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RESEARCH PAPER

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Eigenvalue and stability analysis for transverse vibrations of axially moving strings based on Hamiltonian dynamics

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Abstract The Hamiltonian dynamics is adopted to solve the eigenvalue problem for transverse vibrations of axially moving strings. With the explicit Hamiltonian function the canonical equation of the free vibration is derived. Non-singular modal functions are obtained through a linear, symplectic eigenvalue analysis, and the symplectic-type orthogonality conditions of modes are derived. Stability of the transverse motion is examined by means of analyzing the eigenvalues and their bifurcation, especially for strings transporting with the critical speed. It is pointed out that the motion of the string does not possess divergence instability at the critical speed due to the weak interaction between eigenvalue pairs. The expansion theorem is applied with the non-singular modal functions to solve the displacement response to free and forced vibrations. It is demonstrated that the modal functions can be used as the base functions for solving linear and nonlinear vibration problems.

Keywords Axially moving strings · Symplectic · Modal analysis · Stability · Divergence

1 Introduction

Transverse vibrations of axially moving strings have been extensively studied in the past decades. Here is a brief review. Swope and Ames[1] solved the linear, free vibration of threadlines by the method of characteristic line. Wickert and Mote [2] systematically investigated the classic linear vibration through a complex modal analysis. They used a complex eigenvalue analysis in the state space to determine modal functions of the string. The orthonormal modal functions they obtained become singular when the string transports at a so-called critical speed. The completeness of the modal functions was not discussed. The modal characteristics of the translating system were investigated later by Lee and Renshaw [3]. Perkins and Mote [4] used the Hamiltonian principle to develop the nonlinear equations of motion for three-dimensional traveling cables. For eigenvalues and critical speed stability of axially moving media, Parker [5] adopted a perturbation analysis to determine approximate eigenvalue loci and stability properties near the critical speed. Nonlinear motions can be encountered when complicated material properties are involved for the strings. For instance, Chen, Zhang and Zu [6] investigated bifurcation and chaos of an axially moving viscoelastic string. The Galerkin truncation method was employed and bifurcation diagrams were presented for periodic, quasi-periodic and chaotic motions. Chen, Zu et al. considered geometrically nonlinear vibrations of an axially accelerating viscoelastic string. The method of multiple scales was applied to obtain the existence conditions of nontrivial steady-state response in two-to-one parametric resonance, and the stability of the trivial and nontrivial solutions was analyzed with Lyapunov's linearized stability theory [7]. This paper concerns mainly with eigenvalue analysis of linear vibration, the detailed literature reviews may be found in Ref. [8].

In spite of the fruitful studies in this respect, string dynamics remains an interesting topic. The modal analysis for moving strings leads to a symplectic type of eigenvalue problems due to the gyroscopic term in the governing equation of motion, and the eigenvectors of the system appear in conjugate pairs and constitute a complete set in the Hilbert's space, similar to what was shown by Zhong [9].

In this paper, the Hamiltonian dynamics is adopted to investigate the transverse vibration of the strings in an exact

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modal analysis. Considering the transport speed, the Hamiltonian of the continuum system is obtained as a function of displacement and its dual variable. The Hamilton's canonical equation is obtained with the variational principle, resulting in a symplectic eigenvalue problem for which the eigenvalues and eigenfunctions are determined in closed forms with the symplectic-type orthogonality conditions of the eigenfunctions. The symplectic modal functions do not suffer from singularity problems related with the critical speed like the normalized eigenfunctions obtained by, e.g. Wickert and Mote. The eigenfunctions are complete and are used as base functions in the modal space for determination of both linear and nonlinear vibrations with various transport speeds. The stability of the transverse motion is examined through an eigenvalue analysis for strings transporting with the critical transport speed. It is shown that the eigenvalue pairs become zero and collide to each other at the critical speed. However, the motion is stable and does not possess divergence instability due to the weak interaction between eigenvalue pairs. This is important for understanding stability properties of axially moving strings in transverse vibrations.

2 Mechanical model

A simply supported, axially moving string is shown in Fig. 1, where $\bar{\eta}$ is the curvilinear coordinate of a particle, $\bar{c}$ is the transport speed, $\bar{x}_k^0 = \bar{x}_k^0(\bar{\eta}), k = 1, 2, 3$ are coordinates of the particle resting in its equilibrium position in the Cartesian system $O\bar{x}_1\bar{x}_2\bar{x}_3$. The dynamic configuration of the string is denoted by $\bar{x}_k = \bar{x}_k(\bar{\eta}, \bar{t}), k = 1, 2, 3$, where $\bar{t}$ stands for the temporal variable. For convenience of derivation, the non-dimensionalization in Ref. [10] is used

$$\eta = \bar{\eta}/S, \qquad t = \bar{t}\sqrt{g/S},$$
$$x_k^0 = \bar{x}_k^0/S, \qquad x_k = \bar{x}_k/S,$$
$$c = \bar{c}/\sqrt{gS}, \qquad c_0 = \sqrt{\bar{T}_0/\rho A g S},$$

where S is the total length of the string in the equilibrium configuration, g is the gravitational acceleration, $\bar{T}_0$ is the tension force, A and ρ are the cross-sectional area of the string and the mass density, respectively. t is the non-dimensional temporal variable, $0 \le \eta \le 1$ is the non-dimensional curvilinear coordinate. With the non-dimensionalization, the non-dimensional displacement of the particle in the transverse direction is defined as $U(\eta, t) = x_2(\eta, t) - x_2^0(\eta, t)$ with homogeneous boundary condition

$$U(0, t) = U(1, t) = 0 \tag{1}$$

at both ends. It is assumed that the vibration is small and the tension force does not vary with the curvilinear coordinate, thus the vibration assumes a linear motion. In this case, the in-plane ($O\bar{x}_1\bar{x}_2$ planar) and out-plane ($O\bar{x}_1\bar{x}_3$ planar) motions are decoupled so that only a planar motion (hereby the in-plane motion) needs to be considered.

For axially moving strings, the non-dimensional Hamiltonian is expressed as

$$H(U, p) = \frac{1}{2}\int_0^1 (p^2 + c_0^2(\partial U/\partial\eta)^2 - 2cp(\partial U/\partial\eta))\mathrm{d}\eta, \tag{2}$$

where $c_0 \neq 0$ is the non-dimensional tension force in the string associated with its equilibrium configuration. The dual variable, $p(\eta, t) = \mathrm{D}U(\eta, t)/\mathrm{D}t = \partial U/\partial t + c\partial U/\partial\eta$, is the generalized momentum of the particle. Note that potential energy resulted from the initial tension is constant and has been dropped from Eq. (2) when the following Hamilton's variational principle in phase space[11] is formulated:

$$\delta\int_0^\tau\int_0^1 [p\partial U/\partial t - H(U, p)]\mathrm{d}\eta\mathrm{d}t = 0, \tag{3}$$

which leads to the Hamilton's canonical equations

$$\begin{aligned} \frac{\partial U}{\partial t} &= -c\frac{\partial U}{\partial\eta} + p, \\ \frac{\partial p}{\partial t} &= c_0^2\frac{\partial^2 U}{\partial\eta^2} - c\frac{\partial p}{\partial\eta}. \end{aligned} \tag{4}$$

The foregoing can be expressed compactly with an operator matrix $\mathbf{H}$, as

$$\dot{\boldsymbol{u}}(\eta, t) = \mathbf{H}\boldsymbol{u}(\eta, t), \tag{5}$$

where $\boldsymbol{u}(\eta, t) = (U(\eta, t), p(\eta, t))^{\mathrm{T}}$ and $\mathbf{H} = \begin{bmatrix} -c\partial/\partial\eta & 1 \\ c_0^2\partial^2/\partial\eta^2 & -c\partial/\partial\eta \end{bmatrix}$.

For a comparison of the modal analysis techniques presented in this paper with the previous ones, we go back to the Lagrangian dynamics for a moment. The governing equation of the transverse motion is obtained from the second equation of Eq. (4), as

$$\frac{\partial^2 U}{\partial t^2} + 2c\frac{\partial^2 U}{\partial\eta\partial t} + (c^2 - c_0^2)\frac{\partial^2 U}{\partial\eta^2} = 0, \quad t \ge 0, \quad \eta \in [0, 1], \tag{6}$$

which is essentially the same as the governing equation presented in many existing publications. For instance, Wickert and Mote [2] derived the governing differential equation through a different non-dimensionalization procedure as

$$\frac{\partial^2 U}{\partial t^2} + 2v\frac{\partial^2 U}{\partial x\partial t} + (v^2 - 1)\frac{\partial^2 U}{\partial x^2} = f,$$

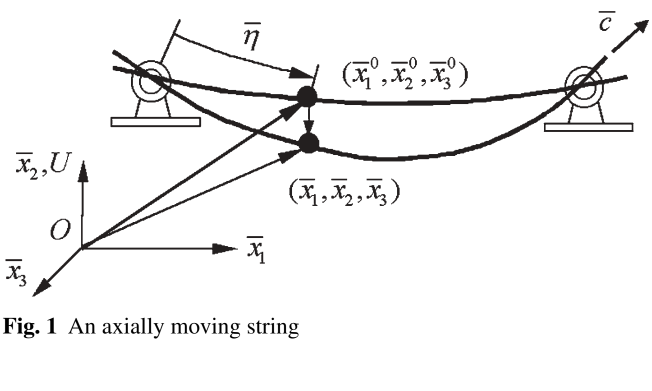

Fig. 1 An axially moving string

where v is the non-dimensional transport speed, x is the curvilinear coordinate and f is the external loading. For free vibration problems the above hyperbolic equation is exactly identical to Eq. (6) if one lets $f = 0$, $c = v$ and $c_0 = 1$.

By using the state space representation and a modal analysis method, Wickert and Mote obtained a pair of complex conjugate modal functions:

$$\psi_n^{\mathrm{R}} = \frac{1}{n\pi}\sqrt{\frac{2}{1-\nu^2}}\sin(n\pi x)\cos(n\pi\nu x),$$
$$\psi_n^{\mathrm{I}} = \frac{1}{n\pi}\sqrt{\frac{2}{1-\nu^2}}\sin(n\pi x)\sin(n\pi\nu x), \tag{7}$$

where the positive integer n is the frequency order. Apparently, this pair of modal functions become singular as the transport speed v approaches unity, i.e. the critical speed by definition. However, when the speed becomes critical, there is a standing wave resulting from the transverse displacement, and the magnitude of the displacement decreases to zero [1]. This will also be demonstrated in Section 4 with a stability analysis based on the eigenvalue bifurcation. As a result, the configuration of the string remains unchanged as the initial configuration. To keep the modal function valid for the whole range of transport speed, the singularity generated from Eq. (7) must be removed. This is realized by a new modal analysis based on the Hamiltonian dynamics in the next section.

3 Modal analysis and symplectic orthogonality conditions

Assuming a solution $\boldsymbol{u}(\eta,t) = \Psi(\eta)\mathrm{e}^{\lambda t}$, the eigenvalue problem of the dual system of Eq. (4) can be derived as

$$\mathbf{H}\Psi = \lambda\Psi, \tag{8}$$

where the eigenvalue, λ, is solved to be one of the two conjugate complex roots (see Appendix A):

$$\lambda_{\pm n} = \pm \mathrm{i} n\pi(c_0^2 - c^2)/c_0 \quad (n = 1,2,3,\ldots) \tag{9}$$

with $\mathrm{i} = \sqrt{-1}$. The eigenfunctions of operational matrix $\mathbf{H}$ are determined explicitly, as

$$\Psi_n(\eta) = \mathrm{e}^{\mathrm{i}n\pi\eta c/c_0}\begin{bmatrix} A \\ B \end{bmatrix},$$
$$\Psi_{-n}(\eta) = \mathrm{e}^{-\mathrm{i}n\pi\eta c/c_0}\begin{bmatrix} -A \\ C \end{bmatrix}, \qquad (n = 1,2,3,\ldots) \tag{10}$$

where

$$A = \sin n\pi\eta,$$
$$B = n\pi(c\cos n\pi\eta + \mathrm{i}c_0 \sin n\pi\eta),$$
$$C = -n\pi(c\cos n\pi\eta - \mathrm{i}c_0 \sin n\pi\eta),$$

The real and imaginary parts of the displacement term in $\Psi_n(\eta)$, referred to as $\Psi_{n,1}(\eta)$ hereafter, have the same shape as the ψ_n^{R} and ψ_n^{I} of Eq. (2), i.e.

$$\mathrm{Re}\Psi_{n,1}/\psi_n^{\mathrm{R}} = \mathrm{Im}\Psi_{n,1}/\psi_n^{\mathrm{I}}$$

if one lets $c_0 = 1$ and replaces c and η with v and x, respectively. However, the coefficients, or amplitudes, of the modal functions of Eq. (10) are different from those of ψ_n^{R} and ψ_n^{I}. It should be pointed out that the eigenvectors in Eq. (10) are complete eigenfunctions of the dual system (cf.[9], where a proof is provided in detail). As modes of the string, the eigenfunctions satisfy the symplectic orthogonality conditions

$$\langle\Psi_m, \Psi_n\rangle = \langle\Psi_{-n}, \Psi_{-n}\rangle = 0,$$
$$\langle\Psi_m, \Psi_{-n}\rangle = \langle\Psi_{-m}, \Psi_n\rangle = \langle\Psi_{-m}, \Psi_{-n}\rangle = 0,$$
$$\langle\Psi_n, \Psi_{-n}\rangle = -\langle\Psi_{-n}, \Psi_n\rangle = \mathrm{i}n\pi c_0,$$
$$(m \neq n;\ m, n = 1,2,3,\ldots) \tag{11}$$

as shown in Ref. [9], where the symplectic inner product of the complex field vectors is defined as

$$\langle\boldsymbol{\mu}_1, \boldsymbol{\mu}_2\rangle = \int_0^1 \boldsymbol{\mu}_1^{\mathrm{T}}(\eta)\boldsymbol{J}\boldsymbol{\mu}_2(\eta)\mathrm{d}\eta, \quad \boldsymbol{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

With the orthogonality conditions of Eq. (11), the normalized modal functions are expressed as

$$\tilde{\Psi}_n(\eta) = \frac{\mathrm{e}^{\mathrm{i}n\pi\eta c/c_0}}{\sqrt{\mathrm{i}n\pi c_0}}\begin{bmatrix} A \\ B \end{bmatrix},$$
$$\tilde{\Psi}_{-n}(\eta) = \frac{\mathrm{e}^{-\mathrm{i}n\pi\eta c/c_0}}{\sqrt{\mathrm{i}n\pi c_0}}\begin{bmatrix} -A \\ C \end{bmatrix}, \qquad (n = 1,2,3,\ldots) \tag{12}$$

which satisfy the symplectic orthogonality condition with respect to the Hamiltonian matrix

$$\langle\tilde{\Psi}_n, \tilde{\Psi}_{-m}\rangle = \delta_{mn},$$
$$\langle\tilde{\Psi}_{-n}, \tilde{\Psi}_m\rangle = -\delta_{mn}, \qquad (m, n = 1,2,3,\ldots) \tag{13}$$

with δ_{mn} being the Kronecker delta. It can be proved that owing to the symplectic orthogonality conditions, the symplectic modal functions of (12) do not suffer from numerical singularity like the modal functions of Eq. (7) near and at the critical speed.

To present the modes of the axially moving string, the first three normalized modal functions are illustrated with transport speed $c = 0.5$ and tension force $c_0 = 1$ in Figs. 2(a) and 2(b), where the real and imaginary parts are both shown.

To observe the influence of the transport speed, the real parts of modal functions are computed with three different c and shown in Figs. 3(a) through 3(c). It can be seen that the shape of modal functions is strongly related to the transport speed. That is why the dynamic configuration of string varies with the transport speed significantly.

4 Eigenvalues and stability at the critical speed

The stability of motion can be determined by analyzing the eigenvalues of the natural vibration. The stability of motion will be lost when at least one eigenvalue comes with a positive real part or becomes multiple zeroes with geometrical multiplicity less than its algebraic multiplicity for both discrete and continuous gyroscopic systems [12, 13]. When the

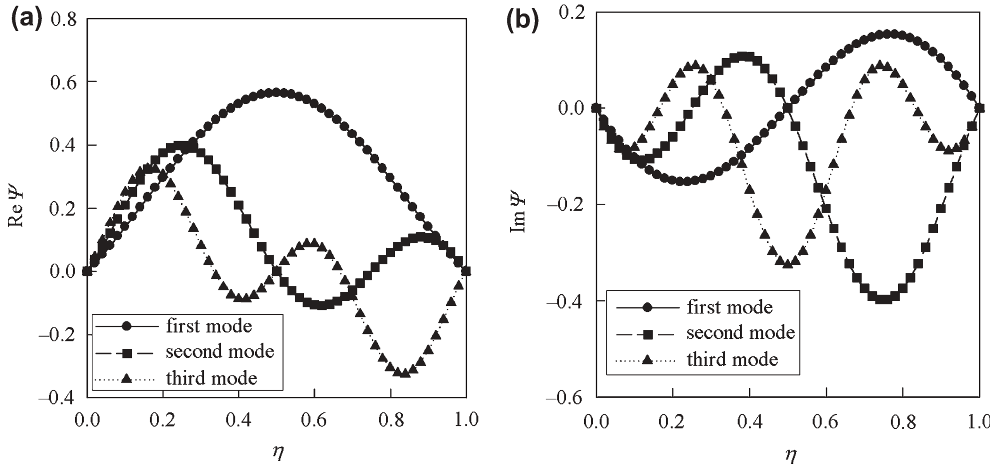

Fig. 2 The first three normalized modal functions. $c_0 = 1$, $c = 0.5$. (a) Real Part; (b) Imaginary part

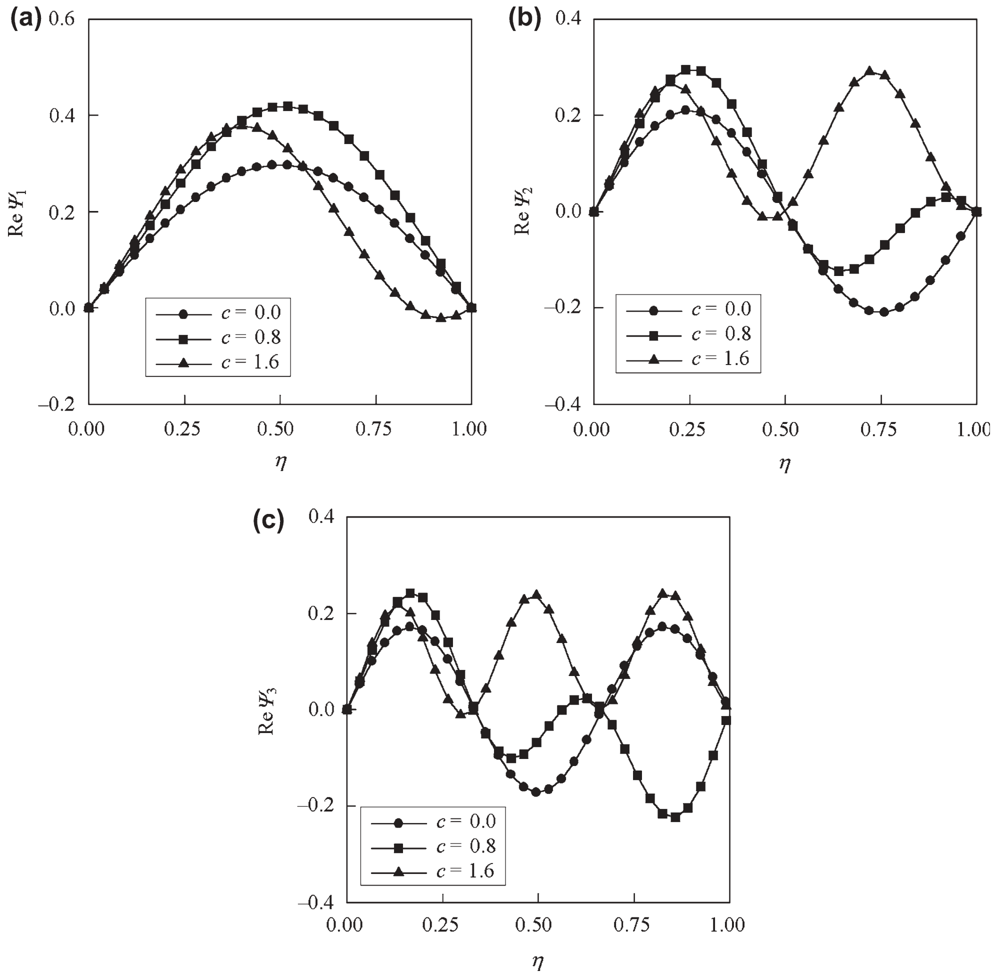

Fig. 3 Real parts of the first three normalized modal functions with different transport speeds. $c_0 = 1.8$. (a) First mode, (b) Second mode, (c) Third mode

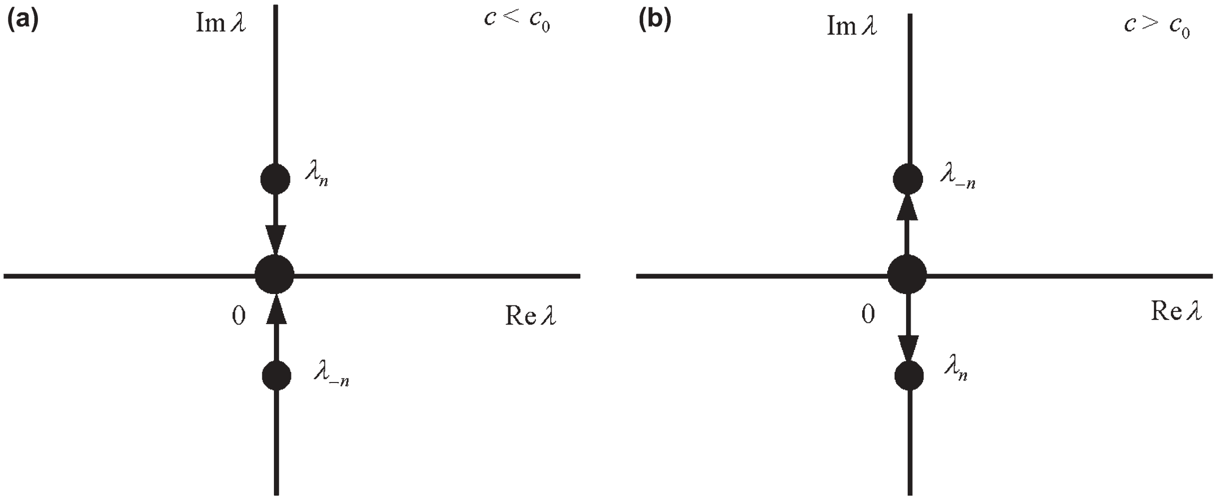

Fig. 4 Bifurcation of eigenvalues. (a) Before the critical speed, (b) After the critical speed

string transports with the critical speed, i.e. $c = c_0$, the eigenvalue pairs of Eq. (9) become $\lambda_n = \lambda_{-n} = 0$. Hence, the two complex conjugate eigenvalues for the nth mode collide and merge at the critical speed, as shown in Fig. 4(a), where the real and imaginary parts of the eigenvalues are depicted. The dot at the origin shows the exact collision of an eigenvalue-pair when $c = c_0$. When $c > c_0$ the two eigenvalues exchange signs and separate from each other again, as shown in Fig. 4(b). Therefore, $\lambda = 0$ serves as a bifurcation point of eigenvalues λ_n and λ_{-n}.

To examine if there exists any divergence instability of the motion at the critical speed as claimed in Ref. [2] and other references, the eigenfunctions are expressed by substituting $c = c_0$ into Eq. (12). One obtains

$$\tilde{\Psi}_n(\eta) = \mathrm{e}^{\mathrm{i}n\pi\eta} \begin{bmatrix} \sin n\pi\eta/\sqrt{\mathrm{i}n\pi c_0} \\ \sqrt{n\pi} c_0 \mathrm{e}^{\mathrm{i}n\pi\eta}/\sqrt{\mathrm{i}c_0} \end{bmatrix},$$

$$\tilde{\Psi}_{-n}(\eta) = \mathrm{e}^{-\mathrm{i}n\pi\eta} \begin{bmatrix} -\sin n\pi\eta/\sqrt{\mathrm{i}n\pi c_0} \\ -\sqrt{n\pi} c_0 \mathrm{e}^{-\mathrm{i}n\pi\eta}/\sqrt{\mathrm{i}c_0} \end{bmatrix}.$$

$$(n = 1, 2, 3, \ldots)$$

Clearly, the modal functions are linearly independent for all modes at the critical speed. As a matter of fact, all eigenvalue pairs collide for each order of the natural motion. However, it can be demonstrated that any single eigenfunction remains linearly independent of other eigenfunctions at the critical speed. Therefore, the eigenvalues are semi-simple and the motion is stable for the cable translating with the critical speed. After the collision the eigenvalues of each pair exchange their signs of imaginary part, and separate from each other again. No real part of the eigenvalues will come up after the collision. In Ref. [12] this is called weak interaction of eigenvalues when stability remains unchanged.

It is concluded that, based on the aforementioned stability analysis, the divergence instability of normalized modal functions suggested by Ref. [2] does not exist and the motion is still bounded at the critical speed. This may be resulted from the fact that the tensile rigidity of the string becomes zero when $c = c_0$, and the restoring force for moving the string from its current configuration vanishes. Consequently, the string moves axially through a fixed configuration determined by the initial configuration. See Appendix B for the derivation.

5 Free vibration response

Using the theorem of the modal expansion [9], the displacement and its dual are expressed as

$$\boldsymbol{u}(\eta, t) = \sum_{n=1}^{\infty} (\xi_n \tilde{\Psi}_n(\eta) \mathrm{e}^{\lambda_n t} + \xi_{-n} \tilde{\Psi}_{-n}(\eta) \mathrm{e}^{\lambda_{-n} t}), \tag{14}$$

where ξ_n and ξ_{-n} are amplitudes of modal coordinates. The initial conditions of the dual variables are denoted by

$$\begin{aligned} &\boldsymbol{u}_0(\eta) = (\phi(\eta), p_0(\eta))^{\mathrm{T}}, \\ &p_0(\eta) = \varphi(\eta) + c \cdot \mathrm{d}\phi(\eta)/\mathrm{d}\eta, \end{aligned} \tag{15}$$

where $\phi(\eta)$ and $\varphi(\eta)$ are inital displacement and velocity functions. Multiplying both sides of Eq. (14) by $\tilde{\Psi}_i^{\mathrm{T}}(\eta)\boldsymbol{J}$ and integrating it over the total length of the string at time $t = 0$, one obtains

$$\sum_{n=1}^{\infty} (\xi_n \langle \tilde{\Psi}_i, \tilde{\Psi}_n \rangle + \xi_{-n} \langle \tilde{\Psi}_i, \tilde{\Psi}_{-n} \rangle) = \langle \tilde{\Psi}_i, \boldsymbol{u}_0 \rangle. \tag{16}$$

Insertion of the normalized, symplectic orthogonality condition of Eq. (13) into the above equation yields the modal-coordinate amplitudes

$$\xi_n = -\langle \tilde{\Psi}_{-n}, \boldsymbol{u}_0 \rangle \text{ and } \xi_{-n} = \langle \tilde{\Psi}_n, \boldsymbol{u}_0 \rangle. \tag{17}$$

Substituting Eq. (12) into Eq. (17), the transverse displacement solution can be explicitly determined

$$\begin{aligned} &U(\eta, t) = \sum_{n=1}^{\infty} [\xi_n F_n(\eta, t) + \xi_{-n} F_{-n}(\eta, t)], \\ &\quad F_n = \mathrm{e}^{\mathrm{i}(n\pi\eta c/c_0 + n\pi t(c_0^2 - c^2)/c_0)} \sin n\pi\eta/\sqrt{\mathrm{i}n\pi c_0}, \\ &\quad F_{-n} = -\bar{F}_n, \end{aligned} \tag{18}$$

where $\tilde{\Psi}_{1,\pm i}(\eta)$ are the first elements of vectors $\tilde{\boldsymbol{\Psi}}_{\pm i}(\eta)$; and the bar denotes the conjugate complex. Figure 5 shows the transverse displacement at the central point with $c_0 = 1.8$ and three different transport speeds. Note that the initial configuration is assumed as a parabola with a 0.1 maximum sag-to-span ratio. Apparently, the period of displacement response increases if the string translates faster.

The string's configuration is symmetric about the central cross section for non-axially moving strings. For a non-zero transport speed, the shape of the configuration distorts and the symmetry is destroyed. Figure 6 shows the displacements at three different locations on the string with $c_0 = 1.8$ and $c = 0.8$.

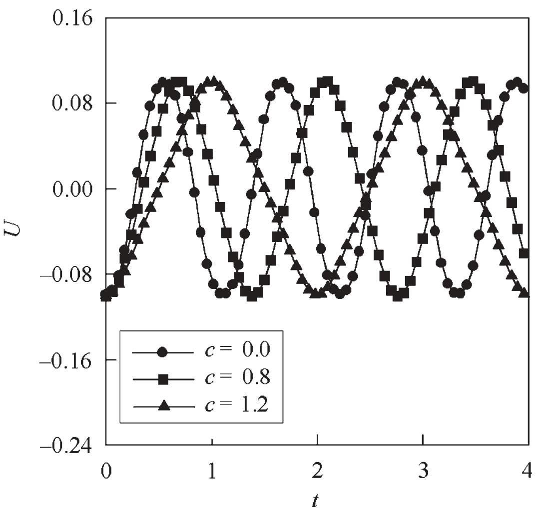

Fig. 5 Displacement at the central point with various transport speeds. $c_0 = 1.8$

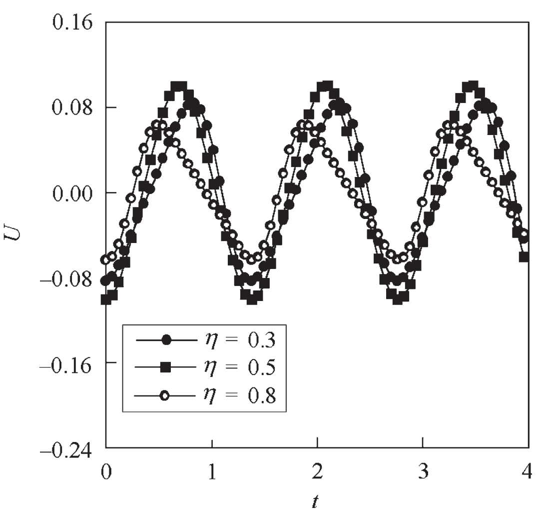

Fig. 6 Displacements at three locations. $c_0 = 1.8$, $c = 0.8$

6 Forced vibration response

Assuming a small elastic deformation, the equation of motion of the string in canonical form with respect the dual vector $\boldsymbol{u}$ is obtained

$$\dot{\boldsymbol{u}}(\eta, t) = \mathbf{H}\boldsymbol{u}(\eta, t) + \boldsymbol{h}(\boldsymbol{u}, \eta, t), \tag{19}$$

where $\boldsymbol{h}(\boldsymbol{u}, \eta, t) = (0, f(U, \dot{U}, \eta, t))^{\mathrm{T}}$. $f(U, \dot{U}, \eta, t)$ is the distributed excitation acting on the string. Note that it is difficult to obtain an analytical solution of the string response for nonlinear vibration problems. In this paper, the expansion theorem of modal functions is used again to determine the approximate solution of displacement. To this end, one selects the normalized eigenvectors as base functions and expands $\boldsymbol{u}$ in the following form

$$\boldsymbol{u}(\eta, t) = \sum_{n=1}^{M}[q_n(t)\tilde{\boldsymbol{\Psi}}_n(\eta) + q_{-n}(t)\tilde{\boldsymbol{\Psi}}_{-n}(\eta)], \tag{20}$$

where M is the truncation number of modes kept in the expansion. Substitution of Eq. (20) into Eq. (19) yields the nonlinear equation in generalized coordinates $q_{\pm 1}, \ldots, q_{\pm M}$

$$\sum_{n=\pm 1}^{\pm M} \dot{q}_n \tilde{\boldsymbol{\Psi}}_n(\eta) = \sum_{n=\pm 1}^{\pm M} q_n \mathbf{H}\tilde{\boldsymbol{\Psi}}_n(\eta) + \boldsymbol{h}(q_{\pm 1}, \ldots, q_{\pm M}, \dot{q}_{\pm 1}, \ldots, \dot{q}_{\pm M}, \eta, t). \tag{21}$$

Multiplying the above equation on both sides with $\tilde{\boldsymbol{\Psi}}_i^{\mathrm{T}}(\eta)\boldsymbol{J}$ and $\tilde{\boldsymbol{\Psi}}_{-i}^{\mathrm{T}}(\eta)\boldsymbol{J}$, respectively, and integrating it over the total length of the string yield first-order ordinary differential equations:

$$\begin{aligned} \dot{q}_i &= \lambda_i q_i + b_i, \\ \dot{q}_{-i} &= \lambda_{-i} q_{-i} + b_{-i}, \end{aligned} \tag{22}$$

where

$$b_{\pm i} = \mp \int_0^1 \tilde{\boldsymbol{\Psi}}_{\mp i}^{\mathrm{T}}(\eta)\boldsymbol{J}\boldsymbol{h}\mathrm{d}\eta \quad (i = 1, 2, \ldots, M) \tag{23}$$

are generalized excitation functions. Note that the orthogonality conditions of normalized modes in Eq. (13) are used to obtain Eq. (22).

6.1 Linear vibration solution

In this case one has $\boldsymbol{h} = (0, f(\eta, t))^{\mathrm{T}}$. The generalized excitation function $b_{\pm i}$ of Eq. (23) can be explicitly determined:

$$b_{\pm i} = \mp \int_0^1 \tilde{\Psi}_{1,\mp i}^{\mathrm{T}}(\eta) f(\eta, t)\mathrm{d}\eta. \quad (i = 1, 2, \ldots, M) \tag{24}$$

The transient response can be solved by the Duhamel's integral

$$U(\eta, t) = \sum_{i=\pm 1}^{\pm M} \int_0^t b_i(\tau)\mathrm{e}^{\lambda_i(t-\tau)}\tilde{\Psi}_{1,i}(\eta)\mathrm{d}\tau, \tag{25}$$

which leads to the exact solution of $U(\eta, t)$ as the truncation number M approaches infinity.

To demonstrate the application of the exact modal functions, a uniformly distributed loading $f(\eta, t) = \sin \omega t$ is applied on the string. Let $c = 0.1$, $c_0 = 1.414$, $\omega = 1.0$, the configuration varying with time is depicted in Fig. 7 over a single period $T = 1.421$. Figure 8 shows the configuration varying with the excitation frequency, where $c = 0.1$, $c_0 = 1.414$ and $t = 1.57$. Figure 9 depicts the string configuration varying with different transport speeds at $t = 1.57$ with $c_0 = 1.414$, $\omega = 1.0$. The effect of transport speed on the configuration can easily be observed. When c is large, the symmetry of the string about its central cross section is noticeably lost. Figure 10 illustrates the configuration solution obtained from expansion of Eq. (20) with different number of modal function terms, where $t = 1.57$, $c = 0.1$, $c_0 = 1.414$, $\omega = 1.0$. The solution obtained with a single, lowest mode in displacement expansion is close to the ones using two and ten modes, and the convergence of the solution is good with term number M equal to two or more.

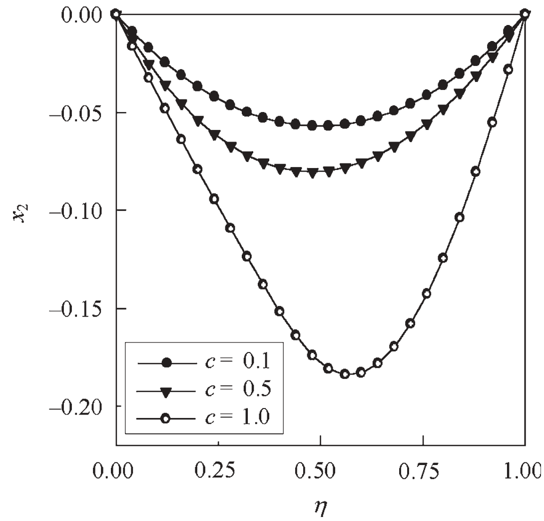

Fig. 9 Configurations vs. transport speeds. $t = 1.57$, $c_0 = 1.414$, $\omega = 1.0$

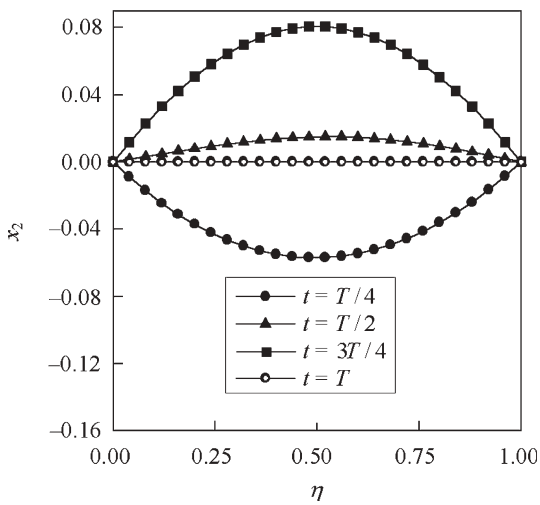

Fig. 7 Configurations in a single period. $T = 1.421$, $c = 0.1$, $c_0 = 1.414$, $\omega = 1.0$

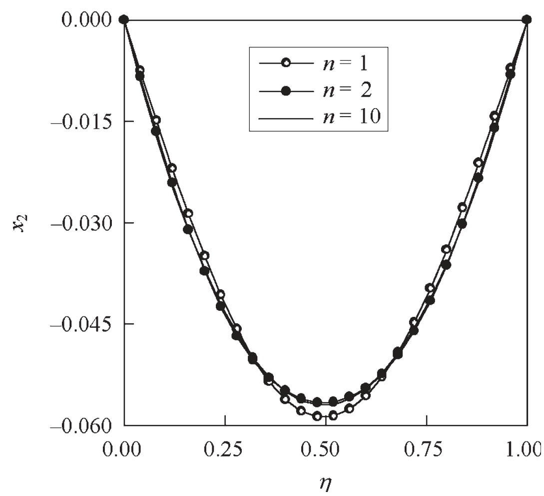

Fig. 10 Configurations vs. numbers of expansion terms. $t = 1.57$, $c = 0.1$, $c_0 = 1.414$, $\omega = 1.0$

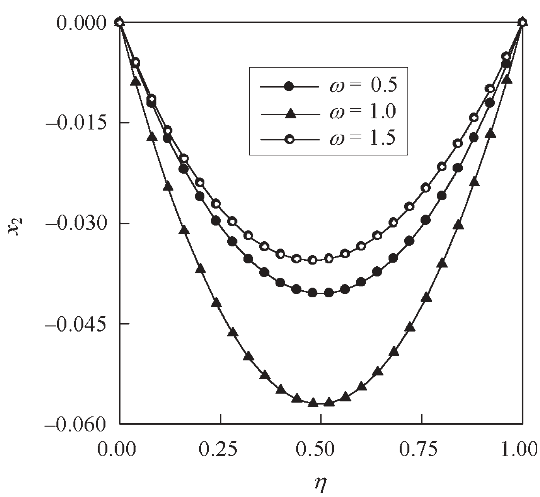

Fig. 8 Configurations vs. excitation frequencies. $t = 1.57$, $c = 0.1$, $c_0 = 1.414$

6.2 Nonlinear vibration: an example

It is now demonstrated that the symplectic modal functions can be used for approximate solutions to nonlinear vibrations with no sigularity at the critical speed. Let the non-dimensional excitation on the string be $f(U, \dot{U}, \eta, t) = a_1 U + a_3 U^3$, where a_1 and a_3 are constant coefficients. The expression of the excitation functions of Eq. (21) becomes

$$\boldsymbol{h} = \left(0, a_1 \sum_{n=\pm 1}^{\pm M} q_n \Psi_{1,n}(\eta) + a_3 \Big[\sum_{n=\pm 1}^{\pm M} q_n \Psi_{1,n}(\eta) \Big]^3 \right)^{\mathrm{T}}. \tag{26}$$

Upon substituting Eq. (26) into Eq. (19), multiplying both sides by $\Psi_i^{\mathrm{T}}(\eta)\boldsymbol{J}$ and $\Psi_{-i}^{\mathrm{T}}(\eta)\boldsymbol{J}$, respectively, and integrating it over the length of the string, equation (19) is decoupled into a series of ordinary differential equations

$$\dot{q}_{\pm n} = \lambda_{\pm n} q_{\pm n} \mp \frac{b'_{\pm n}}{\mathrm{i}n\pi c_0}, \qquad (n = 1, 2, \ldots, M) \tag{27}$$

where

$$b'_{\pm n} = \langle \Psi_{\mp n}, \boldsymbol{h} \rangle. \tag{28}$$

For an approximate solution one may set $M = 1$. The above equation can be expressed in the cubic function of the generalized coordinates as

$$b'_{\pm 1} = Q_{\pm 1} q_1 + Q_{\pm 2} q_{-1} + Q_{\pm 3} q_1^3 + Q_{\pm 4} q_{-1}^3 + Q_{\pm 5} q_1 q_{-1}^2 + Q_{\pm 6} q_1^2 q_{-1}, \tag{29}$$

where the coefficients can be derived explicitly, as

$$\begin{aligned}
Q_1 &= -a_1 \int_0^1 \tilde{\Psi}_{1,-1} \tilde{\Psi}_{1,1} \mathrm{d}\eta = -a_1/2, \\
Q_2 &= a_1 \int_0^1 \Psi_{1,-1}^2 \mathrm{d}\eta = -\mathrm{i} a_1 c_0 (\mathrm{e}^{-2\mathrm{i}\pi c/c_0} - 1) \\
&\quad \times [4\pi c (c^2/c_0^2 - 1)]^{-1}, \\
Q_3 &= a_3 \int_0^1 \Psi_{1,-1} \Psi_{1,1}^3 \mathrm{d}\eta = -3\mathrm{i} a_3 c_0 (\mathrm{e}^{2\mathrm{i}\pi c/c_0} - 1) \\
&\quad \times [4\pi c (c^2/c_0^2 - 1)(c^2/c_0^2 - 4)]^{-1}, \\
Q_4 &= a_3 \int_0^1 \Psi_{1,-1}^4 \mathrm{d}\eta = 3\mathrm{i} a_3 c_0 (\mathrm{e}^{-4\mathrm{i}\pi c/c_0} - 1) \\
&\quad \times [32\pi c (c^2/c_0^2 - 1)(4c^2/c_0^2 - 1)]^{-1}, \\
Q_5 &= 3a_3 \int_0^1 \Psi_{1,1} \Psi_{1,-1}^3 \mathrm{d}\eta = 3\bar{Q}_3, \\
Q_6 &= 3a_3 \int_0^1 \Psi_{1,-1}^2 \Psi_{1,1}^2 \mathrm{d}\eta = 9a_3/8, \\
Q_{-1} &= a_1 \int_0^1 \Psi_{1,1}^2 \mathrm{d}\eta = \bar{Q}_2, \\
Q_{-2} &= a_1 \int_0^1 \Psi_{1,1} \Psi_{1,-1} \mathrm{d}\eta = Q_1, \\
Q_{-3} &= a_3 \int_0^1 \Psi_{1,1}^4 \mathrm{d}\eta = \bar{Q}_4, \\
Q_{-4} &= a_3 \int_0^1 \Psi_{1,1} \Psi_{1,-1}^3 \mathrm{d}\eta = \bar{Q}_3, \\
Q_{-5} &= 3a_3 \int_0^1 \Psi_{1,1}^2 \Psi_{1,-1}^2 \mathrm{d}\eta = Q_6, \\
Q_{-6} &= 3a_3 \int_0^1 \Psi_{1,1}^3 \Psi_{1,-1} \mathrm{d}\eta = 3Q_3.
\end{aligned} \tag{30}$$

Note that the overbar denotes conjugate complex of a quantity. It is observed that

$$\lim_{c/c_0 \to 1} Q_2 = -a_1/4, \quad \lim_{c/c_0 \to 1} Q_3 = -a_3/4, \quad \lim_{c/c_0 \to 1} Q_4 = a_3/16 \tag{31}$$

and

$$\lim_{c/c_0 \to 2} Q_3 = a_3/8, \quad \lim_{c/c_0 \to 1/2} Q_4 = -8a_3. \tag{32}$$

Apparently, all coefficients on the right hand side of Eq. (29) are bounded, which eliminates the singularity of $b'_{\pm 1}$ at the critical speed $c = c_0$. This shows the advantage of the proposed symplectic modal functions. In addition, no divergence occurs for the other two speeds, $c = c_0/2$ and $c = 2c_0$. The governing equations (27) can be solved with perturbation methods or numerical methods, e.g. the method of multiple scale [14].

The exact modal analysis can be combined with approximate methods, e.g. the Galerkin's approach, too. In this case, the symplectic eigenfunctions of Eq. (10) or Eq. (12) are introduced as the trial functions into the discretization scheme, and no divergence of the trial functions will occur when c approaches c_0. Naturally, the precision of the approximate solution of Eq. (27) is mainly determined by the number of modal functions involved in the truncation expression.

7 Conclusions

It is shown in this paper that the symplectic eigenvalue analysis can be adopted to study transverse vibrations of traveling string based on the Hamiltonian dynamics. This method provides not only exact natural frequencies but also complete, non-singular modal functions with transport speeds near or equal to the critical speed, which is an improvement on the traditional complex eigenfunctions. The symplectic modal functions can be used to obtain both linear and nonlinear vibration responses. The convergence of the displacement solution with different number of modal expansion terms is also discussed with examples.

The stability of motion at the critical transport speed is examined based on an eigenvalue analysis. Though the complex conjugate eigenvalue pairs become zeroes, their eigenfunctions remain linear independent at the critical speed, which confirms the stability of the natural motion. Therefore, the divergence instability of the transverse motion at the critical speed does not exist. It is worthwhile to mention that at the critical speed, the stability property of axially moving strings is basically different from translating beams subjected to additional bending forces and governed by a fourth-order differential equation of motion (see, e.g. [13]). A similar comparison of flutter stability characteristics can be found for thin-films and thin-plates excited by aerodynamic loadings [15].

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Appendix A

Based on the Hamilton's canonical equations of (4), a symplectic eigenvalue problem can be formulated as Eq. (8). Assuming $\Psi = (\Psi_1, \Psi_2)^{\mathrm{T}}$ and substituting it into Eq. (7) yield

$$(c^2 - c_0^2)\Psi_1'' + 2c\lambda\Psi_1' + \lambda^2\Psi_1 = 0. \tag{A1}$$

Therefore, one obtains the eigenfunction for displacement as

$$\Psi_1(\eta) = D_1 \mathrm{e}^{\frac{-\lambda\eta}{c-c_0}} + D_2 \mathrm{e}^{\frac{\lambda\eta}{c+c_0}}. \tag{A2}$$

Then, the boundary condition of Eq. (1) leads to

$$\begin{bmatrix} 1 & 1 \\ \mathrm{e}^{\frac{-\lambda}{c-c_0}} & \mathrm{e}^{\frac{\lambda}{c+c_0}} \end{bmatrix} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{A3}$$

Let $\dfrac{2c_0}{c_0^2 - c^2} = \beta \neq 0$, $\lambda = \sigma + \mathrm{i}\omega$. For a non-trivial solution of D_1 and D_2, it must be required that

$$\mathrm{e}^{\beta\sigma}(\cos\beta\omega + \mathrm{i}\sin\beta\omega) = 1. \tag{A4}$$

It gives the natural frequencies of the vibration:

$$\omega_{\pm n} = \pm\beta^{-1}(2n\pi) = \pm n\pi(c_0^2 - c^2)/c_0, \quad (n = 1, 2, 3, \ldots) \tag{A5}$$

where n is the frequency order. The eigenvalue pairs become

$$\lambda_{\pm n} = \mathrm{i}\omega_{\pm n} = \pm\mathrm{i}n\pi(c_0^2 - c^2)/c_0. \quad (n = 1, 2, 3, \ldots) \tag{A6}$$

The eigenvector is determined from (A3) as: $(D_1, D_2)^{\mathrm{T}} = D_{1,\pm n}(1, -1)^{\mathrm{T}}$. Hence, the modal functions of $\Psi_1(\eta)$ are expressed as

$$\Psi_{1,\pm n}(\eta) = D_{1,\pm n}(\mathrm{e}^{-\lambda_{\pm n}\eta/(c-c_0)} - \mathrm{e}^{-\lambda_{\pm n}\eta/(c+c_0)}). \tag{A7}$$

Note that as $c \to c_0$, the limits of the modal functions remain bounded, i.e. $\lim\limits_{c\to c_0} \Psi_{1,\pm n}(\eta) = 2\mathrm{i}\mathrm{e}^{\pm\mathrm{i}n\pi\eta}\sin n\pi\eta$. For convenience, by assigning $D_{1,\pm n} = \pm 1/(2\mathrm{i})$ one obtains

$$\Psi_{1,\pm n}(\eta) = \pm\mathrm{e}^{\pm\mathrm{i}n\pi\eta c/c_0}\sin n\pi\eta. \quad (n = 1, 2, 3, \ldots) \tag{A8}$$

$\Psi_{2,\pm n}(\eta)$ is determined from $\Psi_{2,\pm n} = c\Psi_{1,\pm n}' + \lambda\Psi_{1,\pm n}$. Consequently, the modal functions of the transverse linear vibration are obtained as Eq. (10):

$$\Psi_n(\eta) = \mathrm{e}^{\mathrm{i}n\pi\eta c/c_0}\begin{bmatrix} A \\ B \end{bmatrix},$$

$$\Psi_{-n}(\eta) = \mathrm{e}^{-\mathrm{i}n\pi\eta c/c_0}\begin{bmatrix} -A \\ C \end{bmatrix}, \tag{A9}$$

where

$$A = \sin n\pi\eta,$$

$$B = n\pi(c\cos n\pi\eta + \mathrm{i}c_0\sin n\pi\eta),$$

$$C = -n\pi(c\cos n\pi\eta - \mathrm{i}c_0\sin n\pi\eta), \quad (n = 1, 2, 3, \ldots)$$

It can be easily observed that, when $c = c_0$, the natural frequencies become zero and the motion of the cable will no longer be periodical, but the modal functions remain bounded. So are the normalized modal functions.

Appendix B

Consider the string transporting with the critical speed $c = c_0$. The equation of motion becomes

$$\frac{\partial W}{\partial t} = -2c\frac{\partial W}{\partial \eta}, \tag{A10}$$

where $W(\eta, t) = \partial U/\partial t$ is the velocity function which satisfies the fixed-fixed boundary condition: $W(0, t) = W(1, t) = 0$. Assuming a solution $W(\eta, t) = Z(\eta)\Theta(t)$ and substituting it to (A11) lead to a separation of temporal spatial functions $\dot{\Theta}/\Theta = -2cZ'/Z = \text{const}$ and

$$Z' = \Lambda Z, \tag{A11}$$

where Λ is a constant. Solution $Z(\eta)$ takes the form $Z(\eta) = Z_0 e^{\Lambda\eta}$, where Z_0 is the integration constant. Applying the boundary condition $Z(0) = Z(1) = 0$, it follows that $Z_0 = 0$. As a result, $Z(\eta) = 0$ and, eventually, $W(\eta, t) = 0$. In other words, the velocity function of the string vanishes at the critical speed. And so is the acceleration function if we differentiate $W(\eta, t)$ with respect to time. In conclusion, the string will stay at the initial configuration in this circumstance.