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STABILITY AND LOCAL BIFURCATION IN A SIMPLY-SUPPORTED BEAM CARRYING A MOVING MASS

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ABSTRACT The stability and local bifurcation of a simply-supported flexible beam (Bernoulli-Euler type) carrying a moving mass and subjected to harmonic axial excitation are investigated. In the theoretical analysis, the partial differential equation of motion with the fifth-order nonlinear term is solved using the method of multiple scales (a perturbation technique). The stability and local bifurcation of the beam are analyzed for 1/2 sub harmonic resonance. The results show that some of the parameters, especially the velocity of moving mass and external excitation, affect the local bifurcation significantly. Therefore, these parameters play important roles in the system stability.

KEY WORDS stability, local bifurcation, simply-supported beam, moving mass

I. INTRODUCTION

Owing to their application in industry, the dynamics of flexible structure has been studied for a long time since it always induces the possibility of structural instability. With the rapid development of industry and technology, the stability and local bifurcation of the flexible beam has brought more and more puzzles and attracted considerable attention[1–3]. In addition, the flexible beam carrying a moving mass, which is a fundamental problem both in the field of transportation and in the design of machining processes, has caused greater difficulties than ever before. For the beam system carrying a moving mass, the mass is just assumed to be traversing the continuous beam with a constant velocity resulting in a partial differential equation. In transportation engineering applications, typical examples include the motion of vehicles on bridges, cranes carrying moving loads, robotic arms and space structures[4–6].

To the best of the authors' knowledge, the literature on the topic of flexible beams with moving masses are constantly expanding. Stanišić and Lafayette[7] presented an exact solution of the dynamics behavior of the structures carrying a moving mass which is in perfect agreement with an accurate numerical solution. According to the position of the moving mass upon the beam, they used the form of Green's function to analyze the amplitude coefficient of response. Mofid and Akin[8] introduced parameters to define the type of boundaries. Accordingly, they presented and formulated a new technique for determining the time response of beams carrying a moving mass with different boundary conditions. Lee[9] presented a numerical solution based on integration programs using the Runge-Kutta method for integrating the response of clamped-clamped beam acted upon by a moving mass. He pointed out the possibility of the mass separating from the beam in the course of motion by monitoring the contact force between the mass and the beam. Siddiqui et al.[10, 11] considered a simplified model and retained

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only quadratic nonlinearities in the equations of motion. Then they assumed the motion of a flexible cantilever beam carrying a moving spring-mass system to be a Bernouli-Euler beam using different methods and obtained an approximate solution using the perturbation method of multiple scales. Yau and Fung[12] applied Hamilton's principle and carefully calculated the frequencies of a rotating flexible arm carrying a moving mass under different parameters. Subsequently, they[13] took into account the effect of centrifugal stiffening due to the rotation of the beam and studied a clamped-free flexible arm rotating in a horizontal plane while carrying a moving mass. It is noted that the studies mentioned above have neglected the inertial effect of the moving mass by considering it as a moving force, or simplified the coupling between the moving mass under plane motion and the flexible beam.

To understand the stability and local bifurcation of the beam with a moving mass, the modeling and computational method should be highly accurate. An important aspect of nonlinear vibration is the occurrence of local bifurcation found in many mechanical systems[14].

In this paper, the stability and local bifurcation of a simply-supported flexible beam carrying a moving mass are investigated with special attention to the inertial effect (i.e., the velocity of the moving mass in plane motion), especially the Coriolis acceleration, the centripetal acceleration of the moving mass and the beam acceleration at the point of contact with the moving mass. Accordingly, the partial differential equation of motion with the fifth-order nonlinear term for the beam with simple supports under the action of the moving mass subjected to harmonic axial excitation is obtained.

II. BASIC THEORY

Consider the Bernoulli-Euler damped beam, which is flexible. In Fig.1, the span is L, the bending rigidity and the damping are EI and c, respectively, and the mass per unit length of the uniform beam is m. The moving mass is M, the constant velocity is v, and the harmonic axial excitation is expressed in the form $-p\cos(\Omega t)$. The external force, namely, the moving reaction of the mass upon the beam can be expressed as

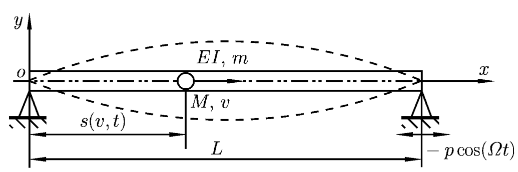

Fig. 1 Structural system.

$$F = -M\ddot{w}(x,t) \tag{1}$$

Taking into account the acceleration due to the moving mass, the partial differential equation with a fifth-order nonlinear term for the bending vibration of the flexible beam can be written as

$$m\frac{\partial^2 w(x,t)}{\partial t^2} + c\frac{\partial w(x,t)}{\partial t} + EI\frac{\partial^4 w(x,t)}{\partial x^4} + \left\{-p\cos(\Omega t) - \frac{EA}{2L}\int_0^L \left[\frac{\partial w(x,t)}{\partial x}\right]^2 \mathrm{d}x + \frac{EA}{8L}\int_0^L \left[\frac{\partial w(x,t)}{\partial x}\right]^4 \mathrm{d}x\right\}\frac{\partial^2 w(x,t)}{\partial x^2} = -M\ddot{w}(x,t)\,\delta(x-s) \tag{2}$$

It should be mentioned that if $x \neq s$, the equation for the vibration of the flexible beam with no moving mass has been obtained before in Ref.[15]. By taking into account the influence of the moving mass, it is appropriate to consider $x = s$ in the following studies. Here, $w(x,t)$ represents the transverse deflection of the beam at the axial coordinate x with the origin at the left end of the beam and time t, $s(t) = vt$ is the displacement of the moving mass at time t and δ is the Dirac delta function.

As the mass is moving on the beam[16], the velocity of the moving mass can be expressed as

$$\dot{w}(x,t) = \frac{\partial w(x,t)}{\partial x}\dot{x} + \frac{\partial w(x,t)}{\partial t} \tag{3}$$

Therefore, the acceleration of the moving mass is

$$\ddot{w}(x,t) = \frac{\partial^2 w(x,t)}{\partial x^2}\dot{x}^2 + 2\frac{\partial^2 w(x,t)}{\partial x \partial t}\dot{x} + \frac{\partial w(x,t)}{\partial x}\ddot{x} + \frac{\partial^2 w(x,t)}{\partial t^2} \tag{4}$$

In Eq.(4), the over dot ($\dot{}$) denotes differentiation with respect to time. The terms on the right-hand side of the equation are the centripetal acceleration of the moving mass, the Coriolis acceleration, the acceleration component in the vertical direction when the moving mass is not a constant, which is invalid if the constant speed of the moving mass and the beam acceleration at the point of contact with the moving mass are considered. It is noted that, however, only the last item is mentioned in Ref.[17]. This paper considers the inertial effect on the moving reaction of the mass.

The following dimensionless quantities are introduced

$$\eta=\frac{w}{L},\quad \xi=\frac{x}{L},\quad \xi_0=\frac{s}{L},\quad C=\frac{cL^2}{\sqrt{EIM}},\quad \alpha=\frac{M}{mL}$$
$$\beta=\frac{AL^2}{I},\quad \tau=\left(\frac{EI}{m}\right)^{1/2}\frac{t}{L^2},\quad u=\sqrt{\frac{m}{EI}}vL,\quad \underline{\Omega}=\Omega L^2\sqrt{\frac{m}{EI}},\quad \bar{p}=\frac{pL^2}{\pi^2 EI}$$

By substituting Eq.(4) into the partial differential Eq.(2), the dimensionless equation is obtained

$$\frac{\partial^2\eta}{\partial\tau^2}+C\frac{\partial\eta}{\partial\tau}+\frac{\partial^4\eta}{\partial\xi^4}+\left[-\bar{p}\cos(\underline{\Omega}\,\tau)-\frac{1}{2}\beta\int_0^1\left(\frac{\partial\eta}{\partial\xi}\right)^2\mathrm{d}\xi+\frac{1}{8}\beta\int_0^1\left(\frac{\partial\eta}{\partial\xi}\right)^4\mathrm{d}\xi\right]\frac{\partial^2\eta}{\partial\xi^2}$$
$$=-\alpha\left(u^2\frac{\partial^2\eta}{\partial\xi^2}+2u\frac{\partial^2\eta}{\partial\xi\partial\tau}+\frac{\partial^2\eta}{\partial\tau^2}\right)\delta\left(\xi-\xi_0\right)\tag{5}$$

Furthermore, by using the assumed mode method, the dimensionless quantity η can be expressed as

$$\eta\left(\xi,\tau\right)=q\left(\tau\right)\sin(\pi\xi)$$

For Eq.(5) using Galerkin integral leads to

$$\ddot{q}\left(\tau\right)+\mu\left(\xi_0,u\right)\dot{q}\left(\tau\right)+\omega^2\left(\xi_0,u\right)q\left(\tau\right)+P\left(\xi_0\right)q\left(\tau\right)\cos(\underline{\Omega}\,\tau)+f\left(\xi_0\right)q^3\left(\tau\right)-\gamma\left(\xi_0\right)q^5\left(\tau\right)=0\tag{6}$$

where

$$\mu\left(\xi_0,u\right)=\frac{C+2u\alpha\pi\sin(2\pi\xi_0)}{1+2\alpha\sin^2(\pi\xi_0)},\ \omega^2\left(u,\xi_0\right)=\frac{\left[\pi^2-2u^2\alpha\sin^2(\pi\xi_0)\right]\pi^2}{1+2\alpha\sin^2(\pi\xi_0)},\ P\left(\xi_0\right)=\frac{\bar{p}}{1+2\alpha\sin^2(\pi\xi_0)}$$
$$f\left(\xi_0\right)=\frac{\beta\pi^4}{4\left[1+2\alpha\sin^2(\pi\xi_0)\right]},\quad \gamma\left(\xi_0\right)=\frac{3\beta\pi^6}{64\left[1+2\alpha\sin^2(\pi\xi_0)\right]}\tag{7}$$

To achieve a system which is suitable for the application of the multiple scales method, the scale transformations can be introduced as[18]

$$\mu\left(\xi_0,u\right)\to\varepsilon\mu\left(\xi_0,u\right),\quad P\left(\xi_0\right)\to\varepsilon P\left(\xi_0\right),\quad f\left(\xi_0\right)\to\varepsilon f\left(\xi_0\right),\quad \gamma\left(\xi_0\right)\to\varepsilon\gamma\left(\xi_0\right)$$

Here, ε is a small perturbation parameter, therefore the dimensionless equation of the nonlinear systems is expressed as

$$\ddot{q}\left(\tau\right)+\varepsilon\mu\left(\xi_0,u\right)\dot{q}\left(\tau\right)+\omega^2\left(\xi_0,u\right)q\left(\tau\right)+\varepsilon P\left(\xi_0\right)q\left(\tau\right)\cos(\underline{\Omega}\,\tau)+\varepsilon f\left(\xi_0\right)q^3\left(\tau\right)-\varepsilon\gamma\left(\xi_0\right)q^5\left(\tau\right)=0\tag{8}$$

To solve the above equation using the method of multiple scales, it is assumed according to Eq.(8) that

$$q\left(\tau,\varepsilon\right)=q_0\left(T_0,T_1\right)+\varepsilon q_1\left(T_0,T_1\right)+\cdots\tag{9}$$

Here $T_0=\tau$ and $T_1=\varepsilon\tau$ are usual fast and slow time scales, respectively. The time derivatives used in the method of multiple scales are defined as

$$\frac{\mathrm{d}}{\mathrm{d}t}=\frac{\partial}{\partial T_0}+\varepsilon\frac{\partial}{\partial T_1}+\cdots=D_0+\varepsilon D_1+\cdots\tag{10}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}=\left(D_0+\varepsilon D_1+\cdots\right)^2=D_0^2+2\varepsilon D_0D_1+\cdots\tag{11}$$

where $D_n=\partial/\partial T_n$, $n=0,1$.

For 1/2 sub harmonic resonance, $\omega^2 = \underline{\Omega}^2/4 + \varepsilon\sigma$ should be assumed, where σ is a detuning parameter. Let $\underline{\Omega} = 2$ for convenience of analysis. Substituting Eqs.(9)-(11) into Eq.(8) and after separating terms at each order of ε, one obtains

$$D_0^2 q_0 + q_0 = 0 \tag{12}$$

$$D_0^2 q_1 + q_1 = -2D_0 D_1 q_0 - \mu D_0 q_0 - 2\sigma q_0 - P q_0 \cos(2T_0) - f q_0^3 + \gamma q_0^5 \tag{13}$$

At the first-order of perturbation, the solution is

$$q_0 = A\left(T_1\right) \mathrm{e}^{\mathrm{i}T_0} + \bar{A}\left(T_1\right) \mathrm{e}^{-\mathrm{i}T_0} \tag{14}$$

At order ε, substituting Eq.(14) into Eq.(13) yields

$$D_0^2 q_1 + q_1 = \left(-2\mathrm{i}D_1 A - \mu \mathrm{i} A - 2\sigma A - \frac{1}{2}P\bar{A} - 3fA^2\bar{A} + 10\gamma A^3 \bar{A}^2\right) \mathrm{e}^{\mathrm{i}T_0} + \mathrm{cc} + \mathrm{NST} \tag{15}$$

where cc stands for complex conjugate of the preceeding terms and NST non-secular terms.

$$D_1 A = -\frac{1}{2}\mu A + \mathrm{i}\sigma A + \frac{1}{4}\mathrm{i}P\bar{A} + \frac{3}{2}\mathrm{i}fA^2\bar{A} - 5\mathrm{i}\gamma A^3\bar{A}^2 \tag{16}$$

To obtain the real solutions, let

$$A = \frac{1}{2}a\mathrm{e}^{\mathrm{i}\varphi}$$

where a and φ are real functions of T_1, respectively. Then Eq.(16) can be expressed as

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}T_1} &= -\frac{1}{2}\mu a + \frac{1}{4}Pa\sin(2\varphi) \\ a\frac{\mathrm{d}\varphi}{\mathrm{d}T_1} &= \sigma a + \frac{1}{4}Pa\cos(2\varphi) + \frac{3}{8}fa^3 - \frac{5}{16}\gamma a^5 \end{aligned} \tag{17}$$

To find the bifurcation response equation, letting $a\mathrm{d}\varphi/\mathrm{d}T_1 = \mathrm{d}a/\mathrm{d}T_1 = 0$ in Eq.(17) and then eliminating φ results in the following algebraic equation

$$4\mu^2 a^2 + \left(\frac{5}{4}\gamma a^5 - 4\sigma a - \frac{3}{2}fa^3\right)^2 = P^2 a^2 \tag{18}$$

Obviously, $a = 0$ is one of the solutions. The others can be given by the roots of the following polynomial

$$\begin{aligned} &25\gamma^2 a^8 - 60f\gamma a^6 - \left(160\gamma\sigma - 36f^2\right)a^4 + 192f\sigma a^2 + 16\left(16\sigma^2 + 4\mu^2 - P^2\right) = 0 \\ &\Delta = 16\sigma^2 + 4\mu^2 - P^2 \end{aligned} \tag{19}$$

Three different cases will be investigated separately (only considering the real solutions).

(i) $\Delta \neq 0$, there are at most four different nonzero solutions simultaneously.

(ii) $\Delta = 0$, there are at most three different nonzero solutions simultaneously.

(iii) $\sigma = 0$ and $\Delta = 0$, there are at most two different nonzero solutions.

To determine the stability of the zero solution, Eq.(17) should be transformed from the polar form into a Cartesian form. Let $A = h + g\mathrm{i}$, where h and g are real functions of T_1, respectively. Hence, Eqs.(17) can be expressed as

$$\begin{aligned} \frac{\mathrm{d}h}{\mathrm{d}T_1} &= -\frac{1}{2}\mu h + \left(\frac{1}{4}P - \sigma\right)g - \frac{3}{2}gf\left(h^2 + g^2\right) + 5h^4 g\gamma + 10h^2 g^3\gamma + 5g^5\gamma \\ \frac{\mathrm{d}g}{\mathrm{d}T_1} &= \left(\frac{1}{4}P + \sigma\right)h - \frac{1}{2}\mu g + \frac{3}{2}fh\left(h^2 + g^2\right) - 5hg^4\gamma - 10h^3 g^2\gamma - 5h^5\gamma \end{aligned} \tag{20}$$

Equation (20) has a zero solution, namely, $(h, g) = (0, 0)$ at which the characteristic equation is

$$\lambda^2 - \mu\lambda + \frac{1}{16}\Delta = 0 \tag{21}$$

where Δ is presented by Eq.(19).

It follows from Eq.(21) that two eigenvalues of the characteristic equation become zero simultaneously, indicating that zero solution are critical points of codimension-3. Thus, the degenerate bifurcations of codimension-3 occur at a certain point. This is the outcome of the fifth-order nonlinear terms.

III. APPROXIMATE SOLUTIONS AND CONCLUSIONS

It can be seen that it is difficult to analytically calculate all the nonzero solutions from the bifurcation response equation (19). Thus, the numerical method will be used to solve the bifurcation response based on Eq.(19). If one considers Eq.(7), it is easy to obtain useful results for several typical cases.

(i) $\xi_0 = 0,\ 0.5,\ 1$

When $\xi_0 = 0, 1$ (here $\alpha = 0$), the case corresponds to one where the moving mass passes though two boundary points; When $\xi_0 = 0.5$, the moving mass is in the middle of the beam. Then several parameters (Eq.(7)) can be written as

$$\mu = \frac{C}{1+2\alpha}, \quad P = \frac{\bar{p}}{1+2\alpha}, \quad f = \frac{\beta\pi^4}{4\,(1+2\alpha)}, \quad \gamma = \frac{3\beta\pi^6}{64\,(1+2\alpha)}$$

Obviously, the damping parameter μ is related to both C and α. In this case, as can be seen, the dimensionless velocity u has no effect on the real function a. However, the detuning parameter σ exerts significant effect on the real function a (see Fig.2, when $\xi_0 = 0, 0.5, 1$).

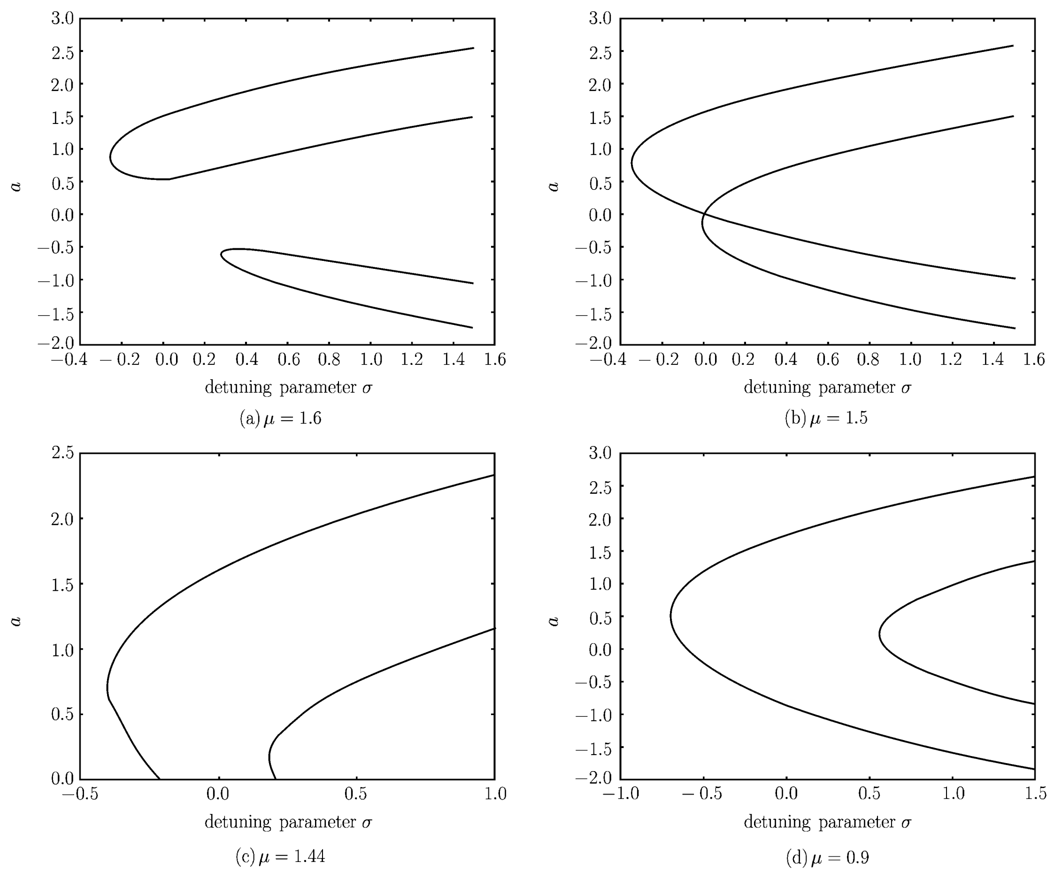

Fig. 2. The bifurcation response curves when $\xi_0 = 0, 0.5, 1,\ f = 1,\ \gamma = 3\pi^2/16$ and $P = 3.0$.

In Fig.2, four different values of damping parameter are chosen. As illustrated in this figure, the frequency range in which either two or four nonzero solutions exist simultaneously changes as the damping parameter μ varies. In addition, the nonzero solutions and zero solutions are close to each other with a decrease of μ. It is observed that the threshold values of nonzero solutions are different if various damping parameters μ are utilized.

Figure 3 represents the effect of forcing amplitude on the bifurcation responses when $\xi_0 = 0, 0.5, 1$. Obviously, the system may have two, three or four nonzero solutions simultaneously. Moreover, the range of the frequency for nonzero solutions is very sensitive to the forcing amplitude P as shown in

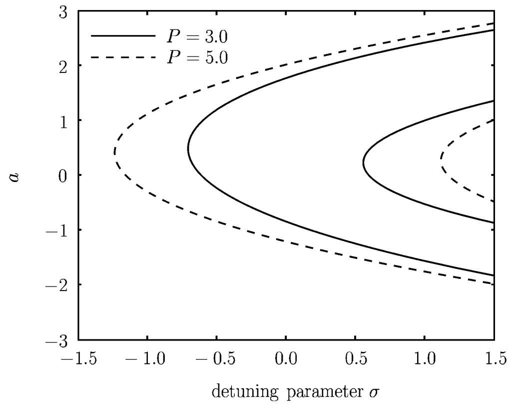

Fig. 3. The variation of bifurcation response curves for different forcing amplitude when $\xi_0 = 0, 0.5, 1$, $f = 1$, $\gamma = 3\pi^2/16$ and $\mu = 0.9$.

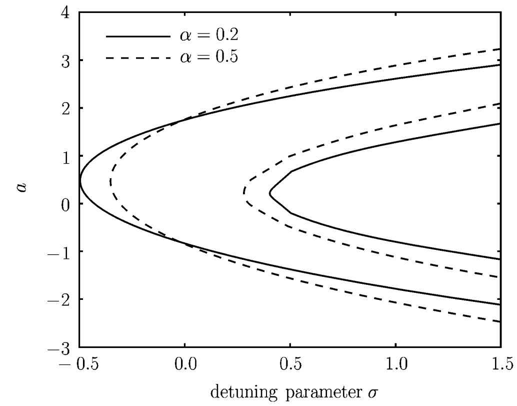

Fig. 4. The variation of bifurcation response curves for different mass ratios when $\xi_0 = 0.5$, $\beta = 4(1+2\alpha)/\pi^4$, $C = 0.9$ and $\bar{p} = 3.0$.

Fig.3. It is also noted that the threshold value of nonzero solutions for large values of P is much smaller than that for the case of small values of P.

Figure 4 shows the effect of the mass ratio on the bifurcation response when $\xi_0 = 0.5$. Similarly, the system may have two, three or four nonzero solutions simultaneously. With the increase of the mass ratio α, the range of the frequency for two nonzero solutions becomes narrow.

(ii) ξ_0 is chosen to be others, for example, $\xi_0 = 1/4$. The parameters (Eq.(7)) can be written as

$$\mu = \frac{C + 2\alpha u\pi}{1+\alpha}, \quad P = \frac{\bar{p}}{1+\alpha}, \quad f = \frac{\beta\pi^4}{4(1+\alpha)}, \quad \gamma = \frac{3\beta\pi^6}{64(1+\alpha)}$$

Hence, the detuning parameter σ and dimensionless velocity u both affect the real function a.

Figure 5 gives the sample results for $\xi_0 = 1/4$ with different dimensionless velocities u. It is shown that the range of the frequency for two nonzero solutions becomes narrow as the dimensionless velocity u is increasing though there are also two, three or four nonzero solutions simultaneously. Similarly, the threshold value of nonzero solutions is sensitive to the system parameter u.

In Figs.6-8, the solutions of the system gradually vary from four to two with increases of u. However, the curves indicate that there exist nonzero solutions, while the zero solution could not be detected. Moreover, in the plane of (u, a), when the moving mass is located at other positions except $\xi_0 = 0, 0.5, 1$, a greater detuning parameter, greater forcing amplitude and smaller damping can expand the range of frequency for nonzero solutions. The numerical results show that, these parameters have appreciable

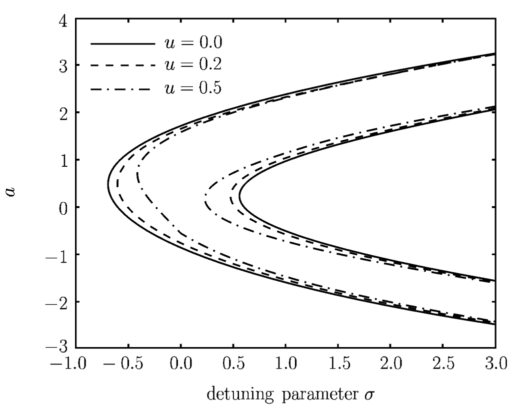

Fig. 5. The variation of bifurcation response curves for different dimensionless velocity when $\xi_0 = 1/4$, $f = 1$, $\gamma = 3\pi^2/16$, $C = 0.9$, $\alpha = 0.2$ and $P = 3.0$.

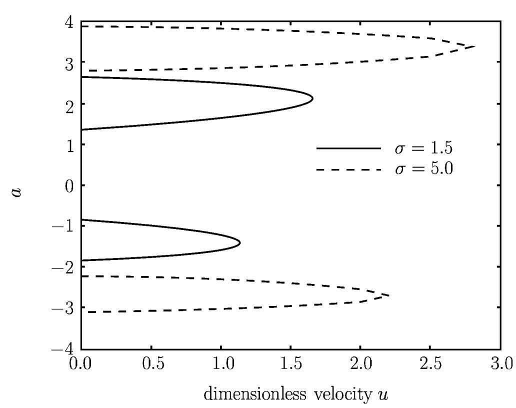

Fig. 6. The variation of a versus u for different detuning parameter when $\xi_0 = 1/4$, $f = 1$, $\gamma = 3\pi^2/16$, $C = 0.9$, $\alpha = 0.2$ and $P = 3.0$.

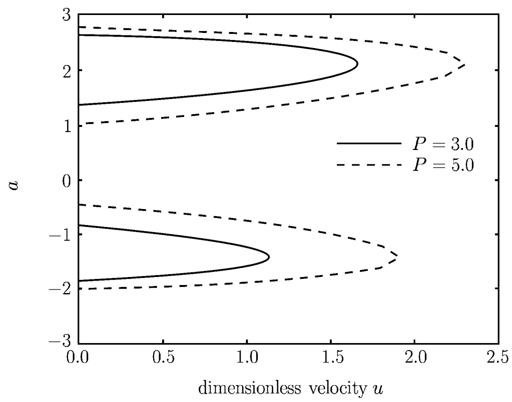

Fig. 7. The variation of a versus u for different forcing amplitudes when $\xi_0 = 1/4$, $f = 1$, $\gamma = 3\pi^2/16$, $C = 0.9$, $\alpha = 0.2$ and $\sigma = 1.5$.

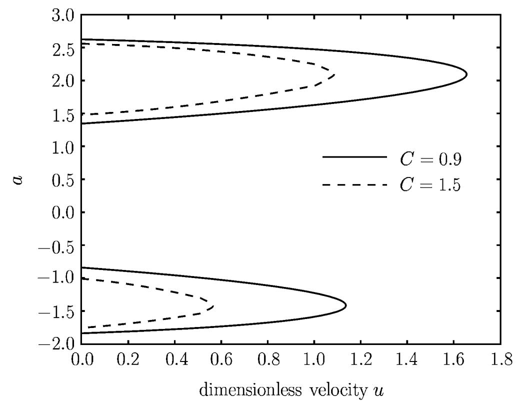

Fig. 8. The variation of a versus u for different damping when $\xi_0 = 1/4$, $f = 1$, $\gamma = 3\pi^2/16$, $\alpha = 0.2$, $P = 3.0$ and $\sigma = 1.5$.

influence on the bifurcation solutions with the various velocity of moving mass. For the analysis, the system stability is directly related to the velocity of the moving mass.

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