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ORIGINAL PAPER

Approximate variational inequalities and equilibria

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Abstract

We study relations between the solution sets of Variational Inequalities, Minty Variational Inequalities, Natural Map problems and Nash Equilibrium Problems. Moreover, motivated by the inherent relevance of inexactness both in modeling non-cooperative games and in algorithms for variational inequalities, we consider inexact versions of such problems and we establish relations to quantify how inexactness propagates from one problem to the other.

Keywords Nash games · Variational inequalities · Natural map · Inexact solutions

Mathematics Subject Classification 90C33 · 91A10 · 49J40 · 90C25

1 Introduction

Nash Equilibrium Problems (NEP) are well-suited to capture the interactions between multiple agents in non-cooperative settings, hence they have been largely used in economics, finance, transportation and engineering (see e.g. Facchinei and Pang 2003; Bigi et al. 2019; Ferris and Pang 1997; Facchinei and Lampariello 2011; Konnov 2007; Nagurney 1998; Facchinei and Sagratella

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2011; Lampariello et al. 2021; Scutari et al. 2012). They are strictly connected to other mathematical models of variational nature. In particular NEPs can be reformulated as Variational Inequalities (VI) and Minty Variational Inequalities (MVI) under standard assumptions (see Facchinei and Pang 2003; Hartman and Stampacchia 1966; Kinderlehrer and Stampacchia 2000; Minty 1962 and the references therein for VIs, MVIs and their fundamental properties). In turn, fixed point techniques have been extensively exploited to tackle such problems by relying on to the Natural Map problem (NM), see (Facchinei and Pang 2003) for an overview. Therefore, it is reasonable to analyze these problems together and to investigate their interplay.

As some degree of inexactness arises quite naturally, both in modeling and in algorithms, we find it convenient to consider also approximate versions of the above problems. In fact, approximation is inherent in numerical algorithms when devising practical stopping criteria and, clearly, complexity guarantees are closely related to the magnitude of inexactness. For example, projection type techniques are widely used to solve VIs, where the fulfillment of a NM-related condition provides a valuable stopping criterion (see He et al. 2002; Bigi et al. 2021; Lampariello et al. 2020). Inexactness on MVI-based stopping criteria and related complexity guarantees arises when relying on averaging techniques on projection iterates or cutting plane type algorithms (see e.g. Bruck 1977; Bigi and Panicucci 2010; Lampariello et al. 2022; Nguyen and Dupuis 1984; Kaushik and Yousefian 2021). On the other hand, introducing inexactness in NEPs amounts to some degree of inaccuracy in each player problem's equilibrium condition (see e.g. Başar and Olsder 1998; Nisan et al. 2007; Morgan 2005; Sagratella 2017a, b). The relations we present are aimed at providing a connection between the inexactness that is inevitably associated to algorithmic procedures commonly used to address NEPs and the corresponding degree of sub-optimality that is achieved in each player's problem.

Theoretical works concerning inexactness on the above problems are mainly focused on existence of approximate solutions (see Brânzei et al. 2003) and their limit behavior as inexactness vanishes (see Gürkan and Pang 2009; Morgan and Raucci 1999; Lignola and Morgan 1994, 2002). Some connections between inexact versions of VIs and MVIs have been investigated in Lignola and Morgan (1994): we aim at deepening this analysis, while expanding it to NEPs and NMs as well. Given some level of inexactness in the solution of a problem, we are able to give an estimate of the degree of inexactness up to which any other problem is solved. These estimates depend only on problem-related constants and on the level of inexactness in the solution of the original problem.

The paper is organized into two sections. In Sect. 2, we introduce exact versions of VI, MVI, NM, NEP. While we review well-known relations, we also obtain some other connections between MVIs and NEPs and MVIs and NMs. In Sect. 3, we address the inexact versions of the above four problems and study the links between them. Just a few of them follow in the footsteps of the exact case, while others require additional assumptions. Furthermore, the corresponding degrees of inexactness in the relations may turn out to be different.

2 The exact case

In this section, we introduce the exact version of the problems and gather some results concerning the correlation between their solution sets. Given $X \subseteq \mathbb{R}^n$ non-empty, closed and convex and $F : \mathbb{R}^n \to \mathbb{R}^n$ continuous, the Variational Inequality problem is the following

$$\text{find } \bar{x} \in X : \quad F(\bar{x})^T(x - \bar{x}) \geq 0 \quad \forall x \in X. \tag{VI}$$

The Minty Variational Inequality problem reads

$$\text{find } \bar{x} \in X : \quad F(x)^T(x - \bar{x}) \geq 0 \quad \forall x \in X. \tag{MVI}$$

The Natural Map problem is as follows

$$\text{find } \bar{x} \in X : \quad \|\bar{x} - P_X(\bar{x} - F(\bar{x}))\| \leq 0. \tag{NM}$$

The presence of the inequality in the above condition, which is actually an equality, is aimed at stressing the connection with the inexact version of the problem that is introduced in the next section.

Consider the case where $X \triangleq X_1 \times \ldots \times X_N$, with $X_\nu \subseteq \mathbb{R}^{n_\nu}$, $\sum_\nu n_\nu = n$, and $\theta_\nu : X \to \mathbb{R}$ are continuously differentiable. The Nash Equilibrium Problem reads

$$\text{find } \bar{x} \in X : \quad \theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}) \leq \theta_\nu(x^\nu, \bar{x}^{-\nu}) \quad \forall x^\nu \in X_\nu, \quad \nu = 1 \ldots N. \tag{NEP}$$

Any such $\bar{x}$ is called equilibrium. In the case $N = 1$, (NEP) boils down to an optimization problem.

The following propositions are common folklore in variational analysis (see e.g. Facchinei and Pang 2003). Nonetheless, we report them for the sake of completeness.

Proposition 2.1 (NM $\longleftrightarrow$ VI) *A point $\bar{x}$ is a solution to* (*NM*) *if and only if it is a solution to* (*VI*).

Proof Consider a solution $\bar{x} \in X$ to (VI). We have

$$(x - \bar{x})^T F(\bar{x}) \geq 0 \quad \forall x \in X,$$

which is equivalent to

$$(x - \bar{x})^T(\bar{x} - (\bar{x} - F(\bar{x}))) \geq 0 \quad \forall x \in X,$$

that, in turn, by the characteristic property of the projection operator, is true if and only if $\bar{x}$ is a solution to (NM). □

Proposition 2.2 (*VI* $\longleftarrow$ *MVI*, *VI* $\stackrel{\text{mono}}{\longrightarrow}$ *MVI*)

(i) *If $\bar{x}$ is a solution to* (*MVI*), *then it is a solution to* (*VI*).

(ii) *Assuming F to be monotone on X, if $\bar{x}$ is a solution to* (*VI*), *then it is a solution to* (*MVI*).

Proof *(i)* For any $x \in X$, we define $u_\lambda = \lambda \bar{x} + (1-\lambda)x$, $\lambda \in (0,1)$. Since $u_\lambda \in X$ by the convexity of X, if $\bar{x}$ is a solution to (MVI), we can write

$$0 \le F(u_\lambda)^T(u_\lambda - \bar{x}) = (1-\lambda)F(u_\lambda)^T(x-\bar{x}).$$

Dividing both sides of the relation by $(1-\lambda)$ and taking the limit for $\lambda \to 1$, by the continuity of F, we have $F(u_\lambda) \to F(\bar{x})$, and, in turn, $\bar{x}$ is a solution to (VI)).

(ii) The proof is obtained by considering Proposition 3.2*(ii)* with $\varepsilon = 0$. □

NEPs turn out to be connected to variational inequalities by considering the operator

$$F(x) = \left[\nabla_{x^\nu}\theta_\nu(x^\nu, x^{-\nu})\right]_{\nu=1}^N, \tag{1}$$

and $X = X_1 \times \ldots \times X_N$, as the following results show.

Proposition 2.3 (NEP $\longrightarrow$ VI, NEP$\overset{\text{conv}}{\longleftarrow}$ VI) *Consider F given by* (1).

(i) *If $\bar{x}$ is a solution to* (*NEP*), *then it is a solution to* (*VI*).

(ii) *Assuming, for all ν, $\theta_\nu(\bullet, x^{-\nu})$ to be convex on X_ν for every $x^{-\nu} \in \prod_{\mu\neq\nu} X_\mu$, if $\bar{x}$ is a solution to* (*VI*), *then it is a solution to* (*NEP*).

Proof *(i)* If $\bar{x}$ is a solution to (NEP), by the minimum principle, for all ν,

$$\nabla_{x^\nu}\theta_\nu\left(\bar{x}^\nu, \bar{x}^{-\nu}\right)^T\left(x^\nu - \bar{x}^{-\nu}\right) \ge 0, \quad \forall x^\nu \in X_\nu.$$

Concatenating these inequalities for all players, $\bar{x}$ turns out to be a solution to (VI) with F given by (1).

(ii) The proof is obtained by considering Proposition 3.3 with $\varepsilon = 0$. □

The proofs to the following propositions can be deduced by considering the results in the next section, specifically Propositions 3.4, 3.6 and 3.7, with $\varepsilon = 0$ and are therefore omitted. While the claim in Proposition 2.4 is well known, to the best of our knowledge, the other two are new.

Proposition 2.4 (NEP $\overset{\text{N=1,conv}}{\longrightarrow}$ MVI) *Consider F given by* (1) *and assume $N = 1$ and θ_1 convex on X_1. If $\bar{x}$ is a solution to* (*NEP*), *then it is a solution to* (*MVI*).

Proposition 2.5 (*NEP $\overset{\text{quad}}{\longleftarrow}$ MVI*) *Consider F given by* (1) *and assume, for all ν, $\theta_\nu(\bullet, x^{-\nu})$ to be quadratic on X_ν for every $x^{-\nu} \in \prod_{\mu\neq\nu} X_\mu$. If $\bar{x}$ is a solution to* (*MVI*), *then it is a solution to* (*NEP*).

Proposition 2.6 (*NM* $\xleftarrow{L<1}$ *MVI*) *Assume F is Lipschitz continuous with modulus $L < 1$. If $\bar{x}$ is a solution to* (*MVI*), *then it is a solution to* (*NM*).

Figure 1 subsumes all the above results. The arrow connecting NEP with MVI is dashed since the relation between the two problems is proven when only a single player is considered. We also remark that the graph in Fig. 1 is not complete because the missing relations have not been established. However, the graph turns out to be connected, so that one can link, possibly not directly, any pair of problems.

3 The inexact case

The results in Sect. 2 bear witness to how the (exact) solutions of the problems we consider are deeply interrelated. However, since inexactness arises naturally, for example when resorting to numerical procedures, it is paramount to understand how the connections between these problems behave when they are solved up to some accuracy. Even more so when dealing with practical situations: suffice it to think to the NEPs modeling the interplay of clients' accounts that are managed by financial service providers in a multi-portfolio context (see Lampariello et al. 2021). Applying numerical procedures to such problems invariably leads to the computation of points satisfying suitable stopping criteria (given in the form of MVI, or NM) up

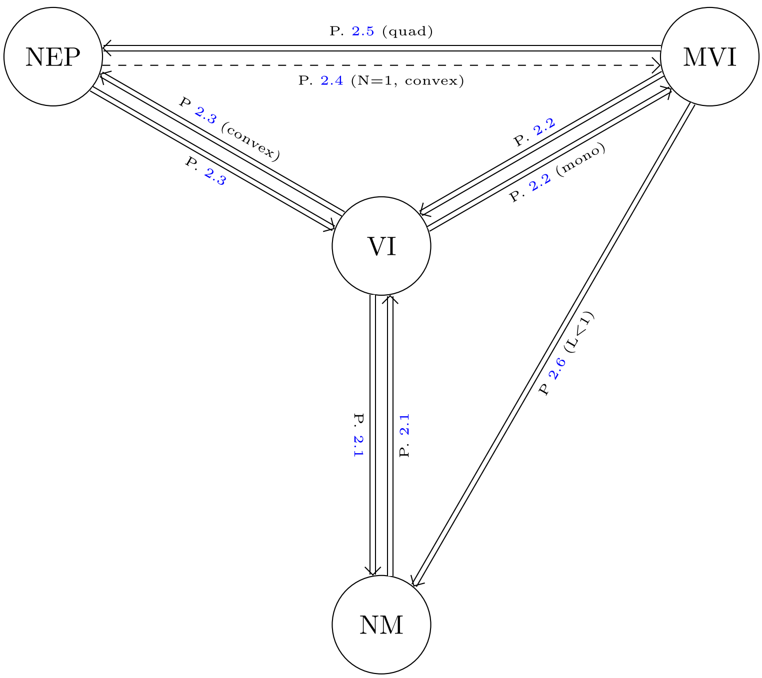

Fig. 1 Exact relations scheme

to some accuracy (see Lampariello et al. 2020, 2022, 2023). In turn, knowing the link between inexact solutions of MVIs or NMs and corresponding NEPs, one can get information on up to what actual amount of money the computed utilities of the clients are optimal.

Consider a degree of inexactness $\varepsilon \geq 0$. The ε-inexact Variational Inequality (ε -VI) problem is the following

$$\text{find } \overline{x} \in X : \quad F(\overline{x})^T(x - \overline{x}) \geq -\varepsilon \quad \forall x \in X.$$

The ε-inexact Minty Variational Inequality (ε-MVI) problem reads

$$\text{find } \overline{x} \in X : \quad F(x)^T(x - \overline{x}) \geq -\varepsilon \quad \forall x \in X.$$

The ε-inexact Natural Map problem (ε-NM) is

$$\text{find } \overline{x} \in X : \quad \|\overline{x} - P_X(\overline{x} - F(\overline{x}))\| \leq \varepsilon.$$

The ε-inexact Nash Equilibrium Problem (ε-NEP) reads

$$\text{find } \overline{x} \in X : \quad \theta_\nu(\overline{x}^\nu, \overline{x}^{-\nu}) \leq \theta_\nu(x^\nu, \overline{x}^{-\nu}) + \varepsilon \quad \forall x^\nu \in X_\nu, \quad \nu = 1 \ldots N.$$

Any such $\overline{x}$ is also called ε-equilibrium, and in particular an ε-minimum when $N = 1$.

Taking $\varepsilon = 0$ in the above (inexact) problems, the corresponding exact versions (see Sect. 2) are recovered. Whenever X is bounded, the quantities

$$D_X \triangleq \max_{x,y \in X} \|x - y\|, \quad \overline{F}_X \triangleq \max_{x \in X} \|F(x)\|$$

are sometimes called for in the forthcoming developments.

Inexactness brings on some difficulties in establishing correspondences between the above problems: relationships turn out to be more complicated and require also additional assumptions that are not needed in the exact case.

The relation between inexact VIs and NMs can be traced back to Lampariello et al. (2022).

Proposition 3.1 (ε-NM $\xrightarrow{\text{bound}}$ $(c_1\varepsilon)$-VI, $\sqrt{\varepsilon}$-NM $\longleftarrow$ ε-VI)

(i) *Assuming X to be bounded, if $\overline{x}$ is a solution to (ε-NM), then it is a solution to (($c_1\varepsilon$)-VI), where $c_1 \triangleq D_X + \overline{F}_X$.*
(ii) *If $\overline{x}$ is a solution to (ε-VI), then it is a solution to ($\sqrt{\varepsilon}$-NM).*

Proof *(i)* Consider a solution $\overline{x} \in X$ to (ε-NM) and let $z = P_X(\overline{x} - F(\overline{x}))$: by the characteristic property of the projection operator, we have

$$(z - (\overline{x} - F(\overline{x})))^T(x - z) \geq 0 \quad \forall x \in X,$$

therefore, for all $x \in X$,

$$\begin{aligned} F(\bar{x})^T(x-\bar{x}) &\geq (\bar{x}-z)^T(x-z) + F(\bar{x})^T(z-\bar{x}) \\ &\geq -\|z-\bar{x}\|(\|x-z\| + \|F(\bar{x})\|) \\ &\geq -\|P_X(\bar{x}-F(\bar{x}))-\bar{x}\|(D_X+\overline{F}_X) \\ &\geq -\varepsilon(D_X+\overline{F}_X), \end{aligned}$$

where the third inequality is due to the boundedness of X, and the last one holds because $\bar{x}$ is a solution to (ε-NM). Therefore, $\bar{x}$ is a solution to (($c_1\varepsilon$)-VI).

(ii) Consider a solution $\bar{x}$ to (ε-VI) and let $z = P_X(\bar{x}-F(\bar{x}))$, we have

$$\begin{aligned} -\varepsilon &\leq F(\bar{x})^T(z-\bar{x}) = (\bar{x}-z-(\bar{x}-z-F(\bar{x})))^T(z-\bar{x}) \\ &= -\|\bar{x}-z\|^2 - (\bar{x}-F(\bar{x})-z)^T(z-\bar{x}) \leq -\|\bar{x}-z\|^2 = -\|\bar{x}-P_X(\bar{x}-F(\bar{x}))\|^2, \end{aligned}$$

where the last inequality is due to the characteristic property of the projection operator. Therefore, $\bar{x}$ is a solution to ($\sqrt{\varepsilon}$-NM). □

While the second implication in the next two results follows in the footsteps of the exact case, the other one requires a more complicated proof as well as additional assumptions.

Proposition 3.2 (($c_2\sqrt{\varepsilon}$)-VI $\overset{\text{Lips, bound}}{\longleftarrow}$ ε-MVI, ε-VI $\overset{\text{mono}}{\longrightarrow}$ ε-MVI)

(i) *Assuming F to be Lipschitz continuous on X with modulus L and X to be bounded, if $\bar{x}$ is a solution to (ε-MVI), then it is a solution to ($c_2\sqrt{\varepsilon}$-VI), where $c_2 \triangleq 2D_X\sqrt{L}$.*

(ii) *Assuming F to be monotone on X, if $\bar{x}$ is a solution to (ε-VI), then it is a solution to (ε-MVI).*

Proof *(i)* Letting $\lambda = 1-\sqrt{\varepsilon}/\left(D_X\sqrt{L}\right)$ with $\varepsilon > 0$, $\lambda \in (0,1)$ because we can consider, without loss of generality, $\sqrt{\varepsilon} < D_X\sqrt{L}$. The proof is now obtained by contradiction. If $\bar{x}$ is not a solution to ($c_2\sqrt{\varepsilon}$-VI), then there must exist $\tilde{x} \in X$ such that

$$-2D_X\sqrt{L}\sqrt{\varepsilon} > F(\bar{x})^T(\tilde{x}-\bar{x}) + \left[F(\lambda\bar{x}+(1-\lambda)\tilde{x}) - F(\lambda\bar{x}+(1-\lambda)\tilde{x})\right]^T(\tilde{x}-\bar{x}).$$

We have

$$\begin{aligned} F(\lambda\bar{x}+(1-\lambda)\tilde{x})^T(\tilde{x}-\bar{x}) &< -2D_X\sqrt{L}\sqrt{\varepsilon} - \left[F(\bar{x}) - F(\lambda\bar{x}+(1-\lambda)\tilde{x})\right]^T(\tilde{x}-\bar{x}) \\ &\leq -2D_X\sqrt{L}\sqrt{\varepsilon} + \|F(\bar{x}) - F(\lambda\bar{x}+(1-\lambda)\tilde{x})\|\|\tilde{x}-\bar{x}\| \\ &\leq -2D_X\sqrt{L}\sqrt{\varepsilon} + (1-\lambda)D_X^2L, \end{aligned}$$

where the last inequality is due to the convexity and boundedness of X, and the Lipschitz continuity of F. Multiplying both sides by $(1-\lambda)$, we have

$$F(\lambda\bar{x}+(1-\lambda)\widetilde{x})^T(\lambda\bar{x}+(1-\lambda)\widetilde{x}-\bar{x}) < -(1-\lambda)2D_X\sqrt{L}\sqrt{\varepsilon}+(1-\lambda)^2D_X^2L = -\varepsilon.$$

Therefore, since $(\lambda\bar{x}+(1-\lambda)\widetilde{x}) \in X$ due to the convexity of X, $\bar{x}$ is not a solution to (ε-MVI), and we have a contradiction.

(ii) By the monotonicity of F, for all $x \in X$, we have

$$F(x)^T(x-\bar{x}) \geq F(\bar{x})^T(x-\bar{x}) \geq -\varepsilon,$$

where the last inequality holds because $\bar{x}$ is a solution to (ε-VI). □

Proposition 3.3 *(ε-NEP $\xrightarrow{\text{Lips,bound,conv}}$ $(c_3\sqrt{\varepsilon})$-VI, ε-NEP $\xleftarrow{\text{conv}}$ ε-VI) Consider F given by (1).*

(i) *Assuming F to be Lipschitz continuous on X with modulus L and X bounded, if $\bar{x}$ is a solution to (ε-NEP) such that, for all ν, $\theta_\nu\left(\bullet,\bar{x}^{-\nu}\right)$ is convex on X_ν, then it is a solution to ($c_3\sqrt{\varepsilon}$-VI), where $c_3 \triangleq 2N\max_\nu(D_{X_\nu})\sqrt{L}$.*
(ii) *Assuming $\theta_\nu(\bullet,x^{-\nu})$ to be convex on X_ν for every $x^{-\nu} \in \prod_{\mu\neq\nu}X_\mu$ and for all ν, if $\bar{x}$ is a solution to (ε-VI), then it is a solution to (ε-NEP).*

Proof *(i)* The proof is obtained by contradiction. If $\bar{x}$ is not a solution to ($c_3\sqrt{\varepsilon}$-VI), where F is given by (1), there must exist $\widetilde{x} \in X$ such that

$$\sum_{\nu=1}^{N}\nabla_{x^\nu}\theta_\nu\left(\bar{x}^\nu,\bar{x}^{-\nu}\right)^T\left(\widetilde{x}^\nu-\bar{x}^\nu\right) < -2N\max_\nu\left(D_{X_\nu}\right)\sqrt{L}\sqrt{\varepsilon},$$

which yields the existence of $i \in \{1\dots N\}$ such that

$$\nabla_{x^i}\theta_i\left(\bar{x}^i,\bar{x}^{-i}\right)^T\left(\widetilde{x}^i-\bar{x}^i\right) < -\frac{2ND_{X_i}\sqrt{L}\sqrt{\varepsilon}}{N}.$$

Relying on the contrapositive of Proposition 3.2 *(i)*, $\widehat{x}^i \in X_i$ exists such that

$$\nabla_{x^i}\theta_i\left(\widehat{x}^i,\bar{x}^{-i}\right)^T\left(\widehat{x}^i-\bar{x}^i\right) < -\varepsilon,$$

and finally, by the convexity of $\theta_i\left(\bullet,\bar{x}^{-\nu}\right)$, we have

$$\varepsilon < \nabla_{x^i}\theta_i\left(\widehat{x}^i,\bar{x}^{-i}\right)^T\left(\bar{x}^i-\widehat{x}^i\right) \leq \theta_i\left(\bar{x}^i,\bar{x}^{-i}\right)-\theta_i\left(\widehat{x}^i,\bar{x}^{-i}\right).$$

Therefore, $\bar{x}$ is not a solution to (ε-NEP), and we get the absurdum.

(ii) If $\bar{x}$ is a solution to (ε-VI), for every $i \in \{1\dots N\}$ we can write

$$\left(\left[\nabla_{x^\nu}\theta_\nu(\bar{x})\right]_{\nu=1}^N\right)^T\left(\left(x^i,\bar{x}^{-i}\right)-\bar{x}\right)\geq-\varepsilon\quad\forall x^i\in X_i,$$

which is equivalent to

$$\nabla_{x^i}\theta_i\left(\bar{x}^i,\bar{x}^{-i}\right)^T\left(x^i-\bar{x}^i\right)\geq-\varepsilon\quad\forall x^i\in X_i.$$

By the convexity of player i's problem, we get

$$-\varepsilon\leq\nabla_{x^i}\theta_i\left(\bar{x}^i,\bar{x}^{-i}\right)^T\left(x^i-\bar{x}^i\right)\leq\theta_i\left(x^i,\bar{x}^{-i}\right)-\theta_i\left(\bar{x}^i,\bar{x}^{-i}\right)\quad\forall x^i\in X_i.$$

□

Remark 1 The boundedness of X is an essential requirement for (i) in Propositions 3.1 and 3.2 to hold.

Taking $X=\mathbb{R}$, the only solutions to (ε-VI) for any $\varepsilon\geq0$, are those points x such that $F(x)=0$. For any given $\varepsilon\geq0$, considering any $\bar{x}$ such that $0<F(\bar{x})\leq\varepsilon$, it is a solution to ($\varepsilon$-NM), but, for any $\varepsilon'\geq0$, it cannot be a solution to *(ε'-VI)*. Taking the cue from this example, the boundedness of X is easily seen to be essential also for (i) in Propositions 3.3.

Taking $F(x)=x$ and $X=\mathbb{R}$, and considering $\varepsilon\geq0$, $\bar{x}=\sqrt{\varepsilon}$ is a solution to (ε-MVI). However, $\bar{x}$ cannot be a solution to *(ε'-VI)* for any $\varepsilon'\geq0$, since $F(\bar{x})\neq0$.

As for the exact case, ϵ-equilibria yield inexact solutions in the corresponding MVI when the NEP boils down to a convex optimization problem (see also [Proposition 2.1 Lignola and Morgan (1994)]).

Proposition 3.4 (ε-NEP $\xrightarrow{N=1,\text{convex}}$ ε-MVI] Consider F given by (1) *and assume $N=1$ and θ_1 to be convex on X_1. If $\bar{x}$ is a solution to* (ε*-NEP*)*, then it is a solution to (ε-MVI).*

Proof The proof is given by the following chain of relations:

$$-\varepsilon\leq\theta_1(x)-\theta_1(\bar{x})\leq\nabla\theta_1(x)^T(x-\bar{x}),$$

where the last inequality is due to the convexity of θ_1. □

To link (ε-MVI) to inexact equilibria, we rely on a parametric version of the classical Mean-value Theorem: considering $x^\nu\in X_\nu$ and $\widetilde{x}\in X$, for the differentiable (parametric in x^ν and $\widetilde{x}$) real-valued function $\varphi_\nu\left(\bullet;x^\nu,\widetilde{x}\right):\mathbb{R}\to\mathbb{R}$ defined as

$$\varphi_\nu\left(\tau;x^\nu,\widetilde{x}\right)\triangleq\theta_\nu\left((1-\tau)\widetilde{x}^\nu+\tau x^\nu,\widetilde{x}^{-\nu}\right),$$

a point $\lambda_\nu\left(x^\nu,\widetilde{x}\right)\in(0,1)$, depending on x^ν and $\widetilde{x}$, exists such that

$$\varphi_\nu\left(1;x^\nu,\widetilde{x}\right)-\varphi_\nu\left(0;x^\nu,\widetilde{x}\right)=\varphi'_\nu\left(\lambda_\nu\left(x^\nu,\widetilde{x}\right);x^\nu,\widetilde{x}\right).$$

Requiring the existence of a point λ_ν satisfying the above relation that is bounded away from zero on $X^\nu \times X$, for all ν, and for every $(x^\nu, \widetilde{x}) \in X^\nu \times X$, we are able to establish a connection between solutions to (ε-MVI) and corresponding inexact equilibria.

Theorem 3.5 *For every ν, and for all $x^\nu \in X_\nu$ and $\bar{x} \in X$, assume $\lambda_\nu(x^\nu, \bar{x})$ satisfying*

$$\varphi_\nu\left(1; x^\nu, \bar{x}\right) - \varphi_\nu\left(0; x^\nu, \bar{x}\right) = \varphi'_\nu\left(\lambda_\nu\left(x^\nu, \bar{x}\right); x^\nu, \bar{x}\right) \tag{2}$$

exists such that

$$\lambda_\nu\left(x^\nu, \bar{x}\right) \in [\bar{\lambda}_\nu, 1) \quad \text{for some} \quad \bar{\lambda}_\nu > 0. \tag{3}$$

Considering F given by (1), *if $\bar{x}$ is a solution to* (ε*-MVI*), *then it is a solution to* ($\left(\varepsilon / \min_\mu \bar{\lambda}_\mu\right)$*-NEP*).

Proof If $\bar{x}$ is a solution to (ε-MVI), choosing $x = \left(x^\nu, \bar{x}^{-\nu}\right)$ for any ν, we get

$$\left(\left[\nabla_{x^\nu} \theta_\nu(x^\nu, \bar{x}^{-\nu})\right]_{\nu=1}^N\right)^T \left(\left(x^\nu, \bar{x}^{-\nu}\right) - \bar{x}\right) \geq -\varepsilon \quad \forall x^\nu \in X_\nu,$$

which is equivalent to

$$\nabla_{x^\nu} \theta_\nu\left(x^\nu, \bar{x}^{-\nu}\right)^T \left(x^\nu - \bar{x}^\nu\right) \geq -\varepsilon \quad \forall x^\nu \in X_\nu.$$

Therefore, exploiting (2), we have, for any $x_\nu \in X_\nu$,

$$\begin{aligned}
\theta_\nu\left(x^\nu, \bar{x}^{-\nu}\right) - \theta_\nu\left(\bar{x}^\nu, \bar{x}^{-\nu}\right) &= \varphi_\nu\left(1; x^\nu, \bar{x}\right) - \varphi_\nu\left(0; x^\nu, \bar{x}\right) = \varphi'_\nu\left(\lambda_\nu\left(x^\nu, \bar{x}\right); x^\nu, \bar{x}\right) \\
&= \nabla_{x^\nu} \theta_\nu\left(\left(1 - \lambda_\nu\left(x^\nu, \bar{x}\right)\right)\bar{x}^\nu + \lambda_\nu\left(x^\nu, \bar{x}\right)x^\nu, \bar{x}^{-\nu}\right)^T \left(x^\nu - \bar{x}^\nu\right) \\
&= \frac{1}{\lambda_\nu\left(x^\nu, \bar{x}\right)} \nabla_{x^\nu} \theta_\nu\left(\left(1 - \lambda_\nu\left(x^\nu, \bar{x}\right)\right)\bar{x}^\nu + \lambda_\nu\left(x^\nu, \bar{x}\right)x^\nu, \bar{x}^{-\nu}\right)^T \\
&\quad ((1 - \lambda_\nu(x^\nu, \bar{x}))\bar{x}^\nu + \lambda_\nu(x^\nu, \bar{x})x^\nu - \bar{x}^\nu) \\
&\geq -\frac{1}{\lambda_\nu(x^\nu, \bar{x})}\varepsilon \geq -\frac{1}{\min_\mu \bar{\lambda}_\mu}\varepsilon,
\end{aligned}$$

where the inequality holds because $((1 - \lambda_\nu(x^\nu, \bar{x}))\bar{x}^\nu + \lambda_\nu(x^\nu, \bar{x})x^\nu) \in X_\nu$, and $\bar{x}$ is a solution to (ε-MVI). □

Exploiting Proposition 3.6 and resorting to the intriguing results in Hiriart-Urruty (2021), we identify player-quadratic NEPs as a broad class of problems for which the sufficient condition (3) in Theorem 3.5 is guaranteed.

Proposition 3.6 ((2ε)-NEP $\overset{\text{quad.}}{\longleftarrow}$ ε-MVI) *Consider F given by* (1) *and assume, for all ν, $\theta_\nu(\bullet, x^{-\nu})$ to be quadratic on X_ν for every $x^{-\nu} \in \prod_{\mu \neq \nu} X_\mu$. If $\bar{x} \in X$ is a solution to* (ε*-MVI*), *then it is a solution to* ((2ε)*-NEP*).

Proof For all ν we can consider, without loss of generality,

$$\theta_\nu(x^\nu, x^{-\nu}) = \frac{1}{2} x^{\nu T} A^\nu(x^{-\nu}) x^\nu + b^\nu(x^{-\nu})^T x^\nu,$$

where, for every $x^{-\nu}$, $A^\nu(x^{-\nu})$ is a symmetric $(n_\nu \times n_\nu)$-dimensional matrix, $b(x^{-\nu}) \in \mathbb{R}^{n_\nu}$. In this case, $\varphi(\tau; x^\nu, \bar{x})$ becomes

$$\varphi_\nu\left(\tau; x^\nu, \bar{x}\right) = \frac{1}{2} \alpha^\nu\left(\bar{x}, x^\nu\right)\tau^2 + \beta^\nu\left(\bar{x}, x^\nu\right)\tau + \gamma(\bar{x}),$$

with

$$\begin{aligned}
\alpha^\nu\left(\bar{x}, x^\nu\right) &= \bar{x}^{\nu T} A^\nu\left(\bar{x}^{-\nu}\right)\bar{x}^\nu - 2\bar{x}^{\nu T} A^\nu\left(\bar{x}^{-\nu}\right) x^\nu + x^{\nu T} A^\nu\left(\bar{x}^{-\nu}\right) x^\nu),\\
\beta^\nu\left(\bar{x}, x^\nu\right) &= -\bar{x}^{\nu T} A^\nu\left(\bar{x}^{-\nu}\right)\bar{x}^\nu + \bar{x}^{\nu T} A^\nu\left(\bar{x}^{-\nu}\right) x^\nu + b^\nu\left(\bar{x}^{-\nu}\right)^T\left(x^\nu - \bar{x}^\nu\right)),\\
\gamma^\nu(\bar{x}) &= \frac{1}{2} \bar{x}^{\nu T} A^\nu\left(\bar{x}^{-\nu}\right)\bar{x} + b^\nu\left(\bar{x}^{-\nu}\right)^T \bar{x}.
\end{aligned}$$

Therefore, observing that, whenever $\alpha^\nu\left(\bar{x}, x^\nu\right) = 0$, relation (2) is trivially satisfied with $\lambda_\nu\left(x^\nu, \bar{x}\right) = 1/2$, and, if this is not the case, one can leverage the results about the Mean-value Theorem in the quadratic case in Hiriart-Urruty (2021), Theorem 3.5 holds with $\min_\mu \overline{\lambda}_\mu = 1/2$, and the claim follows. □

A first connection between inexact MVIs and NMs can be obtained through VIs by combining Propositions 3.2 and 3.1, that leads to a degree of inexactness with a magnitude of order $\sqrt[4]{\varepsilon}$. However, a direct relation holds under a stronger Lipschitz assumption, but the boundedness of X is no longer required. Moreover, it guarantees a better degree of inexactness with a magnitude of order $\sqrt{\varepsilon}$.

Proposition 3.7 (($c_4\sqrt{\varepsilon}$)-NM $\overset{L<1}{\longleftarrow}$ ε-MVI) *Assume F to be Lipschitz continuous on X with modulus $L < 1$. If $\bar{x}$ is a solution to (ε-MVI), then it is a solution to (($c_4\sqrt{\varepsilon}$)-NM), with $c_4 \triangleq 1/\sqrt{1-L}$.*

Proof Let $z = P_X(\bar{x} - F(\bar{x})) \in X$. If $\bar{x}$ is a solution to (ε-MVI), we have

$$\begin{aligned}
\varepsilon &\geq F(z)^T(\bar{x} - z)\\
&= \left[\bar{x} - z - (\bar{x} - z - F(z))\right]^T(\bar{x} - z)\\
&= \|\bar{x} - z\|^2 - \left[\bar{x} - z - F(z) - F(\bar{x}) + F(\bar{x})\right]^T(\bar{x} - z)\\
&= \|\bar{x} - z\|^2 - (\bar{x} - F(\bar{x}) - z)^T(\bar{x} - z) - (F(\bar{x}) - F(z))^T(\bar{x} - z)\\
&\geq (1 - L)\|\bar{x} - z\|^2 = (1 - L)\|\bar{x} - P_X(\bar{x} - F(\bar{x}))\|^2.
\end{aligned}$$

Therefore, $\bar{x}$ is a solution to (($c_4\sqrt{\varepsilon}$)-NM). □

The scheme in Fig. 2, which mirrors the one in Fig. 1 about exact relations, subsumes the previous results by depicting how the magnitude of inexactness

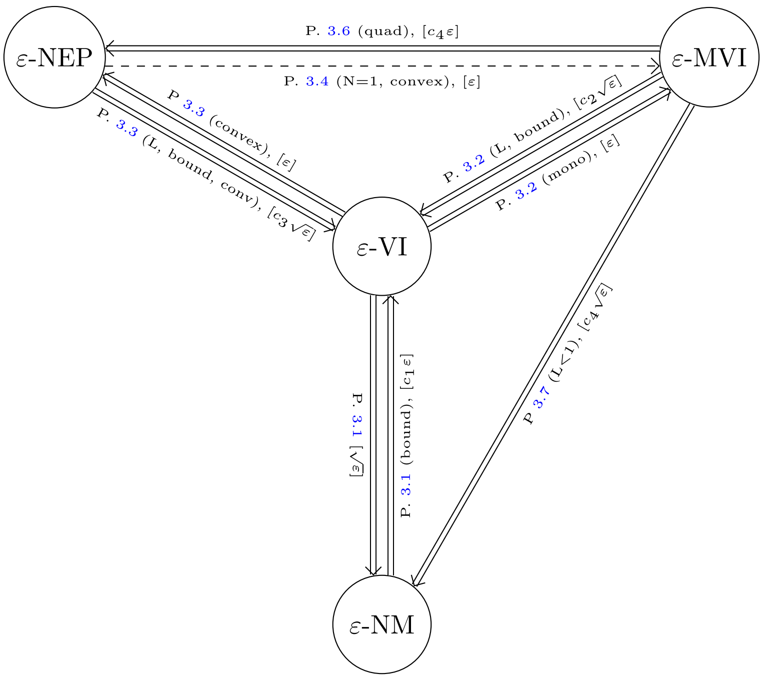

Fig. 2 Inexact relations scheme

propagates from one problem to another. Roughly speaking, the estimates of the amount of propagation are optimal. Indeed, the inexactness bounds provided in the propositions of this section are sharp, meaning that there exist instances where the bounds cannot be improved. These instances are provided and described below.

1. $[\sqrt{\varepsilon}\text{-NM} \longleftarrow \varepsilon\text{-VI}]$ Consider the problems addressed in Proposition 3.1 *(ii)*, with $n = 2$ and

$$F(x) = (1 + x_1,\ \sqrt{\varepsilon} + x_2)^T \quad X = [0, 1] \times [-\sqrt{\varepsilon}, 0].$$

   Then $\bar{x} = (0, 0)$ is a solution to ε-VI, and $\|\bar{x} - P_X(\bar{x} - F(\bar{x}))\| = \sqrt{\varepsilon}$ so that $\bar{x}$ is a solution to $\sqrt{\varepsilon}$-NM, but not to ε'-NM for any $\varepsilon' < \sqrt{\varepsilon}$.
2. $[(c_2\sqrt{\varepsilon})\text{-VI} \xleftarrow{\text{Lips, bound}} \varepsilon\text{-MVI}]$ Consider the problems addressed in Proposition 3.2 *(i)* with $n = 1$ and

$$F(x) = x \quad X = [0, 2\sqrt{\varepsilon}].$$

   Then $c_2 = 4\sqrt{\varepsilon}$ and $\bar{x} = 2\sqrt{\varepsilon}$ is a solution to ε-MVI, furthermore,

$$\min_{x \in X} F(\bar{x})^T(x - \bar{x}) = -4\varepsilon = -c_2\sqrt{\varepsilon}$$

so that $\bar{x}$ is a solution to $(c_2\sqrt{\varepsilon})$-VI, but not to ε'-VI for any $\varepsilon' < c_2\sqrt{\varepsilon}$.

3. [ε-VI $\xrightarrow{\text{mono}}$ ε-MVI] Consider the problems addressed in Proposition 3.2 *(ii)*, with a constant operator F. Then, the two problems coincide and consequently the degree of inexactness is necessarily the same.
4. [ε-NEP $\xrightarrow{\text{Lips,bound,conv}}$ $(c_3\sqrt{\varepsilon})$-VI] Consider the problems addressed in Proposition 3.3 *(i)*, with $N = n = 1$ and

$$\theta(x) = \frac{1}{2}x^2 \quad X = [\sqrt{2\varepsilon}, 2\sqrt{\varepsilon}].$$

Then $F(x) = \nabla\theta(x) = x$, $c_3 = 2(2 - \sqrt{2})\sqrt{\varepsilon}$ and $\bar{x} = 2\sqrt{\varepsilon}$ is a solution to $(c_3\sqrt{\varepsilon})$-VI, furthermore,

$$\min_{x \in X} F(\bar{x})^T(x - \bar{x}) = 2(\sqrt{2} - 2)\sqrt{\varepsilon}\sqrt{\varepsilon} = -c_3\sqrt{\varepsilon}$$

so that $\bar{x}$ is a solution to $(c_3\sqrt{\varepsilon})$-VI, but not to ε'-VI for any $\varepsilon' < c_3\sqrt{\varepsilon}$.

5. [ε-NEP $\xleftarrow{\text{conv}}$ ε-VI] Consider the problems addressed in Proposition 3.3 (ii), with $N = 2, n_1 = n_2 = 1$ and

$$\theta_1(x) = \theta_2(x) = x^1x^2 \quad X_1 = X_2 = [-\sqrt{\varepsilon}, 1].$$

Then $F(x) = \nabla_{x^1}\theta_1(x) \times \nabla_{x^2}\theta_2(x) = (x^2, x^1)^T$, $X = [-\sqrt{\varepsilon}, 1]^2$ and $\bar{x} = (\sqrt{\varepsilon}, 0)$ is a solution to ε-VI, furthermore

$$\theta_1(\sqrt{\varepsilon}, 0) - \min_{x^1 \in X_1} \theta_1(x^1, 0) = 0 \le \varepsilon$$
$$\theta_2(\sqrt{\varepsilon}, 0) - \min_{x^2 \in X_2} \theta_2(\sqrt{\varepsilon}, x^2) = \varepsilon$$

so that $\bar{x}$ is a solution to ε-NEP, but not to ε'-NEP for any $\varepsilon' < \varepsilon$.

6. [ε-NEP $\xrightarrow{\text{N=1,convex}}$ ε-MVI] Consider the problems addressed in Proposition 3.4, with a linear operator θ. Then, the two problems coincide and consequently the degree of inexactness is necessarily the same.
7. [(2ε)-NEP $\xleftarrow{\text{quad.}}$ ε-MVI] Consider the problems addressed in Proposition 3.6. The bound is sharp for any $\bar{x} \in X$: $\min_{x \in X} F(x)^T(x - \bar{x}) = -\varepsilon$, since in this case the proof would require no inequality and therefore no approximation of the inexactness.
8. [$(c_4\sqrt{\varepsilon})$-NM $\xleftarrow{\text{L<1}}$ ε-MVI] Consider the problems addressed in Proposition 3.7, with $n = 1$ and

$$F(x) = \frac{1}{2}x \quad X = [0, 1].$$

Then $c_4 = \sqrt{2}$ and $\bar{x} = 2\sqrt{2\varepsilon}$ is a solution to ε-MVI if $\varepsilon < 1/8$, furthermore $\|\bar{x} - P_X(\bar{x} - F(\bar{x}))\| = \sqrt{2}\sqrt{\varepsilon} = c_4\sqrt{\varepsilon}$ so that $\bar{x}$ is a solution to $c_4\sqrt{\varepsilon}$-NM but not to ε'-NM for any $\varepsilon' < c_4\sqrt{\varepsilon}$.

On the contrary, the inexactness bound in Proposition 3.1 *(i)* seems not to be sharp because it would require the distance between a pair of specific points (namely, $P_X(\bar{x} - F(\bar{x}))$ and the minimizer of $F(\bar{x})^T(x - \bar{x})$ over X) to be exactly the diameter D_X. Anyway, sharpness is missing by an arbitrarily small amount, or it can be recovered by exploiting a solution-dependent constant, as shown by the two following examples.

- Consider $n = 1$,

  $$F(x) = \varepsilon \quad X = [-a, 1].$$

  Then $c_1 = 1 + a + \varepsilon$ and $\bar{x} = 1$ is a solution to ε-NM, furthermore

  $$\min_{x \in X} F(\bar{x})^T(x - \bar{x}) = -(c_1 - \varepsilon)\varepsilon,$$

  while the inexactness bound in Proposition 3.1 is $c_1\varepsilon$. Nonetheless, the ratio $(c_1 - \varepsilon)/c_1$ goes to 1 as $a \to \infty$ or $\varepsilon \to 0$.
- Consider $n = 1$,

  $$F(x) = 1 \quad X = [0, 1].$$

  Then $\bar{x} = \varepsilon$ is a solution to ε-NM, furthermore

  $$\min_{x \in X} F(\bar{x})^T(x - \bar{x}) = -\varepsilon = -\bar{c}_1\varepsilon,$$

  where

  $$\bar{c}_1 = \min\{\|y - (P_X(\bar{x} - F(\bar{x})))\| \,:\, y \in \arg\min_{x \in X} F(\bar{x})^T(x - \bar{x})\} + \|F(\bar{x})\|$$

  depends upon the solution $\bar{x}$. This way, $\bar{x}$ is a solution to $(\bar{c}_1\varepsilon)$-VI but not to ε'-VI for any $\varepsilon' < \bar{c}_1\varepsilon$.

4 Conclusions

We investigated the relations between VIs, MVIs, NMs and NEPs, in both the exact and inexact cases. This study is relevant as inexactness arises naturally when dealing with stopping criteria for numerical procedures and when complexity results are available. In particular, the connections between inexact NEPs and variational problems allow quantifying the quality of the computed approximate equilibria, which is especially relevant in applications. The connections from MVIs to NEPs proved to be the most challenging: restricting to player-quadratic objectives (which are anyhow quite common in concrete applications), one can get a direct implication (see

Fig. 2) that also allows preserving the order of magnitude of the approximation. Overall, the results that we presented show that inexactness behaves somehow well: all the relations for the exact case are maintained in the inexact case, by sometimes paying the price of some additional assumptions; we have been able to quantify how inexactness propagates from a problem to the other ones, and the limit behavior, as inexactness vanishes, turns out to correspond to the exact case.

As regularization is a common technique for variational problems, a similar analysis could be conducted for regularized VIs and MVIs. Other future developments may include the extension of this analysis to generalized NEPs and quasi-variational problems, as well as nonsmooth settings that are likely to call for corresponding generalized variational problems.

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