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RESEARCH PAPER

# Unsteady rotating flows of a viscoelastic fluid with the fractional Maxwell model between coaxial cylinders

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**Abstract** The fractional calculus is used in the constitutive relationship model of viscoelastic fluid. A generalized Maxwell model with fractional calculus is considered. Based on the flow conditions described, two flow cases are solved and the exact solutions are obtained by using the Weber transform and the Laplace transform for fractional calculus.

**Keywords** Viscoelastic fluid · Unsteady flow · Fractional Maxwell model · Exact solution

## 1 Introduction

The non-Newtonian fluids are being considered more important and appropriate in technological applications as compared with the Newtonian fluids. A large class of real fluids do not follow the linear relationship between stress and the rate of strain [1]. Because of the non-linear dependence, it is much more difficult to obtain the exact analytic solutions for the flows of the non-Newtonian fluids.

Recently, fractional calculus has successfully been used in the description of the complex dynamics. In particular, it has been proved to be a valuable tool to handle viscoelastic properties. The starting point of the fractional derivative model of non-Newtonian fluid is usually a classical differential equation being modified by replacing the time derivative of an integer order by the so-called Riemann–Liouville fractional calculus operator. This generalization allows one to define precisely non-integer order integrals or derivatives [2]. Fractional calculus has been found to be quite flexible in describing viscoelastic behavior [3–5]. More recently, Huang et al. [6–11] discussed some unsteady flows of the generalized second grade fluid. The unidirectional flow of viscoelastic fluid with the fractional Maxwell model was studied by Tan et al. [12–14]. The unsteady flow with a generalized Jeffreys model in an annular pipe was studied by Tong et al. [15,16]

The purpose of this paper is to study the flows of a viscoelastic fluid with the fractional derivative Maxwell model between two coaxial cylinders. The flows are generated by a simple harmonic motion or impulsively rotating motion of the outer cylinder. By using the Weber transform and the inverse Laplace transform of fractional derivative, we have obtained the exact solutions of the flows.

## 2 The fractional Maxwell model and basic equation

Generally, the constitutive relationship of a viscoelastic material with the fractional derivative Maxwell model is given by [3,12–14]

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$$\sigma + \lambda^{\alpha} \frac{\mathrm{d}^{\alpha}\sigma}{\mathrm{d}t^{\alpha}} = G\lambda^{\beta} \frac{\mathrm{d}^{\beta}\varepsilon}{\mathrm{d}t^{\beta}}, \tag{1}$$

where $\sigma$ is the shear stress, $\varepsilon$ is the shear strain, $G$ is the shear modulus, $\lambda = \mu/G$ is the relaxation time with the viscosity constant $\mu$, $\alpha$ and $\beta$ are fractional parameters such that $0 \le \alpha \le \beta \le 1$. Also, $\mathrm{d}^{\alpha}/\mathrm{d}t^{\alpha}$, $\mathrm{d}^{\beta}/\mathrm{d}t^{\beta}$ are the Riemann-Liouville fractional derivative operators, and the fractional derivative of order $\alpha$ [2] is defined as

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}} \mathrm{d}\tau, \tag{2}$$

where $\Gamma(\cdot)$ is the Gamma function. It should be noted that this model includes the ordinary Maxwell model as a special case when $\alpha = \beta = 1$ and the Navier-Stokes model when $\alpha = 0$, $\beta = 1$. Friedrich [3] proved that this kind of rheological constitutive equation shows fluid-like behavior only in the case that $\alpha$ takes a value between 0 and 1 and $\beta = 1$, that is,

$$\sigma + \lambda^{\alpha} \frac{\mathrm{d}^{\alpha}\sigma}{\mathrm{d}t^{\alpha}} = \mu\dot{\varepsilon}, \tag{3}$$

where $\dot{\varepsilon} = \mathrm{d}\varepsilon/\mathrm{d}t$ is the shear rate. Equation (3) is used as the constitutive equation of the fractional Maxwell model.

If the fluid is incompressible, then the equation of continuity is

$$\nabla \cdot \boldsymbol{V} = 0, \tag{4}$$

and the equation of motion, in the absence of pressure gradient and body forces, is

$$\rho \frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t} = \nabla \cdot \boldsymbol{T}, \tag{5}$$

where $\rho$ is the density of fluid, $\mathrm{D}/\mathrm{D}t$ the material derivative, $\boldsymbol{T}$ the stress tensor, and $\boldsymbol{V}$ the velocity.

Now we consider the rotating flows of a fluid modeled by Eqs. (3)–(5) and between two very long coaxial cylinders of radius $R_0$ and $R_1 (> R_0)$. We will construct the exact solutions for the following two boundary value problems: (a) the outer cylinder makes a simple harmonic motion and the inner cylinder keeps rest; (b) the outer cylinder rotates at a constant speed and the inner cylinder keeps rest. It is obvious that the motion between the two cylinders is axially symmetric. Under the cylindrical coordinate system $(r, \theta, z)$, the velocity components can be expressed as $V_r = 0$, $V_{\theta} = u(r,t)$, $V_z = 0$. Based on the above conditions, the constitutive equation becomes

$$\sigma_{r\theta} + \lambda^{\alpha} \frac{\partial^{\alpha}\sigma_{r\theta}}{\partial t^{\alpha}} = \mu r \frac{\partial}{\partial r}\left(\frac{u}{r}\right), \tag{6}$$

and the equation of motion is

$$\rho \frac{\partial u}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \sigma_{r\theta}). \tag{7}$$

Substituting Eq. (6) into Eq. (7), we obtain

$$\rho\lambda^{\alpha} \frac{\partial^{\alpha+1}u}{\partial t^{\alpha+1}} + \rho \frac{\partial u}{\partial t} = \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right). \tag{8}$$

Let us introduce dimensionless variables $u^* = u/U_0$, $r^* = r/R_0$, $t^* = U_0^2\rho t/\mu$, where $U_0$, $R_0$ and $\mu/U_0^2\rho$ denote characteristic velocity, length and time, respectively. Using the mean value theorem of the integral, it can be easily proved [7] that the operator $\mathrm{d}^{\alpha}/\mathrm{d}t^{\alpha}$ takes a fractional time dimension $(\mu/U_0^2\rho)^{-\alpha}$. Thus, the dimensionless fractional order equation is obtained as follows (for simplicity, the superscript * is omitted)

$$\eta^{\alpha} \frac{\partial^{\alpha+1}u}{\partial t^{\alpha+1}} + \frac{\partial u}{\partial t} = \zeta\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right), \tag{9}$$

where $\eta = \rho U_0^2/G$, $\zeta = \mu^2/U_0^2\rho^2 R_0^2$ are dimensionless parameters.

## 3 Exact solution of the first problem

We assume that the outer cylinder makes a simple harmonic motion, while the inner cylinder keeps still. Then the dimensionless initial and boundary conditions are

$$u(r,t) = 0, \quad t = 0, \tag{10}$$

$$u(1,t) = 0, \quad t > 0, \tag{11}$$

$$u(b,t) = \cos \omega t, \quad t > 0, \tag{12}$$

where $b = R_1/R_0$, $\omega = \omega_0\mu/U_0^2\rho$, $\omega_0$ is the frequency factor of the simple harmonic motion. Let $\bar{u}(r,s) = L\{u(r,t)\} = \int_0^{\infty} \mathrm{e}^{-st} u(r,t)\mathrm{d}t$ be the image function of $u(r,t)$, where $s$ is the transform parameter. According to the Laplace transform of fractional derivatives [2], we have

$$(\eta^{\alpha}s^{\alpha+1} + s)\bar{u} = \zeta\left(\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} - \frac{\bar{u}}{r^2}\right), \tag{13}$$

$$\bar{u}(1,s) = 0, \tag{14}$$

$$\bar{u}(b,s) = \frac{s}{s^2 + \omega^2}. \tag{15}$$

In order to obtain the exact solution of this problem, we make the Weber transform to Eqs. (13), (14) and (15). The Weber transform [17] is

$$\tilde{\bar{u}}(\rho_i, s) = \int_1^b r\bar{u}(r,s)H(\rho_i, r)\mathrm{d}r. \tag{16}$$

The inverse Weber transform is

$$\bar{u}(r,s)=\sum_{i=1}^{\infty}\tilde{\bar{u}}(\rho_i,s)\frac{H(\rho_i,r)}{N(\rho_i)}, \tag{17}$$

where $H(\rho_i,r)=\mathrm{J}_1(\rho_i r)\mathrm{Y}_1(\rho_i)-\mathrm{J}_1(\rho_i)\mathrm{Y}_1(\rho_i r)$, $\rho_i$ is the positive root of $H(\rho_i,b)=0$ and

$$\frac{1}{N(\rho_i)}=\frac{\pi^2}{2}\frac{\rho_i^2\mathrm{J}_1^2(\rho_i b)}{\mathrm{J}_1^2(\rho_i)-\mathrm{J}_1^2(\rho_i b)}, \tag{18}$$

where $\mathrm{J}_1(x)$ and $\mathrm{Y}_1(x)$ are the Bessel functions of the first kind and second kind of order one. Through the Weber transform, we obtain

$$\tilde{\bar{u}}(\rho_i,s)=\frac{2}{\pi}\frac{\mathrm{J}_1(\rho_i)}{\rho_i^2\mathrm{J}_1(\rho_i b)}\frac{\zeta\rho_i^2}{\eta^\alpha s^{\alpha+1}+s+\zeta\rho_i^2}\frac{s}{s^2+\omega^2}. \tag{19}$$

Here, the Wronskian relationship of Bessel function is used [18],

$$\mathrm{J}_1(x)\mathrm{Y}_1'(x)-\mathrm{J}_1'(x)\mathrm{Y}_1(x)=\frac{2}{\pi x}. \tag{20}$$

Substituting Eq. (19) into Eq. (17), we have

$$\bar{u}(r,s)=\frac{b}{b^2-1}\frac{r^2-1}{r}\frac{s}{s^2+\omega^2}-\sum_{i=1}^{\infty}\bar{A}(\rho_i,s)\times\frac{\pi\mathrm{J}_1(\rho_i)\mathrm{J}_1(\rho_i b)}{\mathrm{J}_1^2(\rho_i)-\mathrm{J}_1^2(\rho_i b)}H(\rho_i,r), \tag{21}$$

where

$$\bar{A}(\rho_i,s)=\frac{\eta^\alpha s^{\alpha+1}+s}{\eta^\alpha s^{\alpha+1}+s+\zeta\rho_i^2}\frac{s}{s^2+\omega^2}. \tag{22}$$

Therefore, we can obtain the exact solution of this problem as long as the inverse Laplace transform of $\bar{A}(\rho_i,s)$ is found. From [2,5], we obtain

$$\begin{aligned}A(\rho_i,t)&=L^{-1}[\bar{A}(\rho_i,s)]\\&=\left\{\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{t^{n\alpha+\alpha-1}}{\eta^{n\alpha+\alpha}}E^{(n)}_{\alpha+1,\alpha-n}(-B_i t^{\alpha+1})\right.\\&\quad\left.+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{t^{n\alpha-1}}{\eta^{n\alpha}}E^{(n)}_{\alpha+1,-n}(-B_i t^{\alpha+1})\right\}*\cos\omega t,\end{aligned} \tag{23}$$

where $B_i=\zeta\rho_i^2/\eta^\alpha$, the sign $*$ represents the convolution integral, $E_{\alpha,\beta}(z)$ is a Mittag–Leffler function [2]. To obtain Eq. (23), we have applied the following formula [2,7]

$$L^{-1}\left(\frac{n!s^{\alpha-\beta}}{(s^\alpha+c)^{n+1}}\right)=t^{\alpha n+\beta-1}E^{(n)}_{\alpha,\beta}(-ct^\alpha), \tag{24}$$

where $\mathrm{Re}(s)>|c|^{1/\alpha}$.

By Eq. (23), the exact solution of the first problem can be obtained as

$$u(r,t)=\frac{b}{b^2-1}\frac{r^2-1}{r}\cos\omega t-\sum_{i=1}^{\infty}A(\rho_i,t)\times\frac{\pi\mathrm{J}_1(\rho_i)\mathrm{J}_1(\rho_i b)}{\mathrm{J}_1^2(\rho_i)-\mathrm{J}_1^2(\rho_i b)}H(\rho_i,r). \tag{25}$$

Particularly, if $\alpha=0$, from Eq. (22) we can easily simplify (23) as

$$A(\rho_i,t)=\frac{4\omega^2\cos\omega t-2\zeta\rho_i^2\omega\sin\omega t+\zeta^2\rho_i^4\mathrm{e}^{-(\zeta\rho_i^2 t/2)}}{\zeta^2\rho_i^4+4\omega^2}. \tag{26}$$

Substituting Eq. (26) into Eq. (25), we get the velocity formula for the Newtonian fluid.

## 4 Exact solution of the second problem

In this section, we assume that the outer cylinder suddenly starts up and rotates at a constant speed $U_0$ at the initial time, while the inner cylinder keeps stationary at all time. Then the governing partial differential equation is Eq. (9) and the dimensionless initial and boundary conditions are Eqs. (10) and (11) and

$$u(b,t)=1,\ t>0. \tag{27}$$

Using the same method in section 3, we have

$$u(r,t)=\frac{b}{b^2-1}\frac{r^2-1}{r}-\sum_{i=1}^{\infty}A(\rho_i,t)\times\frac{\pi\mathrm{J}_1(\rho_i)\mathrm{J}_1(\rho_i b)}{\mathrm{J}_1^2(\rho_i)-\mathrm{J}_1^2(\rho_i b)}H(\rho_i,r), \tag{28}$$

where

$$A(\rho_i,t)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{t^{n\alpha+n}}{\eta^{n\alpha+n}}E^{(n)}_{\alpha+1,\alpha+1-n}(-B_i t^{\alpha+1})+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{t^{n\alpha}}{\eta^{n\alpha}}E^{(n)}_{\alpha+1,1-n}(-B_i t^{\alpha+1}). \tag{29}$$

In particular, when $\alpha=0$, we get the classical solution for the Newtonian fluid

$$u(r,t)=\frac{b}{b^2-1}\frac{r^2-1}{r}-\sum_{i=1}^{\infty}\frac{\pi\mathrm{J}_1(\rho_i)\mathrm{J}_1(\rho_i b)}{\mathrm{J}_1^2(\rho_i)-\mathrm{J}_1^2(\rho_i b)}\times H(\rho_i,r)\exp\left(-\zeta\rho_i^2 t/2\right). \tag{30}$$

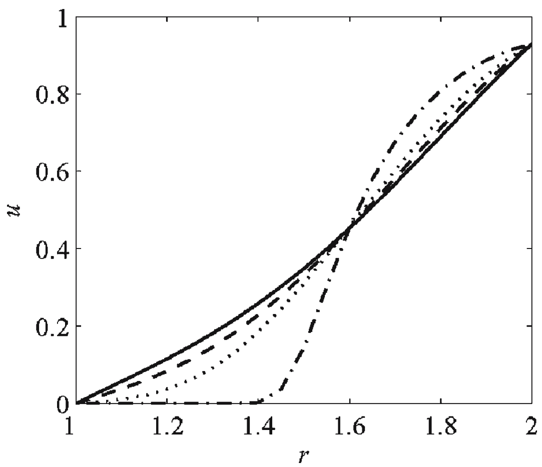


**Fig. 1** Velocity profile $u(r,t)$ corresponding to the first problem. The variations of velocity with $r$, $\omega = 1.5$, $\eta = 0.5$ and $t = 0.25$ for various values of $\alpha$ (*solid line* $\alpha = 0$, *dash line* $\alpha = 0.3$, *dot line* $\alpha = 0.5$, *dash-dot line* $\alpha = 0.8$)

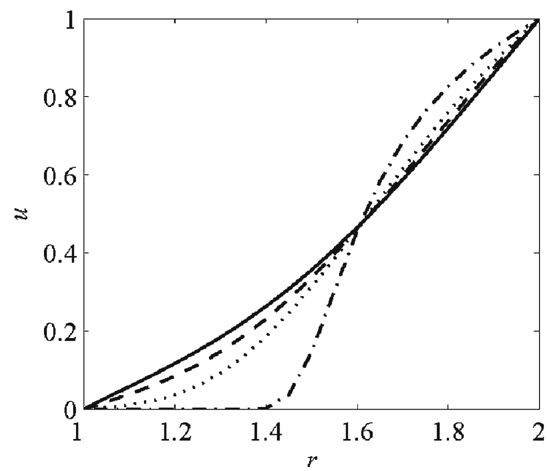


**Fig. 3** Velocity profile $u(r,t)$ corresponding to the second problem. The variations of velocity with $r$, $\eta = 0.5$ and $t = 0.25$ for various values of $\alpha$ (*solid line* $\alpha = 0$, *dash line* $\alpha = 0.3$, *dot line* $\alpha = 0.5$, *dash-dot line* $\alpha = 0.8$)

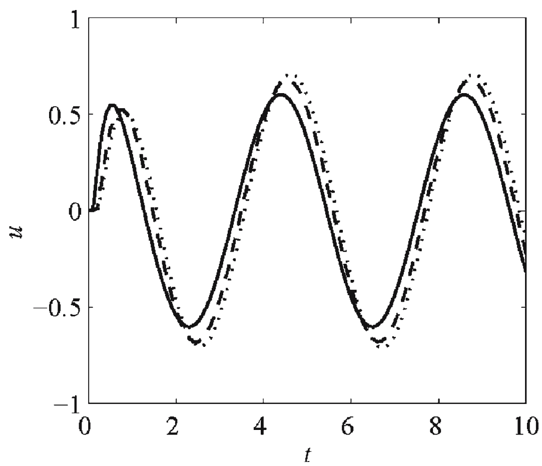


**Fig. 2** Velocity history $u(r,t)$ corresponding to the first problem. The variations of velocity with $t$, $\omega = 1.5$, $r = 1.5$ and $\alpha = 0.5$ for various values of $\eta$ (*solid line* $\eta = 0.5$, *dash line* $\eta = 3$, *dot line* $\eta = 5$)

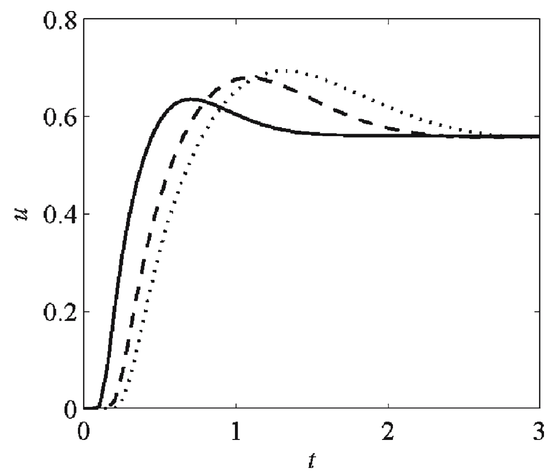


**Fig. 4** Velocity history $u(r,t)$ corresponding to the second problem. The variations of velocity with $t$, $r = 1.5$ and $\alpha = 0.5$ for various values of $\eta$ (*solid line* $\eta = 0.5$, *dash line* $\eta = 3$, *dot line* $\eta = 5$)

## 5 Numerical results and conclusions

In this paper, two types of unsteady rotating flows of the viscoelastic fluid are studied. The fractional calculus is used in the constitutive relationship of a generalized Maxwell fluid. Using the Weber transform and the Laplace transform, we have obtained the exact solutions of velocity profile, given by Eqs. (23), (25) and (28), (29). In particular, when $\alpha \to 0$, our solutions can be simplified to solutions for the Navier-Stokes fluid. Thus, the technique and the fractional Maxwell model used here will be useful in the theory of non-Newtonian fluids.

Further, it is found that the dimensionless constitutive equation of the fractional Maxwell model is governed by the relaxation time $\eta$ and fractional derivative $\alpha$. We plot Figs. 1, 2, 3 and 4 for various values of $\eta$ and $\alpha$, corresponding to the above two problems, when $b = 2$ and $\zeta = 1$. In Fig. 1 and Fig. 3, the variations of velocity field $u(r,t)$ are plotted for different values of $\alpha$ when $t = 0.25$, corresponding to relations (23), (25) and (28), (29), respectively. It is clearly seen from the figures that the propagation of motion for large $\alpha$ is faster than that for smaller one at the neighborhood of the outer cylinder, but it is quite the contrary at the neighborhood of the inner cylinder. This phenomenon is due to the boundary conditions. Figures 2 and 4 are the histories of velocity for selected parameters $\eta$ and a fixed space point. The larger the parameter $\eta$ is, the longer time it is needed such that the velocity approaches the periodic motion state or the steady state.

## References


1. Han, S.F.: Constitutive equation and computational analytical theory Of non-Newtonian fluids. Science Press, Beijing (2000)
2. Podlubny, I.: Fractional differential equations. Academic Press, San Diego (1999)
3. Friedrich, C.: Relaxation and retardation functions of the Maxwell model with fractional derivatives. Rheol. Acta **30**, 151–158 (1991)
4. Hilfer, R.: Applications of Fractional Calculus in Physics. World Scientific Press, Singapore (2000)
5. Xu, M.Y., Tan, W.C.: Representation of the constitutive equation of viscoelastic materials by the generalized fractional element networks and its generalized solutions. Sci. China Ser. G **46**, 145–157 (2003)
6. Huang, J.Q., He, G.Y., Liu, C.Q.: Analysis of general second-order fluid flow in double cylinder rheometer. Sci. China Ser. A **40**, 183–190 (1997)
7. Xu, M.Y., Tan, W.C.: Theoretical analysis of the velocity field, stress field and vortex sheet of generalized second order fluid with fractional anomalous diffusion. Sci. China Ser. A **44**, 1387–1399 (2001)
8. Tan, W.C., Xian, F., Wei, L.: An exact solution of unsteady Couette flow of generalized second grade fluid. Chin. Sci. Bull. **47**, 1783–1785 (2002)
9. Tan, W.C., Xu, M.Y.: The impulsive motion of flat plate in a general second grade fluid. Mech. Res. Comm. **29**, 3–9 (2002)
10. Tan, W.C., Xu, M.Y.: Unsteady flows of a generalized second grade fluid with the fractional derivative model between two parallel plates. Acta Mech. Sin. **20**, 471–476 (2004)
11. Shen, F., Tan, W.C., Zhao, Y.H., Masuoka, T.: Decay of vortex and diffusion of temperature in a generalized second grade fluid. Appl. Math. Mech. **25**, 1151–1159 (2004)
12. Tan, W.C., Xu, M.Y.: Plane surface suddenly set in motion in a viscoelastic fluid with fractional Maxwell model. Acta Mech. Sinica **18**, 342–349 (2002)
13. Tan, W.C., Pan, W.X., Xu, M.Y.: A note on unsteady flows of a viscoelastic fluid with the fractional Maxwell model between two parallel plates. Int. J. Nonlin. Mech. **38**, 645–650 (2003)
14. Hayat, T., Nadeem, S., Asghar, S.: Periodic unidirectional flows of a viscoelastic fluid with the fractional Maxwell model. Appl. Math. Comput. **151**, 153–161 (2004)
15. Tong, D.K., Liu, Y.S.: Exact solutions for the unsteady rotational flow of non-Newtonian fluid in an annular pipe. Int. J. Eng. Sci. **43**, 281–289 (2005)
16. Tong, D.K., Wang, R.H., Yang, H.S.: Exact solutions for the flow of non-Newtonian fluid with fractional derivative in an annular pipe. Sci. China Ser. G **48**, 485–495 (2005)
17. Özisik, M.N.: Heat conduction. Wiley, New York (1980)
18. Watson, G.N.: A Treatise on the Theory of Bessel Functions. Cambridge University Press, Cambridge (1995)