d}x\mathrm{d}y$, integrating along the middle plate surface and taking into account orthogonality conditions (see Appendix B (8)) and corresponding boundary conditions, we obtain mn coupled second order ordinary differential equations for determination of the unknown time functions $T_{(i)nm}(t)$ in the following form:

$$\begin{aligned}&\ddot{T}_{(1)nm}(t)+\omega_{(1)nm}^2T_{(1)nm}(t)-a_{(1)}^2T_{(2)nm}(t)=0,\\&\ddot{T}_{(2)nm}(t)+\omega_{(2)nm}^2T_{(2)nm}(t)-a_{(2)}^2T_{(1)nm}(t)=0.\end{aligned} \tag{18}$$

From Eq. (18), we obtain mn fourth order ordinary differential equations in the form:

$$\begin{aligned}&\ddot{\ddot{T}}_{(1)nm}(t)+\left[\omega_{(1)nm}^2+\omega_{(2)nm}^2\right]\ddot{T}_{(1)nm}(t)\\&\quad+\left[\omega_{(1)nm}^2\omega_{(2)nm}^2-a_{(1)}^2a_{(2)}^2\right]T_{(1)nm}(t)=0,\end{aligned} \tag{19}$$

with the corresponding mn frequency equations in the form of the polynomial biquadratic equation with respect to unknown circular eigenfrequencies $\tilde{\omega}_{nm}^2$:

$$\begin{aligned}&\tilde{\omega}_{nm}^4+[\omega_{(1)nm}^2+\omega_{(2)nm}^2]\tilde{\omega}_{nm}^2\\&\quad+\left[\omega_{(1)nm}^2\omega_{(2)nm}^2-a_{(1)}^2a_{(2)}^2\right]=0.\end{aligned} \tag{20}$$

Solving Eq. (20), we obtain,

$$\tilde{\omega}_{nm(1,2)}^2=\frac{[\omega_{(1)nm}^2+\omega_{(2)nm}^2]}{2}\mp\frac{\sqrt{\left[\omega_{(1)nm}^2-\omega_{(2)nm}^2\right]^2+4a_{(1)}^2a_{(2)}^2}}{2}, \tag{21}$$

or

$$\tilde{\omega}_{nm(1,2)}^2=\frac{\left\{k_{(1)nm}^4\left[c_{(1)}^4+c_{(2)}^4\right]+a_{(1)}^2+a_{(2)}^2\right\}}{2}\mp\frac{\sqrt{\left\{k_{(1)nm}^4\left[c_{(1)}^4-c_{(2)}^4\right]+a_{(1)}^2-a_{(2)}^2\right\}^2+4a_{(1)}^2a_{(2)}^2}}{2}. \tag{22}$$

The solutions of the ordinary differential equations (18), are in the form:

$$\begin{aligned}T_{(1)nm}(t)&=A_{nm}\cos\tilde{\omega}_{nm(1)}t+B_{nm}\sin\tilde{\omega}_{nm(1)}t\\&\quad+C_{nm}\cos\tilde{\omega}_{nm(2)}t+D_{nm}\sin\tilde{\omega}_{nm(2)}t,\\T_{(2)nm}(t)&=A_{(2)nm}^{(1)}[A_{nm}\cos\tilde{\omega}_{nm(1)}t\\&\quad+B_{nm}\sin\tilde{\omega}_{nm(1)}t]\\&\quad+A_{(2)nm}^{(2)}[C_{nm}\cos\tilde{\omega}_{nm(2)}t\\&\quad+D_{nm}\sin\tilde{\omega}_{nm(2)}t],\end{aligned} \tag{23}$$

where the unknown constants $A_{nm}, B_{nm}, C_{nm}, D_{nm}$, are determined by the initial conditions. For more detailed explanations see Appendix C.

Then, the particular solutions of the coupled partial differential equations for free oscillations are:

$$\begin{aligned}&w_1(x,y,t)\\&=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{(1)nm}(x,y)\{A_{nm}\cos\tilde{\omega}_{nm(1)}t\\&\quad+B_{nm}\sin\tilde{\omega}_{nm(1)}t+[C_{nm}\cos\tilde{\omega}_{nm(2)}t\\&\quad+D_{nm}\sin\tilde{\omega}_{nm(2)}t]\},\\&w_2(x,y,t)\\&=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{(2)nm}(x,y)\\&\quad\times\left\{A_{(2)nm}^{(1)}[A_{nm}\cos\tilde{\omega}_{nm(1)}t+B_{nm}\sin\tilde{\omega}_{nm(1)}t]\right.\\&\quad\left.+A_{(2)nm}^{(2)}[C_{nm}\cos\tilde{\omega}_{nm(2)}t+D_{nm}\sin\tilde{\omega}_{nm(2)}t]\right\}.\end{aligned} \tag{24}$$

The initial conditions are

$$\begin{aligned}w_i(x,y,0)&=g_i(x,y),\\\left.\frac{\partial w_i(x,y,t)}{\partial t}\right|_{t=0}&=\tilde{g}_i(x,y),\end{aligned} \tag{25}$$

where the initial condition functions for the middle plate point displacement $g_i(x,y)$ and for the middle plate point velocity $\tilde{g}_i(x,y)$ satisfy boundary conditions. Then, with the previous initial conditions (25), unknown coefficients A_{nm}, B_{nm}, C_{nm}, D_{nm}, can be determined by nonhomogeneous algebraic equations as

$$A_{nm} = \frac{\iint_A \left[A^{(2)}_{(2)nm} g_1(x,y) - g_2(x,y)\right] W_{(1)nm}(x,y)\mathrm{d}x\mathrm{d}y}{\left[A^{(2)}_{(2)nm} - A^{(1)}_{(2)nm}\right] \iint_A [W_{(1)nm}(x,y)]^2 \mathrm{d}x\mathrm{d}y},$$
$$C_{nm} = \frac{\iint_A \left[g_2(x,y) - A^{(1)}_{(2)nm} g_1(x,y)\right] W_{(1)nm}(x,y)\mathrm{d}x\mathrm{d}y}{\left[A^{(2)}_{(2)nm} - A^{(1)}_{(2)nm}\right] \iint_A [W_{(1)nm}(x,y)]^2 \mathrm{d}x\mathrm{d}y},$$
$$B_{nm} = \frac{\iint_A \left[A^{(2)}_{(2)nm} \tilde{g}_1(x,y) - \tilde{g}_2(x,y)\right] W_{(1)nm}(x,y)\mathrm{d}x\mathrm{d}y}{\tilde{\omega}_{nm(1)}\left[A^{(2)}_{(2)nm} - A^{(1)}_{(2)nm}\right] \iint_A [W_{(1)nm}(x,y)]^2 \mathrm{d}x\mathrm{d}y},$$
$$D_{nm} = \frac{\iint_A \left[\tilde{g}_2(x,y) - A^{(1)}_{(2)nm} \tilde{g}_1(x,y)\right] W_{(1)nm}(x,y)\mathrm{d}x\mathrm{d}y}{\tilde{\omega}_{nm(2)}\left[A^{(2)}_{(2)nm} - A^{(1)}_{(2)nm}\right] \iint_A [W_{(1)nm}(x,y)]^2 \mathrm{d}x\mathrm{d}y}. \tag{26}$$

The solutions (24), with constants A_{nm}, B_{nm}, C_{nm}, D_{nm} in the form (26), are the first main analytical result of our research on transversal vibrations of an elastically connected double-plate system. From the analytical solutions (24) and corresponding expressions (26), we can conclude that for every pair of m and n, one eigenamplitude function corresponds to two circular eigenfrequencies and corresponding two-frequency time function $T_{(i)nm}(t)$. We can also conclude that elastic Winkler-type layer introduces the duplication of the number of circular frequencies which correspond to one eigenamplitude function.

5 Particular solutions of the coupled partial differential equations for forced oscillations

Our next step is to derive the analytical solution of the coupled partial differential equations for the forced oscillations:

$$\frac{\partial^2 w_1(x,y,t)}{\partial t^2} + c^4_{(1)}\Delta\Delta w_1(x,y,t) - a^2_{(1)}[w_2(x,y,t) - w_1(x,y,t)] = \tilde{q}_1(x,y,t),$$
$$\frac{\partial^2 w_2(x,y,t)}{\partial t^2} + c^4_{(2)}\Delta\Delta w_2(x,y,t) + a^2_{(2)}[w_2(x,y,t) - w_1(x,y,t)] = -\tilde{q}_2(x,y,t). \tag{27}$$

For solving the corresponding coupled partial differential equations (27) for the forced double-plate oscillations, the eigenamplitude functions $W_{(i)nm}(x,y)$ are expanded into series with time coefficients in the form of the unknown time functions $T_{(i)nm}(t)$ describing their time evolution in the form of Eq. (16), where $W_{(i)nm}(x,y)$ are the same as in the case with the decoupled-plate problem. Then introducing Eq. (16) into Eq. (27), following the same procedure as in the previous section, we obtain the following nonhomogeneous second order ordinary differential equations with respect to the time functions $T_{(i)nm}(t)$ as

$$\ddot{T}_{(1)nm}(t) + \omega^2_{(1)nm} T_{(1)nm}(t) - a^2_{(1)} T_{(2)nm}(t) = f_{(1)nm}(t),$$
$$\ddot{T}_{(2)nm}(t) + \omega^2_{(2)nm} T_{(2)nm}(t) - a^2_{(2)} T_{(1)nm}(t) = f_{(2)nm}(t), \tag{28}$$

where $f_{(1)nm}(t)$ and $f_{(2)nm}(t)$ are defined by the following expressions:

$$f_{(1)nm}(t) = \frac{\int_0^a\int_0^b \tilde{q}_1(x,y,t) W_{(1)nm}(x,y)\mathrm{d}x\mathrm{d}y}{\int_0^a\int_0^b [W_{(1)nm}(x,y)]^2\mathrm{d}x\mathrm{d}y},$$
$$f_{(2)nm}(t) = \frac{\int_0^a\int_0^b \tilde{q}_2(x,y,t) W_{(2)nm}(x,y)\mathrm{d}x\mathrm{d}y}{\int_0^a\int_0^b [W_{(2)nm}(x,y)]^2\mathrm{d}x\mathrm{d}y}. \tag{29}$$

The Lagrange's method of the variations of the constants A_{nm}, B_{nm}, C_{nm}, D_{nm} be applied to the solutions of Eq. (28) in the form Eq. (23). We assume that $A_{nm}(t)$, $B_{nm}(t)$, $C_{nm}(t)$, $D_{nm}(t)$ are time functions and write:

$$T_{(1)nm}(t) = A_{nm}(t)\cos\tilde{\omega}_{nm(1)}t + B_{nm}(t)\sin\tilde{\omega}_{nm(1)}t + C_{nm}(t)\cos\tilde{\omega}_{nm(2)}t + D_{nm}(t)\sin\tilde{\omega}_{nm(2)}t,$$
$$T_{(2)nm}(t) = A^{(1)}_{(2)nm}[A_{nm}(t)\cos\tilde{\omega}_{nm(1)}t + B_{nm}(t)\sin\tilde{\omega}_{nm(1)}t] + A^{(2)}_{(2)nm}[C_{nm}(t)\cos\tilde{\omega}_{nm(2)}t + D_{nm}(t)\sin\tilde{\omega}_{nm(2)}t]. \tag{30}$$

In order to obtain the first and second derivatives with respect to time of the proposed time functions, we assume that first derivatives of the time functions $T_{(i)nm}(t)$ are equal to those which correspond to the constant coefficients and obtain the following equations:

$$\frac{\mathrm{d}A_{nm}(t)}{\mathrm{d}t}\cos\tilde{\omega}_{nm(1)}t + \frac{\mathrm{d}B_{nm}(t)}{\mathrm{d}t}\sin\tilde{\omega}_{nm(1)}t + \frac{\mathrm{d}C_{nm}(t)}{\mathrm{d}t}\cos\tilde{\omega}_{nm(2)}t + \frac{\mathrm{d}D_{nm}(t)}{\mathrm{d}t}\sin\tilde{\omega}_{nm(2)}t = 0,$$
$$A^{(1)}_{(2)nm}\left[\frac{\mathrm{d}A_{nm}(t)}{\mathrm{d}t}\cos\tilde{\omega}_{nm(1)}t + \frac{\mathrm{d}B_{nm}(t)}{\mathrm{d}t}\sin\tilde{\omega}_{nm(1)}t\right] + A^{(2)}_{(2)nm}\left[\frac{\mathrm{d}C_{nm}(t)}{\mathrm{d}t}\cos\tilde{\omega}_{nm(2)}t + \frac{\mathrm{d}D_{nm}(t)}{\mathrm{d}t}\sin\tilde{\omega}_{nm(2)}t\right] = 0. \tag{31}$$

After introducing the second derivatives of $T_{(i)nm}(t)$ into the nonhomogeneous second order ordinary differential equations (28), we obtain the nonhomogeneous algebraic equations with unknown first derivatives of the unknown coefficients $A_{nm}(t)$, $B_{nm}(t)$, $C_{nm}(t)$, $D_{nm}(t)$. We can transform (see Appendix D) the previous nonhomogeneous algebraic equations into the following form:

$$\frac{dA_{nm}(t)}{dt}\cos\tilde{\omega}_{nm(1)}t+\frac{dB_{nm}(t)}{dt}\sin\tilde{\omega}_{nm(1)}t=0,$$
$$\frac{dC_{nm}(t)}{dt}\cos\tilde{\omega}_{nm(2)}t+\frac{dD_{nm}(t)}{dt}\sin\tilde{\omega}_{nm(2)}t=0,$$
$$-\frac{dA_{nm}(t)}{dt}\sin n\tilde{\omega}_{nm(1)}t+\frac{dB_{nm}(t)}{dt}\cos\tilde{\omega}_{nm(1)}t=\frac{\left[A^{(2)}_{(2)nm}f_{(1)nm}(t)-f_{(2)nm}(t)\right]}{\tilde{\omega}_{nm(1)}\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]},$$
$$-\frac{dC_{nm}(t)}{dt}\sin\tilde{\omega}_{nm(2)}t+\frac{dD_{nm}(t)}{dt}\cos\tilde{\omega}_{nm(2)}t=-\frac{\left[A^{(1)}_{(2)nm}f_{(1)nm}(t)-f_{(2)nm}(t)\right]}{\tilde{\omega}_{nm(2)}\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]}. \tag{32}$$

After solving (32), we obtain the first derivatives of the unknown coefficients $A_{nm}(t), B_{nm}(t), C_{nm}(t), D_{nm}(t)$ as:

$$\frac{dA_{nm}(t)}{dt}=\frac{\left[A^{(2)}_{(2)nm}f_{(1)nm}(t)-f_{(2)nm}(t)\right]}{\tilde{\omega}_{nm(1)}\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]}\cos\tilde{\omega}_{nm(1)}t,$$
$$\frac{dB_{nm}(t)}{dt}=-\frac{\left[A^{(2)}_{(2)nm}f_{(1)nm}(t)-f_{(2)nm}(t)\right]}{\tilde{\omega}_{nm(1)}\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]}\sin\tilde{\omega}_{nm(1)}t,$$
$$\frac{dC_{nm}(t)}{dt}=-\frac{\left[A^{(1)}_{(2)nm}f_{(1)nm}(t)-f_{(2)nm}(t)\right]}{\tilde{\omega}_{nm(2)}\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]}\cos\tilde{\omega}_{nm(2)}t,$$
$$\frac{dD_{nm}(t)}{dt}=\frac{\left[A^{(1)}_{(2)nm}f_{(1)nm}(t)-f_{(2)nm}(t)\right]}{\tilde{\omega}_{nm(2)}\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]}\sin\tilde{\omega}_{nm(2)}t. \tag{33}$$

Then integrating these expressions, we obtain the expressions for the coefficients $A_{nm}(t)$, $B_{nm}(t)$, $C_{nm}(t)$, $D_{nm}(t)$ in the following forms:

$$A_{nm}(t)=A_{0nm}+\frac{\int_0^t\left[A^{(2)}_{(2)nm}f_{(1)nm}(\tau)-f_{(2)nm}(\tau)\right]\cos\tilde{\omega}_{nm(1)}\tau d\tau}{\tilde{\omega}_{nm(1)}\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]},$$
$$B_{nm}(t)=B_{0nm}-\frac{\int_0^t\left[A^{(2)}_{(2)nm}f_{(1)nm}(\tau)-f_{(2)nm}(\tau)\right]\sin\tilde{\omega}_{nm(1)}\tau d\tau}{\tilde{\omega}_{nm(1)}\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]},$$
$$C_{nm}(t)=C_{0nm}-\frac{\int_0^t\left[A^{(1)}_{(2)nm}f_{(1)nm}(\tau)-f_{(2)nm}(\tau)\right]\cos\tilde{\omega}_{nm(2)}\tau d\tau}{\tilde{\omega}_{nm(2)}\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]},$$
$$D_{nm}(t)=D_{0nm}+\frac{\int_0^t\left[A^{(1)}_{(2)nm}f_{(1)nm}(\tau)-f_{(2)nm}(\tau)\right]\sin\tilde{\omega}_{nm(2)}\tau d\tau}{\tilde{\omega}_{nm(2)}\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]}, \tag{34}$$

where $A_{0nm}, B_{0nm}, C_{0nm}, D_{0nm}$ are integral constants.

The solutions of $T_{(i)nm}(t)$ for forced vibrations are in the form:

$$\begin{aligned}T_{(1)nm}(t)&=A_{0nm}\cos\tilde{\omega}_{nm(1)}t+B_{0nm}\sin\tilde{\omega}_{nm(1)}t+C_{0nm}\cos\tilde{\omega}_{nm(2)}t+D_{0nm}\sin\tilde{\omega}_{nm(2)}t\\&+\frac{\int_0^t\left[A^{(2)}_{(2)nm}f_{(1)nm}(\tau)-f_{(2)nm}(\tau)\right]\cos\tilde{\omega}_{nm(1)}(t-\tau)d\tau}{\tilde{\omega}_{nm(1)}\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]}\\&-\frac{\int_0^t\left[A^{(1)}_{(2)nm}f_{(1)nm}(\tau)-f_{(2)nm}(\tau)\right]\cos\tilde{\omega}_{nm(2)}(t-\tau)d\tau}{\tilde{\omega}_{nm(2)}\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]}\\T_{(2)nm}(t)&=A^{(1)}_{(2)nm}[A_{0nm}\cos\tilde{\omega}_{nm(1)}t+B_{0nm}\sin\tilde{\omega}_{nm(1)}t]\\&+\frac{\int_0^t\left[A^{(2)}_{(2)nm}f_{(1)nm}(\tau)-f_{(2)nm}(\tau)\right]\cos\tilde{\omega}_{nm(1)}(t-\tau)d\tau}{\tilde{\omega}_{nm(1)}\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]}\\&+A^{(2)}_{(2)nm}[C_{0nm}\cos\tilde{\omega}_{nm(2)}t+D_{0nm}\sin\tilde{\omega}_{nm(2)}t]\\&-\frac{\int_0^t\left[A^{(1)}_{(2)nm}f_{(1)nm}(\tau)-f_{(2)nm}(\tau)\right]\cos\tilde{\omega}_{nm(2)}(t-\tau)d\tau}{\tilde{\omega}_{nm(2)}\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]},\end{aligned} \tag{35}$$

which contain the following set of the unknown constants A_{0nm}, B_{0nm}, C_{0nm}, D_{0nm} to be determined by initial plate-conditions.

Then, we obtain the particular solutions of the coupled partial differential equations for the forced oscillations in the form of corresponding plate displacements

$$\begin{aligned}w_1(x,y,t)&=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{(1)nm}(x,y)\times[A_{0nm}\cos\tilde{\omega}_{nm(1)}t+B_{0nm}\sin\tilde{\omega}_{nm(1)}t\\&+C_{0nm}\cos\tilde{\omega}_{nm(2)}t+D_{0nm}\sin\tilde{\omega}_{nm(2)}t]+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{(1)nm}(x,y)\\&\times\left\{\frac{\int_0^t\left[A^{(2)}_{(2)nm}f_{(1)nm}(\tau)-f_{(2)nm}(\tau)\right]\cos\tilde{\omega}_{nm(1)}(t-\tau)d\tau}{\tilde{\omega}_{nm(1)}\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]}-\frac{\int_0^t\left[A^{(1)}_{(2)nm}f_{(1)nm}(\tau)-f_{(2)nm}(\tau)\right]\cos\tilde{\omega}_{nm(2)}(t-\tau)d\tau}{\tilde{\omega}_{nm(2)}\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]}\right\},\\w_2(x,y,t)&=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{(1)nm}(x,y)\times[A^{(1)}_{(2)nm}(A_{0nm}\cos\tilde{\omega}_{nm(1)}t+B_{0nm}\sin\tilde{\omega}_{nm(1)}t)\\&+A^{(2)}_{(2)nm}(C_{0nm}\cos\tilde{\omega}_{nm(2)}t+D_{0nm}\sin\tilde{\omega}_{nm(2)}t)]+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{(1)nm}(x,y)\\&\times\left\{A^{(1)}_{(2)nm}\frac{\int_0^t\left[A^{(2)}_{(2)nm}f_{(1)nm}(\tau)-f_{(2)nm}(\tau)\right]\cos\tilde{\omega}_{nm(1)}(t-\tau)d\tau}{\tilde{\omega}_{nm(1)}\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]}-A^{(2)}_{(2)nm}\frac{\int_0^t\left[A^{(1)}_{(2)nm}f_{(1)nm}(\tau)-f_{(2)nm}(\tau)\right]\cos\tilde{\omega}_{nm(2)}(t-\tau)d\tau}{\tilde{\omega}_{nm(2)}\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]}\right\}.\end{aligned} \tag{36}$$

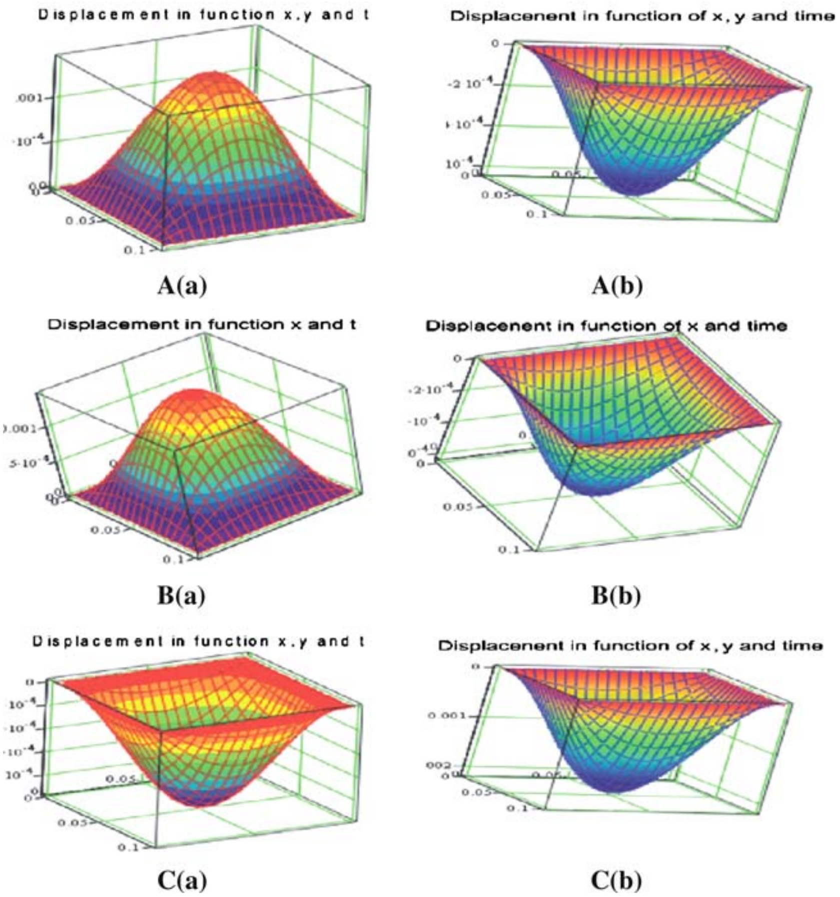

Fig. 5 The characteristic shapes of the double plate system middle surfaces

The solutions (36) are the second main analytical result of our research of transversal forced vibrations of elastically connected double-plates system. From the analytical solutions (36) and corresponding expressions (35), we can conclude that for every pair of m and n, two circular eigenfrequencies and the corresponding number of the forced frequencies as well as corresponding multy-frequency time function $T_{(i)nm}(t)$ correspond to one eigenamplitude function. We can also conclude that elastic Winkler-type layer introduces into system the duplication of the number of circular frequencies which correspond to one eigenamplitude function.

For the initial conditions in the form of Eq. (25), the initial condition functions $g_i(x, y)$ for the middle plate point displacement and $\tilde{g}_i(x, y)$ for the middle plate point velocity satisfy boundary conditions. Then, unknown coefficients A_{0nm}, B_{0nm}, C_{0nm}, D_{0nm} are determined by nonhomogeneous algebraic equations.

5.1 Special case

For the case when the external excitations are one frequency forces distributed along upper plate contour surface, the system of differential equations (28) is in the form:

$$\begin{aligned} &\ddot{T}_{(1)nm}(t) + \omega^2_{(1)nm} T_{(1)nm}(t) - a^2_{(1)} T_{(2)nm}(t) \\ &\quad = h_{(1)nm} \cos \Omega_{nm} t, \\ &\ddot{T}_{(2)nm}(t) + \omega^2_{(2)nm} T_{(2)nm}(t) - a^2_{(2)} T_{(1)nm}(t) = 0, \end{aligned} \tag{37}$$

with corresponding solutions in the form defined by expressions:

$$\begin{aligned} T_{(1)nm}(t) &= A_{0nm} \cos \tilde{\omega}_{nm(1)} t + B_{0nm} \sin \tilde{\omega}_{nm(1)} t \\ &\quad + C_{0nm} \cos \tilde{\omega}_{nm(2)} t + D_{0nm} \sin \tilde{\omega}_{nm(2)} t \\ &\quad + \frac{h_{(1)nm}\left(\omega^2_{(1)nm} - \Omega^2_{nm}\right)}{\left(\omega^2_{(1)nm} - \Omega^2_{nm}\right)\left(\omega^2_{(2)nm} - \Omega^2_{nm}\right) - a^2_{(1)} a^2_{(2)}} \\ &\quad \times \cos \Omega_{nm} t, \\ T_{(2)nm}(t) &= A^{(1)}_{(2)nm}[A_{0nm} \cos \tilde{\omega}_{nm(1)} t + B_{0nm} \sin \tilde{\omega}_{nm(1)} t] \\ &\quad + A^{(2)}_{(2)nm}[C_{0nm} \cos \tilde{\omega}_{nm(2)} t + D_{0nm} \sin \tilde{\omega}_{nm(2)} t] \\ &\quad + \frac{a^2_{(2)} h_{(1)nm}}{\left(\omega^2_{(1)nm} - \Omega^2_{nm}\right)\left(\omega^2_{(2)nm} - \Omega^2_{nm}\right) - a^2_{(1)} a^2_{(2)}} \\ &\quad \times \cos \Omega_{nm} t. \end{aligned} \tag{38}$$

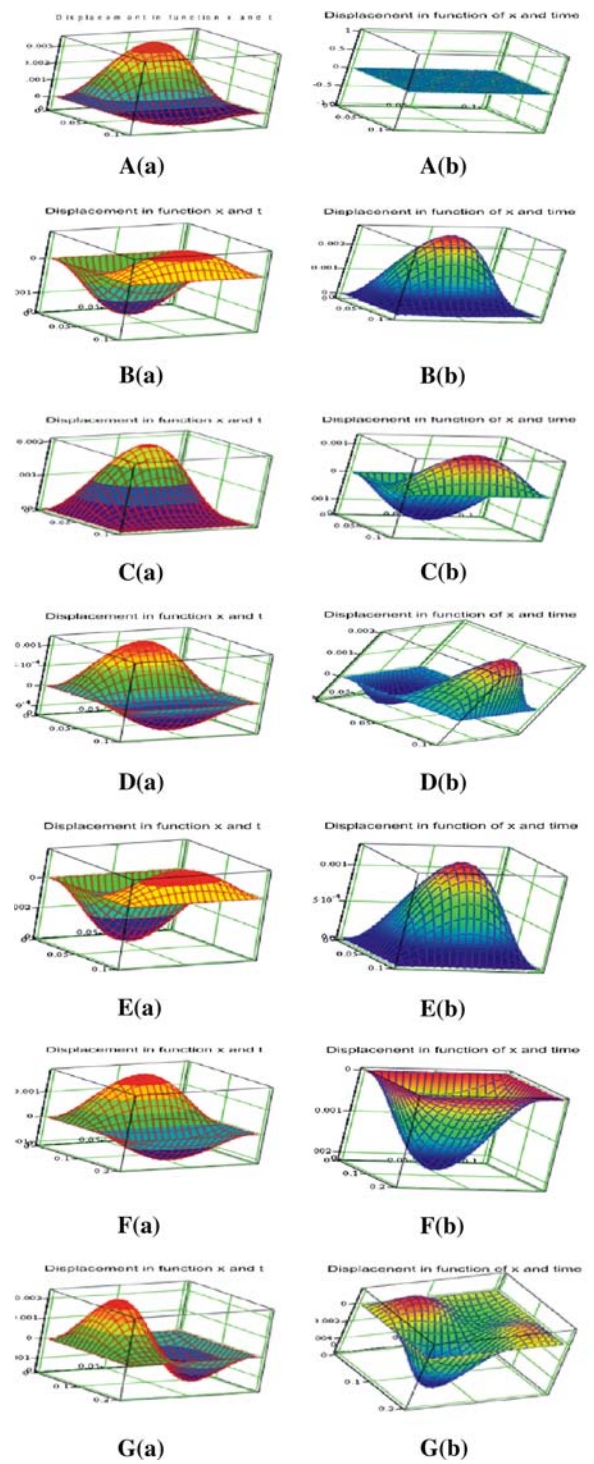

Fig. 6 The characteristic shape of the plate middle surface free vibrations (two first modes 11 and 12)

Then, the particular solutions of the coupled partial differential equations for forced oscillations corresponding to plate displacements under the external excitations by one frequency forces distributed along the upper plate contour surface with forced circular frequency Ω_{nm} are in the form:

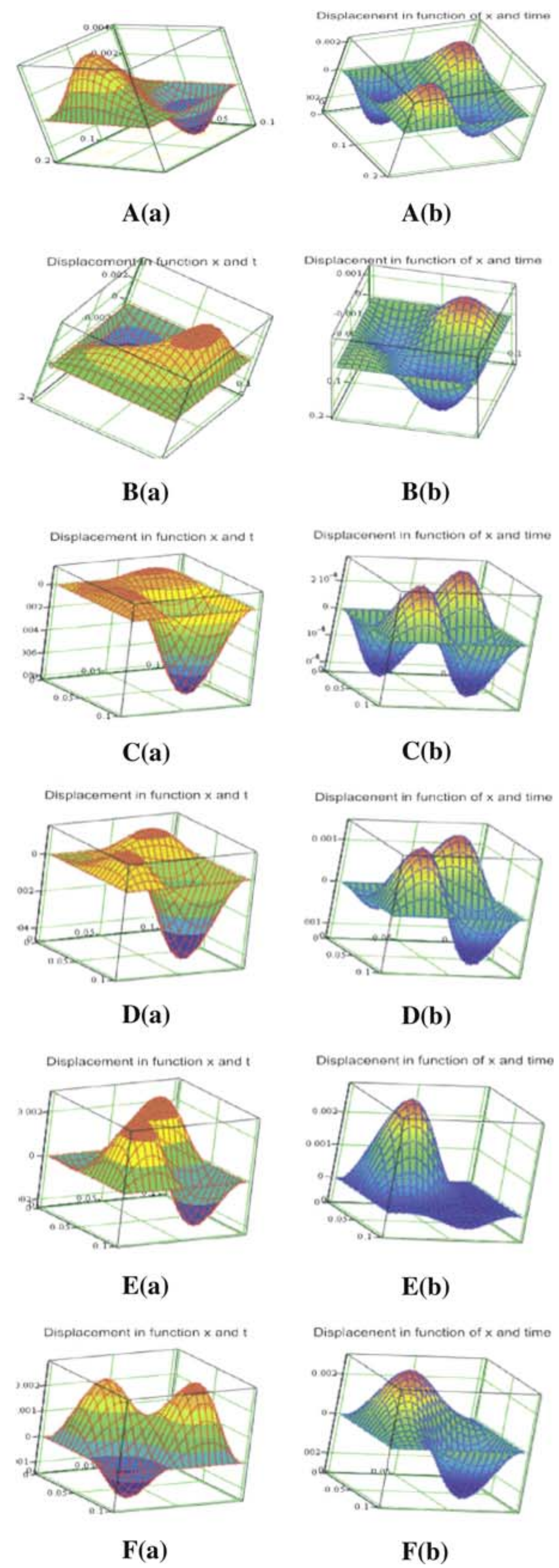

Fig. 7 The characteristic shape of the plate middle surface free vibrations

$$w_1(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{(1)nm}(x,y)\times[A_{0nm}\cos\tilde{\omega}_{nm(1)}t+B_{0nm}\sin\tilde{\omega}_{nm(1)}t+C_{0nm}\cos\tilde{\omega}_{nm(2)}t+D_{0nm}\sin\tilde{\omega}_{nm(2)}t]+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{(1)nm}(x,y)\times\left[\frac{h_{(1)nm}(\omega^2_{(1)nm}-\Omega^2_{nm})}{(\omega^2_{(1)nm}-\Omega^2_{nm})(\omega^2_{(2)nm}-\Omega^2_{nm})-a^2_{(1)}a^2_{(2)}}\times\cos\Omega_{nm}t\right],\tag{39}$$

$$w_2(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{(1)nm}(x,y)\times\left[A^{(1)}_{(2)nm}(A_{0nm}\cos\tilde{\omega}_{nm(1)}t+B_{0nm}\sin\tilde{\omega}_{nm(1)}t)+A^{(2)}_{(2)nm}(C_{0nm}\cos\tilde{\omega}_{nm(2)}t+D_{0nm}\sin\tilde{\omega}_{nm(2)}t)\right]+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{(1)nm}(x,y)\times\left[\frac{a^2_{(2)}h_{(1)nm}}{(\omega^2_{(1)nm}-\Omega^2_{nm})(\omega^2_{(2)nm}-\Omega^2_{nm})-a^2_{(1)}a^2_{(2)}}\times\cos\Omega_{nm}t\right].$$

And,

$$A_{0nm}=\frac{\iint_A\left[A^{(2)}_{(2)nm}g_1(x,y)-g_2(x,y)\right]W_{(1)nm}(x,y)\mathrm{d}x\mathrm{d}y}{\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]\iint_A[W_{(1)nm}(x,y)]^2\mathrm{d}x\mathrm{d}y}-\frac{A^{(2)}_{(2)nm}\left[h_{(1)nm}(\omega^2_{(1)nm}-\Omega^2_{nm})/\left((\omega^2_{(1)nm}-\Omega^2_{nm})(\omega^2_{(2)nm}-\Omega^2_{nm})-a^2_{(1)}a^2_{(2)}\right)\right]}{\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]}+\frac{\left[a^2_{(2)}h_{(1)nm}/\left((\omega^2_{(1)nm}-\Omega^2_{nm})(\omega^2_{(2)nm}-\Omega^2_{nm})-a^2_{(1)}a^2_{(2)}\right)\right]}{\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]},$$

$$C_{0nm}=\frac{\iint_A\left[g_2(x,y)-A^{(1)}_{(2)nm}g_1(x,y)\right]W_{(1)nm}(x,y)\mathrm{d}x\mathrm{d}y}{\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]\iint_A[W_{(1)nm}(x,y)]^2\mathrm{d}x\mathrm{d}y}-\frac{\left[h_{(1)nm}(\omega^2_{(1)nm}-\Omega^2_{nm})/\left((\omega^2_{(1)nm}-\Omega^2_{nm})(\omega^2_{(2)nm}-\Omega^2_{nm})-a^2_{(1)}a^2_{(2)}\right)\right]}{\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]}+\frac{A^{(1)}_{(2)nm}\left[a^2_{(2)}h_{(1)nm}/\left((\omega^2_{(1)nm}-\Omega^2_{nm})(\omega^2_{(2)nm}-\Omega^2_{nm})-a^2_{(1)}a^2_{(2)}\right)\right]}{\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]},\tag{40}$$

$$B_{0nm}=\frac{\iint_A\left[A^{(2)}_{(2)nm}\tilde{g}_1(x,y)-\tilde{g}_2(x,y)\right]W_{(1)nm}(x,y)\mathrm{d}x\mathrm{d}y}{\tilde{\omega}_{nm(1)}\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]\iint_A[W_{(1)nm}(x,y)]^2\mathrm{d}x\mathrm{d}y},\quad D_{0nm}=\frac{\iint_A\left[\tilde{g}_2(x,y)-A^{(1)}_{(2)nm}\tilde{g}_1(x,y)\right]W_{(1)nm}(x,y)\mathrm{d}x\mathrm{d}y}{\tilde{\omega}_{nm(2)}\left[A^{(2)}_{(2)nm}-A^{(1)}_{(2)nm}\right]\iint_A[W_{(1)nm}(x,y)]^2\mathrm{d}x\mathrm{d}y}.$$

Then, the time functions $T_{(i)nm}(t)$ describing their time evolution and satisfying initial and boundary conditions are in the form (38) with coefficients $A_{0nm}, B_{0nm}, C_{0nm}, D_{0nm}$ determined by expressions (40).

The solutions (39) are the third main analytical result of our research. From the analytical solutions (39), and from the corresponding expressions (38) and (40), we can conclude that for every pair of m and n, two circular eigenfrequencies and one forced frequency as well as corresponding three-frequency time functions $T_{(i)nm}(t)$ correspond to one eigenamplitude function. We can also conclude that the elastic Winkler-type layer introduced into the two plate system is the origin of the duplication of the number of circular frequencies which corresponds to one eigenamplitude function. From (38), the following conclusions can be drawn: the time function $T_{(i)nm}(t)$, corresponding to one pair of m and n, contain four terms corresponding to pure free two-frequency vibrations with two circular eigenfrequencies [determined by expression (21) or (22)], four terms of two frequency vibrations also with two corresponding circular eigenfrequencies, but with amplitudes depending on the external force frequency, and one term of the one-frequency forced vibrations with corresponding external force circular frequency Ω_{nm}.

When $\Omega^2_{nm}=\omega^2_{(1)nm}$, from (38) to (40), we can conclude that the value of the external force frequency can provide a condition of the dynamical absorption into the forced vibration mode for the upper plate. It is the case that the forced part of the upper plate forced vibration displacement is equal to zero, when the external excitation is distributed along the upper contour surface of the upper plate. Then, the lower plate is under forced vibration regime, without direct external excitation distributed along the lower contour surface of the lower plate and the upper plate is only in the state of the free two-frequency vibration regime.

When $\left(\omega^2_{(1)nm} - \Omega^2_{nm}\right)\left(\omega^2_{(2)nm} - \Omega^2_{nm}\right) - a^2_{(1)}a^2_{(2)} = 0$, from (38) to (40), we can conclude that the value of the external force frequency can provide the condition of the resonance state for both elastically connected plates and that external force resonant frequency values are in the form:

$$\Omega^2_{\text{rez}(1,2)} = \tilde{\omega}^2_{nm(1,2)} = \frac{\left[\omega^2_{(1)nm} + \omega^2_{(2)nm}\right] \mp \sqrt{\left[\omega^2_{(1)nm} - \omega^2_{(2)nm}\right]^2 + 4a^2_{(1)}a^2_{(2)}}}{2}. \tag{41}$$

Then, we can also conclude that for one pair of m and n, two possibilities for appearance of the resonance states are present for both elastically connected plates.

6 Numerical experiment and visualizations

For the numerical experiment and analysis, we consider a rectangular plate made of steel with dimensions $20 \times 10 \times 1\ \text{cm}^3$. Using the MathCad, we present the numerical results in the form of the plate middle-surface against the time, and also the time-history diagrams of the plate middle-surface point displacements.

Our intention is to make qualitative analysis of the numerical results, so we do not present tables of numerical results, but only graphical presentations.

In Fig. 2, we can see the form of the eigenamplitude functions for the first four shapes among the infinite family with different modes of free vibrations of the rectangular plate with hinged-edge plate contour, for the following mn pairs: 11, 12, 21, 22 and 32 in Fig. 2a–d. These visualizations show also the shapes of the plate middle surface, and the forms of the functional dependence of the plate middle-surface displacement for one frequency vibration of the single plate systems on the ideal elastic foundations unperturbed by initial displacements in the form of corresponding eigenamplitude function at initial time.

In Fig. 3, the time histories of the plate middle surface points corresponding to the time functions $T_{(1)nm}(t)$ and $T_{(2)nm}(t)$ are presented in separate Fig. 3a and b and both in Fig. 3c. We can see that these time functions for free vibrations are in two-frequency regime for every shape of the modes.

In Fig. 4 the time history diagrams of the plate middle surface points corresponding to time functions $T_{(1)}(t)$ and $T_{(2)}(t)$ in the forced regime are presented in separate Fig. 3 a and b and both in Fig. 4c.

We can conclude that the upper contour surface of the upper plate is excited by the external one-frequency forced excitation and the excited oscillation process is in the three-frequency dynamic state of the plate oscillations. These oscillation frequencies are two circular eigenfrequencies and one forced circular frequency of the external forced excitation frequency.

The space surfaces in Fig. 5 present characteristic shapes of the double-plate system middle surfaces at the time A, B and C, with rectangular plates, hinged-edge plate contour and for initial conditions which initiate only 11-family oscillations. We can see that the upper plate middle surface (a) and the lower plate middle surface (b) oscillate in the same direction, as well as in the opposite directions. The regime is a two-frequency one.

Figure 6 shows the characteristic shape of the plate middle surface free vibrations at the time A, B, C, D, E, F and G, which are initiated by initial plate middle surface displacements in the form of the two first modes 11 and 12.

We can see that the plate middle surfaces, at the time A, B, C, D, E, F and G, are in the different resultant forms, when they are in the form of deformed eigenamplitude functions 11 and 12, with the same, or opposite direction displacements of the middle plate surfaces points. We can conclude that the plates oscillate transversally with four frequency regimes, corresponding to the sum of the two shape four-frequency colinear oscillations.

Figure 7 shows characteristic shapes of the plate middle surface free vibrations, at the time A, B, C, D, E, F, initiated by the initial plate middle surface displacements in the form of the four first modes 11, 12, 21 and 22.

We can see that the plate middle surfaces are in different resultant forms, at the time A, B, C, D, E, F, when they are in the form of deformed eigen-amplitude functions 11, 12, 21 and 22, with the same, or opposite direction displacements of the middle plate surface points. We can conclude that the plates oscillate transversally with eight frequency regimes, corresponding to the sum of the eight-frequency co-linear oscillations.

7 Concluding remarks

The analytical solutions of the system of coupled partial differential equations of corresponding dynamical free and forced processes are obtained by using the method of Bernoulli's particular integral and Lagrange's method of variation constants. It is shown that one mode vibrations correspond a two-frequency regime for free vibrations and a three-frequency regime for forced vibrations induced by the initial conditions and one-frequency external excitation. The analytical solutions show that the elastic connection between the plates leads to the appearance of a two-frequency regime of time function corresponding to one eigenamplitude function of one mode, and also that time functions of different vibration modes are uncoupled, for each shape of vibrations. Using the MathCad program the corresponding visualizations of the characteristic forms of the plate middle surfaces through time are presented.

Acknowledgments Parts of this research were supported by the Ministry of Sciences and Environmental Protection of Republic Serbia through Mathematical Institute SANU Belgrade Grants no. 1616 and no. ON144002, Theoretical and Applied Mechanics of Rigid and Solid Body. Mechanics of Materials, and, the Faculty of Mechanical Engineering, University of Niš Grant no. 1828. This support is acknowledged. The author wishes to express her gratitude to the reviewers for useful suggestions to the improvement of the paper and to the younger colleague Ms. Dr. Tamara Nestorović-Trajkov for the help with the review of the English language.

Appendix A

Spatial coordinates, eigenamplitude functions and time function for classical case

Let us consider that the spatial coordinate amplitude functions $W_{(i)}(x,y)$, are expressed in the form $W_{(i)}(x,y) = X_{(i)}(x)Y_{(i)}(y)$, and then we can write:

$$\begin{aligned} &X''_{(i)}(x) + \left(\pm n^2 \pm k^2_{(i)}\right) X_{(i)}(x) = 0, \\ &Y''_{(i)}(y) \mp n^2 Y_{(i)}(y) = 0. \end{aligned} \tag{42}$$

If the plates are rectangular, and when we take into consideration a solution in Descartes' coordinates with different boundary conditions along contours, then

$X(x) :=$ sin mx; cos mx; sinh mx; cosh mx,

where $m^2 = \pm n^2 \pm k^2$, and

$Y(y) :=$ sin ny; cos ny; sinh ny; cosh ny.

If the plates are in the circular form, it is suitable to use the polar-cylindrical coordinate system, and then the set of the partial differential equations in the space cylindrical-polar coordinates r, φ and z is:

$$\Delta W_{(i)}(r,\varphi) \pm k^2 W_{(i)}(r,\varphi) = 0,$$

or

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right) W_{(i)}(r,\varphi) \pm k^2 W_{(i)}(r,\varphi) = 0.$$

We write the solutions in the form $W_{(i)}(r,\varphi) = \Phi_{(i)}(\varphi)R_{(i)}(r)$ and obtain the following system of the ordinary differential equations:

$$\begin{aligned} &R''_{(i)}(r) + \frac{1}{r}R'_{(i)}(r) + \left(\pm k^2_{(i)} \mp \frac{n^2}{r^2}\right) R_{(i)}(r) = 0, \\ &\Phi''_{(i)}(\varphi) \pm n^2 \Phi_{(i)}(\varphi) = 0. \end{aligned}$$

The second equation has particular solutions in the form of Neuman's and Bessel's functions, but Neuman's functions for $r = 0$ have an infinite value, then the particular solutions contain only Bessel's function of the first kind with real argument $\mathrm{J}_n(x)$ as well as with imaginary arguments $\mathrm{I}_n(x)$, where $x = kr$. Modified Bessel's function of the first kind with imaginary arguments $\mathrm{I}_n(x)$, of order n, is in the following form:

$$\mathrm{I}_n(x) = (\mathrm{i})^{-n}\mathrm{J}_n(\mathrm{i}x) = \frac{(-1)^n}{2\pi}\int_{-\pi}^{+\pi} \mathrm{e}^{-x\cos t}\cos nt\mathrm{d}t, \quad \mathrm{i} = \sqrt{-1}.$$

If n is an integer, this function satisfies the following differential equation:

$$\mathrm{I}''_n(\mathrm{i}x) + \frac{1}{(\mathrm{i}x)}\mathrm{I}'_n(\mathrm{i}x) - \left(1 + \frac{n^2}{(\mathrm{i}x)^2}\right)\mathrm{I}_n(\mathrm{i}x) = 0. \tag{43}$$

Using the previous considerations with respect to Eq. (17) for their solutions in the polar coordinates for the circular plate, we can write the following expressions:

$$\begin{aligned} &\Phi_{(i)n}(\varphi) = C_{(i)n}\sin(n\varphi + \varphi_{(i)0n}), \\ &R_{(i)nm}(r) = \mathrm{J}_n(k_{(i)nm}r) + K_{(i)nm}\mathrm{I}_n(k_{(i1)nm}r). \end{aligned} \tag{44}$$

Appendix B

Boundary conditions of the rectangular plates with the hinged edges on the plates contours

Let us, now, study the case of the rectangular plates with basic edges a and b, and with the hinged edges on the middle surface plate contour – simply supported plate. Boundary conditions of these rectangular plates are that the transversal displacements on the corresponding middle surface plate contour points are equal to zero, and also at same points the bending moments are equal to zero. So the boundary conditions are expressed in the following forms:

$$\begin{aligned} &\text{for} \quad x = 0 \quad w_{(i)}\ (0,y,t) = 0, \\ &M'_{(i)(x)}(0,y,t) = M'_{(i)yx}(0,y,t) \\ &= -D_{(i)}\left[\frac{\partial^2 w_{(i)}(0,y,t)}{\partial y^2} + \mu_{(i)}\frac{\partial^2 w_{(i)}(0,y,t)}{\partial x^2}\right] = 0, \end{aligned}$$

$$\begin{aligned}
&\text{for} \quad x=a \quad w_{(i)}\ (a,y,t)=0,\\
&M'_{(i)(x)}(a,y,t) = M'_{(i)yx}(a,y,t)\\
&= -D_{(i)}\Big[\frac{\partial^2 w_{(i)}(a,y,t)}{\partial y^2} +\mu\frac{\partial^2 w_{(i)}(a,y,t)}{\partial x^2}\Big]=0, \qquad (45)\\
&\text{for} \quad y=0 \quad w_{(i)}\ (x,0,t)=0,\\
&M'_{(i)(y)}(x,0,t) = M'_{(i)(xy)}(x,0,t)\\
&= -D_{(i)}\Big[\frac{\partial^2 w_{(i)}(x,0,t)}{\partial x^2} +\mu\frac{\partial^2 w_{(i)}(x,0,t)}{\partial y^2}\Big]=0,\\
&\text{for} \quad y=b \quad w_{(i)}\ (x,b,t)=0,\\
&M'_{(i)(y)}(x,b,t) = M'_{(i)(xy)}(x,b,t)\\
&= -D_{(i)}\Big[\frac{\partial^2 w_{(i)}(x,b,t)}{\partial x^2} +\mu\frac{\partial^2 w_{(i)}(x,b,t)}{\partial y^2}\Big]=0.
\end{aligned}$$

Partial differential equations with boundary conditions are satisfied by the following solutions:

$$W_{(i)mn}(x,y) = C_{(i)mn}\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y, \qquad (46)$$

where

$$k^2_{(i)mn} = k^2_{mn} = \Big[\Big(\frac{m\pi}{a}\Big)^2 + \Big(\frac{n\pi}{b}\Big)^2\Big], \qquad (47)$$

$$w_{(i)}(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} T_{(i)mn}(t)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y. \qquad (48)$$

The space coordinate eigenamplitude functions $W_{(i)nm}(x,y)$, satisfied the following conditions of orthogonality:

$$\int_0^a\int_0^b W_{(i)mn}(x,y)W_{(i)sr}(x,y)\mathrm{d}x\mathrm{d}y = \begin{cases} 0, & nm\neq sr,\\ v_{mnnm} = \dfrac{4ab}{\pi^2}, & nm = sr,\end{cases} \qquad (49)$$
$$s,r = 1,2,3,4,\ldots,\infty,$$

which can be obtained by using the system of Eq. (10).

Appendix C

Time functions

We can write formally the system equation (18) with the following matrices of $\boldsymbol{A}_{nm}$ and $\boldsymbol{C}_{nm}$ of two degrees of freedom:

$$\boldsymbol{A}_{nm} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \boldsymbol{C}_{nm} = \begin{pmatrix}\omega^2_{(1)nm} & -a^2_{(1)}\\ -a^2_{(2)} & \omega^2_{(2)nm}\end{pmatrix}, \qquad (50)$$

and by using the solutions in the form of:

$$\begin{aligned} T_{(1)nm}(t) &= A_{(1)nm}\cos(\tilde{\omega}_{nm}t+\alpha_{nm}),\\ T_{(2)nm}(t) &= A_{(2)nm}\cos(\tilde{\omega}_{nm}t+\alpha_{nm}),\end{aligned} \qquad (51)$$

where $\tilde{\omega}^2_{nm}$, are unknown circular eigenfrequencies, $A_{(i)nm}$ unknown amplitudes, and α_{nm} unknown phases. Then the frequency equation is in the form:

$$f_{nm}\left(\tilde{\omega}^2_{nm}\right) = \left|\boldsymbol{C}_{nm} - \tilde{\omega}^2_{nm}\boldsymbol{A}_{nm}\right| = \begin{vmatrix}\omega^2_{(1)nm}-\tilde{\omega}^2_{nm} & -a^2_{(1)}\\ -a^2_{(2)} & \omega^2_{(2)nm}-\tilde{\omega}^2_{nm}\end{vmatrix} = 0. \qquad (52)$$

The frequency equation may be expanded to obtain Eq. (33) with the sets of the two roots $\tilde{\omega}^2_{nm(s)}$, $s = 1,2$.

The relations of the amplitudes for each set are in the form:

$$\frac{A^{(s)}_{(1)mn}}{a^2_{(1)}} = \frac{A^{(s)}_{(2)mn}}{\left[\omega^2_{(1)nm}-\tilde{\omega}^2_{nm(s)}\right]} = C_{(s)}, \quad s = 1,2. \qquad (53)$$

If we take into account that:

$$A^{(1)}_{(1)nm} = A^{(2)}_{(1)nm} = 1,$$

we obtain:

$$\begin{aligned}
A^{(1)}_{(2)nm} &= \frac{\left[\omega^2_{(1)nm}-\omega^2_{(2)nm}\right]}{2a^2_{(1)}} + \frac{1}{2}\sqrt{\left[\frac{\omega^2_{(1)nm}-\omega^2_{(2)nm}}{a^2_{(1)}}\right]^2 + 4\frac{a^2_{(2)}}{a^2_{(1)}}},\\
A^{(2)}_{(2)nm} &= \frac{\left[\omega^2_{(1)nm}-\omega^2_{(2)nm}\right]}{2a^2_{(1)}} - \frac{1}{2}\sqrt{\left[\frac{\omega^2_{(1)nm}-\omega^2_{(2)nm}}{a^2_{(1)}}\right]^2 + 4\frac{a^2_{(2)}}{a^2_{(1)}}},
\end{aligned} \qquad (54)$$

or in the form:

$$A^{(1,2)}_{(2)nm} = \frac{\left\{k^4_{(1)nm}\left[c^4_{(1)}-c^4_{(2)}\right]+a^2_{(1)}-a^2_{(2)}\right\}}{2a^2_{(1)}} \pm\frac{1}{2}\sqrt{\left[\frac{k^4_{(1)nm}\left[c^4_{(1)}-c^4_{(2)}\right]+a^2_{(1)}-a^2_{(2)}}{a^2_{(1)}}\right]^2 + 4\frac{a^2_{(2)}}{a^2_{(1)}}}, \qquad (55)$$

Appendix D

System determinants are in the forms

$$\tilde{\Delta}_a = \begin{vmatrix} 1 & 1 \\ A^{(1)}_{(2)nm} & A^{(2)}_{(2)nm} \end{vmatrix} = A^{(2)}_{(2)nm} - A^{(1)}_{(2)nm} \neq 0,$$

$$\tilde{\Delta}_b = \begin{vmatrix} \tilde{\omega}_{nm(1)} & \tilde{\omega}_{nm(2)} \\ \tilde{\omega}_{nm(1)} A^{(1)}_{(2)nm} & \tilde{\omega}_{nm(2)} A^{(2)}_{(2)nm} \end{vmatrix} = \tilde{\omega}_{nm(1)}\tilde{\omega}_{nm(2)} \left[A^{(2)}_{(2)nm} - A^{(1)}_{(2)nm}\right] \neq 0,$$

$$\tilde{\Delta}_a = A^{(2)}_{(2)nm} - A^{(2)}_{(2)nm} = \sqrt{\left[\frac{\omega^2_{(1)nm} - \omega^2_{(2)nm}}{a^2_{(1)}}\right]^2 + 4\frac{a^2_{(2)}}{a^2_{(1)}}}, \tag{56}$$

$$\tilde{\Delta}_b = \tilde{\omega}_{nm(1)}\tilde{\omega}_{nm(2)} \left[A^{(2)}_{(2)nm} - A^{(1)}_{(2)nm}\right] = \sqrt{\left[\omega^2_{(1)nm}\omega^2_{(2)nm} - a^2_{(1)}a^2_{(2)}\right]} \times \sqrt{\left[\frac{\omega^2_{(1)nm} - \omega^2_{(2)nm}}{a^2_{(1)}}\right]^2 + 4\frac{a^2_{(2)}}{a^2_{(1)}}}.$$

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