LAMBDA: FAQs

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Abstract Since its introduction in 1993, the LAMBDA method has found widespread use across the world. The method has been employed in many geodetic and navigation applications, with lots of satisfied users. Independent tests show that it is considered the best method for integer carrier phase ambiguity resolution available. But every now and then we still notice some misunderstandings concerning the principles and potential of the method. In this contribution we will briefly summarize the principles underlying the LAMBDA method, go into some of the frequently asked questions on the LAMBDA method and try to clarify some of the existing misunderstandings.

Background

Global navigation satellite system (GNSS) ambiguity resolution is the process of effectively accounting for the integer nature of the unknown cycle ambiguities of double-difference (DD) carrier phase data. The (sole) purpose of ambiguity resolution is to use the integer ambiguity constraints as a means of improving significantly on the precision of the remaining model parameters, such as baseline coordinates and/or atmospheric delays.

For the purpose of ambiguity resolution, GNSS data processing is usually carried out in three steps, as shown in Fig. 1. In the first step no distinction is made between ambiguities and other parameters. The parameter estimation problem is solved without taking into account the special integer nature of the ambiguities. The result so obtained is often referred to as the 'float solution'. The parameters are usually estimated using a standard least-squares (LSQ) algorithm, or in case of moving receivers, a Kalman-filter.

Up to then the fact that the ambiguities are of integer nature is not yet exploited. Two extra steps are necessary to incorporate this information. In the second step, the float solution of the ambiguities is used to estimate the integer ambiguity values. A numerical example is given in Fig. 2. It is this step for which the LAMBDA method has been developed.

Finally, in the third step, the computed integer ambiguities are used to improve the first-step solution for the remaining parameters, like baseline coordinates and/or atmospheric delays. These parameters are recomputed, again in a least-squares sense, but this time with the ambiguities constrained to the integer values as obtained from the second step. This final result is referred to as the 'fixed' solution and it generally inherits a much higher precision than the previously obtained 'float solution'.

Question 1. How do I compute baseline coordinates using the LAMBDA-method?

The LAMBDA method on its own is not meant to compute baseline coordinates. Instead, it is a single, important step in the process of determining very precise coordinates and/or other parameters of interest from raw GNSS observations. As stated above, a three-step procedure has to be applied. First you have to compute the float solution. Any suitable method can be used here, for example an ordinary least-squares, or in recursion a Kalman filter approach. Next the LAMBDA method can be used to determine the integer values for the ambiguities. The LAMBDA method takes the estimated float solution $\hat{a}$, together with its variance–covariance matrix $Q_{\hat{a}}$ as input, and delivers estimated integer ambiguities $\check{a}$. Finally, again using your own favourite algorithm, the eventual fixed solution can be computed, using the values output by the LAMBDA method.

The LAMBDA algorithm is an autonomous module for the middle arrow in Fig. 1. As such it can easily be embedded in any of the existing GPS baseline/network software packages. In fact, it has been implemented in a number of commercially available software packages, for instance SKI-Pro by Leica Geosystems.

Received: 15 February 2002 / Accepted: 7 March 2002
Published online: 11 October 2002

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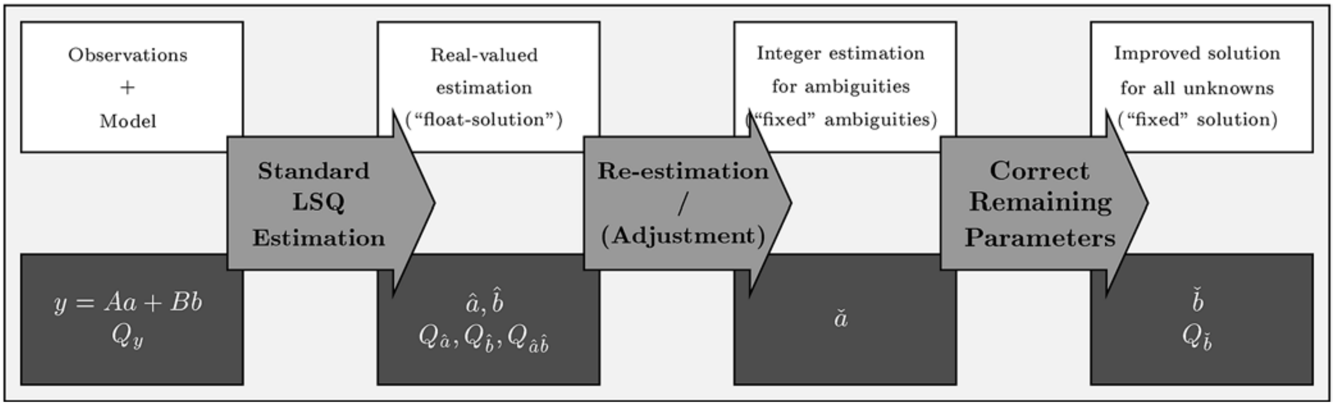

Fig. 1
Procedure to solve a problem in which some of the parameters are known to be integer, like double-difference combinations of ambiguities in the case of GNSS carrier phase observations. y represents the vector of carrier phase observations and optionally pseudo-range code observations, Q_y represents the variance–covariance matrix of these observations, a represents the vector of unknown integer carrier phase ambiguities, b represents the vector of unknown baseline coordinates, atmospheric delays and possibly other parameters, the matrices A and B relate the observations y to the unknowns a and b, and are referred to as 'design matrices'. $\hat{a}$ and $\hat{b}$ are the float estimates for a and b, $Q_{\hat{a}}$ and $Q_{\hat{b}}$ are the corresponding variance–covariance matrices. $Q_{\hat{a}\hat{b}}$ represents the covariances between $\hat{a}$ and $\hat{b}$. $\check{a}$ represents the integer estimate of the carrier phase ambiguities. Finally, $\check{b}$ represents the fixed solution, with $Q_{\check{b}}$ its variance–covariance matrix

Estimation principle

In data processing one should distinguish between the estimation principle and the implementation in an algorithm. The estimation principle is the theoretical mandate that prescribes how estimation values for unknown parameters shall be obtained from observed data values. Over more than 200 years the least-squares principle has been in use. It tells that the (weighted) sum of squared residuals is to be minimized. It is a very versatile criterion to obtain a solution that presents a best fit to observed data. A typical example is fitting a regression line through a 'cloud' of points in a two-dimensional graph (see Fig. 3); the two real-valued parameters to be determined are the offset and slope of the line.
The outcome, the least-squares estimate, generally consists of real-valued numbers for all of the desired parameters. Recently, in Teunissen (1993), least-squares theory has been extended to deal also with *integer* parameters. The resulting integer estimator, analogously to the 'ordinary' least-squares estimator, has distinct and well-defined optimality properties.

Fig. 2
Numerical example of the integer least-squares estimator; notice that the result ($\check{a}$) is not equal to the float solution ($\hat{a}$) rounded to the nearest integers. The transformation matrix Z^T is discussed in the implementation section of this paper

$$\hat{a} = \begin{bmatrix} 5.450 \\ 3.100 \\ 2.970 \end{bmatrix}; Q_{\hat{a}} = \begin{bmatrix} 6.290 & 5.978 & 0.544 \\ 5.978 & 6.292 & 2.340 \\ 0.544 & 2.340 & 6.288 \end{bmatrix} \xrightarrow[LAMBDA]{} \check{a} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}; Z^T = \begin{bmatrix} -2 & 3 & 1 \\ 3 & -3 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Question 2. Is the LAMBDA method the best method around for ambiguity resolution?

The LAMBDA method is a strict implementation of the integer least-squares principle. How you rank the LAMBDA method therefore depends on your appreciation of the optimality property of integer least-squares. This principle qualifies as 'best' as it offers the *highest probability* of coming up with the *correct* integer values for the ambiguities. 'Best' is measured by a probability, and is thereby a statistical property.
In order to understand this, one has to be familiar with the fact that ambiguity estimators are stochastic quantities, simply because they are determined from noisy data. Only in the hypothetical case of perfect observations without any noise or errors would the float solution always yield the correct integer ambiguity values. In reality, however, this is not the case. Any uncertainty in the observations will propagate and manifest itself as uncertainty in the integer ambiguity values.
Figure 4 shows a repeated single-frequency experiment based on a geometry-free GNSS model. The figure illustrates empirically how uncertainty in the data (left) propagates into the ambiguity float estimate (middle) and finally into the integer ambiguity estimate (right). The correct integer for the ambiguity is value 4 in this case, but, as one can see from the graph at right, also other integer values are frequently obtained. A more extensive discussion of this can be found in Joosten and Tiberius (2000).
This concept is formalized in a probabilistic measure, referred to as the *ambiguity success-rate*. The success-rate is a number between 0 and 1, or 0 and 100%, and it expresses the chance, or probability, that the whole set of integer ambiguities is correctly estimated. This ambiguity success-rate depends on three contributing factors: the observation equations (functional model), the precision of the observables (the stochastic model) and the chosen principle

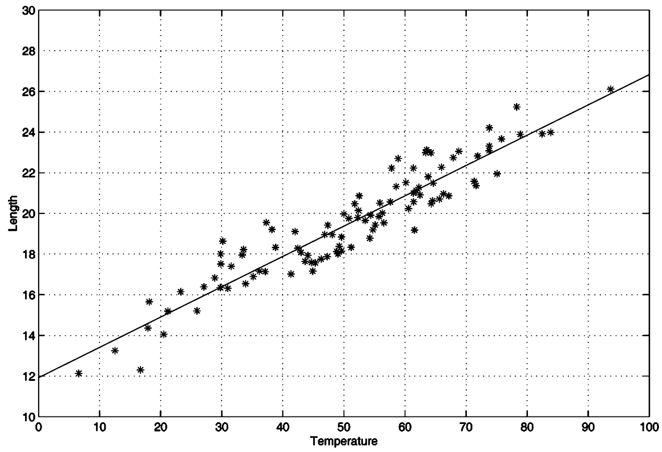

Fig. 3
One physical parameter (length) depends on another (temperature) and the relationship is known or assumed to be linear. By means of least-squares a regression line is fitted through a set of observed values

of integer estimation. Changes in any one of these will affect the success-rate. The first two contributing factors reflect the strength of the data model and they are given once the measurement setup is known. As to the principle of integer estimation, one has a variety of options available, but integer least-squares maximizes this success-rate, and can thus be qualified as 'best' (see Teunissen 1999a, 1999b).

Question 3. I thought the LAMBDA method is the best method available, but still I get wrong results, how is this possible, and what can I do about it?

The LAMBDA method provides the highest probability of estimating the integer ambiguities correctly, and as such is indeed the best possible method available. But 'best' does still not imply perfect, i.e. a 100% success-rate. As long as our data are subject to noise, the success-rate will never achieve the full 100%. It is still possible to get wrong results. Some possible causes for this are:

- Quality of data: the less precise (the more noisy) the measurements are, the less precise the resulting (float) estimates will be.
- Amount of data (observations/measurements): not enough data have been collected to estimate the unknown parameters with sufficient accuracy. More data would yield more precise (float) estimates. Going back to Fig. 4, the average of all 1,800 pseudo-range residuals in Fig. 4 has a much better precision than each of the

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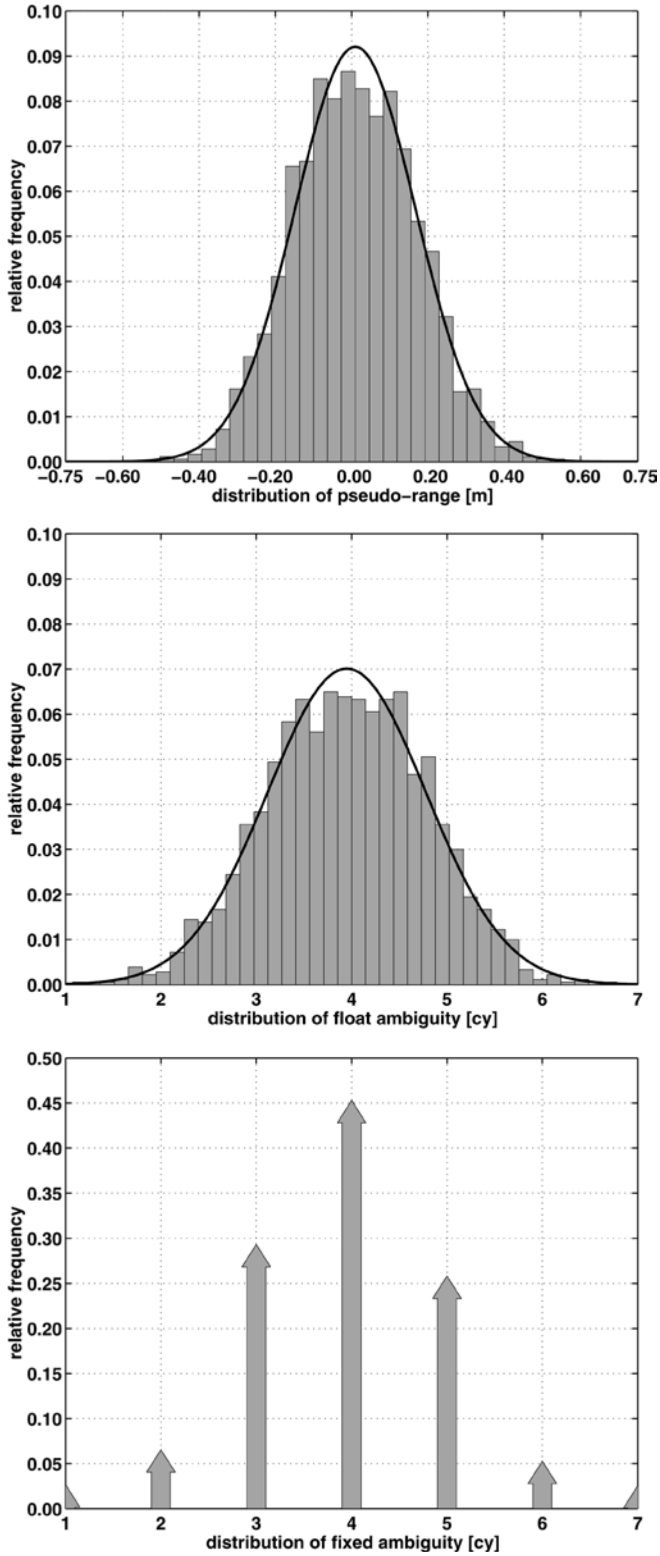


Fig. 4
Using single-frequency pseudo-range and carrier phase data, the phase ambiguity of the geometry-free GPS model is estimated in 1,800 single epoch experiments at a 1-s interval. The *histogram at left* shows the residuals of the (double-difference) pseudo-range measurements. Repeating the experiment yields slightly different outcomes each time; the noise is at the decimetre level. The *histogram in the middle* concerns the float ambiguity. It is primarily the noise in the pseudo-range which is reflected in the noise of the float ambiguity, and as the L1 wavelength is about 2 dm, the corresponding uncertainty in the float ambiguity is at the cycle level. In both the graph at the left and in the middle, the formal Gaussian probability distribution is also shown. Finally, the integer ambiguity was computed for each experiment (just by nearest integer rounding, in this simple case with just one ambiguity), and yields the *'histogram' at right*. In this case the integer ambiguity is estimated correctly (value 4) in only 43% of the experiments

individual observations. This average is pretty close to 0.00 m, and when using this average to estimate the ambiguity as integer, the success-rate would be much closer to 100%.

- Strength of model: data are fused into parameter estimates according to a certain (functional) relationship that is assumed to exist between observations and unknown parameters. The weaker the relationship represented by the model, the less precise the resulting (float) estimates will be.

The success-rate depends on the strength of the (mathematical) model. A simple bridge across a river constructed from just a few trees is not to be taken by a 40-tonne truck. Don't ask for the unachievable. Don't expect a 99.99% success-rate at a permanently moving rover with five satellites over a 100 km baseline with a cheap, single-frequency engine. When the rover receiver is in permanent motion, new position coordinate unknowns have to be introduced every epoch and this weakens the mathematical model. On long distances (differential) atmospheric delays start to play a role, and they are to be accounted for. As an example, Fig. 5 shows the impact of using *dual*-frequency data instead of single-frequency data on ambiguity resolution. For visualization purposes the fail-rate is shown instead of the success-rate. The fail-rate represents the probability of estimating the *wrong* integer values for the carrier phase ambiguities, and thus equals 1 (one) minus the success-rate. The example is based on the GPS satellite configuration as of 1 January 2002, as seen in Delft. The figure clearly shows the advantage of using dual-frequency data. In case of dual-frequency data the fail-rate very rarely reaches a level of 0.01 (or 1%), whereas in the single-frequency case the fail-rate is above 0.1 (or 10%) for most of the day. This clearly shows that instantaneous ambiguity resolution even over short baselines with a single-frequency receiver is unlikely to be successful.

Fig. 5
Ambiguity fail-rates in case of single- (*left*) and dual- (*right*) frequency data, geometry-based, single epoch, short baseline over a full 24 h period. The 'fail-rate' is the probability of *wrong* integer ambiguity estimation, and thus equals 1 minus the success-rate

Implementation

An estimation principle, as for instance least-squares, provides a mathematical formulation that tells how estimation values are to be obtained from observed data. This formula can commonly be evaluated in different ways. The formula '*14+3*5*' is equivalent to '*14+15*', first evaluating the multiplication, but also to '*14+5+5+5*', evaluating the addition three times.

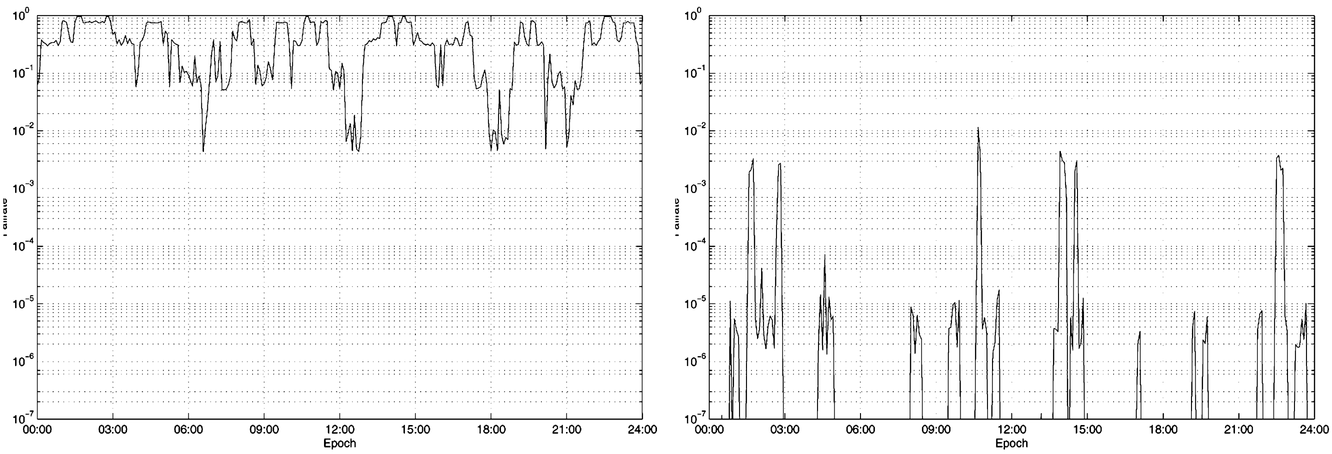
Solving a linear system of equations $N\,x=r$ for unknown vector x, with square and full rank matrix N and given vector r, can be done by computing the matrix inverse. Multiplication of the right hand side r yields the solution $x=N^{-1}r$. A much more elegant (and efficient) algorithm is provided by Gaussian elimination. The system $N\,x=r$ is handled by solving, in two steps, an equivalent triangular system by forward and back substitution, with triangular matrices L and U, as $N=L\,U$. Similarly to Gaussian elimination to solve a general linear system of equations for unknown real-valued parameters, the LAMBDA method is an efficient implementation to solve *integer* least-squares problems.

Unfortunately, the integer least-squares principle does not present us with a formula for the integer estimate as an *explicit* function of the observed pseudo-range and carrier phase measurements. The (integer) least-squares norm, sometimes referred to as the cost-function, that needs to be minimized has an implicit formulation and reads

$$\min_a (\hat{a}-a)^T\, Q_{\hat{a}}^{-1}(\hat{a}-a)$$

with $\hat{a}$ the float estimate and $Q_{\hat{a}}$ the corresponding variance–covariance matrix. The minimization is solved by a search over grid points a, each representing an ambiguity vector with all integer values, in an ellipsoidal region in the ambiguity parameter space (Fig. 6). The vector found is the integer least-squares estimate $\check{a}$.

In practice, the estimated (float) ambiguities are highly correlated and the ellipsoidal region stretches over a considerable range of cycles. A search for integer vectors inside this region can be terribly inefficient. To improve the computational efficiency of the search, the float ambiguities are transformed, whereby the elongated ellipsoid turns into a sphere-like search space. The search is



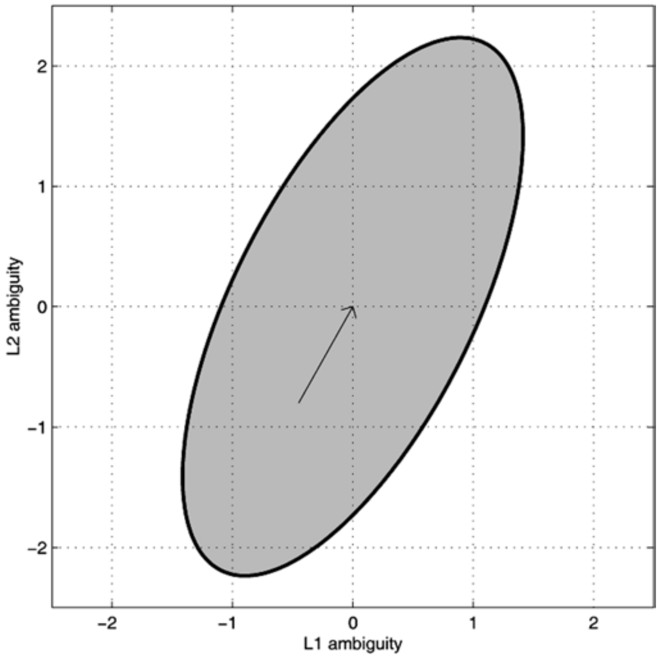


Fig. 6
Two-dimensional example of ellipsoidal search region in ambiguity parameter space. The boundary of the ellipse represents locations a that all have equal norm

1	2		−1	−2	−2		−3	1	−1	1	2
	−1	−1	2			1		1	−3	−1	
−1			1	1	−1	1		−1	−1	−2	2
1	−2	−1			1	1	−1	1	1		−2
−1	1	−1	1	−1		−1	2	1	−2	1	1
−1		−1		−1		1					1
	1		−1	−1			−1	−1		−1	1
−1			1		1	2	−2		1	2	−2
	1	1	−1	1	−1			1			
		2	−2		−1	1	1		−1	−1	
1	−1	1		1	−1	−1	1	−1		−1	1
1	−1	1			−1					1	

Fig. 7
Example of a decorrelating transformation matrix Z^T that brings original ambiguities a into decorrelated ambiguities z, according to $z=Z^Ta$. This example is based on a single epoch of dual-frequency GPS data to seven satellites on a short baseline. Consequently there are six L1 double-difference ambiguities, followed by six L2 double-difference ambiguities in the same order. Zero entries in the matrix Z^T have been left blank

executed and the eventual result is transformed back. The LAMBDA method includes both steps, the transformation and the actual search (see e.g. Joosten and Tiberius 2000). It should be noted that the transformation is not required by the (theoretical) estimation principle; it is only to achieve a considerable gain in speed in the computation process. Counting the exact amount of money in a big bag of coins is more efficient once you first sort the different pieces, instead of taking them piece by piece and adding their value to the overall running sum. The eventual outcomes of the transformation, search and back-transformation together, and of just the search alone, are identical, but for the latter approach a 'little' more patience has to be exercised.

Question 4. Lots of people are always talking about 'wide-laning', how is this different from the LAMBDA method?

Both the LAMBDA method and 'wide-laning' techniques aim at creating linear combinations of ambiguities that have better precision and are less correlated than the original ambiguities. The way this goal is achieved, however, is different. Where wide-laning makes use of certain *predefined* linear combinations, the LAMBDA method creates linear combinations based on the measurement precision and the structure of the (mathematical) model employed, for instance the receiver–satellite geometry.

As an example, Fig. 7 shows the decorrelating transformation matrix Z^T for a case with a single epoch of dual-frequency GPS data to seven satellites. Consequently, there are 12 ambiguities to be estimated. For example, in the last-but-one row, one finds a '1' for the L1 ambiguity and a '–1' for the corresponding L2 ambiguity, or vice versa, like the traditional wide-lane combination. But here the total transformed ambiguity is eventually a combination of five wide-lane combinations across satellites. Other, similar, combinations occur in the transformation matrix Z^T. A linear combination can be formed across frequencies and across satellites, thus exploiting the actual satellite–receiver geometry in order to achieve a better decorrelation than could be achieved by traditional wide-lane combinations.

In short: wide-laning is just one special (and usually suboptimal) case of the LAMBDA Z-transformation of ambiguities.

Question 5. I heard/believe LAMBDA is not suitable for [my application], is this true?

No, this is not true. The LAMBDA method can intrinsically handle any integer estimation problem, as long as you can provide a float solution for your problem. In other words, as long as you can manage to solve the first step of the procedure outlined in Fig. 1, the LAMBDA method is suitable for solving the second step of this procedure, independent of your application. This implies, for example, that the LAMBDA method will also be suitable for future triple-frequency systems like modernized GPS and the envisioned European Galileo. In fact, the use of the LAMBDA method is not even restricted to satellite navigation, as it is currently being used to solve the problem of phase-unwrapping as encountered in the field of INSAR [Interferometric Synthetic Aperture Radar, see Hanssen et al. (2001)]. LAMBDA is suitable for any problem in which all or part of the unknown parameters in the model are of an integer nature. Maybe there will exist applications even in fields like chemistry or biology.

Question 6. With the future introduction of modernized triple-frequency GPS and/or Galileo, do we need to invent a new method of carrier ambiguity resolution?

No, on the contrary. Compared with present dual-frequency GPS, ambiguity success-rates will go up, when more signals and more satellites become available (see e.g. Eissfeller et al. 2001). Just provide a float solution, integral of both the GPS and Galileo ambiguities, and feed it to LAMBDA to produce the optimal integer estimate for them. LAMBDA is ready for it, and will still be the best method around for ambiguity resolution when performance of the method is measured in terms of success-rates.

LAMBDA method: information and feedback

Question 7. I have an idea to improve the LAMBDA method, are you interested?

Of course we would be interested, as there is always a possibility for improvements. However, before you claim to have improved the LAMBDA method, or to have found a better way of solving the integer ambiguity estimation problem, make sure you actually did. By improving the mathematical model, by, for example, finding a great way of dealing with multipath or atmospheric disturbances, you will find that the performance of your integer ambiguity resolution module will increase. This might suggest you have improved the LAMBDA method, or even found a better way of solving the integer ambiguity estimation problem. In such case, you would not have improved the integer resolution module itself, but you would have served it with higher quality input. Although this certainly would be very valuable, it would not be an improvement of the LAMBDA method.

Question 8. I have received the LAMBDA software, but it seems to be incomplete, what is wrong?

If you have received the software directly from the Mathematical Geodesy and Positioning Group of the Delft University of Technology, it will be complete. It consists of several routines, which will perform an integer least-squares estimation using a float solution, together with its quality description in the form of a variance–covariance matrix as input. Computation of this float solution is not part of the LAMBDA method, as explained with Fig. 1.

Question 9. I want to know more, where do I get more information?

First, there is an extensive list of papers available dealing with the LAMBDA method. The original paper on integer least-squares and the LAMBDA method (Teunissen 1993) dates from the early 1990s. An extensive explanation of the algorithm can be found in De Jonge and Tiberius (1996). Joosten and Tiberius (2000) is not directly related to the LAMBDA method, but gives an explanation of the success-rate and its importance. Also, most of the textbooks dealing with GPS positioning either mention or explain the LAMBDA method (e.g. Teunissen and Kleusberg 1998; Strang and Borre 1997; Hofmann-Wellenhof 1997; Misra and Enge 2001). On the website of the department of Mathematical Geodesy and Positioning of the Delft University of Technology, an extensive list of literature is available, and most papers are also available for download. The website can be found at the following URL: http://www.geo.tudelft.nl/mgp. Finally, you can contact the authors/maintainers of the LAMBDA method via email at mgp@geo.tudelft.nl, but please make sure your questions are specific rather than general, and indicate in the ‘subject’ of your email that your question concerns the LAMBDA method.

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