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Analytical and approximate expressions predicting post-failure landslide displacement using the multi-block model and energy methods

Abstract A multi-block sliding model has been proposed in order to simulate the actual geometry of landslides and their rotation with displacement. The governing equation of motion was formulated with the force equilibrium approach and solved by numerical integration in terms of time. The present work derives the formulation of the multi-block model based on another perspective, the energy conservation principle. This approach, in contrast to the force equilibrium approach, has the ability to derive analytical equations predicting the distance moved of masses sliding with resistance exhibiting both cohesional and frictional components. The most general geometry, where analytical solution predicting post-failure displacement can be obtained, is considered. Then, and as this equation is complex, a simple special case geometry is considered in order to derive easy-to-apply simple expressions which predict post-failure landslide displacement in terms of soil resistance and geometric parameters of the sliding mass. The accuracy of this approximate for general geometries expression is validated by extensive parametric analyses.

Keywords Landslides · Ground displacement · Multi-block model · Analytical solution · Energy method · Residual soil strength

Introduction

A multi-block sliding model has been proposed by Stamatopoulos et al. (2011), Stamatopoulos and Di (2014) and Stamatopoulos (2015) in order to simulate the actual geometry of landslides and their rotation with displacement. These studies formulate the governing equation of motion with the force equilibrium approach, and they solve it by numerical integration in terms of time. The present work derives the formulation of the multi-block model based on another perspective, the energy conservation principle. This approach, in contrast to the force equilibrium approach, has the ability to derive analytical equations predicting the distance moved of masses sliding with constant resistance exhibiting both cohesional and frictional components for some specific cases. Such analytical expressions can be used not only to validate the numerical code associated with the force equilibrium and its numerical convergence for the case of post-failure movement but also to obtain exact solutions predicting post-failure landslide displacement for some cases of great interest. Furthermore, these analytical solutions for simple special case geometries can be used in order to derive simple expressions which predict post-failure landslide displacement in terms of soil resistance and geometric parameters of the sliding mass. These expressions can provide easy-to-apply factors predicting the effect of topography and soil strength on post-failure landslide displacement for mapping landslide risk.

The paper first describes briefly the multi-block model and the associated methodology. Then, it derives, validates, and applies a general analytical equation predicting the post-failure displacement of the multi-block model using the energy approach. The most general geometry where analytical solution can be obtained is considered. Then, as this equation is complex, simple geometries are considered in order to derive easy-to-apply simple expressions which predict post-failure landslide displacement. The accuracy of this approximate for general geometries expression is validated by extensive parametric analyses.

The multi-block sliding system model and methodology and validation

Similarly to the Sarma (1979) stability method, shown in Fig. 1a, a general mass sliding on a slip surface which consists of n linear segments with inclinations β_i is considered. In order for the mass to move, at the nodes between the linear segments, interfaces inside the sliding mass must be formed, where resisting forces are exerted. They are situated at angles δ_i, measured from the vertical, positive clockwise (Fig. 1a). The manner which the angles δ_i are estimated is described below. Thus, the mass is divided into n blocks sliding in different inclinations. The forces which are exerted in each block "i" are given in Fig. 1a. According to the Mohr-Coulomb criterion, the forces resisting motion at segment "i" of the slip surface (F_i) and at interface "i" (T_i) are equal to

$$\begin{aligned} F_i &= (R_i - P_i)\tan\varphi_i + c_i l_i \\ T_i &= (N_i - Pb_i)\tan\varphi b_i + cb_i b_i \end{aligned} \tag{1}$$

where φ_i and c_i, l_i, and P_i are the frictional and cohesional components of resistance, the slip length, and the pore water pressure at segment "i" of the slip surface counting uphill, while φb_i and cb_i, b_i and Pb_i are the frictional and cohesional components of resistance, the length and pore water pressure at interface "i."

When the slide moves, two options exist regarding the relative movement of blocks: (a) no separation and (b) separation. Case (b) is not very common and is not considered in the present work. When blocks are not separated, the velocity must be continuous at the interfaces. This means that the displacement of the n blocks is related to each other as:

$$u_i = u_n q_i \qquad \text{where} \quad q_i = \prod_{j=i}^{n-1} \cos\left(\beta_{j+1} + \delta_j\right) / \cos\left(\beta_j + \delta_j\right) \tag{2}$$

where u is shear displacement along the slip surface and the subscript i was defined above.

Taking equilibrium for each block, $2n$ equations are formulated, where n is the number of blocks. The unknown variables are the (n) normal forces to the slip surface R_i, the (n-1) interslice normal forces N_i, and the distance moved by the system. Thus, the system has (2n) equations and (2n) unknowns and thus it can be solved.

As shear displacement develops, mass transfer between blocks occurs according to Eq. (2). In addition, it is assumed that when each block is displaced by $d\tilde{u}_i$, each point of the block is also displaced by $d\tilde{u}_i$ and the masses and lengths of the governing

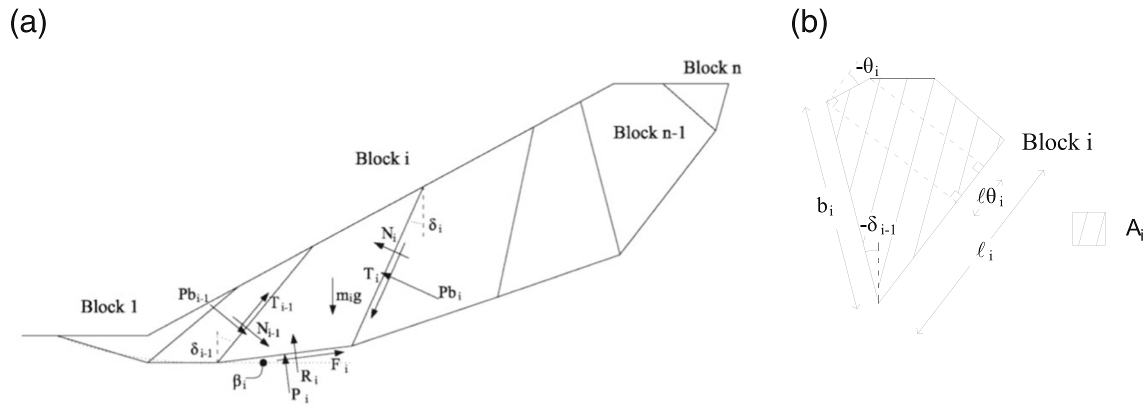

Fig. 1 **a** The multi-block stability method proposed by Sarma (1979). **b** Definition of the lengths b_i, l_i and $l\theta_i$, the area A_i and the angle θ_i of each block, that affect the solution. The x-axis gives the horizontal distance, while the y-axis gives the elevation

equation of motion are updated. It is illustrated in Fig. 2(iii) for actual landslide geometries. A computer program which estimates the displacement of the multi-block model by solving numerically the equations based on the force equilibrium described above has been developed. Details are given by Stamatopoulos et al. (2011).

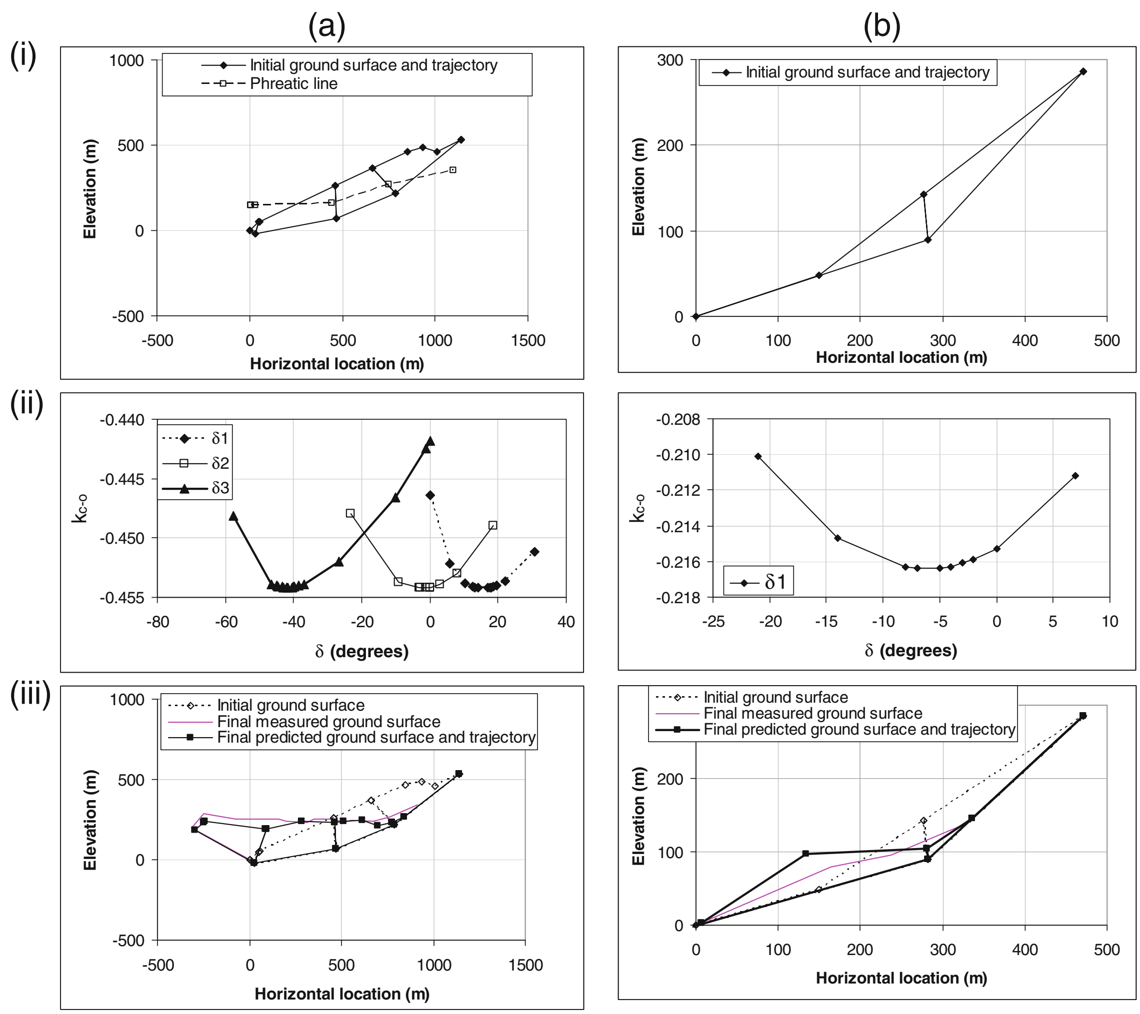

Fig. 2 Applications of Eq. (13): (*i*) Initial slide configuration and phreatic line assumed, (*ii*) critical acceleration normalized by the acceleration of gravity for relative motion at the initial configuration in terms of the interface angles and (*iii*) computed final configuration and comparison with measured final configuration for (**a**) the Vaiont slide (Ciabati 1964; Hedron and Patton, 1985) and (**b**) the Wangjiayan slide (Wu et al. 2010). For the landslide (**b**), the water table line was not measured and it is assumed either to coincide with the slip surface or to be located at mid-depth between the ground and slip surfaces

For a given slip surface, the steps required to apply the proposed model are as follows: (a) the slip, ground and water table surfaces are simulated as a series of linear segments; (b) the inclination of the interfaces is obtained according to the condition of minimum critical horizontal acceleration value for stability of the initial slide configuration (Sarma 1979). If more than one interface exists, iteration is needed. (c) The potential slide deformation is estimated. For steps (b) and (c) along the slip surface, the residual value of soil strength should be used because we deal with post-failure very large displacement. At the interfaces, the peak values of strength must be used because the velocity is assumed continuous and thus relative movement is zero there.

The application procedure above assumes that the residual soil strength is known. In cases where the residual soil strength is not known, for back analyses of these slides, the following procedure is recommended: (1) guess a soil strength; (2) estimate the inclinations of the interfaces in terms of the soil strength, based on the criterion of minimum critical acceleration value and using these inclinations back-estimate the soil strength which best predicts the final deformed geometry; and (3) compare the back-estimated strength with the strength assumed in (1), and if it is different, perform again steps (2) and (3) until convergence is achieved.

Derivation of analytical solution using energy equilibrium

In order to derive an analytical solution predicting landslide displacement, the changes in geometry must be continuous functions of the distance moved. This means that during motion, the cross-sectional area of the mass which moves from block "i" to its neighbor one, "i-1," must be a trapezium, or referring to Fig. 1b

$$u_i < l\theta_i \tag{3}$$

where the distance $l\theta_i$ is the projection along the linear segment "i" of the slip surface of the linear segment of the ground surface just to the right of the interface (i-1). Then, the change in dimensions and masses (m_i) of the blocks can be expressed in terms of the distance moved as:

$$\begin{aligned} l_1 &= l_{1,o} + q_1 u_n \\ l_n &= l_{n,o} - u_n \end{aligned} \tag{4}$$

$$b_i = b_{i,o} + \left(q_{i+1} u_n\right) \left[\sin\left(\theta_{i+1}\right) / \cos\left(\theta_{i+1} + \beta_{i+1} + \delta_i\right)\right] \tag{5}$$

$$\begin{aligned} m_i = m_{i,o} + (\gamma_i/g)\Big\{ &\left(u_n q_{i+1}\right) b_{i,o}\cos\left(\beta_{i+1}+\delta_i\right) - \left(u_n q_i\right) b_{i-1,o}\cos\left(\beta_i + \delta_{i-1}\right) \\ &+ \frac{0.5\cdot\cos\left(\beta_{i+1}+\delta_i\right)\cdot\sin\theta_{i+1}}{\cos\left(\theta_{i+1}+\beta_{i+1}+\delta_i\right)}\cdot\left(u_n q_{i+1}\right)^2 - \frac{0.5\cdot\sin\theta_i\cdot\cos\left(\beta_i+\delta_{i-1}\right)}{\cos\left(\theta_i+\beta_i+\delta_{i-1}\right)}\cdot\left(u_n q_i\right)^2 \Big\} \end{aligned} \tag{6}$$

where the subscript "o" indicates initial value and γ and g indicate the total unit weight and the acceleration of gravity.

The change in the kinetic energy of each of the moving blocks should equal the change in the potential energy (Timoshenko and Young 1948). In addition, the change in the kinetic energy between the initial and the final configuration is zero. Thus,

$$\int_0^{u_{i-m}} \sum_{j=1}^{n} \left[EF_{j-i}\cdot\cos\left(EF_{j-i}, \nu\right)\right] du_i = 0 \tag{7}$$

where for each block i, $u_{i\text{-}m}$ is the maximum (and final) slide displacement, $EF_{j\text{-}i}$ are the acting external forces, and ($EF_{j\text{-}i}$,ν) is the angle that the force $EF_{j\text{-}i}$ makes with the direction of motion.

According to the forces of Fig. 1a, for each block, Eq. (7) gives

$$0 = \int_0^{u_{i-m}} \left\{m_i\, g \sin\, \beta_i + T_i\, \sin(-\delta_i + \beta_i) - T_{i-1}\, \sin(-\delta_{i-1} + \beta_i) - N_i\cos(-\delta_i + \beta_i) + T_{i-1}\cos(-\delta_{i-1} + \beta_i) - F_i\right\} du_i \tag{8}$$

In addition, equilibrium at the direction perpendicular to motion of each block i gives

$$R_i = \left\{m_i\, g \cos\, \beta_i + T_i\, \cos(-\delta_i + \beta_i) - T_{i-1}\, \cos(-\delta_{i-1} + \beta_i) - N_i\, \sin(-\delta_i + \beta_i) + T_{i-1}\, \sin(-\delta_{i-1} + \beta_i) - F_i\right\} \tag{9}$$

Finally, combination of Eqs. (8), (9), and (1) gives

$$\begin{aligned} 0 = \int_0^{u_{i-m}} \Big\{ &-(m_i\, g)v_i + xx_i\, (c_i\, l_i\cos\varphi_i - P_i\sin\varphi_i) \\ &+ N_{i-1}\cos(\varphi b_i + \varphi b_{i-1} - \delta_{i-1} + \beta_i) - N_i\cos(\varphi_i + \varphi b_i - \delta_i + \beta_i) \\ &+ \sin(\varphi b_{i-1} - \delta_{i-1} - \beta_i)\left[cb_{i-1} b_{i-1}\cos(\varphi b_{i-1}) - Pb_{i-1}\sin(\varphi b_{i-1})\right] \\ &- \sin(\varphi b_i - \delta_i - \beta_{i+1})\left[cb_i b_i\cos(\varphi b_i) - Pb_i\sin(\varphi b_i)\right] \Big\}\, du_i \end{aligned} \tag{10a}$$

where

$$\begin{aligned} xx_i &= \cos(\varphi_i - \beta_i) \\ v_i &= \sin(\varphi_i - \beta_i) \end{aligned} \tag{10b}$$

To produce cancelation of the forces N_i and the same displacement, we multiply Eq. (10) by $\left(q_i \prod_{k=i}^{n-1} \mu_k\right)$ where:

$$\mu_k = \cos\left(\varphi_{k+1} + \varphi b_k - \beta_{k+1} - \delta_k\right) / \cos\left(\varphi_k + \varphi b_k - \beta_k - \delta_k\right) \tag{11}$$

Addition of all equations (10) for every block i and insertion of the masses and lengths in terms of the distance moved (u_n), according to Eqs. (4–6), gives

$$\int_0^{u_{n-m}} \left[A1 + B1 u_n + C1 (u_n)^2\right] du_n = 0 \tag{12a}$$

where

$$A1 = \sum_{i=1}^{n-1}\left(\left[-cb_i b_{i,o}\cos(\varphi b_i) + Pb_i\sin(\varphi b_i)\right]\cdot\frac{ss_i}{ff_i}\cdot\prod_{j=i+1}^{n-1}\mu_j\right) + \sum_{i=1}^{n}\left(\left\{-v_i\left[m_{i,o}g\right] + c_i l_{i,o}\cos(\varphi_i) - P_i\sin(\varphi_i)\right\}\cdot\prod_{j=i}^{n-1}\mu_j\right) \tag{12b}$$

$$B1 = \left\{q_2 b_{1,o}\gamma_2\cos(\delta_1+\beta_2)v_1\prod_{j=1}^{n-1}\mu_j - b_{(n-1),o}\gamma_n\cos(\delta_{n-1}+\beta_n)\cdot v_n + \right.$$

$$\left. + \sum_{i=2}^{n-1}\left[q_{i+1}b_{(i+1),o}\gamma_{i+1}\cos(\delta_i+\beta_{i+1}) - q_i b_{i,o}\gamma_i\cos(\delta_{i-1}+\beta_i)\right]v_i\prod_{j=i}^{n-1}\mu_j\right\}$$

$$+ c_1 \; q_n\prod_{j=1}^{n-1}\mu_j - c_n \tag{12c}$$

$$C1 = 0.5\cdot\left\{q_1^{\;2}\prod_{j=1}^{n-1}\mu_j\cdot\frac{\cos(\delta_1+\beta_2)}{\cos(\delta_1+\beta_2+\theta_2)}\cdot\gamma_2 v_1\sin\theta_2 - \cdot\frac{\cos(\delta_n+\beta_n)}{\cos(\delta_{n-1}+\beta_n+\theta_n)}\cdot\gamma_n v_n\cdot\sin\theta_n + \right.$$

$$\left. + \sum_{i=2}^{n-1}\prod_{j=1}^{n-1}\mu_j\cdot\left[-q_i^2\cdot\frac{\cos(\delta_{i-1}+\beta_i)}{\cos(\delta_{i-1}+\beta_i+\theta_i)}\cdot\gamma_i\cdot\sin\theta_i + q_{i+1}^2\frac{\cos(\delta_i+\beta_{i+1})}{\cos(\delta_{i+1}+\beta_{i+1}+\theta_{i+1})}\cdot\gamma_{i+1}\sin\theta_{i+1}\right]\cdot\sin(\beta_i-\varphi_i)\right\} \tag{12d}$$

where $u_{\text{n-m}}$ is the maximum distance moved by the upper block and

$$ss_i = \sin\left(\beta_{i+1} - \beta_i + \varphi_i + \varphi b_{i+1}\right)$$
$$ff_i = \cos\left(\varphi_i + \varphi b_i - \beta_i - \delta_i\right) \tag{12e}$$

For $C1$ different than zero, Eq. (12) predicts that

$$u_{\text{n-m}} = 1.5\left(-0.5\,B1 + \left(0.25B1^2 - 4\,A1\,C1/3\right)^{0.5}/\,A1\right. \tag{13a}$$

When $C1$=0, Eq. (12) gives

$$u_{\text{n-m}} = -2\,A1/B1 \tag{13b}$$

It should be noted that Eqs. (13) are valid only for the case where static instability occurs, or, equivalently, as illustrated below, when $A1$<0.

Discussion, validation, and application of Eq. (13)

The factor $A1$ of Eq. (12b) can be expressed as

$$A1 = g\,k_{c-o}\left(\sum_{i=1}^{n}xx_i m_{i,o}\prod_{j=i}^{n-1}\mu_j\right) \tag{14}$$

where $k_{\text{c-o}}$ is the critical horizontal acceleration for limit stability of the geometry of Fig. 1a at its initial configuration, normalized by the acceleration of gravity. It is given by Stamatopoulos et al. (2011) (their equation (3)). Thus, the factor $A1$ is proportional to the critical acceleration of the sliding mass at the initial configuration. In the present study, it is negative, as instability exists initially. Furthermore, the factors $B1$ and $C1$ represent the effect of the mass transfer during motion in increasing the critical acceleration and thus slowing down the slide. In particular, the factor $B1$ represents the transfer of the mass at blocks 2 to n with cross-sectional area with constant height at each block "i" governed by the interface height, $b_{i,o}$, and length u_i. The factor $C1$ represents the effect of the remaining transferred mass, governed by the angles θ_i. Both factors $B1$ and $C1$ increase as the factor ($\beta_n-\beta_1$) increases. Additionally, the factor $C1$ increases as the angles θ_i increase and diminishes when θ_i=0.

A computer program was written solving the analytical Eq. (13). In this program, the slip, ground, and water table surfaces, as well as the strength parameters along the segments of the slip surface and at the interfaces are given as input. From this input, the geometric parameters and pore pressures needed in Eq. (13) are estimated and then the final displacement ($u_{\text{n-m}}$) is computed. In addition, from $u_{n\text{-m}}$ the program applies Eq. (2) to estimate the displacements $u_{\text{i-m}}$ and plots the final slide configuration.

For the particular case of a slope consisting of two blocks (n=2) and only cohesional resistance (φ_1=φb_1=φ_2=0), it is easy to show that Eqs. (13) are identical to equations (C3) of the previous analytical solution given by Stamatopoulos et al. (2000). In addition, validation of Eq. (13) was performed by comparing the displacement predicted by the two numerical codes, the one which predicts the multi-block displacement numerically, described in "The multi-block sliding system model and methodology and validation" section, and the one using Eq. (13). The geometries of actual slides of natural slopes, dams, and embankments studied by Stamatopoulos et al. (2011) and described in Table 1 were considered. Figure 9 of Stamatopoulos et al. (2011) gives the multi-block representation of these slides. For all these geometries, for $k_{\text{c-o}}$=−0.1 and for both cohesional and frictional components of resistance along the slip surface, it was observed that the difference of the displacement $u_{\text{n-m}}$ computed by the two numerical codes was less than 10 %. In addition, in the geometries of Table 1, the limit displacement ($u_{\text{n-lim}}$) for which Eq. (13) can be applied, based on the restriction (3), is given in Table 1. It can be observed that $u_{\text{n-lim}}$ normalized by the slip length (l_t) equals to 0.1 and 1, with the larger values corresponding to slides with relatively planar slip surfaces.

Furthermore, Eq. (13) was applied to back-estimate the residual friction angle in two cases of actual landslides with relatively planar slip surfaces: (a) the Vaiont slide, which occurred on 9 October 1963 due to change in water table as a result of dam construction (Ciabati 1964, Hendron and Patton 1985) and (b) the Middle School of Beichuan County landslide, triggered by the powerful 12 May 2008 Wenchuan earthquake (Wu et al. 2010). For these slides, the initial and deformed slide geometries and water table elevation at a typical cross-section are given in Fig. 2(i and iii). For the second landslide, the water table line was not measured and it is assumed either to coincide with the slip surface or to be located at mid-depth between the ground and slip surfaces because (i) as the slide was mobilized presumably by a decrease of the residual strength caused by excess pore pressure build-up, the water table surface was on or above the slip surface and (ii) as the

Table 1 Actual slides considered to validate Eq. (13) and estimate the accuracy of Eqs. (19) and (21)

N o	Name	β_1	β_n	l_t (m)	$u_{n\text{-lim}}$ (m)	Nu	Eq. (19) X'_{Rc}	Eq. (19) σ_{Rc}	Eq. (21) X'_{Rc}	Eq. (21) σ_{Rc}
1	Alaska	1	51	237	28	11	0.70	0.23	1.19	0.63
2	Malakassa	−15	45	329	37	16	0.82	0.16	0.78	0.36
3	Nikawa	6	32	89	18	25	1.20	0.16	0.82	0.49
4	Choan	−2	54	24	5.1	15	1.01	0.13	0.97	0.15
5	Sifnos-1	0	42	12	2.9	15	1.11	0.14	1.28	0.15
6	Sifnos-2	0	18	82	82	17	0.92	0.19	0.73	0.14
7	Rimnio	0	72	57	10	17	0.64	0.22	0.77	0.68
8	Kushiro	0	36	21	4.4	20	0.89	0.09	1.25	0.46
9	Wachusset	0	43	90	8.1	21	1.13	0.18	0.70	0.37
10	Lamarquessa-upstream	0	45	20	6.9	14	0.66	0.08	0.88	0.52
11	Lamarquessa-downstream	0	45	12	5.4	24	1.34	0.08	0.70	0.06
12	Lapalma	10	88	21	3.4	25	0.77	0.09	1.19	0.58
13	Lower San-Fernando	−3	62	119	35	21	0.74	0.08	0.61	0.32
14	Kabutono	0	63	28	7.0	21	0.87	0.03	0.73	0.06
				ALL		262	0.94	0.18	0.89	0.43

Figure 9 of Stamatopoulos et al. (2011) gives the multi-block representation of these slides. The values of the parameters β_1, β_n, and l_t for each geometry are provided. Regarding the estimation of the accuracy of Eqs. (19) and (21), the number of cases studied (Nu) and the mean and standard deviation values of the factor Rc ($=u_{ma\text{-}ap}/u_{ma}$) is given

earthquake occurred just at the start of the annual rainfall season and as just prior or during the earthquake it did not rain, the water table was not presumably near the ground surface. For both slides, uniform strength is taken along the slip surface, the unit weight of the soil is taken as 2T/m^3 and at the interfaces $\varphi=\varphi_{max}=35°$ and zero cohesion is assumed. The back analysis procedure described in "The multi-block sliding system model and methodology and validation" section was applied. For both slides, convergence was achieved after two iterations and Fig. 2(ii) gives the computed critical acceleration for relative motion at the initial configuration, in terms of the interface angles. For the interface angles which produce minimum critical acceleration, the final slide configurations obtained which best predict the measured deformed geometries is given in Fig. 2(iii). The corresponding strength $\varphi(=\varphi_{res})$ equals for the first slide 9° and for the second slide 25 and 28° if the water table is at the slip surface or at mid-depth between the ground and slip surfaces respectively .

It can be observed that the proposed method worked well for both slides: convergence was achieved when estimating the interface angles. In addition, the computed deformed geometries agree reasonably with the measured. The discrepancy which exists in the second slide, where the cross-sectional area of the deformed slide is larger than the initial, is presumably a result of lateral spread in the direction not shown in the figure, which is not modeled in the two-dimensional model used. Last but not least, for the first slide, the back-estimated residual friction angle is in the range of the measured values (6–10°) in ring shear tests performed in samples from the slip surface of the slide by Tika and Hutchinson (1999).

Derivation of simplified expressions predicting landslide displacement

As the analytical expression (13) is complex, and its application requires knowledge of the interface angles which are usually not readily known, at this section, simpler expressions are derived. In particular, the governing equation of motion of the multi-block model greatly simplifies when (a) $\varphi b_i=0$ and $\varphi_i=\varphi$, (b) $cb_i=0$ and $c_i=c$, (c) $\delta_i=-(\beta_i+\beta_{i-1})/2$, (d) $\theta_i=0$, (e) $\gamma_i=\gamma$, and (f) $P_i=Pb_i=0$. These assumptions are applied in the present section. Figure 3a gives the geometry corresponding to assumptions (c) and (d). As a result of provisions (a), (b), and (c), $q_i=1$, $\mu_i=1$, $\nu_i=-\sin(\varphi-\beta_i)$, $xx_i=\cos\beta_i$. Thus, in Eqs. (13), $C1=0$ and

$$u_{m-n} = -2\,k_{c-o}\sum_{i=1}^{n}\left(A_{i,o}\cos\beta_i\right) / \left[h\left(\sin\beta_1-\sin\beta_n\right)\right] \tag{15a}$$

where h is the height of the blocks (Fig. 3a) and

$$k_{c-o} = \frac{cl_t + g\sum_{i=1}^{n}\left\{\sin(\varphi-\beta_i)m_{i,o}\right\}}{g\sum_{i=1}^{n}\left(\cos\beta_i m_{i,o}\right)} \tag{15b}$$

In order to apply Eq. (15) for the case of a general multi-block geometry where blocks with different height slide with non-uniform displacement along a slip surface, we define the average height (h_{av}) and the average displacement (u_{av}) as

$$h_{av} = (h_2 + \ldots + h_n)/(n\text{-}1) \tag{16}$$

$$u_{av} = (u_1 + \ldots + u_n)/n \tag{17}$$

where h_i is the height of interface "i-1," given in Fig. 3b. In addition, to further simplify Eq. (15), it can be assumed that

$$\sum_{i=1}^{n}(A_{i,o}\cos\beta_i)\approx\cos\left[(\beta_1+\beta_n)/2\right]\sum_{i=1}^{n}A_{i,o} = A_t\cos\left[\left(\beta_1+\beta_n\right]/2\right) \tag{18}$$

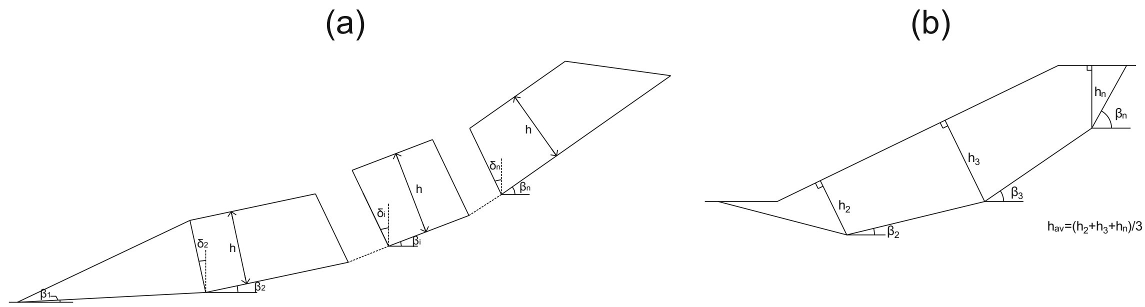

Fig. 3 **a**. Simplified multi-block geometries considered in "Derivation of simplified expressions predicting landslide displacement" section, **b** Illustration of the manner to obtain h_{av} in the case of a general multi-block geometry

where A_t is the total cross-sectional area of the sliding mass. Then, Eqs. (15), (16), (17), and (18) give

$$u_{\text{ma-ap}} = -2\,k_{\text{c-o}}\ (A_t/\,h_{av})\cos((\beta_1+\beta_n)/2)\ /\ (\sin\beta_1\text{-}\sin\beta_n) \quad (19)$$

where the subscripts "ap" and "ma" indicate "approximate" and "maximum average" displacement of the sliding mass respectively.

Furthermore, we define the angle β_{av} as the "average inclination" of the slope under consideration. The angle β_{av} corresponds to the "limit friction angle for stability" of the potentially unstable mass and can be estimated by soil stability analyses, as the friction angle corresponding to a factor of safety equal to unity. Alternatively, as a first approximation, it can be estimated as the average of the inclinations of the top and bottom linear segments of the slip surface.

Then, it can be assumed that

$$\sum_{i=1}^{n}\left[A_{i,o}\sin(\varphi-\beta_i)\right]\approx\sum_{i=1}^{n}\left[A_{i,o}\sin(\varphi-\beta_{av})\right]=\sin(\varphi-\beta_{av})\,A_t \quad (20)$$

When the cohesional resistance is zero, Eqs. (15b), (19), and (20) give

$$u_{\text{ma-ap}}\approx -2\sin(\varphi\text{-}\beta_{av})(A_t/\,h_{av})\ /\ [(\sin\beta_1\text{-}\sin\beta_n)] \quad (21)$$

Finally, as a first approximation, it can be assumed that the expressions (19) and (20) can be applied at very large displacements, even in cases where the restriction (3) does not hold, by replacing the angle β_n with the angle β'_n that equals

$$\beta'_n = [\,(l_n\text{-}u_n)\beta_n + \beta_{n\text{-}1}(u_n)\,]\ /\ l_n \quad (22)$$

Discussion and error analysis Eqs. (19), (21), and (22)

The accuracy of Eqs. (19), (21), and (22) predicting shear displacement for general slide geometries was studied (a) using the actual slide geometries of Table 1 described above and (b) a two-block triangular slope, which was also studied by Stamatopoulos et al. (2011). Figure 5 of Stamatopoulos et al. (2011) gives the geometry. The parameters defining this triangular slide are the angles giving the inclinations of the linear segments of the slip surface β_1 and β_2 (=β_n), the interface angle (δ), the inclination of the ground surface ($\theta+\beta_1$), the slip length l_t (=l_1+l_2), and the water table elevation (wte). All these parameters were varied in a parametric manner. The initial and main case considered had the following characteristics: β_1=5°, β_n=40°, δ=−10°, θ=30°, l_t=20m, wte=0. Then, only one geometric parameter changed per analysis, as indicated in Table 2.

The methodology of analysis in all cases was the following: estimate by trial-and-error a limit minimum value of uniform strength, which gives critical acceleration value, between 0 and −0.01 g. Then, decrease the value of strength in small increments by dividing each time by 1.05. When the critical acceleration is less than −0.2 g, the analysis stops. Equation (19) was validated for both the c=0 and ϕ=0 cases and Eq. (21), by definition, for only the c=0 case. Regarding Eq. (19), for the critical acceleration, the

Table 2 Cases that were considered in the parametric analyses of a two-block triangular slide

Parameter	Cases considered	Nu	Eq. (19) X'_{Rc}	σ_{Rc}	Eq. (21) X'_{Rc}	σ_{Rc}
β_1 (°)	−30, −20, −10, 0, 5, 10, 20	74	1.06	0.10	0.83	0.11
β_n (°)	31, 40, 50, 60	37	1.03	0.12	0.83	0.13
δ (°)	−40, −30, −20, −10, 0, 10, 20, 30	96	1.02	0.16	0.81	0.14
θ (°)	15, 20, 25, 30, 35	65	0.93	0.10	1.15	0.28
l_t (m)	20, 60, 100, 300, 1000	47	0.99	0.06	1.35	0.12
wte (m)	0, 1, 2, 3	81	1.02	0.05	0.77	0.15
c (kPa)	0, 2, 4, 6, 8	47	0.91	0.12	0.90	0.13
ALL		447	1.00	0.10	0.95	0.15

Figure 5 of Stamatopoulos et al. (2011) gives the geometry of this slide. The number of cases studied (Nu) and the accuracy of Eqs. (19) and (21) are given in terms of the factor Rc and the parameter that was varied actual value for the slide case considered each time is used. Regarding Eq. (21), for β_{av} the friction angle corresponding to a critical acceleration value equal to zero is used. The analyses performed corresponded to a range of u_n/l_t values between 0 and 1 for the slides of Table 1 and 0 to 0.5 for the triangular slides. Tables 1 and 2 indicate the total number of analyses performed.

From all the results of these analyses, first, it was found that the factors defined as

$$
\begin{aligned}
E1 &= \cos \beta_{av} A_t \Big/ \sum_{i=1}^{n} \left(A_{i,o} \cos \beta_i \right) \\
E2 &= \left(A_t \gamma \sin\left(\varphi - \beta_{av}\right) + c\, l_t \right) / \left(k_{c-o} A_t\, \gamma \cos \beta_{av} \right)
\end{aligned}
\tag{23}
$$

have an average value equal to 0.92 and 0.93 and a standard deviation equal to 0.03 and 0.31, respectively. The fact that the average values are near unity and the standard deviation values are small, verifies the approximations (18) and (20).

Then, the accuracy of Eqs. (19) and (21) was studied by comparing the predicted displacement with that computed by the numerical code described in "The multi-block sliding system model and methodology and validation" section. The ratio of predicted to the actual computed values of displacement ($Rc = u_{ma\text{-}ap}/u_{ma}$), was estimated in all cases. The average value of this ratio (X'_{Rc}) is an estimate of accuracy and its standard deviation (σ_{Rc}) is an estimate of consistency. Tables 1 and 2 give X'_{Rc} and σ_{Rc} in terms for each case considered, and totally. For all results, the factor Rc has (a) a mean value near unity (between 0.7 and 1.2) and (b) a standard deviation less than 0.3 and 0.7 for Eqs. (19) and (22) respectively. Thus, considering the simplicity of the proposed equations, it can be inferred that their accuracy and consistency for general slide geometries is satisfactory.

Conclusions

The post-failure displacement of slides is studied using a recently developed sliding system model. First, the general analytical expression (13) predicting the displacement of slopes consisting of n blocks and exhibiting both frictional and cohesional components of resistance is derived, validated, and applied. The energy approach is used for this purpose. Then, and as this expression is complex, simpler geometries and strength assumptions are considered in order to derive Eqs. (i) (19) and (ii) (21), where the slide displacement depends only on simple geometrical parameters of the sliding mass and (i) the critical horizontal acceleration for stability at the initial configuration of the sliding mass or (ii) the limit friction angle for stability at the initial configuration of the sliding mass and the residual soil strength. The accuracy of these expressions for general geometries was studied by extensive parametric analyses.

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