Computational Management Science (2024) 21:41
https://doi.org/10.1007/s10287-024-00523-0

ORIGINAL PAPER

Connection between higher order measures of risk and stochastic dominance

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Received: 8 February 2024 / Accepted: 21 August 2024

Abstract
Higher order risk measures are stochastic optimization problems by design, and for this reason they enjoy valuable properties in optimization under uncertainties. They nicely integrate with stochastic optimization problems, as has been observed by the intriguing concept of the risk quadrangles, for example. Stochastic dominance is a binary relation for random variables to compare random outcomes. It is demonstrated that the concepts of higher order risk measures and stochastic dominance are equivalent, they can be employed to characterize the other. The paper explores these relations and connects stochastic orders, higher order risk measures and the risk quadrangle. Expectiles are employed to exemplify the relations obtained.

Keywords Higher order risk measure · Higher order stochastic dominance · Risk quadrangle

Mathematics Subject Classification 62G05 · 62G08 · 62G20

1 Introduction

Risk measures are considered in various disciplines to assess and quantify risk. Similarly to assigning a premium to an insurance contract with random losses after appraising its risk, risk measures assign a number to a random variable, which itself has stochastic outcomes.

This paper focuses on higher order risk measures, as these risk measures naturally combine with stochastic optimization problems or in 'learning' objectives, as they are the result of optimization problems. In addition, these risk measures relate to the risk quadrangle.

The paper derives explicit representations of higher order risk measures for general, elementary risk measures in a first main result. These characterizations are employed

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Published online: 05 September 2024

to characterize stochastic dominance relations, which are built on general norms. The second main result is a verification theorem. This is a characterization of higher order stochastic dominance relations, which is numerically tractable.

For the norm in Lebesgue spaces, stochastic dominance relations have been considered for example in Dupačová and Kopa (2014), Kopa et al. (2016, 2023), Post and Kopa (2017) and Consigli et al. (2023), in portfolio optimization involving commodities (cf. Frydenberg et al. (2019)), and by Dentcheva and Martinez (2012) and Maggioni and Pflug (2016, 2019) in a multistage setting. The paper employs the characterizations obtained to establish relations for general norms. A comparison of these methods is given in Gutjahr and Pichler (2013). The paper illustrates these connections for expectiles (Bellini et al. 2016; Bellini and Caperdoni 2007) and adds a comparison with other risk measures.

Outline of the paper The following Sect. 2 recalls the mathematical framework for higher order risk measures. Section 3 addresses the higher order risk measure associated with the spectral risks, as these risk measures constitute an elementary building block for general risk measures. This section develops the first main result, which is an explicit representation of a spectral risk's higher order risk measure. As a special case, the subsequent Sect. 4 links and relates stochastic dominance and higher order risk measures. This section presents the second main result, which allows verifying a stochastic dominance relation by involving only finitely many risk levels. The final Sect. 5 addresses the expectile and establishes the relations of the preceding sections for this specific risk measure. Section 6 concludes.

2 Mathematical framework

Higher order risk measures are a special instance of *risk measures*, often also termed *risk functionals*. To introduce and recall their main properties we consider a space $\mathcal{Y}$ of $\mathbb{R}$-valued random variables on a probability space with measure P containing at least all bounded random variables, that is, $L^\infty(P) \subseteq \mathcal{Y}$. A risk measure then satisfies the following axioms, originally introduced by Artzner et al. (1999).

Definition 2.1 (*Risk functional*) Let $\mathcal{Y}$ be a space of $\mathbb{R}$-valued random variables on a probability space (Ω, Σ, P). A mapping $\mathcal{R} : \mathcal{Y} \to \mathbb{R}$ is

(i) monotone, if $\mathcal{R}(X) \le \mathcal{R}(Y)$, provided that $X \le Y$ almost everywhere;
(ii) positively homogeneous if $\mathcal{R}(\lambda Y) = \lambda \mathcal{R}(Y)$ for all $\lambda > 0$;
(iii) translation equivariant, if $\mathcal{R}(c + Y) = c + \mathcal{R}(Y)$ for all $c \in \mathbb{R}$;
(iv) subadditive, if $\mathcal{R}(X + Y) \le \mathcal{R}(X) + \mathcal{R}(Y)$ for all X and $Y \in \mathcal{Y}$.

A mapping satisfying (i)–(iv) is called a *risk functional*, or a *risk measure*.

The risk quadrangle (cf. Rockafellar and Uryasev (2013)) relates risk measures with the measure of regret by

$$\mathcal{R}(Y) = \inf_{c\in\mathbb{R}} c + \mathcal{V}(Y-c), \tag{2.1}$$

where $\mathcal{V}$ is called *regret function*. Equation (2.1) was first introduced for the conditional value-at-risk in Rockafellar and Uryasev (2000). For the expectation type function, i.e., $\mathcal{V}(X) = \mathbb{E}\, v(X)$, the relationship (2.1) is studied in Ben-Tal and Teboulle (2007), where $\mathcal{V}$ was called *optimized certainty equivalent*; also, Krokhmal (2007) study the relation (2.1).

It follows from relation (2.1) that $\mathcal{R}$—if given as in (2.1)—is translation equivariant, i.e, $\mathcal{R}$ satisfies $\mathcal{R}(Y+c) = c + \mathcal{R}(Y)$ for any $c \in \mathbb{R}$ (cf. (iii) above). In an economic interpretation, the amount c in (2.1) corresponds to an amount of cash spent today, while the remaining quantity $Y-c$ is invested and consumed later, thus subject to $\mathcal{V}$.

The risk functional $\mathcal{R}$ is positively homogeneous, if the regret function $\mathcal{V}$ is positively homogeneous. If $\mathcal{V}$ is not positively homogeneous, then one may consider the positively homogeneous envelope

$$\mathcal{V}_{\tilde\beta}(Y) = \inf_{t>0} t\Big(\tilde\beta + \mathcal{V}\Big(\frac{Y}{t}\Big)\Big),$$

where $\tilde\beta \geq 0$ is a risk aversion coefficient. The combined functional

$$\begin{aligned}\mathcal{R}_\beta(Y) &= \inf_{c\in\mathbb{R}} c + \mathcal{V}_\beta(Y-c)\\ &= \inf_{\substack{t>0\\ q\in\mathbb{R}}} t\Big(\tilde\beta + q + \mathcal{V}\Big(\frac{Y}{t} - q\Big)\Big)\end{aligned} \tag{2.2}$$

is positively homogeneous and translation equivariant (cf. (ii) and (iii)). The φ *-divergence risk measure* is an explicit example of a risk measure, which is defined exactly as (2.2), cf. Dommel and Pichler (2021).

The paper suggests a regret for a higher-order risk starting from a given risk $\mathcal{R}$. To this end consider a space $\mathcal{Y} \subset L^1(P)$ endowed with norm $\|\cdot\|$. We shall assume the norm to be monotone, that is, $\|X\| \leq \|Y\|$ provided that $0 \leq X \leq Y$ almost everywhere. We associate the following family of risk measure with a given norm.

Definition 2.2 (*Higher order risk measure*) Let $\|\cdot\|$ be a monotone norm on $\mathcal{Y} \subset L^1(P)$ with $\|\mathbb{1}\| = 1$, where $\mathbb{1}(\cdot) = 1$ is the identically one function on $\mathcal{Y}$. The *higher order risk measure* at risk level $\beta \in [0,1)$ associated with the norm $\|\cdot\|$ is

$$\mathcal{R}_\beta^{\|\cdot\|}(Y) = \inf_{t\in\mathbb{R}} t + \frac{1}{1-\beta}\|(Y-t)_+\|, \tag{2.3}$$

where $\beta \in [0,1)$ is the *risk aversion coefficient* and $x_+ := \max(0,x)$.

We shall also omit the superscript and write $\mathcal{R}_\beta$ instead of $\mathcal{R}_\beta^{\|\cdot\|}$ in case the norm is unambiguous given the context. We shall demonstrate first that the higher order risk measure is well-defined for any $\beta \geq 0$.

Proposition 2.3 *Let $(\mathcal{Y}, \|\cdot\|)$ be a normed space of random variables. For the functional $\mathcal{R}_\beta$ defined in (2.3) it holds that*

$$-\|Y\| \leq \mathcal{R}_\beta(Y) \leq \frac{1}{1-\beta}\|Y\|, \tag{2.4}$$

so that $\mathcal{R}_\beta(\cdot)$ is indeed well-defined on $(\mathcal{Y}, \|\cdot\|)$ for every $\beta \in [0,1)$.

Proof The upper bound follows trivially from the definition by choosing $t=0$ in the defining equation (2.3).

For $t \leq 0$, it holds that $-t = -Y + (Y-t) \leq -Y + (Y-t)_+$. It follows from the triangle inequality that $-t \leq \|Y\| + \|(Y-t)_+\|$ and thus

$$-\|Y\| \leq t + \|(Y-t)_+\| \quad \text{for all } t \leq 0.$$

To establish the relation also for $t \geq 0$, we start by observing the following monotonicity property of the objective in (2.3) in addition: for $\Delta t \geq 0$, it follows from the reverse triangle inequality that

$$\|Y_+\| - \|(Y-\Delta t)_+\| \leq \|Y_+ - (Y-\Delta t)_+\| \leq \|\Delta t\,\mathbb{1}\| = \Delta t,$$

where we have used that $0 \leq Y_+ - (Y-\Delta t)_+ \leq \Delta t$ together with monotonicity of the norm. Replacing Y by $Y-t$ in the latter expression gives

$$t + \|(Y-t)_+\| \leq t + \Delta t + \|(Y-(t+\Delta t))_+\|;$$

that is, the function $t \mapsto t + \|(Y-t)_+\|$ is non-decreasing, which finally establishes that

$$-\|Y\| \leq t + \|(Y-t)_+\| \quad \text{for all } t \in \mathbb{R}.$$

The lower bound in (2.4) thus follows from the latter inequality, as $\mathcal{R}_0(Y) \leq \mathcal{R}_\beta(Y)$ for any $\beta \geq 0$. □

Example 2.4 For Lebesgue spaces $L^p(P)$ and norm $\|Y\|_p := (\mathbb{E}\,|Y|^p)^{1/p}$, $p \geq 1$, the higher order risk measure has been introduced in Krokhmal (2007) and studied in Dentcheva et al. (2010). For the norm $\|\cdot\|_\infty$, the higher order risk measure is

$$\mathcal{R}_\beta^{\|\cdot\|_\infty}(Y) = \operatorname{ess\,sup} Y, \quad \beta > 0; \tag{2.5}$$

indeed, it follows from (2.3) that

$$0 \in \left[1 - \frac{1}{1-\beta}, 1\right] = \partial_t \left(t + \frac{1}{1-\beta}\|(Y-t)_+\|_\infty \right)\Bigg|_{t=\operatorname{ess\,sup} Y},$$

the subgradient of the convex function in the latter expression at $t = \operatorname{ess\,sup} Y$. The infimum in (2.3) is attained at $t = \operatorname{ess\,sup} Y$, and thus (2.5).

Lemma 2.5 $\mathcal{R}_\beta(\cdot)$ *is a risk functional, provided that the norm is monotone. Further,* $\mathcal{R}_\beta$ *is Lipschitz continuous with respect to the norm, the Lipschitz constant is* $\frac{1}{1-\beta}$.

Proof The assertions (ii)–(iv) in Definition 2.1 are straight forward to verify; to verify (i) it is indispensable to assume that the norm is monotone.

As for continuity, it follows from subadditivity together with (2.4) that $\mathcal{R}_\beta(Y)-\mathcal{R}_\beta(Z)\le\mathcal{R}_\beta(Y-Z)\le\frac{1}{1-\beta}\|Y-Z\|$, and $|\mathcal{R}_\beta(Y)-\mathcal{R}_\beta(Z)|\le\frac{1}{1-\beta}\|Y-Z\|$ after interchanging the roles of Y and Z. Hence, the assertion. □

Note that the higher order risk measure as defined in (2.3) defines a risk functional based on a norm. In contrast to this construction, a risk functional $\mathcal{R}$ defines a norm via

$$\|Y\|:=\mathcal{R}(|Y|) \tag{2.6}$$

and a Banach space with $\mathcal{Y}=\left\{Y\in L^1\,:\,\mathcal{R}(|Y|)<\infty\right\}$ (cf. Pichler (2013)). Its natural dual norm for $Z\in\mathcal{Z}:=\mathcal{Y}^*$ is

$$\begin{aligned}\|Z\|^*:&=\sup\{\,\mathbb{E}\,YZ\,:\,\|Y\|\le 1\}\\&=\sup\{\,\mathbb{E}\,YZ\,:\,\mathcal{R}(|Y|)\le 1\}.\end{aligned} \tag{2.7}$$

The following relationship allows defining a regret functional to connect a risk functional $\mathcal{R}$ with the higher-order risk quadrangle.

Proposition 2.6 (Duality) *Let* $\mathcal{R}$ *be a risk functional with associated norm* $\|\cdot\|$ *and dual norm* $\|\cdot\|^*$. *For the higher order risk functional it holds that*

$$\mathcal{R}_\beta(Y)=\sup\left\{\,\mathbb{E}\,YZ\,:\,Z\ge 0,\ \ \mathbb{E}\,Z=1\text{ and }\|Z\|^*\le\frac{1}{1-\beta}\right\} \tag{2.8}$$

$$=\inf_{t\in\mathbb{R}}\ t+\frac{1}{1-\beta}\|(Y-t)_+\|, \tag{2.9}$$

where $\beta\in[0,1)$.

Remark 2.7 By the interconnecting formula (2.1), the higher order risk functional $\mathcal{R}_\beta^{\|\cdot\|}$ associated with the norm $\|\cdot\|$ is the regret function $\mathcal{V}_\beta^{\|\cdot\|}(\cdot):=\frac{1}{1-\beta}\|(\cdot)_+\|$.

Proof It holds by the Hahn–Banach theorem and as $(Y-t)_+\ge 0$ that

$$\frac{1}{1-\beta}\cdot\|(Y-t)_+\|=\sup_{\|Z\|^*\le\frac{1}{1-\beta}}\mathbb{E}\,Z(Y-t)_+\ge\sup_{\substack{\mathbb{E}\,Z=1,\ Z\ge 0,\\ \|Z\|^*\le\frac{1}{1-\beta}}}\mathbb{E}\,Z(Y-t)_+.$$

This establishes the first inequality '≤' in (2.9) with $t+(Y-t)_+\ge Y$, as

$$t + \frac{1}{1-\beta} \cdot \|(Y-t)_+\| \geq \sup_{\substack{\mathbb{E}\, Z = 1 \\ Z \geq 0,\ \|Z\|^* \leq \frac{1}{1-\beta}}} \mathbb{E}\left(t + (Y-t)_+\right) Z$$
$$\geq \sup_{\substack{\mathbb{E}\, Z = 1 \\ Z \geq 0,\ \|Z\|^* \leq \frac{1}{1-\beta}}} \mathbb{E}\, YZ.$$

As for the converse inequality assume first that Y is bounded. Note, that

$$\inf_{t\in\mathbb{R}} t + \mathbb{E}\,(Y-t)Z = \mathbb{E}\, YZ + \inf_{t\in\mathbb{R}} t \cdot (1 - \mathbb{E}\, Z) = \begin{cases} \mathbb{E}\, YZ & \text{if } \mathbb{E}\, Z = 1, \\ -\infty & \text{else,} \end{cases}$$

so that it follows that

$$\sup_{\substack{\mathbb{E}\, Z = 1 \\ Z \geq 0,\ \|Z\|^* \leq \frac{1}{1-\beta}}} \mathbb{E}\, YZ = \sup_{\substack{Z \geq 0, \\ \|Z\|^* \leq \frac{1}{1-\beta}}} \inf_{t\in\mathbb{R}} t + \mathbb{E}\,(Y-t)Z.$$

Further, it holds that $\mathbb{E}\, YZ = t^* + \mathbb{E}\, Z(Y-t^*)_+$ for $t^* \leq Y$ a.s. and thus

$$\sup_{\substack{\mathbb{E}\, Z = 1, Z \geq 0, \\ \|Z\|^* \leq \frac{1}{1-\beta}}} \mathbb{E}\, YZ = \sup_{\substack{Z \geq 0, \\ \|Z\|^* \leq \frac{1}{1-\beta}}} t^* + \mathbb{E}\, Z(Y-t^*)_+ = t^* + \frac{1}{1-\beta}\|(Y-t^*)\| \geq \inf_{t\in\mathbb{R}} t + \frac{1}{1-\beta}\|(Y-t)_+\|,$$

thus the desired converse inequality, provided that Y is bounded; if Y is not bounded, then there is a bounded Y_ε with $Y \leq Y_\varepsilon$ ($\varepsilon > 0$) and $\|Y_\varepsilon - Y\| < \varepsilon$, so that

$$\mathbb{E}\, Z(Y_\varepsilon - t)_+ - \varepsilon\, \mathbb{E}\, Z \leq \mathbb{E}\, Z(Y-t)_+ \leq \mathbb{E}\, Z(Y_\varepsilon - t)_+,$$

so that we may conclude that (2.9) holds for every $Y \in \mathcal{Y}$. □

Example 2.8 (Lebesgue spaces) The dual norm of the genuine norm $\|X\|_p :=(\mathbb{E}\, |X|^p)^{1/p}$ in the Lebesgue space $L^p(P)$ is $\|Z\|^* = (\mathbb{E}\, |Z|^q)^{1/q}$ for the Hölder conjugate exponent q with $\frac{1}{p} + \frac{1}{q} = 1$. With Proposition 2.6 it follows that

$$\mathcal{R}_\beta^{\|\cdot\|_p}(Y) = \inf_{t\in\mathbb{R}} t + \frac{1}{1-\beta}\|(Y-t)_+\|_p$$
$$= \sup\left\{ \mathbb{E}\, YZ : \|Z\|_q \leq \frac{1}{1-\beta},\ Z \geq 0 \text{ and } \mathbb{E}\, Z = 1 \right\},$$

cf. also Pichler and Shapiro (2015) and Pichler (2017).

In what follows, we shall elaborate the higher order risk measure and the associated regret function for specific risk measures, specifically the spectral risk measure.

3 Higher order spectral risk

By Kusuoka's theorem (cf. Kusuoka (2001)), every law invariant risk functional can be assembled by elementary risk functionals, each involving the average value-at-risk.

The following section develops the explicit representations of the higher order risk measures associated with spectral risk measures first. The explicit representation then is extended to general risk functionals.

Definition 3.1 (*Spectral risk measures*) The function $\sigma : [0,1) \to \mathbb{R}$ is called a *spectral function*, if

(i) $\sigma(\cdot) \geq 0$,
(ii) $\int_0^1 \sigma(u)\,\mathrm{d}u = 1$ and
(iii) $\sigma(\cdot)$ is non-decreasing.

The *spectral risk measure* with spectral function σ is

$$\mathcal{R}_\sigma(Y) := \int_0^1 \sigma(u) F_Y^{-1}(u)\,\mathrm{d}u,$$

where

$$F_Y^{-1}(u) := \mathsf{V@R}_u(Y) := \inf\{x \in \mathbb{R} : P(Y \leq x) \geq u\}$$

is the *value-at-risk*, the *generalized inverse* or *quantile function*.

The higher order risk measure of the spectral risk measure is a spectral risk measure itself. The following theorem presents the corresponding spectral function explicitly and generalizes (Pflug 2000). The result is central towards the main characterization presented in the next sections.

Theorem 3.2 (Higher order spectral risk) *Let $\beta \in [0,1)$ be a risk level. The higher order risk functional of the risk functional $\mathcal{R}_\sigma$ with spectral function $\sigma(\cdot)$ has the representation*

$$\inf_{t\in\mathbb{R}} t + \frac{1}{1-\beta}\mathcal{R}_\sigma\big((Y-t)_+\big) = \mathcal{R}_{\sigma_\beta}(Y), \tag{3.1}$$

where σ_β is the spectral function

$$\sigma_\beta(u) := \begin{cases} 0 & \text{if } u < u_\beta, \\ \frac{\sigma(u)}{1-\beta} & \text{else;} \end{cases} \tag{3.2}$$

here, $u_\beta \in \mathbb{R}$ is the β-quantile with respect to the density σ, that is, the solution of

$$\int_0^{u_\beta} \sigma(u)\,\mathrm{d}u = \beta, \tag{3.3}$$

which is unique for $\beta > 0$.

Proof We remark first that σ_β indeed is a spectral function, as $\int_0^1 \sigma_\beta(u)\,\mathrm{d}u = \frac{1}{1-\beta}\int_{u_\beta}^1 \sigma(u)\,\mathrm{d}u = \frac{1-\beta}{1-\beta} = 1$ by the defining property (3.3) and (ii) in Definition 3.1. The quantile u_β is uniquely defined for $\beta > 0$, as the function σ is non-decreasing by (iii). In what follows we shall demonstrate that the infimum in (3.1) is attained at $t^* := F_Y^{-1}(u_\beta)$. Note first that

$$F^{-1}_{(Y-t)_+}(u) = \begin{cases} 0 & \text{if } u < F_Y(t), \\ F_Y^{-1}(u) - t & \text{else,} \end{cases}$$

so that

$$\mathcal{R}_\sigma\big((Y-t)_+\big) = \int_0^1 \sigma(u) F^{-1}_{(Y-t)_+}(u)\,\mathrm{d}u = \int_{F_Y(t)}^1 \sigma(u)\big(F_Y^{-1}(u) - t\big)\,\mathrm{d}u$$

and

$$(\mathcal{R}_\sigma)_\beta(Y) = \inf_{t\in\mathbb{R}} t + \frac{1}{1-\beta}\int_{F_Y(t)}^1 \sigma(u)\big(F_Y^{-1}(u) - t\big)\,\mathrm{d}u. \tag{3.4}$$

Assume first that $t \le t^*$. The inequality $u \le F_Y(t)$ is equivalent to $F_Y^{-1}(u) \le t$ (cf. van der Vaart (1998); this relation of functions F_Y and F_Y^{-1} is occasionally called a *Galois connection*), and thus

$$\int_{F_Y(t)}^{F_Y(t^*)} \sigma(u)\big(F_Y^{-1}(u) - t\big)\,\mathrm{d}u \le 0,$$

or equivalently

$$\int_{F_Y(t)}^1 \sigma(u)\big(F_Y^{-1}(u) - t\big)\,\mathrm{d}u \le \int_{F_Y(t^*)}^1 \sigma(u)\big(F_Y^{-1}(u) - t\big)\,\mathrm{d}u.$$

Assume next that $u_\beta \le F_Y(t^*)$, then $\int_{F_Y(t^*)}^1 \sigma(u)\,\mathrm{d}u \le 1-\beta$ so that

$$\frac{t-t^*}{1-\beta}\int_{F_Y(t^*)}^1 \sigma(u)\,\mathrm{d}u \le t - t^*.$$

Combining the inequalities in the latter displays gives

$$t^* + \frac{1}{1-\beta}\int_{F_Y(t^*)}^1 \sigma(u)\big(F_Y^{-1}(u) - t^*\big)\,\mathrm{d}u \le t + \frac{1}{1-\beta}\int_{F_Y(t)}^1 \sigma(u)\big(F_Y^{-1}(u) - t\big)\,\mathrm{d}u \tag{3.5}$$

and thus the assertion, provided that $u_\beta \le F_Y(t^*)$ and $t^* \le t$.

Conversely, assume that $t \le t^*$. Then the inequality $u \le F_Y(t^*)$ is equivalent to $F_Y^{-1}(u) \le t^*$ and thus

$$\int_{F_Y(t)}^{F_Y(t^*)} \sigma(u)\big(F_Y^{-1}(u) - t^*\big)\, \mathrm{d}u \le 0,$$

which is equivalent to

$$\int_{F_Y(t)}^{1} \sigma(u)\big(F_Y^{-1}(u) - t^*\big)\, \mathrm{d}u \le \int_{F_Y(t^*)}^{1} \sigma(u)\big(F_Y^{-1}(u) - t^*\big)\, \mathrm{d}u.$$

Assume further that $F_Y(t^*) \le u_\beta$, then $\int_{F_Y(t^*)}^1 \sigma(u)\,\mathrm{d}u \ge 1-\beta$ so that

$$t^* - t \le \frac{t^* - t}{1-\beta}\int_{F_Y(t^*)}^{1} \sigma(u)\,\mathrm{d}u.$$

Combining the latter inequalities gives

$$t^* + \frac{1}{1-\beta}\int_{F_Y(t)}^{1} \sigma(u)\big(F_Y^{-1}(u) - t^*\big)\,\mathrm{d}u \le t + \frac{1}{1-\beta}\int_{F_Y(t^*)}^{1} \sigma(u)\big(F_Y^{-1}(u) - t\big)\,\mathrm{d}u. \tag{3.6}$$

It follows from (3.5) and (3.6) that $t^* := F_Y^{-1}(u_\beta)$ is optimal in (3.4). That is,

$$\begin{aligned}(\mathcal{R}_\sigma)_\beta(Y) &= t^* + \frac{1}{1-\beta}\int_{u_\beta}^{1} \sigma(u)\big(F_Y^{-1}(u) - t^*\big)\,\mathrm{d}u \\ &= \frac{1}{1-\beta}\int_{u_\beta}^{1} \sigma(u) F_Y^{-1}(u)\,\mathrm{d}u \qquad (3.7)\\ &= \int_0^1 \sigma_\beta(u) F_Y^{-1}(u)\,\mathrm{d}u\end{aligned}$$

and thus the assertion. □

The following statement expresses the higher order risk functional by at the base value u_β, and the random variable's aberrations to the right, involving the survival function instead of its inverse distribution function.

Corollary 3.3 *The higher order spectral risk measure is*

$$\big(\mathcal{R}_\sigma\big)_\beta(Y) = \mathsf{V@R}_{u_\beta}(Y) + \frac{1}{1-\beta}\int_{\mathsf{V@R}_{u_\beta}(Y)}^{\infty} \Sigma\big(F_Y(y)\big)\mathrm{d}y \tag{3.8}$$

(*with* u_β *as in* (3.3)) *or*, *provided that* Y *is bounded*,

$$\left(\mathcal{R}_\sigma\right)_\beta(Y) = \operatorname{ess\,inf} Y + \int_{\operatorname{ess\,inf} Y}^{\infty} \Sigma_\beta\big(F_Y(y)\big)\mathrm{d}y, \tag{3.9}$$

where

$$\Sigma_\beta(u) := \min\left(1,\ \frac{1}{1-\beta}\int_u^1 \sigma(p)\,\mathrm{d}p\right)$$

is the cumulative spectral function and $\Sigma(u) := \Sigma_0(u) = \int_u^1 \sigma(p)\,\mathrm{d}p$.

Proof Notice first that $\Sigma_\beta(u) = 1$ for $u \le u_\beta$, where u_β is given in (3.3). By Theorem 3.2, Riemann-Stieltjes integration by parts and changing the variables it holds that

$$\begin{aligned}\left(\mathcal{R}_\sigma\right)_\beta(Y) &= \mathcal{R}_{\sigma_\beta}(Y)\\ &= \frac{1}{1-\beta}\int_{u_\beta}^1 \sigma(u)F_Y^{-1}(u)\,\mathrm{d}u\\ &= -\int_0^1 F_Y^{-1}(u)\,\mathrm{d}\Sigma_\beta(u)\end{aligned} \tag{3.10}$$

$$\begin{aligned}&= -F_Y^{-1}(u)\Sigma_\beta(u)\Big|_{u=0}^1 + \int_0^1 \Sigma_\beta(u)\,\mathrm{d}F_Y^{-1}(u)\\ &= \operatorname{ess\,inf} Y + \int_{\operatorname{ess\,inf} Y}^{\infty} \Sigma_\beta\big(F_Y(y)\big)\,\mathrm{d}y,\end{aligned} \tag{3.11}$$

where we have used that $F_Y^{-1}(0) = \operatorname{ess\,inf} Y$ and $\Sigma_\beta(1) = 0$ in (3.11). This gives (3.9).

The equation (3.8) results from sticking to the lower bound u_β (instead of 0) in (3.10). That is,

$$\begin{aligned}\left(\mathcal{R}_\sigma\right)_\beta(Y) &= -\int_{u_\beta}^1 F_Y^{-1}(u)\,\mathrm{d}\Sigma_\beta(u)\\ &= -F_Y^{-1}(u)\Sigma_\beta(u)\Big|_{u=u_\beta}^1 + \int_{u_\beta}^1 \Sigma_\beta(u)\,\mathrm{d}F_Y^{-1}(u)\\ &= \mathsf{V@R}_{u_\beta}(Y) + \int_{\mathsf{V@R}_{u_\beta}(Y)}^{\infty} \Sigma_\beta\big(F_Y(y)\big)\,\mathrm{d}y,\end{aligned}$$

which is assertion (3.8). □

Corollary 3.4 *The higher order spectral risk measure has the representation*

$$\mathcal{R}_{\sigma_\beta}(Y) = \sup\{\,\mathbb{E}\,[Y\cdot\sigma_\beta(U)]\,:\,U\in[0,1] \text{ is uniformly distributed}\}.$$

Proof Recall first that $Y \sim F_Y^{-1}(U)$ for U uniformly distributed. By the rearrangement inequality, $\mathbb{E}\, Y\sigma_\beta(U) \le \mathbb{E}\, F_Y^{-1}(U)\sigma_\beta(U)$, because $F_Y^{-1}(U)$ and $\sigma_\beta(U)$ are comonotone and both, $F_Y^{-1}(\cdot)$ and $\sigma_\beta(\cdot)$ are non-decreasing functions. The assertion follows with (3.7). □

The celebrated formula (cf. Pflug (2000), Rockafellar and Uryasev (2000), Ogryczak and Ruszczyński (2002))

$$\mathsf{AV@R}_\alpha(Y) = \frac{1}{1-\alpha}\int_\alpha^1 \mathsf{V@R}_u(Y)\,\mathrm{d}u = \inf_{t\in\mathbb{R}}\ t + \frac{1}{1-\alpha}\,\mathbb{E}\,(Y-t)_+$$

for the average value-at-risk is a special case of Theorem 3.2 for the spectral function $\sigma(\cdot) = \frac{1}{1-\alpha}\mathbb{1}_{[\alpha,1]}(\cdot)$.

The following corollary estabilshes this risk functional's higher order variant.

Corollary 3.5 (Average value-at-risk) *The higher order average value-at-risk is*

$$(\mathsf{AV@R}_\alpha)_\beta(Y) = \mathsf{AV@R}_{1-(1-\alpha)(1-\beta)}(Y), \tag{3.12}$$

where $Y \in L^1$; *equivalently,*

$$\mathsf{AV@R}_\beta(Y) = \inf_{t\in\mathbb{R}}\ t + \frac{1}{1-\frac{\beta-\alpha}{1-\alpha}}\,\mathsf{AV@R}_\alpha\big((Y-t)_+\big), \tag{3.13}$$

where $\beta \ge \alpha$.

Proof The spectral function of the average value-at-risk is $\sigma_\alpha(\cdot) = \frac{\mathbb{1}_{\cdot\ge\alpha}}{1-\alpha}$. It follows from (3.3) that $u_\beta = \alpha + \beta(1-\alpha) = 1-(1-\alpha)(1-\beta)$ and $(\sigma_\alpha)_\beta = \begin{cases} 0 & \text{if } u \le u_\beta, \\ \frac{1}{(1-\alpha)(1-\beta)} & \text{else.}\end{cases}$ This is the spectral function of the average value-at-risk at risk level u_β.

The assertion (3.13) follows by replacing β with $\frac{\beta-\alpha}{1-\alpha}$ in (3.12). □

Corollary 3.6 (*Kusoka representation spectral risk measures*)*Suppose the risk functional is*

$$\mathcal{R}(Y) = \int_0^1 \mathsf{AV@R}_\gamma(Y)\,\mu(\mathrm{d}\gamma), \tag{3.14}$$

where μ *is a probability measure on* $[0, 1]$. *Then the higher order risk measure is*

$$\mathcal{R}_\beta(Y) = \int_0^1 \mathsf{AV@R}_\gamma(Y)\,\mu_\beta(\mathrm{d}\gamma),$$

where $\mu_\beta(\cdot)$ *is the measure*

$$\mu_\beta(A) := p_0 \cdot \delta_{u_\beta}(A) + \frac{1}{1-\beta}\mu\big(A \cap (u_\beta, 1]\big) \tag{3.15}$$

and u_β and p_0 are determined by the equation and definition

$$\int_0^{u_\beta} \frac{u_\beta - \alpha}{1-\alpha}\mu(\mathrm{d}\alpha) = \beta \text{ and } p_0 := \frac{1-u_\beta}{1-\beta}\int_0^{u_\beta}\frac{\mu(\mathrm{d}\alpha)}{1-\alpha}. \tag{3.16}$$

Proof Above all, μ_β is a probability measure, because $p_0 \geq 0$ and

$$\begin{aligned}\mu_\beta([0,1]) &= p_0 + \frac{1}{1-\beta}\int_{u_\beta+}^1 \mu(\mathrm{d}\alpha)\\ &= \frac{1}{1-\beta}\int_0^{u_\beta}\frac{1-\alpha-(u_\beta-\alpha)}{1-\alpha}\mu(\mathrm{d}\alpha) + \frac{1}{1-\beta}\int_{u_\beta+}^1 \mu(\mathrm{d}\alpha)\\ &= \frac{1}{1-\beta}\int_0^1 \mu(\mathrm{d}\alpha) - \frac{\beta}{1-\beta} = 1.\end{aligned}$$

The spectral function of the average value-at-risk at risk level α is $\sigma_\alpha(\cdot) = \frac{\mathbb{1}_{\cdot\geq\alpha}}{1-\alpha}$. The quantile condition (3.3) thus is

$$\beta = \int_0^1 \frac{\max(0, u_\beta - \alpha)}{1-\alpha}\mu(\mathrm{d}\alpha)$$

and thus (3.16).

For $u < u_\beta$, the spectral function corresponding to the measure $\mathcal{R}_\beta$ in (3.14) is 0, which coincides with (3.2). For $u > u_\beta$, the spectral function for $\mathcal{R}_\beta$ is

$$\begin{aligned}\frac{p_0}{1-u_\beta}\mathbb{1}_{u\geq u_\beta} + \int_{u_\beta}^1 \frac{1}{1-\beta}\frac{\mathbb{1}_{u\geq\alpha}}{1-\alpha}\mu(\mathrm{d}\alpha) &= \frac{1}{1-\beta}\int_0^{u_\beta}\frac{\mathbb{1}_{u\geq u_\beta}}{1-\alpha}\mu(\mathrm{d}\alpha) + \int_{u_\beta}^1 \frac{1}{1-\beta}\frac{\mathbb{1}_{u\geq\alpha}}{1-\alpha}\mu(\mathrm{d}\alpha)\\ &= \frac{1}{1-\beta}\int_0^1 \frac{\mu(\mathrm{d}\alpha)}{1-\alpha},\end{aligned}$$

which is the desired result in light of (3.2). □

In situations of practical interest, the risk measure is often given as finite combination of average values-at-risk at varying levels. The following corollary addresses this situation explicitly.

Corollary 3.7 *Suppose that*

$$\mathcal{R}(Y) = \sum_{i=1}^n p_i \cdot \mathsf{AV@R}_{\alpha_i}(Y) \tag{3.17}$$

with $p_i \geq 0$, $\sum_{i=1}^n p_i = 1$ and $\alpha_i \in [0,1]$ for $i = 1, \ldots, n$. Then

$$\mathcal{R}_\beta(Y) = p_0 \cdot \mathsf{AV@R}_{u_\beta}(Y) + \sum_{i:\alpha_i > u_\beta} \frac{p_i}{1-\beta} \mathsf{AV@R}_{\alpha_i}(Y), \tag{3.18}$$

where u_β *satisfies* $\beta = \sum_{i=1}^n p_i \frac{\max(0, u_\beta - \alpha_i)}{1-\alpha_i}$ *and* $p_0 := \sum_{i:\alpha_i \le u_\beta} \frac{p_i}{1-\alpha_i} \frac{1-u_\beta}{1-\beta}$.

For large risk levels β, *specifically if*

$$\beta \ge 1 - \left(1 - \max_{i=1,\dots,n} \alpha_i\right) \cdot \sum_{i=1}^n \frac{p_i}{1-\alpha_i}, \tag{3.19}$$

the involved risk measure (3.18) *collapses to the average value-at-risk*, *it holds that*

$$\mathcal{R}_\beta(Y) = \mathsf{AV@R}_{1-(1-\tilde{\alpha})(1-\beta)}(Y),$$

where $\tilde{\alpha}$ *is the weighed risk quantile* $\tilde{\alpha} := \frac{\sum_{i=1}^n \frac{p_i}{1-\alpha_i}\alpha_i}{\sum_{i=1}^n \frac{p_i}{1-\alpha_i}}$.

Proof The result corresponds to the measure $\mu = \sum_{i=1}^n p_i \delta_{\alpha_i}$ in (3.14), which is a special case in Corollary 3.6.

For $u_\beta \ge \alpha_i$, $i = 1, \dots, n$, it holds that $\beta = \sum_{i=1}^n p_i \frac{u_\beta - \alpha_i}{1-\alpha_i} = \sum_{i=1}^n p_i \frac{1-\alpha_i-(1-u_\beta)}{1-\alpha_i} = 1 - (1-u_\beta)\sum_{i=1}^n \frac{p_i}{1-\alpha_i}$, so that $u_\beta \ge \max_{i=1,\dots,n} \alpha_i$ is equivalent to (3.19). It follows that

$$\begin{aligned} u_\beta &= 1 - \frac{1-\beta}{\sum_{i=1}^n \frac{p_i}{1-\alpha_i}} \\ &= 1 - (1-\beta)\left(1 - \frac{\sum_{i=1}^n \frac{p_i}{1-\alpha_i} - \sum_{i=1}^n \frac{p_i(1-\alpha_i)}{1-\alpha_i}}{\sum_{i=1}^n \frac{p_i}{1-\alpha_i}}\right) \\ &= 1 - (1-\beta)(1-\tilde{\alpha}), \end{aligned} \tag{3.20}$$

and $p_0 = \sum \frac{p_i}{1-\alpha_i} \frac{1-u_\beta}{1-\beta} = 1$, thus the result with (3.18). □

Remark 3.8 Corollary 3.5 is a special case of (3.18) in the preceding corollary, as $\tilde{\alpha} = \alpha$ in this case.

The following statement generalizes the statements from and provides the higher order risk functional for general risk measures.

Theorem 3.9 (Kusuoka representation of higher order risk measures) *Let* $\mathcal{R}$ *be a law invariant risk measure with Kusuoka representation*

$$\mathcal{R}(Y) = \sup_{\mu \in \mathcal{M}} \mathcal{R}_\mu(Y). \tag{3.21}$$

The higher order risk measure is

$$\mathcal{R}_\beta(Y) = \sup_{\mu\in\mathcal{M}} \mathcal{R}_{\mu_\beta}(Y),$$

where the truncated measures μ_β are given in (3.15).

Proof For the risk functional defined in (3.21) it follows from the min-max inequality that

$$\begin{aligned}\mathcal{R}_\beta(Y) &= \inf_{t\in\mathbb{R}} t + \sup_{\mu\in\mathcal{M}} \mathcal{R}_\mu\big((Y-t)_+\big)\\ &\geq \sup_{\mu\in\mathcal{M}} \inf_{t\in\mathbb{R}} t + \frac{1}{1-\beta}\mathcal{R}_\mu\big((Y-t)_+\big)\\ &= \sup_{\mu\in\mathcal{M}} (\mathcal{R}_\mu)_\beta(Y) =\\ &= \sup_{\mu\in\mathcal{M}} \mathcal{R}_{\mu_\beta}(Y),\end{aligned} \tag{3.22}$$

where we have used Corollary 3.6.

For the reverse inequality in (3.22) consider the function

$$(t,\mu)\mapsto t + \mathcal{R}_\mu\big((Y-t)_+\big)$$

on $\mathbb{R}\times\mathcal{M}([0,1])$, where $\mathcal{M}([0,1])$ collects the probability measures on [0, 1] (with its Borel σ-algebra). By its definition (3.14), this function is linear in μ, and convex in t, where $t\in\mathbb{R}$ and μ is a measure on [0, 1]. By Prokhorov's theorem, the set $\mathcal{M}([0,1])$ of probability measures is sequentially compact, as [0, 1] is compact. From Sion's minimax theorem (cf. Sion (1958)) it follows that equality holds in (3.22). Thus, the result. □

Theorem 3.9 provides an explicit characterization for the general higher order risk measure. The following section exploits this representation to characterize general stochastic dominance relations.

4 General stochastic dominance relations

As Sect. 2 mentions above, the risk measure $\mathcal{R}$ defines a norm via the setting $\|\cdot\| := \mathcal{R}(|\cdot|)$ (cf. (2.6)), and conversely, the norm $\|\cdot\|$ defines a risk measure via $\mathcal{R}_\beta^{\|\cdot\|}$, cf. (2.3). In what follows we connect a specific stochastic dominance relation with the norm. This stochastic dominance relation can be described by higher order risk measures, developed in the preceding Sect. 3.

We start by defining the stochastic dominance relation based on a monotone norm and consider the Lebesgue norm and stochastic dominance for integer orders in below.

Definition 4.1 (*Stochastic dominance*) Let $X, Y \in \mathcal{Y}$ be $\mathbb{R}$-valued random variables in a Banach space $(\mathcal{Y}, \|\cdot\|)$. The random variable X is *dominated* by Y, denoted

$$X \preccurlyeq^{\|\cdot\|} Y,$$

if

$$\|(t-X)_+\| \geq \|(t-Y)_+\| \quad \text{for all } t \in \mathbb{R}. \tag{4.1}$$

If the norm is unambiguous from the context, we shall also simply write $\preccurlyeq$ instead of $\preccurlyeq^{\|\cdot\|}$.

The cone of random variables triggered by a single variable is convex.

Lemma 4.2 (Convexity of the stochastic dominance cone) *For $X \in \mathcal{Y}$ given, the set*

$$\{Y \in \mathcal{Y} : X \preccurlyeq Y\}$$

is convex.

Proof The map $y \mapsto (t-y)_+$ is convex, as follows from reflecting and translating the convex function $x \mapsto x_+$. Suppose that $X \preccurlyeq Y_0$ and $X \preccurlyeq Y_1$. Then it follows for $Y_\lambda := (1-\lambda)Y_0 + \lambda\, Y_1$, together with monotonicity of the norm and (4.1), that

$$\begin{aligned}
\|(t-Y_\lambda)_+\| &\leq \left\|\big((1-\lambda)(t-Y_0)+\lambda(t-Y_1)\big)_+\right\| \\
&\leq (1-\lambda)\|(t-Y_0)_+\| + \lambda\|(t-Y_1)_+\| \\
&\leq (1-\lambda)\|(t-X)_+\| + \lambda\|(t-X)_+\| \\
&= \|(t-X)_+\|.
\end{aligned}$$

That is, it holds that $X \preccurlyeq Y_\lambda$ and thus the assertion. □

4.1 Characterization of stochastic dominance relations

Stochastic dominance relations can be fully characterized by higher order risk measures. The following theorem presents this main result, which integrates the details developed above for these risk functionals and stochastic dominance relations.

Theorem 4.3 (Characterization of stochastic dominance, cf. Gómez et al. (2022)) *The following are equivalent*:

(i) $X \preccurlyeq^{\|\cdot\|} Y$,
(ii) $\mathcal{R}_\beta(-X) \geq \mathcal{R}_\beta(-Y)$ *for all* $\beta \in [0,1)$, *and*

(iii) $\inf_{Z\in\mathcal{Z}_\beta} \mathbb{E}\, ZX \le \inf_{Z\in\mathcal{Z}_\beta} \mathbb{E}\, ZY$ *for every* $\beta \in (0,1)$, *where*

$$\mathcal{Z}_\beta := \left\{ Y \in \mathcal{Y}^* : \|Z\|_* \le \frac{1}{1-\beta},\ \mathbb{E}\, Z = 1,\ Z \ge 0 \right\}$$

is the positive cone ($Z \ge 0$) in the dual ball with radius $\frac{1}{1-\beta}$ ($\|Z\|_* \le \frac{1}{1-\beta}$), intersected with the simplex ($\mathbb{E}\, Z = 1$).

Proof Suppose that $X \preccurlyeq^{\|\cdot\|} Y$, then, by definition, $\|(t-X)_+\| \ge \|(t-Y)_+\|$ for every $t \in \mathbb{R}$. It follows that $t + \frac{1}{1-\beta}\|(-X-t)_+\| \ge t + \frac{1}{1-\beta}\|(-Y-t)_+\|$ for all $t \in \mathbb{R}$, and thus assertion (ii) after passing to the infimum.

As for the contrary, assume that (ii) holds. To demonstrate (i) note first that $q \mapsto \|(q-X)_+\|$ is convex; indeed, with $q_\lambda := (1-\lambda)q_0 + \lambda\, q_1$ and $(a+b)_+ \le a_+ + b_+$ it holds that

$$(q_\lambda - X)_+ = \big((1-\lambda)(q_0 - X) + \lambda(q_1 - X)\big)_+ \le (1-\lambda)(q_0 - X) + \lambda(q_1 - X)_+$$

and thus

$$\|(q_\lambda - X)\| \le (1-\lambda)\cdot \|(q_0 - X)_+\| + \lambda \cdot \|(q_1 - X)_+\|$$

by the triangle inequality of the norm.

For $q \in \mathbb{R}$ fixed, choose

$$\alpha \in \partial_\eta\ \|(\eta - Y)_+\|\Big|_{\eta=q},$$

that is, the subdifferential (of the convex function $\eta \mapsto \|(\eta - Y)_+\|$) evaluated at $\eta = q$, and note that $\alpha \in [0,1]$. Set $\beta := 1 - \alpha$, and observe that

$$0 \in \partial_q\ -q + \frac{1}{1-\beta}\|(q-Y)_+\|$$

so that

$$\mathcal{R}_\beta(-Y) = -q + \frac{1}{1-\beta}\|(q-Y)_+\|$$

by (2.3). Employing the definition (2.3) again and assumption (ii), it follows that

$$\begin{aligned} -q + \frac{1}{1-\beta}\|(-X+q)_+\| &\ge \mathcal{R}_\beta(-X) \\ &\ge \mathcal{R}_\beta(-Y) \\ &= -q + \frac{1}{1-\beta}\|(q-Y)_+\|, \end{aligned}$$

or equivalently

$$\|(q-X)_+\| \ge \|(q-Y)_+\|.$$

The assertion (i) follows, as $q \in \mathbb{R}$ was arbitrary; this establishes equivalence of (i) and (ii).

Finally, let $\beta \in (0,1)$. With (ii) and Proposition 2.6 we have that

$$\inf_{Z \in \mathcal{Z}_\beta} \mathbb{E}\, ZX \le \inf_{Z \in \mathcal{Z}_\beta} \mathbb{E}\, ZY,$$

where the infimum in both expressions is among $Z \in \mathcal{Z}_\beta = \left\{ Z \in \mathcal{Z} :\ \|Z\|_* \le \frac{1}{1-\beta} \right\}$, as the set $\mathcal{Z}_\beta$ collects the constraints in (2.8). This establishes equivalence between (ii) and (iii). □

Remark 4.4 The quantity $-\mathcal{R}(-Y) =: \mathcal{A}(Y)$ arising naturally in Theorem 4.3 (ii) above is often called an *acceptability functional*, cf. Pflug and Römisch (2007).

Corollary 4.5 *Suppose that*

$$\mathbb{E}\, ZX \le \mathbb{E}\, ZY \text{ for all } Z \in \mathcal{Z} := \bigcup_{\beta \in (0,1)} \mathcal{Z}_\beta, \tag{4.2}$$

then X is dominated by Y, $X \preccurlyeq^{\|\cdot\|} Y$. Further, the assertion (4.2) is equivalent to

$$\mathcal{R}_\beta(X - Y) \le 0 \quad \text{for all } \beta \in (0,1). \tag{4.3}$$

Proof Fix $\beta \in (0,1)$, then $\inf_{Z \in \mathcal{Z}_\beta} \mathbb{E}\, ZX \le \inf_{Z \in \mathcal{Z}_\beta} \mathbb{E}\, ZY$ by (4.2). With (iii) in the preceding Theorem 4.3 it follows that $X \preccurlyeq Y$.

With (2.7), the statement (4.3) is equivalent with $\mathbb{E}\, Z(X - Y) \le 0$ for $Z \in \mathcal{Z}$ and hence the assertion. □

Remark 4.6 The assertion (4.3), however, is *strictly stronger* than (ii) in Theorem 4.3. Indeed, it follows with convexity and (4.3) that

$$\mathcal{R}(-Y) \le \mathcal{R}(X - Y) + \mathcal{R}(-X) \le \mathcal{R}(-X),$$

and hence (ii), the assertion, although the reverse implication does not hold true.

Example 4.7 (Uniform norm) For the uniform norm $\|\cdot\|_\infty$, the defining relation (4.1) is equivalent to

$$X \preccurlyeq^{\|\cdot\|_\infty} Y \iff \operatorname{ess\,inf} X \le \operatorname{ess\,inf} Y;$$

this relation derives from the characterization (i) in Theorem 4.3 as well.

4.2 Higher order stochastic dominance

A traditional way of introducing stochastic dominance relations is by iterating integrals of the cumulative distribution function. This is a special case of the Lebesgue norm $\|\cdot\|_p$, $p \in [1,\infty)$, with $p \in \mathbb{N}$.

Definition 4.8 (*Higher order stochastic dominance, cf.* Müller and Stoyan (2002)) The random variable X is dominated by Y in *first order stochastic dominance*, if

$$F_X(x) \geq F_Y(x) \text{ for all } x \in \mathbb{R},$$

where $F_X(x) := P(X \leq x)$ is the cumulative distribution function. We shall write $X \preccurlyeq^{(1)} Y$. For $p \in [1,\infty]$, the random variable X is stochastically dominated by Y in p^{th}-*stochastic order*, if

$$\mathbb{E}\,(x-X)_+^{p-1} \geq \mathbb{E}\,(x-Y)_+^{p-1} \text{ for all } x \in \mathbb{R}; \tag{4.4}$$

we write $X \preccurlyeq^{(p)} Y$.

By (4.1) in Definition 4.1,

$$X \preccurlyeq^{(p+1)} Y \text{ is equivalent to } X \preccurlyeq^{\|\cdot\|_p} Y, \qquad p \geq 1,$$

where $\|\cdot\|_p$ is the usual norm in the Lebesgue space L^p. It is for historical—although unfortunate—reasons that the p-indici in the preceding display do not match. The higher order stochastic dominance of integral orders has been indtroduced and considered in earlier publications.

Lemma 4.9 (Cf. Ogryczak and Ruszczyński (1999, 2001)) *With* $F_X^{(1)}(\cdot) := F_X(\cdot)$, *the kth* ($k = 2, 3, \ldots$) *repeated integral is* $F_X^{(k)}(x) := \int_{-\infty}^x F_X^{(k-1)}(y)\,\mathrm{d}y$. *The following two points are equivalent, they characterize stochastic dominance of integer orders* ($k = 1, 2, \ldots$) *by repeated integrals*:

(i) $X \preccurlyeq^{(k)} Y$,
(ii) $F_Y^{(k)}(x) \geq F_X^{(k)}(x)$ *for all* $x \in \mathbb{R}$.

Proof It holds with Cauchy's formula for repeated integration that

$$F_X^{(k)}(x) = \frac{1}{(k-2)!}\int_{-\infty}^x (x-y)^{k-2} F_X(y)\,\mathrm{d}y.$$

By integration by parts, the latter is

$$F_X^{(k)}(x) = \frac{1}{(k-1)!}\int_{-\infty}^x (x-y)^{k-1}\,\mathrm{d}F_X(y),$$

so that

$$F_X^{(k)}(x) = \frac{1}{(k-1)!}\int_{-\infty}^{\infty}(x-y)_+^{k-1}\mathrm{d}F_X(y) = \frac{1}{(k-1)!}\,\mathbb{E}\,(x-X)_+^{k-1},$$

from which the assertion follows from the defining condition (4.1) in Definition 4.1. □

Remark 4.10 It follows from the iterated integral and (ii) in Lemma 4.9 that $X \preccurlyeq^{(k)} Y \implies X \preccurlyeq^{(k+1)} Y$ for all *natural* numbers $k = 1, 2, \ldots$. We notice next that

$$X \preccurlyeq^{(p)} Y \implies X \preccurlyeq^{(p')} Y \text{ for all real numbers } 1 \le p \le p' \in \mathbb{R}. \tag{4.5}$$

To this end note first that the characterization (4.4) is equivalent to

$$\int_{-\infty}^{x}(x-z)^{p-1}\mathrm{d}F_X(z) \ge \int_{-\infty}^{x}(x-z)^{p-1}\mathrm{d}F_Y(z) \text{ for all } x \in \mathbb{R}. \tag{4.6}$$

With $\int_z^x (x-y)^{\alpha-1}(y-z)^{\beta-1}\,\mathrm{d}y = B(\alpha,\beta)(x-z)^{\beta+\alpha-1}$ (B is Euler's integral of the first kind) and integration by parts it follows that

$$\begin{aligned}\int_{-\infty}^{x}(x-z)^{p'-1}\,\mathrm{d}F_X(z) &= \frac{1}{B(p,p'-p)}\int_{-\infty}^{x}\int_{z}^{x}(x-y)^{p'-p-1}(y-z)^{p-1}\,\mathrm{d}y\,\mathrm{d}F_X(z)\\ &= \frac{1}{B(p,p'-p)}\int_{-\infty}^{x}(x-y)^{p'-1-p}\int_{-\infty}^{y}(y-z)^{p-1}\,\mathrm{d}F_X(z)\,\mathrm{d}y\\ &\ge \frac{1}{B(p,p'-p)}\int_{-\infty}^{x}(x-y)^{p'-1-p}\int_{-\infty}^{x}(y-z)^{p-1}\,\mathrm{d}F_Y(z)\,\mathrm{d}y\\ &= \int_{-\infty}^{x}(x-z)^{p'-1}\,\mathrm{d}F_Y(z),\end{aligned} \tag{4.7}$$

where we have used the characterization (4.6) in (4.7), as $x - y \ge 0$ and that $B(p, p'-p)$ is well-defined and positive for $p' > p$. The assertion again follows with (4.6).

4.3 Characterization of stochastic dominance for spectral risk measures

The following builds on the spectral risk measure $\mathcal{R}_\sigma(\cdot)$ introduced in Definition 3.1 and considers the norm

$$\|\cdot\|_\sigma := \mathcal{R}_\sigma(|\cdot|)$$

for the spectral function σ. Theorem 4.3 and the characterization of higher order spectral risk measures (Theorem 3.2) give rise to the following result.

Theorem 4.11 *The stochastic dominance relation*

$$X \preccurlyeq^{\|\cdot\|_\sigma} Y$$

with respect to the norm associated with the spectral risk measure $\mathcal{R}_\sigma$ *is equivalent to*

$$\begin{aligned}&-\sigma_p \cdot \mathsf{V@R}_p(Y) + \int_{-\infty}^{\mathsf{V@R}_p(Y)} \Sigma\big(S_Y(y)\big)\,\mathrm{d}y\\ &\le -\sigma_p \cdot \mathsf{V@R}_p(X) + \int_{-\infty}^{\mathsf{V@R}_p(X)} \Sigma\big(S_X(x)\big)\,\mathrm{d}x \quad \text{for all } p \in (0,1),\end{aligned}$$

where $\sigma_p := \int_{1-p}^1 \sigma(u)\,\mathrm{d}u$ *and* $S_X(x) := 1 - F_X(x) = P(X > x)$ *is the survival function of the random variable* X.

Proof We argue with the norm $\|Y\|_\sigma := \mathcal{R}_\sigma(|Y|)$. Note, that $(Y-t)_+ \ge 0$, hence the defining equation (2.3) is

$$\begin{aligned}\mathcal{R}_\beta^{\|\cdot\|_\sigma}(Y) &= \inf_{t\in\mathbb{R}}\; t + \frac{1}{1-\beta}\|(Y-t)_+\|_\sigma\\ &= \inf_{t\in\mathbb{R}}\; t + \frac{1}{1-\beta}\mathcal{R}_\sigma\big((Y-t)_+\big)\\ &= \mathcal{R}_{\sigma_\beta}(Y),\end{aligned} \tag{4.8}$$

where we have used Theorem 3.2 in (4.8).

From (3.8) we have that

$$\begin{aligned}\mathcal{R}_\beta(-Y) &= \mathsf{V@R}_{u_\beta}(-Y) + \frac{1}{1-\beta}\int_{\mathsf{V@R}_{u_\beta}(-Y)}^{\infty} \Sigma\big(F_{-Y}(y)\big)\,\mathrm{d}y\\ &= -\mathsf{V@R}_{1-u_\beta}(Y) + \frac{1}{1-\beta}\int_{-\mathsf{V@R}_{1-u_\beta}(Y)}^{\infty} \Sigma\big(S_Y(-y)\big)\,\mathrm{d}y\\ &= -\mathsf{V@R}_{1-u_\beta}(Y) + \frac{1}{1-\beta}\int_{-\infty}^{\mathsf{V@R}_{1-u_\beta}(Y)} \Sigma\big(S_Y(y)\big)\,\mathrm{d}y,\end{aligned}$$

where we have used that $F_{-Y}(y) = P(-Y \le y) = P(Y \ge -y) = 1 - F_Y(-y) = S_Y(-y)$ and $\mathsf{V@R}_\alpha(-Y) = -\mathsf{V@R}_{1-\alpha}(Y)$ at points of continuity of $F_Y(\cdot)$.

Now set $1 - u_\beta =: p$. Then, by employing the characterizing relation (3.3) for the β -quantile of σ, it holds that

$$1-\beta = \int_{u_\beta}^1 \sigma(u)\,\mathrm{d}u = \int_{1-p}^1 \sigma(u)\,\mathrm{d}u = \sigma_p,$$

so that

$$\mathcal{R}_\beta(-Y) = -\mathsf{V@R}_p(Y) + \frac{1}{\sigma_p}\int_{-\infty}^{\mathsf{V@R}_p(Y)} \Sigma\big(S_Y(y)\big)\,\mathrm{d}y.$$

By Theorem 4.3, the relation $X \preccurlyeq^{\|\cdot\|_\sigma} Y$ is equivalent to $\mathcal{R}_\beta^{\|\cdot\|_\sigma}(-Y) \le \mathcal{R}_\beta^{\|\cdot\|_\sigma}(-X)$ for all $\beta \in (0,1)$. With that, the assertion follows. □

4.4 Comparison of stochastic order relations

Different stochastic dominance relations may vary in strength (the implication (4.5) in the preceding Remark 4.10 is an example). In what follows, we provide an explicit relation to compare stochastic dominance relations, which are built on different spectral functions.

Proposition 4.12 (Comparison of spectral stochastic orders) *Suppose that*

$$\sigma_\mu(u) = \sigma(u) \cdot \int_0^{u_\beta} \frac{\mu(\mathrm{d}\beta)}{1-\beta} \tag{4.9}$$

for some probability measure μ, where u_β is as defined in (3.3). Then the stochastic order associated with σ_μ is weaker than the genuine stochastic order associated with σ. Specifically, for different spectral functions σ and σ_μ, it holds that

$$X \preccurlyeq^{\|\cdot\|_\sigma} Y \implies X \preccurlyeq^{\|\cdot\|_{\sigma_\mu}} Y.$$

Remark 4.13 The function σ_μ in (4.9) is indeed a spectral function. It is positive, as μ is a positive measure (thus (i) in Definition 3.1). The function is non-decreasing, as u_β is non-decreasing for β increasing. Finally, the function σ_μ is a density: indeed, it holds that

$$\int_0^1 \sigma_\mu(u)\,\mathrm{d}u = \int_0^1 \sigma(u) \cdot \int_0^{u_\beta} \frac{\mu(\mathrm{d}\beta)}{1-\beta}\mathrm{d}u = \int_0^1 \int_{\beta_u}^1 \sigma(u)\,\mathrm{d}u\, \frac{\mu(\mathrm{d}\beta)}{1-\beta} = \int_0^1 \mu(\mathrm{d}\beta) = 1$$

by integration by parts, where we have used the definition of u_β in (3.3).

Proof of Proposition 4.12 Since $x \preccurlyeq^{\|\cdot\|_\sigma} Y$, it holds with Theorem 4.3 that $\mathcal{R}_{\sigma_\beta}(-X) \ge \mathcal{R}_{\sigma_\beta}(-Y)$ for all $\beta \in (0,1)$, where σ_β is defined in (3.2). By the characterization (3.1), this is

$$\int_{u_\beta}^1 \frac{\sigma(u)}{1-\beta} F_{-X}^{-1}(u)\,\mathrm{d}u \ge \int_{u_\beta}^1 \frac{\sigma(u)}{1-\beta} F_{-Y}^{-1}(u)\,\mathrm{d}u, \qquad \beta \in (0,1).$$

Integrating the latter expression with respect to $\mu(\mathrm{d}\beta)$ establishes the inequality

$$\int_\beta^1 \int_{u_{\beta'}}^1 \frac{\sigma(u)}{1-\beta'} F_{-X}^{-1}(u)\,\mathrm{d}u\, \mu(\mathrm{d}\beta') \ge \int_\beta^1 \int_{u_{\beta'}}^1 \frac{\sigma(u)}{1-\beta'} F_{-Y}^{-1}(u)\,\mathrm{d}u\, \mu(\mathrm{d}\beta'), \qquad \beta \in (0,1).$$

Interchanging the order of integration together with (3.17) gives that

$$\int_{u_\beta}^{1}\int_{\beta}^{\beta_u}\frac{\sigma(u)}{1-\beta'}\mu(\mathrm{d}\beta')F_{-X}^{-1}(u)\,\mathrm{d}u \geq \int_{u_\beta}^{1}\int_{\beta}^{\beta_u}\frac{\sigma(u)}{1-\beta'}\mu(\mathrm{d}\beta)F_{-Y}^{-1}(u)\,\mathrm{d}u, \qquad \beta\in(0,1),$$

which in turn is

$$\int_{u_\beta}^{1}\sigma_\mu(u)F_{-X}^{-1}(u)\,\mathrm{d}u \geq \int_{u_\beta}^{1}\sigma_\mu(u)F_{-Y}^{-1}(u)\,\mathrm{d}u, \qquad \beta\in(0,1).$$

This is the assertion. □

5 Example: the expectile

The expectile risk measure, originally introduced by Newey and Powell (1987), has recently gained additional interest (cf. Malandii et al. (2024), Balbás et al. (2023) or Farooq and Steinwart (2018) for conditional regressions). A main reason for the additional interest in this risk measure is because it is the only elicitable risk functional (cf. Ziegel (2014)).

As Proposition 2.6 indicates, the higher order risk measure can be based on the dual norm. For this reason, the following section establishes the dual norm of expectiles first, as it is crucial in understanding its regret function in the risk quadrangle. Next, we provide an explicit characterization of the higher order expectiles, that is, the higher order risk measure based on the expectile risk measure.

The expectile is defined as a *minimizer*. Its Kusuoka representation is central in elaborating the corresponding higher order risk functional.

Definition 5.1 For $\alpha\in(0,1)$, the expectile is

$$e_\alpha(Y)=\operatorname*{arg\,min}_{x\in\mathbb{R}}\ \mathbb{E}\,\ell_\alpha(Y-x), \tag{5.1}$$

where

$$\ell_\alpha(x)=\begin{cases}\alpha\,x^2 & \text{if } x\geq 0,\\ (1-\alpha)x^2 & \text{else}\end{cases}$$

is the asymmetric loss, or quadratic error function.

The expectile satisfies the first order condition

$$(1-\alpha)\,\mathbb{E}\,(x-Y)_+=\alpha\,\mathbb{E}\,(Y-x)_+, \tag{5.2}$$

and $e_\alpha(\cdot)$ is a risk measure for $\alpha\in[1/2,1]$. We mention that condition (5.2) provides a definition for $Y\in L^1$, it is thus more general than (5.1), which requires $Y\in L^2$. The Kusuoka representation of the expectile (cf. Bellini et al. (2014, Proposition 9)) is given by

$$e_\alpha(Y) = \max_{\gamma\in[0,1-\eta]} (1-\gamma)\cdot \mathbb{E}\, Y + \gamma\cdot \mathsf{AV@R}_{1-\frac{\gamma}{1-\gamma}\frac{\eta}{1-\eta}}(Y), \tag{5.3}$$

where $\eta = \frac{1-\alpha}{\alpha}$, so that the risk level in (5.3) is $1-\frac{\gamma}{1-\gamma}\frac{\eta}{1-\eta} = \frac{\alpha(2-\gamma)-1}{(2\alpha-1)(1-\gamma)}$. Involving spectral risk measures, the expectile can be recast as

$$e_\alpha(Y) = \sup\left\{\mathcal{R}_{\sigma_\gamma}(Y) : \sigma_\gamma \in \mathcal{S}\right\},$$

where $\mathcal{S} = \left\{\sigma_\gamma : \gamma\in[0,1-\eta]\right\}$ collects the spectral functions

$$s_\gamma(u) = \begin{cases} 1-\gamma & \text{if } u \le 1-\frac{\gamma}{1-\gamma}\frac{\eta}{1-\eta},\\ \frac{1-\gamma}{\eta} & \text{else.}\end{cases}$$

The higher order expectile can be described by involving its dual norm (cf. (2.9)), as well as its Kusuoka representation (cf. Corollary 3.6). The following two (sub)sections elaborate these possibilities for the expectile.

5.1 The dual norm of expectiles

The higher order expectile can be described with the dual representation (2.8), for which the dual norm of the expectile is necessary.

By the characterization of the loss function (5.2) it holds that $e_\alpha(Y)$ is well-defined for $Y\in L^1(P)$. This is enough to conclude that $\mathbb{E}\,|Y| \le C_\alpha \cdot e_\alpha(|Y|)$ for some constant $C_\alpha > 0$ (Lakshmanan and Pichler 2023, Corollary 2.16) elaborate the tight bound $C_\alpha = \frac{\alpha}{1-\alpha}$). It follows that $\mathcal{Y}^* = L^\infty$, so that $\|Z\|_\infty$ is well-defined for $Z\in\mathcal{Y}^*$.

The following result provides the dual norm of the expectile explicitly.

Proposition 5.2 (Dual norm of the expectile) *For* $\alpha \ge 1/2$, *the dual norm is*

$$\|Z\|_\alpha^* := \sup\left\{\mathbb{E}\, YZ : e_\alpha(|Y|) \le 1\right\} \tag{5.4}$$

(*cf.* (2.7)) . *It holds that*

$$\|Z\|_\alpha^* = \sup_{\beta\in(0,1)} (1-\beta)\cdot \mathsf{AV@R}_\beta(|Z|) + \beta\frac{1-\alpha}{\alpha}\|Z\|_\infty. \tag{5.5}$$

Notably, the norm $\|\cdot\|_\alpha^*$ is *not* a risk measure itself, and (5.5) is *not* a Kusuoka representation; indeed, the total weight in the representation (5.5) is

$$(1-\beta) + \beta\frac{1-\alpha}{\alpha} < 1$$

for $\alpha\in(1/2,1]$.

Proof of Proposition 5.2 We may assume that $Z\ge 0$, as otherwise we may consider $\operatorname{sign}(Z)\cdot Y$ instead of Y. For arbitrary sets B and G with $B\subset G$ and $P(G) < 1$ define the random variable

$$\tilde{Y}_{B,G}(\omega) := \begin{cases} 0 & \text{if } \omega \in B, \\ 1 & \text{if } \omega \in G \setminus B, \text{ and} \\ \frac{1-\alpha}{\alpha} \cdot \frac{P(B)}{1-P(G)} + 1 & \text{else.} \end{cases} \tag{5.6}$$

Note, that

$$(1-\alpha) \cdot P(B)(1-0) = \alpha \cdot \big(1 - P(G)\big) \left(\frac{(1-\alpha)P(B)}{\alpha(1-P(G))} + 1 - 1 \right),$$

and hence $e_\alpha(\tilde{Y}_{B,G}) = 1$ by the defining equation (5.2). It follows with (5.4) that

$$\|Z\|_\alpha^* \geq \mathbb{E}\, Z\, Y_{B,G}.$$

As $B \subset G$ are arbitrary, we conclude in particular that

$$\|Z\|_\alpha^* \geq \big((1 - P(B)\big) \cdot \mathsf{AV@R}_{P(B)}(Z) + P(B)\frac{1-\alpha}{\alpha} \cdot \mathsf{AV@R}_{P(G)}(Z),$$

because the random variables

$$\tilde{Y}_{B,G} = \big(1 - P(B)\big) \cdot \frac{1}{1-P(B)} \mathbb{1}_{[P(B),1]}(U) + P(B)\frac{1-\alpha}{\alpha} \cdot \frac{1}{1-P(G)} \mathbb{1}_{[P(G),1]}(U)$$

satisfy all conditions from above for any uniform variable U. Now let $P(G) \to 1$ and by denoting $\beta = P(B)$ it follows that

$$\|Z\|_\alpha^* \geq \sup_{\beta \in (0,1)} (1-\beta) \cdot \mathsf{AV@R}_\beta(Z) + \beta\frac{1-\alpha}{\alpha} \operatorname{ess\,sup} Z,$$

as $\mathsf{AV@R}_\gamma(Z) \to \operatorname{ess\,sup} Z$ for $\gamma \to 1$.

As for the converse observe that we may assume $e_\alpha(Y) = 1$ for the optimal random variable in (5.4). Consider the Lagrangian

$$L(Y; \lambda, \mu) := \mathbb{E}\, ZY - \lambda\big((1-\alpha)\,\mathbb{E}\,(1-Y)_+ - \alpha\,\mathbb{E}\,(Y-1)_+\big) + \mathbb{E}\,\mu Y, \tag{5.7}$$

where the Lagrangian multiplier $\lambda \in \mathbb{R}$ is associated with the equality constraint $e_\alpha(Y) = 1$, i.e., (5.2), and the measurable variable $\mu \in L^1$, $\mu \geq 0$, is associated with the inequality constraint $Y \geq 0$. Provided That the derivative exists, the first order conditions are

$$0 = \frac{\partial}{\partial Y} L(Y; \lambda, \mu),$$

or

$$Z = \lambda\big(-(1-\alpha)\mathbb{1}_{\{Y<1\}} - \alpha\,\mathbb{1}_{\{Y>1\}}\big) - \mu \cdot \mathbb{1}_{\{Y=0\}}. \tag{5.8}$$

Now note that the left-hand side of (5.8) involves the *variable* Z, while the right-hand side only involves *constants*, except on $\{Y = 0\}$, where μ is not necessarily constant. The first order conditions (5.8) thus hold true on plateaus of Z, if they coincide with $\{Y < 1\}$ or $\{Y > 1\}$; for $\{Y = 0\}$, equation (5.8) is $\mu = -Z - \lambda(1-\alpha)$; for $\{Y = 1\}$, the derivative of (5.7) does not exist or depends on the direction.

It follows, that the optimal Y in (5.4) exactly is of form (5.6) and hence the assertion. □

5.2 Higher order expectiles

The Kusuoka representation (5.3) is the basis for the expectile's higher order risk measure.

Proposition 5.3 *For $\beta \in (0,1)$, the higher order expectile is*

$$(e_\alpha)_\beta(Y) = \max_{\gamma\in[0,1-\eta]} \begin{cases} \left(1-\frac{\gamma}{1-\beta}\right) \mathsf{AV@R}_{\frac{\beta}{1-\gamma}}(Y) + \frac{\gamma}{1-\beta}\mathsf{AV@R}_{1-\frac{\gamma}{1-\gamma}\frac{\eta}{1-\eta}}(Y) & \text{if } \frac{\gamma}{1-\beta} < 1-\eta, \\ \mathsf{AV@R}_{1-(1-\beta)(1-\tilde{\alpha})}(Y) & \text{else,} \end{cases} \tag{5.9}$$

where $\eta = \frac{1-\alpha}{\alpha}$ (as above) and $\tilde{\alpha} := 1 - \frac{1}{1-\gamma}$.

Proof The measure in the Kusuoka representation (5.3) is $\mu(\cdot) = (1-\gamma)\delta_0 + \gamma\cdot\delta_{1-\frac{\gamma}{1-\gamma}\frac{\eta}{1-\eta}}$. To apply Corollary 3.7 we set $p_1 := 1-\gamma$ and $p_2 = \gamma$, the corresponding risk levels are $\alpha_1 = 0$ and $\alpha_2 = 1 - \frac{\gamma}{1-\gamma}\frac{\eta}{1-\eta}$. The mixed risk level is $\tilde{\alpha} := \frac{\alpha_1\frac{p_1}{1-\alpha_1} + \alpha_2\frac{p_2}{1-\alpha_2}}{\frac{p_1}{1-\alpha_1} + \frac{p_2}{1-\alpha_2}} = \frac{\alpha(2-\gamma)-1}{\alpha(1-\gamma)} = 1 - \frac{\eta}{1-\gamma}$.

We distinguish the cases $\frac{\gamma}{1-\beta} < 1-\eta$ and $\frac{\gamma}{1-\beta} < 1-\eta$, which are equivalent to $u_\beta \lesseqgtr \alpha_2$, i.e., $1 - \frac{1-\beta}{\frac{\gamma}{1-\alpha_1} + \frac{1-\gamma}{1-\alpha_2}} \lesseqgtr \alpha_2$ in view of (3.20). In the first case, the critical equation (3.16) is $(1-\gamma)u_\beta = \beta$, while it is $(1-\gamma)u_\beta + \gamma\frac{u_\beta - \alpha_2}{1-\alpha_2} = \beta$ in the other case; the solutions thus are $u_\beta = \frac{\beta}{1-\gamma}$ and $u_\beta = \frac{\alpha(2-\beta-\gamma)-1+\beta}{\alpha(1-\gamma)}$. The corresponding weights p_0 (cf. (3.16) again) are $p_0 = \frac{1-u_\beta}{1-\beta}(1-\gamma)$, or $p_0 = \frac{1-u_\beta}{1-\beta}\left(\frac{1-\gamma}{1-0} + \frac{\gamma}{1-\alpha_2}\right) = 1$. Finally, note that $u_\beta = 1-(1-\beta)(1-\tilde{\alpha})$.

The assertion follows with (3.17) and (3.18) in Corollary 3.7. □

The average value-at-risk is 'closed under higher orders', as its higher order variant is an average value-at-risk as well (cf. (3.12)). This is not the case for the expectile, as the first term in (5.9) is not an expectation as in the genuine Kusuoka representation (5.3). Repeating the construction and passing to higher order expectiles leads to more complicated risk measures.

Remark 5.4 The results in Mafusalov and Uryasev (2016) on stochastic properties of the average value-at-risk reveal relations and risk measures, which are similar to the risk measures exposed in (5.9).

6 Summary

Higher order risk measures naturally integrate with stochastic optimization, as they are stochastic optimization problems themselves. This paper presents and derives explicit forms of higher order risk measures, specifically for spectral risk measures. These risk measures constitute the central building block of general law invariant risk measures.

Extending these results result it is demonstrated that stochastic dominance relations can be characterized by employing higher order risk measures, and vice versa. We provide a verification theorem, which makes higher stochastic dominance relations accessible to numerical computations.

The results are exemplified for expectiles, a specific risk measure with unique properties.

Author contributions Everything by AP.

Funding Open Access funding enabled and organized by Projekt DEAL.

Data availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare no competing interests.

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