



# Non-tradability interval for heterogeneous rational players in the option markets

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## Abstract

This study uses theoretical and empirical approaches to analyze a number of phenomena observed in trading floors, such as the changes in trading volumes and Bid–Ask spreads as a function of the moneyness level and the remaining time until the option’s expiration. A mathematical model for pricing options is developed that assumes two players with heterogeneous beliefs, where the objective of each player is to maximize their profit on the expiration day. By solving a system of algebraic equations, which takes into consideration the subjective beliefs of the players regarding the price of the underlying asset on expiration day, a feasible price domain is constructed that defines the boundaries within which a transaction may be executed. The developed model is applied to the special case in which the distribution of the underlying asset price on expiration day is uniform, and a sensitivity analysis for selected parameters is presented. An interesting theoretical result that emerges from the proposed model is the existence of an interval under which there is no tradability near the expiration day. The existence of this interval offers an explanation for the decrease in the apparent trading volumes of out-of-money (OTM) options, together with an increase in Bid–Ask spreads, as the expiration day approaches. The main parameters that affect the point of time after which there will be no trading are those that represent the players’ subjective beliefs about the distribution of the expiration values, and the cost of trading.

**Keywords** Option pricing · Non-tradability interval · Heterogeneous players · Bid–Ask spreads

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# 1 Introduction

## 1.1 Motivation and related literature

Empirical evidence shows a decrease in the trading volumes of out-the-money (OTM) options, together with an increase in Bid–Ask spreads, as expiration day approaches. Research literature discussing the link between actual option prices and the associated trading volumes is very limited (Muravyev 2016). Trading volumes are an indicator of market liquidity in security markets and have an important influence on market players. Asset pricing could be affected by liquidity if market players demand a premium for bearing the risk of illiquidity (Chaudhury 2015). Bid–Ask spreads relative to the mid-quote price are considered to be a measure of liquidity in the options market (Chaudhury 2015).

As mentioned, considerable empirical evidence shows that Bid–Ask spreads in options markets are also linked to the remaining time until the expiration of the option and to the moneyness level. For example, Wei and Zheng (2010) demonstrated that Bid–Ask spreads increase by 31% during the month prior to expiration (in comparison to the previous month) for out-the-money options, while the corresponding increase was 66% for in-the-money (ITM) options. George and Longstaff (1993) showed that for the S&P 100 index, the Bid–Ask spreads grow with the moneyness level and the proximity to the expiration day.

Existing literature disputes the origins of Bid–Ask spreads in option prices. Muravyev (2016) claims that they arise from three sources: transaction costs; asymmetric information between investors (e.g., due to additional private information gained by some individuals), especially with regards to anticipated changes in the price of the underlying asset; and the cost of the inventory risk of the market maker (i.e., the loss created by holding inventory of non-hedged options, whereby the market maker is exposed to changes in the option price). Most studies have focused on the last two factors. In this paper, we mainly examine the first two factors, both empirically and analytically. With regards to the third factor, i.e., the inventory risk, existing literature that has addressed this issue mainly dealt with market makers, who hedge against the inventory risk, while in this article we examine players who maximize their expected profit.

### 1.1.1 The effect of the remaining time until expiration day on trading volumes and on Bid–Ask spreads

Many studies, such as Hollifield et al. (2017) and Ajina et al. (2015), have shown that there is a negative relationship between trading volumes and the size of the Bid–Ask spreads. Hsieh and Jarrow (2017) examined the “maturity effect”, which refers to the increase in Bid–Ask implied-volatility spreads at an increasing rate as the option’s expiration day approaches. They showed that although the inventory risk decreases as the option’s maturity date approaches, an increased risk premium (in terms of implied volatility) is needed to compensate the market makers (who hold an inventory of options, and are obligated, according to the stock exchange’s

regulation, to hold specific minimal amounts of options and to quote Bid and Ask prices) for bearing the unhedgeable risk. This increased risk premium for both Bid and Ask prices results in widening of the option's Bid–Ask implied volatility spread as the maturity date approaches. Hsieh and Jarrow (2017) also examined the “maturity effect” empirically, for options on the S&P 500 index in 2001–2017, and documented the fact that the implied volatility Bid–Ask spread increases at an increasing rate as the option's expiration day approaches. Similar results were obtained for other specific stocks in the period 2004–2012 (Christoffersen et al. 2017).

### 1.1.2 The impact of informed traders on trading volumes and on Bid–Ask spreads

The analysis of informed traders in options markets has been the subject of numerous theoretical and empirical studies. Informed traders use the extraneous information to seek statistical arbitrage opportunities, while at the same time accommodating the additional risk (Brody et al. 2008). The asymmetry of information among traders, and the preference for particular trading venues, can cause option transactions to convey information to market participants about imminent changes in stock and option prices (Easley et al. 1998). Pan and Poteshman (2006) showed that informed traders prefer to trade through stock options, and not through stocks directly, since the cost of the option is smaller than that of the stock, and also because if the option expires, its value will be zero; that is, there will be no loss beyond the option price. Traders who have private information prefer to buy cheaper options, especially at-the-money (ATM) options. According to Pan and Poteshman's findings, the effect of private information is mainly associated with options of specific stocks, rather than with options of stock indices. Chatrath et al. (2015) carried out an empirical comparison between monthly and weekly options on the S&P 500 index between the years 2011 and 2012. They found that the impact of private information was greater for weekly options than for monthly options, since weekly options have greater tradability, and informed traders will prefer to purchase options with greater tradability. Bernales et al. (2018) showed evidence of a liquidity searching behavior of informed investors in option listings. This behavior was also observed by Collin-Dufresne and Fos (2015) using private trading stock markets data. Nevertheless, unlike these authors, Bernales et al. (2018) argued that the option's Bid–Ask spread is a good proxy for informed trading, despite the aforementioned liquidity searching behavior. Furthermore, Bernales et al. (2018) showed an upward trend in the option's Bid–Ask spread after option introductions (as informed traders avoid trading in earlier periods after listing dates due to the low liquidity environment)—a trend that was steeper for options with a higher chance of information asymmetries.

Additional empirical evidence of the impact of private information on trading volume exists for other trading floors. Lin et al. (2018) found, using data from the TXO index in Taiwan between the years 2009 and 2012, that there is an impact of private information on institutional players. They showed that trading mainly occurs in options around the money, for which the tradability is highest. In another empirical study, Chung and Ryu (2016) found that for players who do not have private information, when the trading volume is too high or too low, it is best to avoid

trading ITM and OTM options, in order to reduce the advantage of players with private information.

The previous paragraphs described two phenomena—the effect of the remaining time to expiration and the effect of informed traders on the size of the Bid–Ask spreads and the trading volumes. In this article, these two phenomena are brought together under a general theory, which assumes heterogeneous beliefs of the two players.

## 1.2 Contribution of this article

Existing models explain the changes in daily trading volumes, as well as the existence of Bid–Ask spreads, through a particular market failure, in the sense that one of the players has the advantage of private information over the other players, or, alternatively, that a market maker is obligated to quote the Bid and Ask prices, and thus also to affect the scope of the option's tradability (by increasing or decreasing the Bid–Ask spreads) mainly before expiration day. The proposed model offers a different approach, based on heterogeneous beliefs of players regarding the distribution of the underlying asset's price on expiration day. The model generates a theoretical explanation for the formation of the Bid–Ask gap and the decline in the trading volumes of (mainly) in-the-money options before expiration day.

In addition to the development of a real-time option pricing model, the current study makes a number of contributions to the existing literature:

- By means of the developed model, we demonstrate the existence of a single point of time (at least for the special case in which the underlying asset is uniformly distributed) that defines a time interval up to the expiration day during which there will be no trade between two rational players. We suggest that (at least for the uniform distribution), as the remaining time to expiration decreases, the likelihood of a non-tradability interval increases. This theoretical result, which is derived from the model, is compatible with the evidence from real trading systems, in the sense that there is no trading very close to the expiration day, since at least one of the players estimates that the option's execution is not worthwhile.
- The Ask and Bid prices observed in real trading systems are given significance in the theoretical framework of the developed model. Specifically, the Ask and Bid prices reflect the non-existence of a feasible price domain in which a transaction may be executed; thus, players enter a state of waiting for a change in external conditions or for a change initiated by the players themselves.
- We show that when the goal is to maximize the expected profit, the decision of each player regarding buying or selling an option does not depend on the portfolio exposed to the index prior to making the transaction. This decreases the amount of information required about the opposite player in order to determine option prices.

The remainder of this article is as follows. In the second section, we present the feasible domain of prices, and define its boundaries. In the third section, we

present the impact of the heterogeneous beliefs of the players, as well as the impact of transaction costs, on the trading dynamics. Section 4 examines a special case of the model, whereby the price of the underlying asset is uniformly distributed as of the expiration day. In the fifth section, the existence of a non-tradability interval is shown both theoretically and empirically, still for the special case where the distribution of the price of the underlying asset at expiration day is uniform. Section 6 provides a summary and draws conclusions.

## 2 The feasible price domain

In this section, according to the suggested model, and assuming that the players' subjective estimates of the distribution of the underlying asset price at expiration day are given, the boundaries of the feasible domain for the price of an option are derived. This corresponds to the range in which a transaction between the two parties is possible, assuming that both players are rational. As mentioned above, the proposed model assumes that each player's goal is to maximize the expected profit on expiration day. In this study we focus specifically on European options. We consider a single European *Call option* on the index  $s_T$  (the value of the underlying asset at expiration day  $T$  is  $s \in s_T$ ), as has also been considered in many other analyses, among them those of Black and Scholes (1973) and Merton (1973). We define the point of time  $t$  as the moment at which the two players estimate the value of the index at expiration day  $T$ , according to their subjective view, and accordingly,  $\Delta t = T - t$  is the length of time until the option expires. We also define  $x$ , the exercise price, as is customary in the trading of stock options, for the purchase of the option at the examined time  $t$ . The (deterministic) future profit function at expiration day  $T$  for the buyer (without the transaction cost) is:

$$c_t^x(s) = \begin{cases} -c_t^x(1+r)^{\Delta t} & s \leq x \\ -c_t^x(1+r)^{\Delta t} + (s-x) & s > x \end{cases} \tag{1.1}$$

where  $r$  denotes annual risk-free interest rate.

According to the definition of options trading, for the sale of the option at an examined time  $t$ , the future profit function at expiration day  $T$  (without the transaction cost) for the seller is:

$$c_t^x(s) = \begin{cases} c_t^x(1+r)^{\Delta t} & s \leq x \\ c_t^x(1+r)^{\Delta t} - (s-x) & s > x \end{cases} \tag{1.2}$$

By the very definition of an option's transaction (purchase and sale), assuming such a transaction is executed, the price  $c_t^x$  is also the price of the sale. The meaning of exercising the option according to (1) at expiration day is a payment  $s - x$  from the option's seller to the buyer, when the price of the underlying asset at expiration day is higher than the exercise price of the option, i.e.,  $s > x$  (otherwise there is no payment). The profit, in general, is not deterministic due to the fact the underlying asset price at expiration is random. Therefore, as computed later in (4), we consider

this volatility by taking the expectation (the distribution of the underlying asset price on expiration day).

### 2.1 Description of the feasible price domain

A transaction between two players means that one of the players sells the option, and the other player buys the same option at the same price. From the assumption of rationality, it follows that not every price can be a price under which a transaction is possible; there are prices under which the expected profit at expiration day will decrease due to executing the action, for one or for both players. On the other hand, even if the price is such that the expected profit for both players increases, this still does not guarantee the execution of the transaction. A possible reason for this phenomenon is that one or both players may subjectively assess that a potential better offer from the opposing player can be obtained by waiting (Miao and Wang 2011). Therefore, at time  $t$  for each given option with exercise price  $x$ , and for each pair of players A and B, there is, in principle, a possible price domain  $D_t^x$  (which may be empty) in which a transaction may occur. This domain is the set of all option prices  $c_t^x \in D_t^x$  that allow a transaction between the two players, where each price would subjectively improve the expected profit of both players.

In order to describe the feasible price domain at time  $t$ , we define  $c_{A,t}^x$  as the possible buying (or selling) price of an option with exercise price  $x$  from player A's perspective. Accordingly, player A has a maximum buying price,  $c_{\max,A,t}^x$ , and a minimum selling price,  $c_{\min,A,t}^x$ . Similarly, player B has a maximum buying price  $c_{\max,B,t}^x$  and a minimum selling price  $c_{\min,B,t}^x$  (see Fig. 1 below). The feasible price domain,

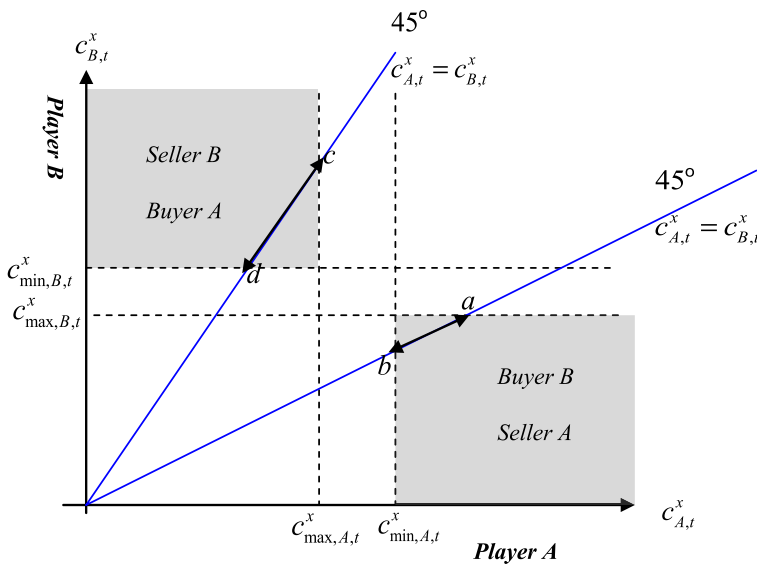


Fig. 1 Feasible domain for an option's price at time  $t$  with exercise price of  $x$

at any observation time  $t$  for each option given an exercise price of  $x$ , is displayed schematically in Fig. 1.

Figure 1 shows a two-dimensional view of both possible transaction domains (i.e., where each player takes on a different role—either buyer or seller). Each of the two domains is represented by a diagonal line that starts at the origin and has a slope of 45 degrees. This is because any transaction between the two players implies a single agreed price  $c_t^x$  (that is,  $c_{A,t}^x = c_{B,t}^x \equiv c_t^x$ ), thus only lines at an angle of 45 degrees are possible. The feasible domain is the collection of points that are located along the lines that cross one of the rectangles and are bounded by it. In any given transaction, only one of the two rectangles can be intersected. Two diagonal lines are drawn, since only one of the two surfaces (rectangles) can be intersected. In the case where player A is the buyer and B is the seller, the feasible domain is

$$c_{\min,B,t}^x \leq c_t^x \leq c_{\max,A,t}^x \tag{2}$$

If domain defined in (2) does not exist when player B is the seller and player A is the buyer, then the Bid–Ask range of prices comes into operation, implying that a transaction will not take place at time  $t$ , since the minimum selling price from player B’s perspective is higher than the maximum price at which player A is willing to buy. Similarly, for the existence of a transaction in which player A sells and player B buys, the feasible domain is

$$c_{\min,A,t}^x \leq c_t^x \leq c_{\max,B,t}^x \tag{3}$$

Part of the proposed methodology involves finding the boundaries of the feasible domains described in Fig. 1, i.e., the parameters  $c_{\max,A,t}^x$ ,  $c_{\min,A,t}^x$ ,  $c_{\max,B,t}^x$ ,  $c_{\min,B,t}^x$  as detailed below. We note that even feasible points do not necessarily indicate a transaction (e.g., when players wait for a better offer from a counter player.) Points that are not included in the feasible domain indicate (in the case where the players are rational) that a transaction between the players will not take place.

## 2.2 The expiration day as a basis for evaluating pricing strategies

In this section, we present a proposition that, together with a major result presented in the following Sect. (2.3), allows simplification of the analysis of the model to find an option price in real time. This proposition indicates that transactions at any time  $t$  can be deferred and described, without losing generality, on the expiration day  $T$ . This proposition enables the postponement of any option clearing action (and any resulting change in cash) until the expiration day, thereby significantly simplifying the analysis. Player A has an external portfolio  $O_{A,t}^{out}(s)$ , which is exposed to the value of the index  $s$ , and an options portfolio  $O_{A,t}^{op}(s)$ , which includes  $n$  options.

**Proposition 1** *If  $r = 0$  then, buying (positive sign) an option with exercise price  $x$  at time  $t < T$ , whose sign is the opposite of an option (purchased at time  $t_1 < t$ ) that is in portfolio  $O_{i,t_1}^{op}(s)$  (i.e., zeroing out the option while yielding positive or negative*

cash), where  $t_1 < t < T$ , is identical to adding the zeroing option with inverse sign to the portfolio  $O_{i,t}^{op}(s)$  at time  $t$  and keeping it until the expiration day.

If  $r = 0$  then, selling (negative sign) an option with exercise price  $x$  at time  $t < T$ , whose sign is the opposite of an option (sold at time  $t_1 < t$ ) that was in portfolio  $O_{i,t_1}^{op}(s)$  (i.e., zeroing out the option while yielding positive or negative cash), where  $t_1 < t < T$ , is identical to adding the zeroing option with inverse sign to the portfolio  $O_{i,t}^{op}(s)$  at time  $t$  and keeping it until the expiration day.

**Proof** See “Appendix A” section. □

### 2.3 Boundaries of the feasible domain

Assuming rationality of the two players, the boundaries of the feasible domain require that any sale/purchase action not subjectively reduce the expected profit of each player in comparison to the alternative of non-execution of the action. The boundaries of the feasible price domain are obtained based on this requirement. The total value of player A’s portfolio before the transaction is  $O_{A,t}(s)$ . This player considers, at time-point  $t$ , buying (or selling) an option with an exercise price  $x$ , which would yield a (deterministic) profit on expiration day of  $c_t^x(s)$ . Adding the transaction cost  $k$ , the total value of player A’s portfolio on expiration day is  $O_{A,t}(s) + c_t^x(s) - k(1 + r)^{\Delta t}$ . We denote the distribution of the underlying asset price on expiration day from player A’s perspective by  $f_{A,t}(s)$ . From the viewpoint of player A, a necessary condition for adding the option to the overall portfolio (as stated, by buying or selling it) is to improve the expected profit on the expiration day  $U_A(s)$ ,

$$U_A(O_{A,t}(s) + c_t^x(s) - k(1 + r)^{\Delta t}, f_{A,t}(s)) \geq U_A(O_{A,t}(s), f_{A,t}(s)), \tag{4}$$

where

$$U_A(O_{A,t}(s), f_{A,t}(s)) = \int_0^\infty O_{A,t}(s) \cdot f_{A,t}(s) ds.$$

Given a specific option with an expected profit at expiration  $c_t^x(s)$ , a player can use (4) to determine a threshold condition for buying or selling the option at time  $t$ . Even if the specific option is not tradable at time  $t$  (i.e., not available for sale or purchase by an opposing player, B), player A will be able to describe its maximum purchase price and its minimum sale price schematically (parametrically). When (4) is satisfied as an equality for buying an option [according to (1.1)], we obtain  $c_{\max,A,t}^x$ , and when (4) is satisfied as an equality for selling an option [according to (1.2)], we obtain  $c_{\min,A,t}^x$ . Satisfying an equation that is equivalent to (4) as an equality yields  $c_{\min,B,t}^x$  and  $c_{\max,B,t}^x$ .



respectively, where the corresponding expected profit from adding the option to the portfolio is  $U_B\left(O_{B,t}(s) + c_t^x(s) - k(1+r)^{\Delta t}, f_{B,s}(s)\right)$ . Accordingly, we obtain the feasible domain  $c_{\min,A,t}^x \leq c_t^x \leq c_{\max,B,t}^x$  for executing a transaction in which player A is the seller and player B is the buyer, and the feasible domain  $c_{\min,B,t}^x \leq c_t^x \leq c_{\max,A,t}^x$  for executing a transaction in which player A is the buyer and player B is the seller.

The market conditions in real time and the heterogeneous characteristics of the players (which are represented by their subjective assessments of the distribution of the index at expiration day), determine the identities of the buyer and the seller, and accordingly specify the feasible domain. In this article, we assume that each player subjectively maximizes her own expected profit at expiration day, as opposed to many other studies that assume the goal of the players is to hedge against the risk of changes in the index affecting the existing portfolio that is exposed to the index. Our model indirectly considers risk premium (i.e., the gap between invest in risk free assets and buying or selling an option). In particular, risk premium is heterogeneous in our model. The following general conclusion simplifies the mathematical analysis of the problem of finding the boundaries of the feasible domain.

**Theorem 1** *When the objective function is to maximize the expected profit, each player’s decision about whether to execute a transaction at all, does not depend on the portfolio exposed to the index before the transaction is made.*

**Proof** When the objective function is to maximize profit expectancy, then inequality (4) can also be written in the following manner:

$$\begin{aligned}
 U_A\left(O_{A,t}(s) + c_t^x(s) - k(1+r)^{\Delta t}, f_{A,s}(s)\right) &\geq U_A\left(O_{A,t}(s), f_{A,t}(s)\right) \\
 U_A\left(O_{A,t}(s) + c_t^x(s), f_{A,s}(s)\right) - k(1+r)^{\Delta t} &\geq U_A\left(O_{A,t}(s), f_{A,t}(s)\right) \\
 U_A\left(O_{A,t}(s), f_{A,s}(s)\right) + U_A\left(c_t^x(s), f_{A,s}(s)\right) - k(1+r)^{\Delta t} &\geq U_A\left(O_{A,t}(s), f_{A,t}(s)\right) \\
 U_A\left(c_t^x(s), f_{A,s}(s)\right) - k(1+r)^{\Delta t} &\geq 0
 \end{aligned} \tag{5}$$

Since the expected profit is additive with respect to its components, we obtain the result that the portfolio  $O_{A,t}(s)$  does not appear in the left-hand side of the last inequality; that is, the decision of each player regarding whether to execute a transaction at all is independent of the portfolio exposed to the index before the transaction is performed. □

A necessary (but not sufficient) condition for a transaction between two rational players is the existence of a feasible domain. In accordance with condition (5), which is a simpler version of condition (4), it is possible to calculate, in general, the boundaries of the feasible domain for the buyer and the seller of the option. Inequality (5), in relation to buyer  $i$ , according to (1.1), can be expressed as:

$$\begin{aligned}
 & \int_0^x -c_t^x(1+r)^{\Delta t} \cdot f_{i,t}(s)ds + \int_x^\infty (-c_t^x(1+r)^{\Delta t} + (s-x)) \cdot f_{i,t}(s)ds - k(1+r)^{\Delta t} \geq 0 \\
 & -c_t^x(1+r)^{\Delta t} + \int_x^\infty (s-x)f_{i,t}(s)ds \geq k(1+r)^{\Delta t} \\
 c_{\max,i,t}^x &= \frac{1}{(1+r)^{\Delta t}} \left[ \int_x^\infty (s-x)f_{i,t}(s)ds \right] - k
 \end{aligned} \tag{6}$$

Inequality (5), in relation to seller  $i$ , according to (1.2), becomes

$$\begin{aligned}
 & c_t^x(1+r)^{\Delta t} - \int_x^\infty (s-x)f_{i,t}(s)ds \geq k(1+r)^{\Delta t} \\
 c_{\min,i,t}^x &= \frac{1}{(1+r)^{\Delta t}} \left[ \int_x^\infty (s-x)f_{i,t}(s)ds \right] + k
 \end{aligned} \tag{7}$$

The boundaries of the feasible domain,  $c_{\min,A,t}^x, c_{\max,A,t}^x$  for player A and  $c_{\min,B,t}^x, c_{\max,B,t}^x$  for player B, constitute points at which the corresponding player has a minimal subjective expected profit, whether she is the buyer or the seller. Formulas (6) and (7) emphasize the importance of the consideration of the transaction cost in the model, since only in the special case where there is no transaction cost ( $k = 0$ ) are  $c_{\max,A,t}^x$  and  $c_{\min,A,t}^x$  identical, and at this price, both players are indifferent in regard to the action of buying or selling.

Each of the players is interested in executing a transaction (assuming that one occurs) at the price positioned at the opposite end of the feasible interval. Since a transaction requires the consent of both parties, at this stage there is no guaranteed price at which a transaction will be executed. An immediate transaction is only possible if the players take the initiative to execute one in the light of (a) their knowledge of the feasible domain and (b) their subjective assessment of the opponent’s eagerness to compromise. Alternatively, the feasible domain will be reduced, resulting in a lower likelihood of a future transaction (since the domain of the total possibilities for executing a transaction would be smaller).

### 3 The impact of heterogeneous beliefs and of the transaction cost on tradability

In this section, we present several theoretical results that reflect the effects of these parameters on tradability.

### 3.1 The impact of heterogeneous beliefs on tradability

The following existence and uniqueness theorem further specifies the necessary condition for a transaction between the two players, by taking into consideration the heterogeneity of the players with regards to their estimates of the distribution of the underlying asset price at expiration day.

**Theorem 2** *Let us define the condition*

$$\int_x^\infty s(f_{B,t}(s) - f_{A,t}(s)) ds \geq 2k(1 + r)^{\Delta t} + x[F_{A,t}(x) - F_{B,t}(x)] \tag{8}$$

where  $F_{i,t}(s)$  is the distribution function of the value of the index at expiration day for player  $i$  as estimated at time  $t$ . If condition (8) is satisfied, then

*There is a non-empty feasible domain  $c_t^x \in [c_{\min,A,t}^x, c_{\max,B,t}^x]$  that is a necessary condition for executing a transaction, in the case where player A is the seller and player B is the buyer.*

*The feasible domain  $c_t^x \in [c_{\min,A,t}^x, c_{\max,B,t}^x]$  is unique.*

**Proof** See “Appendix B” section. □

From Theorem 2, we obtain an interesting conclusion whereby, when player A is the seller and player B is the buyer, a necessary condition for executing a transaction is that player B is more “optimistic” than player A regarding a transaction of a Call option. We define the concept of “optimism” as a player’s relative estimation that the value of the index at expiration day is higher than the exercise price,  $x$ , which characterizes the option. This conclusion can be obtained from observation of both sides of inequality (8). The left-hand side expresses the difference in the partial expectations associated with players B and A, respectively, for the higher index values (greater than  $x$ ). The right-hand side (even without the transaction cost) expresses the difference in the partial expectations associated with player B and player A for the lower index values (smaller than  $x$ ). For a Put option, the belief of each player regarding the distribution of the index value at expiration day does not change; however the payoff received for the option will change, since for a Put option, the buyer will receive the difference between the exercise price and the price of the underlying asset at expiration day, if the price of the underlying asset will be lower than the exercise price. The condition for executing a transaction will be that player B (the Put option’s buyer) will be more “optimistic” than player A (the option’s seller). The resulting conclusion is that both for a Call option and for a Put option, the condition for the existence of a non-empty feasible domain implies that the option’s buyer will be more “optimistic” than the option’s seller. From Theorem 2 we also conclude that for every positive value of the transaction cost, where the two players are homogeneous in their beliefs regarding the value of the index at expiration day (have the same

distribution), a Put or Call purchase or sale will not be executed—a conclusion that also receives support from the study of Kang and Luo (2016).

### 3.2 The impact of the transaction cost on tradability

In this subsection, two commonsense propositions that are derived from the rationality of the two players and that are related to the impact of the transaction cost, are presented.

The significance of Proposition 2 is that for any option  $x$ , offered at time  $t$  (in a market of only two players), associated with profit  $c_t^x(s)$  to the buyer on expiration day, only one of the following two transactions is possible: a transaction in which player A buys the option and player B sells it, or alternatively, a transaction in which player A sells the option and player B buys it. Graphically (Fig. 1), a potential transaction can only belong to one of the two gray rectangular price surfaces with which the (blue) diagonal line intersects.

**Proposition 2** *For any player  $i = A, B$ , and for any point in time  $t$ ,*

$$c_{\min,i,t}^x - c_{\max,i,t}^x = 2k.$$

**Proof** From expressions (6) and (7), for each player, respectively, we obtain:

$$c_{\min,i,t}^x - c_{\max,i,t}^x = \frac{1}{(1+r)\Delta t} \left[ \int_x^\infty (s-x)f_{i,t}(s)ds \right] + k - \left\{ \frac{1}{(1+r)\Delta t} \left[ \int_x^\infty (s-x)f_{i,t}(s)ds \right] - k \right\}.$$

□

From Proposition 2, we conclude that any rational player is willing, as a buyer, to pay no more than he is willing to receive as a seller. In the situation in which the transaction costs are insignificant, and assuming neither side’s insistence on their identity as a seller or as a buyer in order to achieve the goal of increasing expected profit, it is more likely than the case of transaction costs are not insignificant, according to Proposition 2, that the parties will execute a transaction (higher tradability), since there is a higher likelihood that the 45° line in Fig. 1 will intersect with one of the two possible surfaces. The obvious conclusion is that the transaction cost has an effect on the level of tradability, an effect we illustrate in Sect. 5. The practical meaning of the following proposition is to justify the removal of those options whose prices are too low (lower than the transaction cost) from the calculation, when measuring tradability in the option markets.

**Proposition 3** *A rational player  $i$  will not sell (or buy) an option  $x$  if its price  $c_t^x$  is lower than the transaction cost,  $k$ .*

**Proof** By contradiction, let us assume that player  $i$ , who sells a given option  $x$  at a price  $c_t^x < k$ , has an overall portfolio that yields a profit  $O_{i,t}(s) = O_{i,t}^{op}(s) + O_{i,t}^{out}(s)$

on the expiration day. After the sale, the overall portfolio's value is changed to  $O_{i,t}(s) = O_{i,t}^{op}(s) + O_{i,t}^{out}(s) + c_t^x(s) - k(1+r)^{\Delta t}$ , where the price  $c_t^x(s)$  is obtained by (1.2). Following the assumption,  $c_t^x(s) < k(1+r)^{\Delta t}, \forall s$ , the overall profit value of the portfolio,  $O_{i,t}(s)$ , after the sale is carried out, is less than the transaction cost for every  $s \in S$ . Therefore, the value of the portfolio at expiration day after the sale is carried out is lower, irrespective of the distribution  $f_{i,t}(s)$ . Thus, the value of the expected profit is lower, which is in contradiction to the assumption of rationality. Since no rational player would consent to sell at price  $c_t^x$ , which is lower than the transaction cost  $k$ , a buy transaction is not executed.  $\square$

### 4 The feasible domain for the case in which the distribution of the underlying asset is uniform

In this section, a special case of the proposed model is presented, in which the price of the underlying asset has a uniform distribution within a domain that is subjectively defined by each player. In the existing research literature, a uniform distribution reflects the greatest amount of entropy (Buchen and Kelly 1996) when compared to other distributions of the underlying asset price, since in a uniform distribution there is no preference for any one particular index value over any other within the boundaries of the available data.

#### 4.1 Description of the distribution of the underlying asset price on the expiration day

For the purpose of implementing the proposed model, we begin by finding the feasible domain for executing a transaction between two rational players, when the value of the index at expiration day,  $S_T$ , which is used by each of the two players to evaluate the portfolio's value, is distributed continuously and uniformly. The pdf of the index value at expiration day  $t = T$ , evaluated at time-point  $t$  from the viewpoint of player  $i$ , is described by:

$$f_{i,t}(s) = \begin{cases} \frac{1}{s_t(1+2u_{i,t})-s_t(1-2d_{i,t})} & s \in [s_t(1 + 2u_{i,t}), s_t(1 - 2d_{i,t})] \\ 0 & otherwise \end{cases} \tag{9}$$

where player  $i$  assumes that on the expiration day, the index values will be between  $s_t(1 - 2d_{i,t})$  and  $s_t(1 + 2u_{i,t})$ . Figure 2 shows a numerical example of the density function  $f_{i,t}(s)$  for a single set of parameters:  $s_t = 100$  and  $\Delta t = 1$ , where  $\Delta t$  is the remaining time until expiration day,  $\Delta t = T - t$ . We define, without losing generality, and similarly to existing models assuming a binomial distribution (e.g., Kim et al. 2016),  $u_{i,t} = \sigma_i \sqrt{\Delta t}$  and  $d_{i,t} = 1 - \frac{1}{1+u_{i,t}}$ , where  $\sigma_i$  is the standard deviation (for the purposes of the illustration,  $\sigma_i$  is assigned the value 0.1). This parameter is independent of time and it reflects the dispersion of the index values, subjectively from the viewpoint of player  $i$ . Inserting factor  $\Delta t$  into the distribution formula of the index values allows each player  $i$  to create a subjective distribution image of the

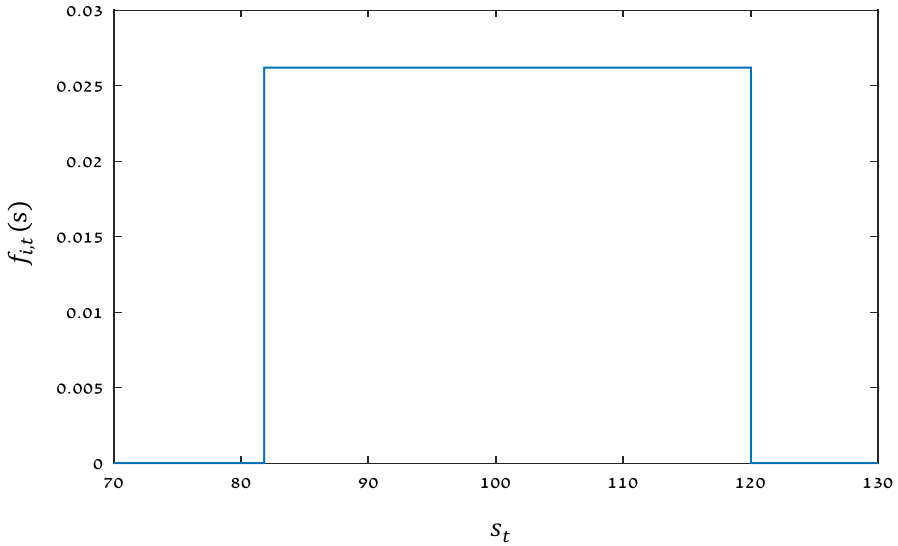


Fig. 2 Density function of  $S_T$  for the case of a uniform distribution ( $s_t = 100, \Delta t = 1$ )

index value to the expiration day, which becomes concentrated (eventually into a single point) around the current index value, as the remaining time until the expiration day becomes shorter. According to the set of points in this example, at the current time  $t$ ,  $d_{i,t} = 0.0910$  and  $u_{i,t} = 0.1$  (see Fig. 2).

### 4.2 Finding a feasible price interval

Since the value of the index  $s_T$  has an upper limit  $s_t(1 + 2u_{i,t})$ , we obtain, in accordance with (6) for player  $i$  as the option buyer,

$$\begin{aligned}
 c_{\max,i,t}^x &= \frac{1}{(1+r)^{\Delta t}} \left[ \int_x^{s_t(1+2u_{i,t})} (s-x)f_{i,t}(s)ds \right] - k \\
 c_{\max,i,t}^x &= -k + \frac{1}{(1+r)^{\Delta t}} \begin{cases} 0 & s_t \leq \frac{x}{1+2u_{i,t}} \\ \int_x^{s_t(1+2u_{i,t})} (s-x)f_{i,t}(s)ds & \frac{x}{1+2u_{i,t}} < s_t \leq \frac{x}{1-2d_{i,t}} \\ \int_{s_t(1-2d_{i,t})}^x sf_{i,t}(s)ds - x & \frac{x}{1-2d_{i,t}} < s_t \end{cases} \quad (10)
 \end{aligned}$$

After integration

$$c_{\max,i,t}^x = -k + \frac{1}{(1+r)^{\Delta t}} \begin{cases} 0 & s_t \leq \frac{x}{1+2u_{it}} \\ \frac{[x-s_i(1+2u_{it})]^2}{2[s_i(1+2u_{it})-s_i(1-2d_{it})]} & \frac{x}{1+2u_{it}} < s_t \leq \frac{x}{1-2d_{it}} \\ \frac{[s_i(1+2u_{it})]^2 - [s_i(1-2d_{it})]^2}{2[s_i(1+2u_{it})-s_i(1-2d_{it})]} - x & \frac{x}{1-2d_{it}} < s_t \end{cases} \quad (11)$$

where  $u_{i,t} = \sigma_i \sqrt{\Delta t}$  and  $d_{i,t} = 1 - \frac{1}{1+u_{i,t}}$ ,  $d_{i,t} = 1 - \frac{1}{1+u_{i,t}}$  and  $\Delta t = T - t$ . In the same way, according to Proposition 2, for player  $i$  selling the option, we get

$$c_{\min,i,t}^x = 2k + c_{\max,i,t}^x, \quad \forall x. \quad (12)$$

By (11) and (12), and an additional identical pair of equations for player B, a possible range  $c_t^x \in [c_{\min,A,t}^x, c_{\max,B,t}^x]$  is obtained for the case where player A is the seller and player B is the buyer, and  $c_t^x \in [c_{\min,B,t}^x, c_{\max,A,t}^x]$  for the case where player A is the buyer and player B is the seller, where  $c_t^x$  is the price of the Call option with exercise price  $x$  traded at time  $t$ . In these intervals, transactions that match the interests of both players may be realized, and hence the players' identities are determined, as at most a single transaction in which player A buys and player B sells, or vice versa, is possible. From (11) for player A as the option seller and from (12) for player B as the option buyer, we conclude that there are five possible price intervals in the case where player A sells the option and player B buys it:  $s_t \leq \frac{x}{(1+2u_{B,t})}, \frac{x}{(1+2u_{B,t})} < s_t \leq \frac{x}{(1+2u_{A,t})}, \frac{x}{(1+2u_{A,t})} < s_t \leq \frac{x}{(1-2d_{A,t})}, \frac{x}{(1-2d_{A,t})} < s_t \leq \frac{x}{(1-2d_{B,t})}$ , and  $\frac{x}{(1-2d_{B,t})} < s_t$ . These domains are obtained by combining the three domains in expression (11) for player A as the option seller with the three domains in expression (12) for player B as the option buyer. Not all of the possible price intervals are feasible. For example, in the first interval,  $s_t \leq \frac{x}{(1+2u_{B,t})}$ , the feasible trading interval is negative because  $c_{\max,B,t}^x - c_{\min,A,t}^x = -2k$ , and therefore no feasible interval exists.

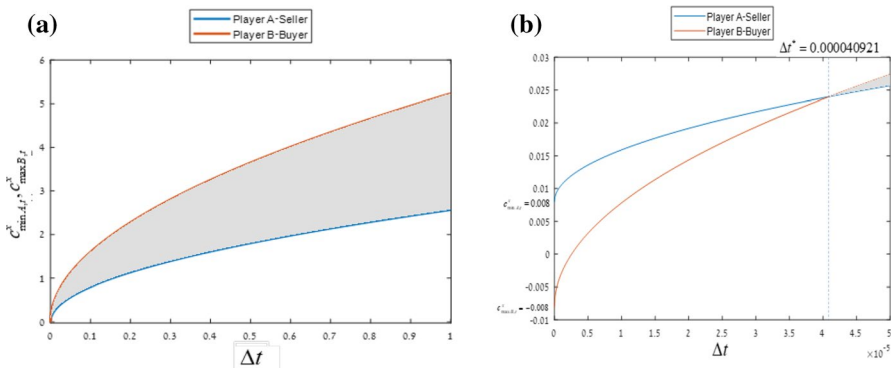
### 5 The non-tradability interval

In this section, we use the special case of the suggested model, in which the price of the underlying asset on the expiration day is distributed uniformly, to show the existence of a non-tradability interval before the expiration day. Finding a non-tradability interval is important, because if such an interval exists, it is not advisable for a group of rational players seeking to increase their expected profit to operate in the trading floor from the beginning of that interval until the expiration day. First, we will present a numerical example that illustrates the reduction of the feasible domain in proximity to the expiration day, and then we will discuss, numerically and analytically, the existence of a single time-point at which trade between two rational players will cease.

### 5.1 A numerical illustration

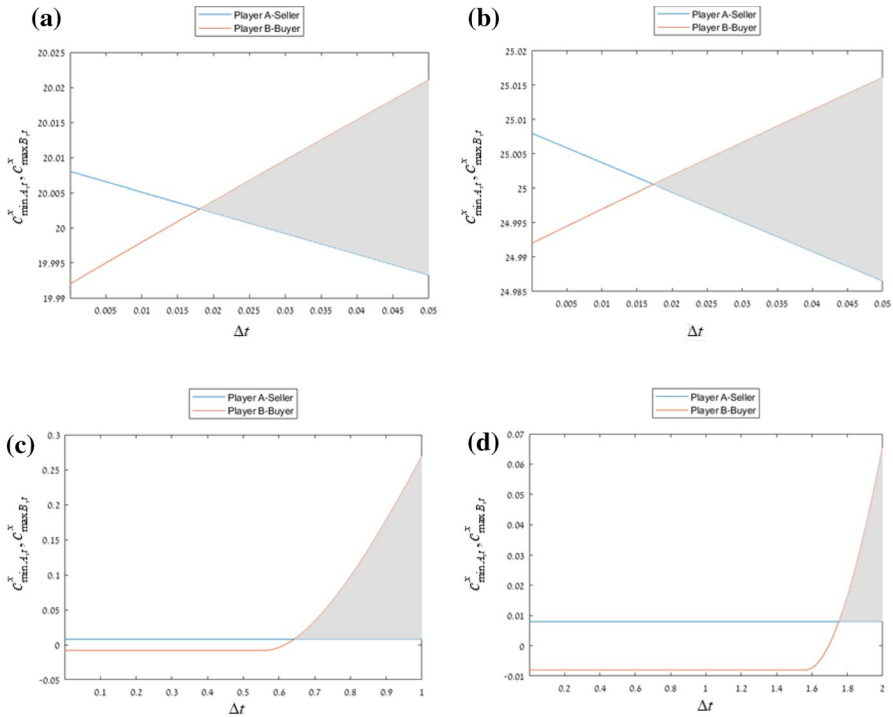
In Fig. 3a we present the physical feasible domain as a function of time (in years) until the option expires for the parameters:  $k = 0.008$ ,  $r = 0.03$ ,  $\sigma_A = 0.05$ ,  $\sigma_B = 0.1$ ,  $x = 100$  and  $s_t = 100$ , where player A sells the option and player B buys it. According to (11) and (12), these data correspond to the second domain in (11), namely the domain  $\frac{x}{(1+2u_{A,t})} < s_t \leq \frac{x}{(1-2d_{A,t})}$ . Figure 3b shows exactly the same data and relationship as Fig. 3a but zoomed in. We see from Fig. 3a that as the expiration day of this option approaches, the feasible domain is reduced. A possible explanation is the increase in the certainty of the value of the index at expiration day, since  $u_{i,t} = \sigma_i \sqrt{\Delta t}$  and  $d_{i,t} = 1 - \frac{1}{1+u_{i,t}}$ . Figure 3a also shows that player B's (selling) price declines faster than player A's (buying) price. A possible explanation relates to the fact that  $\sigma_B > \sigma_A$ . Setting  $\sigma_A = 0.05$  in (12) and  $\sigma_B = 0.1$  in (11), and then differentiating with respect to  $\Delta t$ , yields this outcome. This process continues up to a specific point of time,  $\Delta t^* = 0.00004092085$ , which is the turning point before which  $C_{\max B,t}^x = C_{\min A,t}^x$  is satisfied, while immediately afterwards, the inequality  $C_{\max B,t}^x < C_{\min A,t}^x$  holds, i.e., an immediate trade between the two rational players is no longer possible (see Fig. 3b, which focuses on time-points close to this turning point, in close proximity to the expiration day). Thus, in this example, trading between players operating according to the suggested model stops about 53 min prior to expiration. The existence of the turning point defines the “non-tradability interval”. This phenomenon, under which the feasible domain is reduced as the option's expiration day approaches, also occurs for the other four domains (see Fig. 4).

Figure 4 displays the feasible domains as a function of the period of time (in years) until the expiration of the option for the remaining intervals introduced in Sect. 4.2. In particular, as in Fig. 3b, we focus only on the turning point, after which



**Fig. 3** **a** The feasible domain (gray area)  $\left[ C_{\min A,t}^x, C_{\max B,t}^x \right]$  as a function of the remaining time until the expiration of the option (in years), for the interval  $\frac{x}{(1+2u_{A,t})} < s_t \leq \frac{x}{(1-2d_{A,t})}$ . **b** A zoomed in view of part of the data in **a** with the x-axis dramatically magnified to illustrate the time point after which the feasible domain is empty, for the interval  $\frac{x}{(1+2u_{A,t})} < s_t \leq \frac{x}{(1-2d_{A,t})}$





**Fig. 4** **a** The non-tradability point (in years) for the interval  $\frac{x}{(1-2d_{A,t})} < s_t \leq \frac{x}{(1-2d_{B,t})}$ . **b** The non-tradability point (in years) for the interval  $\frac{x}{(1-2d_{B,t})} < s_t$ . **c** The non-tradability point (in years) for the interval  $\frac{x}{(1+2u_{B,t})} < s_t \leq \frac{x}{(1+2u_{A,t})}$ . **d** The non-tradability point (in years) for the interval  $s_t \leq \frac{x}{(1+2u_{B,t})}$

there is no tradability for the remaining time along the interval. Two significant differences exist between the two domains shown in Fig. 4a, b and those shown in Fig. 4c, d. Firstly, for the intervals  $\frac{x}{(1-2d_{A,t})} < s_t \leq \frac{x}{(1-2d_{B,t})}$  and  $\frac{x}{(1-2d_{B,t})} < s_t$ , in which there is a high likelihood of exercising the option, the effect of time on the feasible domain is linear, while for the intervals  $\frac{x}{(1+2u_{B,t})} < s_t \leq \frac{x}{(1+2u_{A,t})}$  and  $s_t \leq \frac{x}{(1+2u_{B,t})}$ , in which the likelihood of exercising an option is low, the effect of time on the feasible domain is non-linear, as well as being slower. The second difference relates to the length of the non-tradability interval. For the intervals  $\frac{x}{(1-2d_{A,t})} < s_t \leq \frac{x}{(1-2d_{B,t})}$  and  $\frac{x}{(1-2d_{B,t})} < s_t$ ,  $\Delta t^* = 0.018$  (about 6.5 days) and  $\Delta t^* = 0.0173$  (about 6.3 days) respectively. In contrast, for the intervals  $\frac{x}{(1+2u_{B,t})} < s_t \leq \frac{x}{(1+2u_{A,t})}$  and  $s_t \leq \frac{x}{(1+2u_{B,t})}$ , the length of the non-tradability interval is  $\Delta t^* = 0.642$  (about 234 days) and  $\Delta t^* = 1.752$  (about 640 days) respectively. In the case of  $s_t \leq \frac{x}{(1+2u_{B,t})}$ , the non-tradability interval starts very early, which may be explained by the level of certainty with which the players estimate the likelihood of exercising the option at expiration day. The low probability of exercising the option, as in Fig. 4c, d, will lead buyers who are maximizing their expected profit to prefer buying the underlying asset rather than the option (and similarly for sellers). Within the non-tradability interval, every offer that

a player makes to their opponent will be “unattractive”, and thus no transaction will be executed in the case where both players are rational. The fact that the feasible domain ceases to exist from a time-point that is nearly one and a half years before the expiration day stems from the fact that the buyer estimates that there is a possibility, although negligible, that within the remaining (long) time until expiration, the option will be exercised (i.e., its value will be positive at expiration day). Therefore, the buyer will offer a price reflecting this likelihood, i.e., a price that is slightly higher than the transaction cost. As the option’s expiration day approaches, the buyer estimates that the likelihood of exercising is so small that, from his perspective, the value of exercising the option is lower than the transaction cost. Beyond this point of time, the feasible domain is empty, as the buyer would only offer a negative price,  $-k$ . It is worth noting that in real trading floors, options are traded within the money, even in the days approaching the option’s expiration, probably within the theoretical non-tradability interval. Possible reasons for observing such trade are due to considerations that are different from the theoretical considerations described above, such as hedging options and irrational behavior among some players.

In summary, for in-the-money (ITM) options, as is illustrated in Fig. 3, the closer the option’s expiration day, the faster the rate at which the feasible domain is reduced. For out-the-money (OTM) options, as is illustrated in Fig. 4c, d, the closer the option’s expiration day, the slower the rate at which the feasible domain is reduced. For options at-the-money (ATM), as is illustrated in Fig. 4a, b, the non-tradability interval is very narrow, and the feasible domain reduces at a constant rate with the remaining time until expiration. The reason for the differences between the sizes of the non-tradability intervals presented in Fig. 4 is related to the certainty of exercising the option from each player’s viewpoint. In the following subsection, we proceed to a more general discussion that generalizes this example.

## 5.2 The effect of key parameters on the non-tradability interval

While, in Sect. 5.1, we presented a numerical example that illustrated the existence of a non-tradability point, the following proposition demonstrates its existence more generally.

**Theorem 3** *Given an existing feasibility domain and for the special case that the distribution of the price of the underlying asset on the expiration day is uniform, there is a single point in time,  $\Delta t^*$ , the “non-tradability point”, which satisfies  $c_{\max B,t}^x(\Delta t^*) = c_{\min A,t}^x(\Delta t^*)$ , and immediately after which the inequality  $c_{\max B,t}^x < c_{\min A,t}^x$  holds, a condition which does not allow a transaction between the players.*

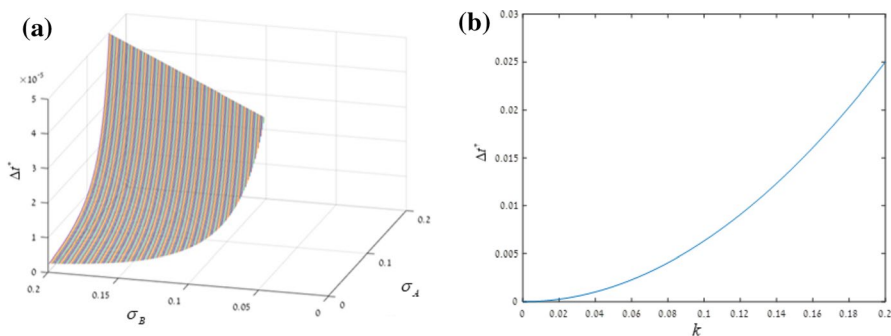
**Proof** Assume, without losing generality, a current point in time  $t$ , for which  $\Delta t$  is sufficiently large such that the domain  $c_t^x \in [c_{\min B,t}^x, c_{\max A,t}^x]$  is not empty. Therefore condition (8) is satisfied for  $t = T - \Delta t$ ; that is

$$\int_x^\infty s(f_{B,t}(s) - f_{A,t}(s)) ds \geq 2k(1+r)^{\Delta t} + x[F_{A,t}(x) - F_{B,t}(x)].$$

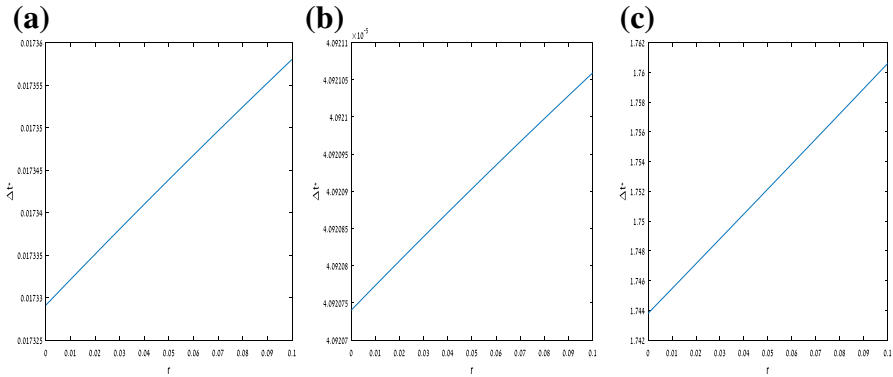
The distribution function  $f_{i,t}(s)$ , which is uniform [see (9)] by definition, includes  $u_{i,t} = \sigma_i \sqrt{\Delta t}$  and also  $d_{i,t} = 1 - \frac{1}{1+u_{i,t}}$ ; thus for  $\Delta t \rightarrow 0$ , it is concentrated with mass 1 at the point of the value of the underlying asset  $s_t$ . Therefore, there are two possibilities,  $s_t < x$  and  $s_t \geq x$ , each of which yields  $0 \geq 2k(1+r)^{\Delta t}$  for inequality (8); that is, the inequality is not satisfied. Therefore, (8) as an equality must have at least one solution. If (8) as an equality has more than one solution,  $\Delta t_n^*$ ,  $n = 1, 2, \dots, V$ , the smallest solution among them is defined as the single “non-tradability point”; that is,  $\Delta t^* = \arg \min_{n=1,2,\dots,V} \Delta t_n^*$ . □

The non-tradability point  $\Delta t^*$  also defines the non-tradability interval,  $\psi^* \equiv [T - \Delta t^*, T]$ . From (11) and (12) we can conclude that the main parameters that have an impact on the point in time after which trade stops are the estimations of the players regarding the dispersion of the expiration values,  $\sigma_A$  and  $\sigma_B$ , and the transaction cost  $k$ . Figure 5 presents a sensitivity analysis for these parameters. Figure 5a illustrates that the higher each player’s estimate of the dispersion of the expiration values, and the more similar it is to that of the opposing player, the earlier the non-tradability interval starts.

From Fig. 5a we notice, for the given input data, that when the standard deviation associated with player A is large and the standard deviation of player B is small, the non-tradability interval occupies virtually the entire timeline before the option expires. In this case, no transactions are carried out because the price offered by the buyer will be lower than that of the seller, and accordingly, by (11) and (12), condition (8) is not satisfied at any time until the expiration day. According to Fig. 5a, the non-tradability interval grows rapidly as the standard deviations increase. For example, for  $x = s_t = 100, k = 0.008, \sigma_A = 0.049995, \sigma_B = 0.05$ , the non-tradability



**Fig. 5** **a** The non-tradability point  $\Delta t^*$  (z axis) for various dispersion values of the price of the underlying asset at expiration day from the viewpoint of the two players (x and y axes) when  $x = s_t = 100, k = 0.008$ . **b** The non-tradability point  $\Delta t^*$  for various values of the transaction cost



**Fig. 6** **a** The non-tradability point  $\Delta t^*$  for various values of the interest rate levels for the interval  $\frac{x}{(1-2d_{B,t})} < s_t$ . **b** The non-tradability point  $\Delta t^*$  for various values of the interest rate levels for the interval  $\frac{x}{(1+2u_{A,t})} < s_t \leq \frac{x}{(1-2d_{A,t})}$ . **c** The non-tradability point  $\Delta t^*$  for various values of the interest rate levels for the interval  $s_t \leq \frac{x}{(1+2u_{B,t})}$

interval is  $\Delta t^* = 1000$ . Figure 5b shows that the greater the transaction cost, the earlier the non-tradability interval starts.

It should be noted that for the world’s largest stock exchanges, it is impossible to execute a transaction on the expiration day, since the trading in options ends 17 h before expiration. The numerical result and the subsequent theoretical one obtained by the proposed model appear to reflect this aspect of trading systems in the sense that there should not be any trade very close to the expiration day. However, if trading systems were to technically enable trading during the non-tradability interval, the suggested model’s players could utilize this circumstance to their advantage when trading against non-rational players who continue to make purchases or sales, while the player using this model improves their expected profits.

Figure 6 presents the non-tradability point for different values of the annual risk-free interest rate in three different moneyness level. Although a nearly linear increase of the non-tradability interval with the interest rate is observed, the absolute value of the non-tradability barely changes.

## 6 Conclusions

### 6.1 Summary and conclusions

This article provides theoretical analysis of a number of empirical phenomena that are observed in trading floors, such as trading volumes and Bid–Ask spreads that vary depending on the proximity to the money and the time remaining until the options expire. Through the solution of a system of algebraic equations, and in accordance with the players’ subjective beliefs regarding the price of the underlying asset on the expiration day, we developed a feasible price domain that defines the

boundaries of the interval for executing a transaction. Each point within the domain constitutes a possible price of the option that improves each player's expected profit on the expiration day. Through the developed model, we have proven (at least for the uniform distribution) the existence of a single point of time near the expiration day after which there will be no trade between two rational players.

The theoretical framework of the developed model lends significance to the Ask and Bid prices observed in real trading systems. Their existence means the non-existence of a feasible domain; thus the players wait for a change in external conditions, without executing a transaction. Since real trading systems have a positive, and sometimes not insignificant, transaction cost, a description of the feasible domain reflects reality better than the standard accepted description in numerous studies of a single representative price. The result, a somewhat surprising finding, is that the "non-tradability interval", during which it is not worthwhile for rational players to trade with the market, is not negligible and may even span many months. From a numerical analysis of the model, it can be concluded that (a) the larger each player's estimate of the dispersion of values of the exercise price, the more similar their dispersion estimate will be to that of the opposing player, (b) the higher the transaction cost, the longer the non-tradability interval approaching the expiration day, and (c) the greater the interest rate, the longer the non-tradability interval approaching the expiration day.

## 6.2 Advantages and theoretical/practical implications of the model

The uniqueness of the core model is expressed in its generality; it does not rely on a specific distribution. To the best of our knowledge, a theoretical explanation for non-tradability conditions prior to the expiration day does not exist in the research literature. The existence of this interval, whose theoretical and numerical basis is obtained from the model, seems to correctly reflect the dynamics of real trading systems, in the sense that there is no economic incentive for rational players to trade very close to the expiration day. If trading systems were to enable trading near the expiration date, traders operating according to the suggested model could take advantage of this situation when facing non-rational players who buy or sell in a way that does not improve their expected profit at expiration. The actual existence of trading within the non-tradability interval indicates the possibility that some of the players operate in a non-rational manner, or that their goal is not to maximize expected profit.

This work suggests a multitude future research directions on the subject of determining options prices. One of the main results of this study, the existence of a "non-tradability point" near the expiration date, was obtained based on the assumption that the price of the underlying asset at expiration day is uniformly distributed. The generalization of this result to other distributions would strengthen this conclusion. In this paper, we assumed profit maximization players. Different risk preferences (e.g., hedgers) would affect the purely additive utility function. A future research

on the effect of different risk preferences on option price might explain non-rational transactions made within the non-tradability interval.

### Appendix A. Proof of Proposition 1

We distinguish between two cases:

1. In the first case, the player is interested in removing an option that was sold at time  $t_1 < t$  at price  $c_{A,t_1}^x$  from the portfolio (assume a quantity of 1 without losing generality), thus leaving a portfolio of  $n - 1$  options. To remove the option from the portfolio at time-point  $t$ , the player must buy one option of the same type that yields a current profit on expiration day  $c_{A,t}^x(s)$  and must pay the transaction cost. The value of the portfolio at expiration day, as measured after the purchase is executed, is  $O_{A,t}^{(n+1)}(s) = O_{A,t}^{out}(s) + O_{A,t}^{op}(s) + c_{A,t}^x(s) - k$ . The options portfolio after a purchase at time-point  $t$  consists of  $n + 1$  options, where the specified option exists with a quantity of 2: the one that was sold at a price  $c_{A,t_1}^x$  at time  $t_1 < t$  and the one that was bought at a price  $c_{A,t}^x(s)$  at time-point  $t$ , as described in the expression:

$$O_{A,t}^{(n+1)}(s) = \left[ O_{A,t}^{out}(s) + O_{A,t}^{op}(s) - \left( c_{A,t_1}^x(s) - k \right) + \left( c_{A,t}^x(s) - k \right) \right] + c_{A,t}^x(s) - k$$

Since the value of the two options with the same exercise price and expiry date does not depend on the value of the index  $s$ , it is possible to determine the value of the portfolio on expiration day using the value of the  $n - 1$  other options plus the fixed (positive or negative) monetary difference  $v_1$  between the two options, i.e.,

$$O_{A,t}^{(n-1)}(s) = \left[ O_{A,t}^{out}(s) + O_{A,t}^{op}(s) + \left( c_{A,t_1}^x(s) - k \right) \right] + v_1$$

where  $v_1 \equiv c_{A,t}^x(s) - c_{A,t_1}^x(s)$  is positive or negative monetary difference. The expression in square brackets describes the portfolio that includes the  $n - 1$  options without the two specified options (the one that was previously sold at  $t_1 < t$  and the one that is now bought at time point  $t$ ).

2. In the second case, the player is interested in removing an option that was bought at time  $t_2 < t$  at price  $c_{A,t_2}^x$  from the portfolio (with a quantity of 1), leaving a portfolio of  $n - 1$  options. The player wants to remove a specific option bought at a price  $c_{A,t_2}^x$  at time  $t_2 < t$  from the portfolio at current time-point  $t$ . In this case, the player must sell one option of the same type with a current profit at expiration day  $c_{A,t}^x(s)$  and must pay the transaction cost. The value of the portfolio on expiration day, as measured at time-point  $t$  after the sale is executed, is  $O_{A,t}^{(n+1)}(s) = O_{A,t}^{out}(s) + O_{A,t}^{op}(s) + c_{A,t}^x(s) - k$ . The option's portfolio actually has  $n + 1$  options, where the same option exists with a quantity of 2: the one bought at a

price  $c_{A,t_2}^x$  at time  $t_2 < t$  and the one sold at a price  $c_{A,t}^x(s)$  at time-point  $t$ , as described in the expression:

$$O_{A,t}^{(n+1)}(s) = \left[ O_{A,t}^{out}(s) + O_{A,t}^{op}(s) - \left( c_{A,t_2}^x(s) - k \right) + \left( c_{A,t_2}^x(s) - k \right) \right] + c_{A,t}^x(s) - k$$

Since the value of the two options with the same exercise price and expiration day does not depend on the value of the index  $s$ , it is possible to determine the value of the portfolio on expiration day using the value of the  $n - 1$  other options plus the fixed monetary difference (positive or negative) between the two options, i.e.,

$$O_{A,t}^{(n-1)}(s) = \left[ O_{A,t}^{out}(s) + O_{A,t}^{op}(s) + \left( c_{A,t_2}^x(s) - k \right) \right] + v_2$$

where  $v_2 \equiv c_{A,t}^x(s) - c_{A,t_2}^x(s)$  is positive or negative monetary difference. The expression in square brackets describes the portfolio that includes the  $n - 1$  options without the two specified options (the one that was previously bought at  $t_2 < t$  and the one now sold). □

### Appendix B. Proof of Theorem 2

From the requirement for a transaction in which player A sells and player B buys,  $c_t^x \in [c_{\min,A,t}^x, c_{\max,B,t}^x]$  (see Fig. 1), for a Call option with an exercise price  $x$  at time  $t$  and in light of expressions (6) and (7),

$$\begin{aligned} c_{\max,B,t}^x &\geq c_{\min,A,t}^x \\ &\equiv \frac{1}{(1+r)^{\Delta t}} \left[ \int_x^\infty (s-x)f_{B,t}(s)ds \right] - k \geq \frac{1}{(1+r)^{\Delta t}} \left[ \int_x^\infty (s-x)f_{A,t}(s)ds \right] + k \\ &\int_x^\infty (s-x)f_{B,t}(s)ds - k(1+r)^{\Delta t} \geq \int_x^\infty (s-x)f_{A,t}(s)ds + k(1+r)^{\Delta t} \\ &\int_x^\infty (s-x)(f_{B,t}(s) - f_{A,t}(s))ds - 2k(1+r)^{\Delta t} \geq 0 \\ &\int_x^\infty s(f_{B,t}(s) - f_{A,t}(s))ds \geq 2k(1+r)^{\Delta t} + x[F_{A,t}(x) - F_{B,t}(x)] \end{aligned}$$

where  $F_{i,t}(s)$  is the distribution function of the value of the index at expiration day for player  $i$ . A  $\infty$  condition for executing a transaction by player B, as stated, is  $-c_t^x(1+r)^{\Delta t} + \int_x^\infty (s-x)f_{B,t}(s)ds \geq k(1+r)^{\Delta t}$ . We now show that this condition holds

for  $c_t^x = 0$ . When  $c_t^x = 0$ , the left-hand side of the inequality becomes  $\int_x^\infty (s-x)f_{B,t}(s)ds$ . Reverting to the existence condition  $c_{\max,B,t}^x \geq c_{\min,A,t}^x$ ,

$$\int_x^\infty (s-x)f_{B,t}(s)ds - k(1+r)^{\Delta t} \geq \int_x^\infty (s-x)f_{A,t}(s)ds + k(1+r)^{\Delta t}$$

$$\int_x^\infty (s-x)f_{B,t}(s)ds \geq \int_x^\infty (s-x)f_{A,t}(s)ds + 2k(1+r)^{\Delta t} > k(1+r)^{\Delta t}.$$

Since for  $-c_t^x(1+r)^{\Delta t} + \int_x^\infty (s-x)f_{B,t}(s)ds \geq k(1+r)^{\Delta t}$ , the derivative with respect to the option price  $c_t^x$  is negative (value of  $-(1+r)^{\Delta t}$ ), thus a single intersection point of the linear line with solution (6) at level  $k(1+r)^{\Delta t}$  must exist. A condition for player A to execute a transaction, as stated, is  $c_t^x(1+r)^{\Delta t} - \int_x^\infty (s-x)f_{i,t}(s)ds \geq k(1+r)^{\Delta t}$ . When  $c_t^x = 0$ , the left-hand side of the inequality is negative. Since the derivative with respect to the option price  $c_t^x$  is positive (value of  $(1+r)^{\Delta t}$ ), a single intersection point of the linear line with solution (7) at level  $k(1+r)^{\Delta t}$  must exist.  $\square$

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