

Pricing and hedging GMWB in the Heston and in the Black–Scholes with stochastic interest rate models

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Abstract In this paper, we approach the problem of valuing a particular type of variable annuity called GMWB when advanced stochastic models are considered. As remarked by Yang and Dai (Insur Math Econ 52(2):231–242, 2013), and Dai et al. (Insur Math Econ 64:364–379, 2015), the Black–Scholes framework seems to be inappropriate for such a long maturity products. Also Chen et al. (Insur Math Econ 43(1):165–173, 2008) show that the price of GMWB variable annuities is very sensitive to the interest rate and the volatility parameters. We propose here to use a stochastic volatility model (the Heston model) and a Black–Scholes model with stochastic interest rate (the Black–Scholes Hull–White model). For this purpose, we consider four numerical methods: a hybrid tree-finite difference method, a hybrid tree-Monte Carlo method, an ADI finite difference scheme and a Standard Monte Carlo method. These approaches are employed to determine the no-arbitrage fee for a popular version of the GMWB contract and to calculate the Greeks used in hedging. Both constant withdrawal and dynamic withdrawal strategies are considered. Numerical results are presented, which demonstrate the sensitivity of the no-arbitrage fee to economic and contractual assumptions as well as the different features of the proposed numerical methods.

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1 Introduction

Variable annuities are investment contracts with insurance coverage. In recent years, they have become a source of attraction for many investors, because of their specific features: they are tax-deferral products able to guarantee a minimum return for a long period and to take advantage of favorable market movements. As observed by Horneff et al. (2015), this kind of investments have been one of the most rapidly growing financial products over the past few decades: in 2013, the variable annuities sold in the US market totaled more than \$140 billions, while the assets invested in variable annuities contracts by the US investors were more than double than the assets invested in fixed annuities.

The majority of the models in the literature present the analysis of these products in the Black–Scholes model, usually disregarding possible changes in the interest rate and the volatility of the underlying, which drives the value of the policy. Some recent works begin to address the problem of pricing in more comprehensive models.

In this article, we consider a Guaranteed Minimum Withdrawal Benefit (GMWB) annuity which is one of the most popular contracts for both practitioners and academics. We restrict our attention to a simplified form of a GMWB which is initiated by making a lump sum payment to an insurance company. This lump sum is then invested in risky assets, usually a mutual fund. The benefit base, or guarantee account balance, is initially set to the amount of the premium payed. The holder of the policy (hereinafter, the *PH*) is entitled to withdraw a fixed amount at some dates specified by the contract, even if the actual investment in the risky asset declines to zero. The *PH* may withdraw more than the guaranteed amount, including complete surrender of the contract, upon payment of a penalty. In most cases, this penalty for full or partial surrender declines to zero after 5–7 years. During contract execution, a death benefit may come with the *PH*'s death: in this case, her heirs receive the remaining amount in the risky asset account.

The hedging costs for this guarantee are offset by deducting a proportional fee from the risky asset account. From an insurance point of view, these products are treated as financial ones: they are hedged as if they were pure financial and the mortality risk is hedged using the law of large numbers [see Lin et al. (2016) and Bernard and Kwak (2016) for a description of move-based and semi-static hedging of variable annuities].

Moreover these products have long maturities that could last almost 25 years, and thus the Black–Scholes model, which assumes the interest rate and the volatility to be constant, seems to be unsuitable. In order to shed light on this matter, we present the considered pricing methods in two stochastic models which provide stochastic volatility (Heston 1993) and stochastic interest rate (Hull–White model 1994) respectively.

There have been several recent articles on pricing GMWBs. Some of them focus on the mathematical properties of optimal withdrawals. Chen and Forsyth (2008) use an impulse stochastic control formulation to price GMWB, assuming the *PH* to

optimally withdraw money from the account value continuously or only at anniversaries. Chen et al. (2008) analyze the impact of several product and model parameters using a PDE approach, which demonstrates to be very fast and accurate. In a recent work, Huang and Kwok (2014) provide a characterization of the pricing properties of the GMWB products and perform a full mathematical analysis of the optimal dynamic withdrawal policies, reducing the pricing formulation to an optimal stopping problem.

Some authors prefer Monte Carlo methods. In their seminal work, Bacinello et al. (2011) apply a Monte Carlo approach to evaluate different types of variable annuities including GMWB products. In particular, they assume the PH's behavior to be semi-static (i.e. the holder withdraws at the contract rate or surrenders the contract). Also Bauer et al. (2008) consider a universal pricing framework to evaluate different variable annuities by using both Monte Carlo methods and a generalization of a finite mesh discretization approach. Concerning the GMWB policies, they consider the PH to withdraw at a constant rate or to follow a simplified optimization approach.

Other works focus on the research of efficient numerical methods to evaluate GMWB contracts. In particular, Donnelly et al. (2014) apply partial derivative equations to efficiently price GMWB, considering stochastic interest rate and stochastic volatility and assuming a simplified PH behavior. Later, Luo and Shevchenko (2015a, b) compute the value of a GMWB contract in the Black–Scholes model under optimal PH behavior by means of Gauss-Hermite integration, and more recently, in Shevchenko and Luo (2017), they prove the efficiency of their approach in evaluating GMWB contracts when the Vasicek interest rate model is assumed. Similarly, Ignatieva et al. (2018) introduce the Fourier Space Time-Stepping algorithm to evaluate the contracts embedding GMWB rider in the Black–Scholes model, assuming static or dynamic withdrawals. Some other authors propose pricing methods based on the use of trees. Costabile (2017) considers a trinomial tree method to evaluate GMWB contracts in a regime-switching model. Yang and Dai (2013) use a method based on a flexible tree and more recently Dai et al. (2015) improve such a tree based model to include stochastic interest rate and mortality.

Other authors focus on the effects of stochastic interest rates, stochastic volatility and jumps of the fund value. For example, Peng et al. (2012) price variable annuities such as GMWB contracts, under the Vasicek stochastic interest rate model. In particular, under the assumption of deterministic withdrawal rates, they develop the pricing formulation of the value function of a GMWB contract, comparing their results with a Monte Carlo benchmark. Huang et al. (2012) investigate the singular stochastic control problem which arises when pricing GMWB products whose underlying fund is assumed to follow a jump diffusion process.

In this paper, we price a simplified version of a GMWB contract. Specifically, we compute the no-arbitrage fee in the Heston model and the Black–Scholes model with stochastic interest rate following the Hull–White model (Black–Scholes Hull–White model). The term *no-arbitrage fee* designates the fee which is required to maintain a replicating portfolio in a complete market [the interested reader can find a description of the replicating portfolio for these types of guarantees in Chen and Forsyth (2008) and Bélanger et al. (2009)]. The GMWB products described in the literature are not

always standardized and they exhibit some differences in the mechanisms that define them. Therefore, we consider the contract proposed by Chen and Forsyth (2008) which embeds a GMWB type rider. First, we treat a static withdrawal strategy in which the PH withdraws money from the risky asset account exactly at the rate specified in the contract. Then, we price the guarantees assuming the PH to follow a dynamic withdrawal strategy, in which the PH selects the amount to withdraw in order to maximize her total wealth. In order to evaluate the considered contracts, we employ four numerical methods: a hybrid tree-finite difference method and a Hybrid Monte Carlo method [both introduced by Briani et al. (2017a, b)], an ADI finite difference scheme (Haentjens and In't Hout 2012) and a Standard Monte Carlo method which employs some common simulation techniques [see Alfonsi (2010) and Ostrovski (2013)]. In particular, both the two Monte Carlo methods consider a Longstaff-Schwartz least squares regression type (Longstaff and Schwartz 2001) to deal with the optimal withdrawal case. We accentuate that these methods have already been employed to evaluate the Guaranteed Lifelong Withdrawal Benefit (GLWB) variable annuities in Goudenège et al. (2016) but, in this case, the achievement is harder since the problem dimension is greater. In fact, the dimension of the GLWB pricing problem is equal to 2, while it is equal to 3 in the GMWB case. Moreover, the problem in the dynamic case is more difficult since a simple Bang-Bang strategy is not enough (see Luo and Shevchenko 2015b): the optimal withdrawal is often different from standard choices (no-withdrawal, withdraw the contract guaranteed amount and surrender) which work for GLWB contracts. These differences in the problem under examination force us to improve the algorithms introducing new features.

The main results of this paper are the following ones. We investigate the impact of considering stochastic volatility or stochastic interest rate on pricing and Greeks calculation, and the sensitivity of the GMWB fee to various modeling parameters. To this aim, we compute the cost of maintaining a replicating hedging portfolio in the Heston model and in the Black–Scholes Hull–White model using different pricing methods. In particular, one of the most important outcomes is the introduction of an innovative procedure to evaluate the GMWB contracts under dynamic withdrawals with Monte Carlo procedure. We would like to remark that, to the best of our knowledge, this is the first time that a Monte Carlo method is proposed to evaluate GMWB policies under dynamic withdrawals.

The remainder of the paper is organized as follows. In Sect. 2, we describe the main features of the contract such as event times, withdrawals and penalties. In Sect. 3, we provide a brief review of the stochastic models used afterwards. In Sect. 4, we describe the numerical methods that we propose to solve the GMWB pricing problem. In Sect. 5, we provide some numerical results and we study the sensitivity of the no-arbitrage fee to economic and contractual assumptions. Finally, in Sect. 6, some conclusions are drawn.

2 The GMWB contract

In our framework, we refer to the contract described in the paper of Chen and Forsyth (2008). Let us present a brief summary of the main features of such a contract.

2.1 Mortality

Similar to the work of Chen and Forsyth (2008), Dai et al. (2015), and Milevsky and Salisbury (2006), we ignore the mortality effects.

2.2 Contract state parameters

At time $t = 0$ the PH pays with a lump sum the premium P to the insurance company. The premium P is invested in a fund whose price at time t is denoted by S_t .

We suppose that the PH can withdraw only during a set of discrete times $\{t_i = i\Delta t, i = 1, \dots, N\}$, which we term *event times*. Immediately after the last event time, the PH receives a final payoff and the contract ends. Usually, withdrawals happen on an annual or semi-annual basis, i.e. $\Delta t = 1$ or $\Delta t = \frac{1}{2}$. We stress out that the first withdrawal is available at t_1 and not at $t = 0$.

The contract state parameters at a given time t are the account value ($A_t, A_0 = P$) and the base benefit ($B_t, B_0 = P$). Moreover, in the Heston model, also the volatility of the underlying fund v_t affects the option value, and so does the interest rate r_t in the Black–Scholes Hull–White model. To be brief, let u_t denote the volatility v_t as far as the Heston model is concerned and the interest rate r_t as far as the Black–Scholes Hull–White model is concerned.

Finally, let $\mathcal{V}(A, B, u, t)$ denote the fair value of the considered GMWB contract at time t with account value equal to A , base benefit equal to B and interest rate or volatility equal to u .

2.3 Evolution of the contract in the deferred period and between the event times

During the deferred period (i.e. the time between 0 and t_1) and during the time between two consecutive event times (t_i and t_{i+1}), the account value A_t follows the same dynamics of the underlying fund S_t , with the exception that some fees, determined by the parameter α_{tot} , may be subtracted from A_t :

$$dA_t = \frac{A_t}{S_t} dS_t - \alpha_{tot} A_t dt. \quad (2.1)$$

Specifically, we suppose that the total annual fees are charged to the PH (the insurance company pays no fees to a third party) and withdrawn proportionally and continuously only from the account value. These fees include the mutual fund management fee α_m and the fee α_g charged to fund the guarantee, so that

$$\alpha_{tot} = \alpha_m + \alpha_g.$$

The only portion used by the insurance company to hedge the contract is that coming from α_g : the other part of the fees has to be considered as an outgoing money flow as well as the PH's withdrawals are.

2.4 Event times and final payoff

During the event times, the PH is entitled to withdraw a guaranteed amount G from her account. The amount G is set in the contract statements and usually it is equal to the initial premium P divided by the number of event times N (i.e. $G = P/N$). Let us present how the contract state variables change at one of these event times. Let us denote $(A_{t_i^{(-)}}, B_{t_i^{(-)}}, u_{t_i^{(-)}}, t_i)$ the contract state variables just before the event time t_i occurs, and $(A_{t_i^{(+)}} , B_{t_i^{(+)}} , u_{t_i^{(+)}} , t_i)$ the same state variables just after t_i occurs. Moreover, let W_i represent the amount withdrawn at time t_i , which is required to be non-negative and smaller than the base benefit $B_{t_i^{(-)}}$.

If the amount withdrawn satisfies $W_i \leq G$, then there is no penalty imposed, whereas, if $W_i > G$, a proportional penalty charge $\kappa_i (W_i - G)$ is imposed, which reduces the amount actually received by the PH. Therefore, the PH may not receive all the money she withdraws from the risky asset account: let $f_i (W_i) : [0, B_{t_i^{(-)}}] \rightarrow \mathbb{R}$ be the function of W_i denoting the cash flow received by the PH due to the withdrawal at time t_i , that is given by the following expression:

$$f_i (W_i) = \begin{cases} W_i & \text{if } W_i \leq G \\ W_i - \kappa_i (W_i - G) & \text{if } W_i > G. \end{cases}$$

The contract state variables just after the withdrawal are given by:

$$(A_{t_i^{(+)}} , B_{t_i^{(+)}} , u_{t_i^{(+)}} , t_i) = (\max (A_{t_i^{(-)}} - W_i, 0) , B_{t_i^{(-)}} - W_i, u_{t_i^{(-)}}, t_i). \tag{2.2}$$

After the last event time t_N has occurred, the PH receives the final payoff, which is worth

$$FP = \max (A_T, (1 - \kappa_N) B_T), \tag{2.3}$$

and the contract terminates.

As far as the determination of W_i is concerned, we consider two possible approaches. In the first case, the so called static withdrawal, the PH withdraws at each event time exactly the guaranteed amount G , which is stated in the contract. In the second case, the so called dynamic withdrawal, the PH chooses W_i in order to maximize her total wealth, that is

$$W_i = \operatorname{argmax}_{w_i \in [0, B_{t_i^{(-)}}]} \mathcal{V} (\max (A_{t_i^{(-)}} - w_i, 0) , B_{t_i^{(-)}} - w_i, u_{t_i}, t_i^+) + f_i (w_i). \tag{2.4}$$

Furthermore, numerical tests prove that the optimization problem in (2.4) can be solved by simply comparing the objective function at the multiples of G , that is the elements of the set $\{nG : n = 0, \dots, N, \text{ and } nG \leq B_{t_i^{(-)}}\}$. This fact is particularly useful for the numerical evaluation of GMWB products, since it permits to quickly compute the

best withdrawal, as well as it reduces the actual possible values of B to the elements of the set $\{nG : n = 0, \dots, N\}$.

2.5 Last event time

Let $T = t_N$ denote the last event time. In this particular case, it is possible to prove that the optimal last withdrawal in the dynamic case is given by

$$W_N = \min(G, B_{T^{(-)}}), \quad (2.5)$$

and the value of the contract before the withdrawal is given by

$$\mathcal{V}(A_{T^{(-)}}, B_{T^{(-)}}, u_{T^{(-)}}, T^{(-)}) = \max(A_{T^{(-)}}, (1 - \kappa_N) B_{T^{(-)}} + \kappa \min(G, B_{T^{(-)}})), \quad (2.6)$$

which again simplifies the computation of the optimal withdrawal.

3 The stochastic models

To understand the different impacts of stochastic volatility and stochastic interest rate over such a long maturity contract, we price the GMWB according to two models: the Heston model, which provides stochastic volatility, and the Black–Scholes Hull–White model, which provides stochastic interest rate.

3.1 The Heston model

The Heston model (1993) is one of the most known and used models in finance to describe the evolution of an underlying asset and of its volatility. In order to fix the notation, we report its dynamics under a risk neutral probability:

$$\begin{cases} dS_t = rS_t dt + \sqrt{v_t} S_t dZ_t^S \\ dv_t = k(\theta - v_t) dt + \omega \sqrt{v_t} dZ_t^v, \end{cases} \quad (3.1)$$

where Z^S and Z^v are Brownian motions and $d\langle Z_t^S, Z_t^v \rangle = \rho dt$.

3.2 The Black–Scholes Hull–White model

The Hull–White model (1994) is one of historically most important interest rate models, which is nowadays often used by insurance companies. The important advantage of the Hull–White model is the existence of closed formulas which allow one to calculate the prices of bonds, caplets and swaptions. In order to fix the notation, we report the model dynamics under a risk neutral probability:

$$\begin{cases} dS_t = r_t S_t dt + \sigma S_t dZ_t^S, \\ dr_t = k(\theta_t - r_t) dt + \omega dZ_t^r, \end{cases} \tag{3.2}$$

where Z^S and Z^r are Brownian motions and $d\langle Z_t^S, Z_t^r \rangle = \rho dt$.

The process r is a generalized Ornstein-Uhlenbeck process. In fact, θ_t is not constant but it is a deterministic function which is completely determined by the market values of the zero-coupon bonds by calibration (see Brigo and Mercurio 2007) so that the theoretical prices of the zero-coupon bonds match exactly the market prices. It is well known that the short rate process r can be written as

$$r_t = \omega X_t + \beta(t), \tag{3.3}$$

where X is a stochastic process given by the following relation

$$dX_t = -kX_t dt + dZ_t^r, \quad X_0 = 0, \tag{3.4}$$

and $\beta(t)$ is a function given as follows:

$$\beta(t) = f^M(0, t) + \frac{\omega^2}{2k^2} (1 - \exp(-kt))^2. \tag{3.5}$$

Then, using relations (3.3) and (3.4), the Black–Scholes Hull–White model (3.2) can be rewritten as follows:

$$\begin{cases} dS_t = r_t S_t dt + \sigma S_t dZ_t^S \\ dX_t = -kX_t dt + dZ_t^r \\ r_t = \omega X_t + \beta(t). \end{cases} \tag{3.6}$$

If we assume that the market price of a zero-coupon bond at time t with maturity \bar{t} is given by $P^M(t, \bar{t}) = e^{-r_0(\bar{t}-t)}$, then we obtain the so-called *flat curve* case. In this very particular case the function $\beta(t)$ is given by the following expression

$$\beta(t) = r_0 + \frac{\omega^2}{2k^2} (1 - \exp(-kt))^2,$$

and no additional calibration is required.

4 Numerical methods of pricing

In this section, we describe the four pricing methods proposed: a Hybrid Monte Carlo method, a Standard Monte Carlo method, a Hybrid PDE method, and an ADI PDE method. The numerical methods proposed here resume the same principles employed by Goudenège et al. (2016) in the case of GLWB variable annuities and thus we only provide a brief description of them [the interested reader can find more details about the employed numerical procedures in Goudenège et al. (2016)]. Moreover, we focus on the algorithm improvements specifically introduced for the GMWB contracts, since

the evaluation of these products presents new numerical obstacles with reference to the GLWB contracts.

We remember that our aim is to find the fair value for α_g : this is the charge that makes the initial value of the policy $\mathcal{V}(P, P, u_0, 0)$ equal to the initial premium P . To this aim, we employ the secant method to approach the fair value of α_g , calculating at each step the value of the contract by employing one of the considered methods. Therefore, the main goal is to compute the value $\mathcal{V}(P, P, u_0, 0)$ assuming a given value for α_g .

Finally, as far as the Greeks computation is concerned, we apply the same methods described in Goudenège et al. (2016), which still rely on the computation of $\mathcal{V}(P, P, u_0, 0)$.

4.1 Hybrid Monte Carlo

The Hybrid Monte Carlo method employs a set of Monte Carlo simulations to compute an approximation $\mathcal{V}^{HMC}(A, B, u, t)$ of the value $\mathcal{V}(A, B, u, t)$ of the policy.

First of all, we select a positive integer n_s and we simulate n_s random trajectories of the stochastic process (S, u) by employing the Hybrid Monte Carlo simulation approach, introduced by Briani et al. (2017a, b). This method is called *hybrid* because it combines trees and Monte Carlo techniques to simulate the paths of the stochastic process (S, u) . Moreover, it is particularly appealing because of its easiness of implementation, efficiency and flexibility (it can easily be applied to different stochastic models with a small computational cost).

First of all, a simple tree needs to be built: this can be done according to the procedure described in Goudenège et al. (2016). Then, using a vector of Bernoulli random variables, we move from the root along the tree, describing the path of the volatility or the interest rate process. Finally, the values of the underlying S at each time step are obtained by using an Euler scheme (see Goudenège et al. 2016).

Once the n_s random trajectories are simulated, the initial value of the contract $\mathcal{V}(P, P, u_0, 0)$ can be approximated as the average of the initial policy values determined by the trajectories. In particular,

$$\mathcal{V}^{HMC}(P, P, u_0, 0) = \sum_{k=1}^{n_s} \frac{V_k(P, P, u_0, 0)}{n_s},$$

where $V_k(P, P, u_0, 0)$ denotes the value of the policy when the k th trajectory is considered, that is the sum of the particular discounted cash flows (withdrawals plus final payoff).

We stress out that, once the PH behavior is stated, a trajectory of (S, u) fully determines the withdrawals performed by the PH. In order to present how the cash flows are computed in the Monte Carlo approaches, we distinguish between the two possible methods of withdrawing, that is static or dynamic.

4.1.1 Static withdrawal

In the static withdrawal case, the PH withdraws at each event time the guaranteed amount G . The initial value of the policy can be simply calculated as the sum of discounted cash flows as follows:

$$V_k(P, P, u_0, 0) = \sum_{i=1}^N G e^{-\int_0^{t_i} r_s^k ds} + \max\left(A_{t_N}^k, (1 - \kappa_N) B_{t_N}^k\right) e^{-\int_0^{t_N} r_s^k ds}, \quad (4.1)$$

where $A_{t_N}^k, B_{t_N}^k$ and r_s^k denote respectively the final account value, the final base benefit and the interest rate at time s when the k th trajectory is considered.

We stress out that, if the Heston model is assumed, the discount factor $e^{-\int_0^{t_i} r_s ds}$ in relation (4.1) is equal to e^{-rt_i} , while if the Black–Scholes Hull–White model is assumed, the discount factor can be approximated via the trapezoidal rule from the simulated values of the interest rate r (see Stoer and Bulirsch 2013).

4.1.2 Dynamic withdrawal

In the dynamic withdrawal case, the PH performs optimal withdrawals, i.e. she chooses at each event time how much to withdraw in order to maximize her total wealth. In a recent article [see section 3, “Numerical Algorithm” in Luo and Shevchenko (2015a)] Luo and Shevchenko assert that it is not possible to use the Least Squares Monte Carlo method to price GMWB contracts in case of optimal withdrawal, because of the dynamic behavior of the PH affecting the paths of the underlying risky asset account. Conversely, in this section we show that such a computation is possible and we explain how to do it.

Let us suppose that, at each event time t_i , the PH withdraws the amount W_i determined as follows:

$$W_i = \operatorname{argmax}_{w \in [0, B_{t_i}^{(-)}]} \mathcal{V}^{HMC} \left(\max\left(A_{t_i}^{(-)} - w, 0\right), B_{t_i}^{(-)} - w, u_{t_i}, t_i \right) + f_i(w). \quad (4.2)$$

As we observed in Sect. 2.5, the choice of the last withdrawal W_N is simple and determined by Eq. (2.5). To deal with the search for the optimal withdrawal in equation (4.2) in a generic event time t_i , we have to solve two issues: how to compute $\mathcal{V}^{HMC}(A, B, u, t_i)$ and how to maximize the expression in (4.2). As already observed in Sect. 2.4, the latter question can be solved by simply evaluating the objective function at the multiples of G which are smaller than $B_{t_i}^{(-)}$. On the other hand, the calculation of $\mathcal{V}^{HMC}(A, B, u, t_i)$ can be successfully performed by proceeding backwards in time and by employing a least squares polynomial approximation based on the simulated trajectories. In fact, if the value $\mathcal{V}^{HMC}(A, B, u, t_j)$ is already known for all $j = i + 1, \dots, N$, then the value $\mathcal{V}^{HMC}(A, B, u, t_i)$ can be approximated as the (empirical) expected discounted value of future cash flows, assuming A, B and u as the

initial values at time t_i and assuming the PH to withdraw dynamically at the upcoming event times t_j , for $j = i + 1, \dots, N$.

Of course, we can not compute $\mathcal{V}^{HMC}(A, B, u, t_i)$ via Monte Carlo simulations for all the possible values A, B and u , and therefore we perform such a calculation only for the values of A and B which belong to a set $\mathcal{G} = \{(A_k, B_k), 0 \leq k \leq K\}$, and considering the values of u determined at time t_i by the n_s simulated trajectories. These points are then employed to estimate a three variates polynomial function $Q_i(A, B, u)$ by performing a least squares regression. The value of $\mathcal{V}^{HMC}(A, B, u, t_i)$ for some generic values of A, B and u is then approximated by $Q_i(A, B, u)$ and the optimization problem (4.2) can be handled.

We stress out that the use of a set \mathcal{G} lets us be sure that at each event time, the initial values for the account value and the base benefit are well distributed and useful for polynomial regression (this is not necessary for u since its value is not affected by the PH particular withdrawals). Anyway, the approximation via polynomials proves to be hard: this is due to the fact that at each event time t_i , the function \mathcal{V}^{HMC} depends on three variates, it is very curved when the values A and B are close and is very straight elsewhere. To face this issue, the use of $Q_i(A, B, u)$ as a piecewise polynomial rather than a polynomial is advisable. The choice of the type of the function $Q_i(A, B, u)$ used in the regression must be made taking into account the computational time, the number of data available and the particular set \mathcal{G} employed. We propose here two possible choices, called respectively *full regression* and *regression by levels*, which are conceived in order to improve the computational time and the convergence rate respectively.

4.2 Full regression

In this case, we consider $\mathcal{G} = \mathcal{A} \times \mathcal{B}$, where \mathcal{A} is a set of uniform nodes from 0 to $3P$ and \mathcal{B} as a set of Chebychev nodes from 0 to P (see Fig. 1). The function $Q_i(A, B, u)$ is defined as follows:

$$Q_i(A, B, u) = \begin{cases} Q_i^{up}(A, B, u) & \text{if } A \geq B \\ Q_i^{dw}(A, B, u) & \text{otherwise,} \end{cases}$$

where $Q_i^{up}(A, B, u)$ and $Q_i^{dw}(A, B, u)$ are two polynomials estimated by employing the values of $\mathcal{V}^{HMC}(A, B, u, t_i)$ at the points (A, B) of the set \mathcal{G} which satisfy $A \geq B$ and $A < B$ respectively.

4.3 Regression by levels

The regression by levels approach is based on the observation that, as already pointed out, the optimal withdrawal is always a multiple of G and therefore we have to evaluate (and approximate) the function $\mathcal{V}^{HMC}(A, B, u, t_i)$ only for those values of B which are multiples of G (in fact $B_0 = NG$ and B is progressively reduced by the withdrawals W_i which are assumed to be multiples of G).

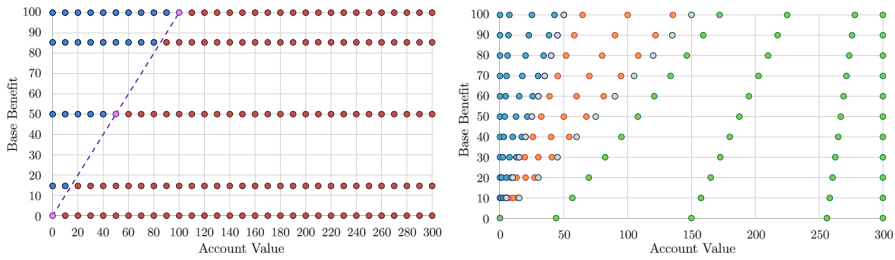


Fig. 1 The grids used in the full regression method and in the regression by levels method for a GMWB with $T = 10$ and annual withdrawals ($WF = 1$). In the first picture, purple points are used in both the two regions. In the second picture, for each B level, the gray points border the different sectors

In this case, the set of the points for the polynomial regression is $\mathcal{G} = \{(A, B) | B \in \mathcal{B} \text{ and } A \in \mathcal{A}(B)\}$ where $\mathcal{B} = \{nG, \text{ for } n = 0, \dots, N\}$ is the set of all multiples of G from zero to P and

$$\mathcal{A}(B) = \mathcal{A}_1(B) \cup \mathcal{A}_2(B) \cup \mathcal{A}_3(B)$$

where

$$\begin{aligned} \mathcal{A}_1(B) &= \left\{ n \frac{B}{2N_{RBL}}, \text{ for } n = 0, \dots, N_{RBL} \right\} \\ \mathcal{A}_2(B) &= \left\{ \frac{B}{2} + n \frac{B}{N_{RBL}}, \text{ for } n = 0, \dots, N_{RBL} \right\} \\ \mathcal{A}_3(B) &= \left\{ \frac{3B}{2} + n \frac{6P - 3B}{2N_{RBL}}, \text{ for } n = 0, \dots, N_{RBL} \right\}. \end{aligned}$$

In particular, for each $B \in \mathcal{B}$, the set $\mathcal{A}(B)$ is the union of three sets, each of them containing $N_{RBL} + 1$ uniformly distributed points in $[0, \frac{1}{2}B]$, $[\frac{1}{2}B, \frac{3}{2}B]$ and $[\frac{3}{2}B, 3P]$ respectively. The particular form of the set $\mathcal{A}(B)$ permits to increase the density of points (A, B) which satisfy $A \leq B$ (this condition is very common during the life of a GMWB contract) and also to have a relevant amount of points which satisfy $A \approx B$ (for such those points the curvature of \mathcal{V}^{HMC} is the highest). An example of the set \mathcal{G} is shown in Fig. 1. In particular, we can see that the set \mathcal{G} is structured by levels, that is the points of \mathcal{G} can be divided in $N + 1$ subsets of points having all the same ordinate B .

Since we need to calculate $\mathcal{V}^{HMC}(A, B, u, t_i)$ only for those values of B which are multiples of G , in order to approximate such a function, we can employ a different piecewise polynomial for each value of B . Specifically, we define

$$Q_i(A, nG, u) = \begin{cases} Q_{i,n}^{up}(A, u) & \text{if } A \leq \frac{nG}{2} \\ Q_{i,n}^{md}(A, u) & \text{if } \frac{nG}{2} < A < \frac{3nG}{2} \\ Q_{i,n}^{dw}(A, u) & \text{if } A \geq \frac{3nG}{2} \end{cases}$$

where $Q_{i,n}^{up}(A, u)$, $Q_{i,n}^{md}(A, u)$ and $Q_{i,n}^{dw}(A, u)$ are polynomials estimated by employing the values of $\mathcal{V}^{HMC}(A, B, u, t_i)$ at the points (A, B) of the set \mathcal{G} which satisfy $B = nG$, and $A \in [0, \frac{1}{2}B]$, $A \in [\frac{1}{2}B, \frac{3}{2}B]$ and $A \in [\frac{3}{2}B, 3P]$ respectively.

We remark that the regression by levels method implies more polynomial regressions than full regression method, which entails a better quality of the results but also an higher computational cost.

4.4 Standard Monte Carlo method

The Standard Monte Carlo method is similar to the Hybrid Monte Carlo one. The only difference consists in the generation of the random trajectories. Specifically, the trajectories for the Heston model (underlying and volatility) are simulated using a third order scheme described by Alfonsi (2010), while the trajectories for the Black–Scholes Hull–White model (underlying and interest rate) are obtained using an exact scheme described by Ostrovski (2013), with a few changes in order to incorporate the correlation between the underlying and the interest rate.

These two numerical approaches are standard methods to generate trajectories for the Heston model and the Black–Scholes Hull–White model and therefore we call such a Monte Carlo method the Standard one.

4.5 Hybrid PDE method

The Hybrid PDE approach is a PDE pricing method introduced by Briani et al. (2017a, b) both for the Heston and the Hull–White models. Here we simply give a brief description of how this approach can be employed to compute an approximation $\mathcal{V}^{HPDE}(A, B, u, t)$ of the value $\mathcal{V}(A, B, u, t)$, and for further details about this numerical method, we refer the interested reader to Goudenège et al. (2016).

In order to develop the algorithm, we consider a bi-dimensional grid $\mathcal{G} = \mathcal{A} \times \mathcal{B}$, whose points represent a couple of values for the account value and the benefit base. In particular, the set \mathcal{A} is given by $\mathcal{A} = \{A_0\} \cup \mathcal{A}_{exp}$, where $A_0 = 0$ and $\mathcal{A}_{exp} = \{A_k, k = 1, \dots, K\}$ is a mesh of K exponentially distributed points around the value P . Again, the set \mathcal{B} is given by $\mathcal{B} = \{nG, n = 0, \dots, N\}$, as in the regression by levels method (see Sect. 4.3). Moreover, in order to implement the hybrid component of the method, we also consider a quadrinomial tree $\mathcal{U} = \{(\tau_i, u_{i,j}), \text{ for } i = 0, \dots, MN, \text{ and } j = 0, \dots, 3i\}$. Specifically, $u_{i,j}$ represents the j -th possible state for the process u at time $\tau_i = i\Delta\tau$, where $\Delta\tau = \Delta t/M$ is the tree time step and M is a positive integer. We stress out that, since $\Delta t = M\Delta\tau$, the event times are included in the time mesh of \mathcal{U} .

The Hybrid PDE method for GMWB pricing consists in employing the Hybrid PDE approach during the deferred period and between the event times, and in applying the changes due to the withdrawals performed at each event time by the PH. Moreover, the algorithm computes the function \mathcal{V}^{HPDE} only for those values of A, B, u, t such that (A, B) is a point of \mathcal{G} and (t, u) is a point of \mathcal{U} (the value $\mathcal{V}^{HPDE}(P, P, u_0, 0)$ which defines the initial price of the policy is therefore computed).

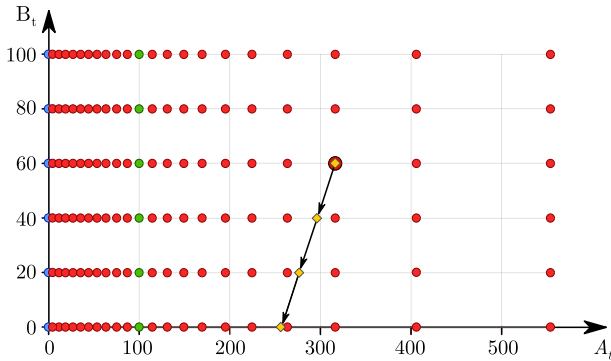


Fig. 2 The grid employed for the Hybrid PDE method

First of all, we set the final condition, that is we compute $\mathcal{V}^{HPDE}(A_k, B_h, u_{MN,j}, T)$ according to equation (2.6). Then, we proceed backwards: for each event time $t_i, i = 1, \dots, N - 1$, we employ the Hybrid PDE approach to compute $\mathcal{V}^{HPDE}(A_k, B_h, u_{i,j}, t_i^{(+)})$ from the values of \mathcal{V}^{HPDE} at time $t_{i+1}^{(-)}$ and then we apply the relations (2.2) and (2.4) to compute $\mathcal{V}^{HPDE}(A_k, B_h, u_{i,j}, t_i^{(+)})$. Finally, we apply a last step of the Hybrid PDE approach to compute $\mathcal{V}^{HPDE}(P, P, u_0, 0)$ from the values of \mathcal{V}^{HPDE} at time $t_1^{(-)}$.

We observe that, since we employ a discrete grid \mathcal{G} , an approximation of the function \mathcal{V}^{HPDE} in certain point outside \mathcal{G} is required. To this aim, we employ natural cubic splines (see Powell 1981): for each event time t_i , for each node $u_{i,j}$ of \mathcal{U} and for each value of B_h of \mathcal{B} , we compute a natural cubic spline $sp(A)$ calibrated on the values $\mathcal{V}^{HPDE}(A_k, B_h, u_{i,j}, t_i^{(+)})$, $k = 1, \dots, K$. As already observed, we consider only the withdrawals which are multiples of G : this assures that the only values of B to be considered in order to solve (2.4) are those in \mathcal{B} . Therefore, the size of the interpolation problem can be reduced from 2 to 1, and then natural cubic splines can be successfully employed.

In Fig. 2 we represent an example of the grid \mathcal{G} (red, green and blue points) employed to price a product with $G = 20$, and we outline how to perform the optimal withdrawal search (see the yellow points) for a particular point (the red bold one).

We stress out that, during the accumulation phase and between the event times, the Hybrid PDE approach works diffusing alternately the underlying and the additional stochastic process (the interest rate or the volatility process) as described in Goudenège et al. (2016) for a GLWB contract. The computational cost of such an algorithm is low and the use of curtailing techniques speeds up the algorithm (see again Goudenège et al. 2016 for more details).

4.6 ADI PDE method

The ADI PDE method is similar to the Hybrid PDE one. In this case we do not consider a bi-dimensional grid \mathcal{G} and a tree \mathcal{U} as in the Hybrid PDE method, but we employ

a tri-dimensional grid $\mathcal{G} = \mathcal{A} \times \mathcal{B} \times \mathcal{U}$. The sets \mathcal{A} and \mathcal{U} are defined according to Haentjens and In't Hout (2012), while \mathcal{B} is the same employed by the Hybrid PDE method. The ADI PDE method employs the same steps of the Hybrid PDE method. First of all, it sets the final condition; then, proceeding backwards, for each event time $t_i, i = N - 1, \dots, 1$ it solves the model PDE within $[t_i, t_{i+1}]$ and reproduce withdrawals effects; finally, it solves the PDE within $[0, t_1]$.

In the ADI framework, the intermediate steps need modified boundary conditions, because at each step we only consider a reduced part of the partial differential equation - only one direction, most of the time. This is a technical and numerical question, because the speed of convergence could be impacted by a mis-specified boundary condition. In the Black–Scholes model with stochastic interest rate and in the Heston model, we have assumed every time that the boundary conditions are given by homogeneous Neumann conditions for the full partial differential equation. Thus, at each intermediate step, this Neumann condition is compatible with intermediate steps. Moreover it simplifies the numerical procedure because it is straightforward to implement Neumann conditions on linear systems.

For further details about the numerical method, we refer the interested reader to Goudenège et al. (2016).

4.7 The secant method for the computation of the fair fee

The secant method is commonly used to compute the fair value of the parameter α_g as far as variable annuities are concerned. For example, Huang and Kwok (2016) use such a method in the case of GLWB. Actually we can prove that the value function \mathcal{V} is Lipschitz with respect to α_g in various Sobolev norms and spaces. Indeed, between two event times, it can be proved using classical estimation on partial differential equations that the solution is smooth with respect to α_g in various Sobolev norms and spaces. Moreover, at an event time, we can prove that the jump transformation preserves the continuity with respect to α_g , but furthermore a Lipschitz regularity in the following sense: the function

$$\Omega_\alpha : A \mapsto \operatorname{argmax}(w \mapsto \mathcal{V}(\alpha; \max(A - w, 0), B - w, u, t) + f(w))$$

is such that

$$\bar{\mathcal{V}}_\alpha : A \mapsto \mathcal{V}(\alpha; \max(A - \Omega_\alpha(A)), B - \Omega_\alpha(A), u, t) + f(\Omega_\alpha(A))$$

and

$$\|\bar{\mathcal{V}}_\alpha - \bar{\mathcal{V}}_\beta\|_\infty \leq \|\mathcal{V}_\alpha - \mathcal{V}_\beta\|_\infty$$

since the control variable W_i realizes this function Ω . Thus, if the initial point of the secant method is close enough to the solution, then the secant method, together with one of the proposed methods for the computation of the value function, should converge to the solution.

5 Numerical results

In this section we compare pricing and Greeks computation, assuming static or dynamic withdrawal, by using the numerical methods introduced in Sect. 4: Hybrid Monte Carlo (*HMC*), Standard Monte Carlo (*SMC*), Hybrid PDE (*HPDE*) and ADI PDE (*APDE*). Moreover, we employ the Standard Monte Carlo method with 10^8 independent scenarios (doubled by the antithetic variables technique) as a benchmark (*BM*).

We employ the numerical methods according to 4 configurations (*A*, *B*, *C*, *D*), each of them with an increasing number of steps, determined in order to achieve approximately these run times using an ordinary computer: (*A*) 30 s, (*B*) 120 s, (*C*) 480 s, (*D*) 1920 s. Table 1 reports the parameters considered for the 4 configurations using the following notation: (time steps per year \times space steps \times u steps) for the ADI PDE method, (time steps per year \times space steps) for the Hybrid PDE method and (time steps per year \times number of simulations) for the Monte Carlo methods. Moreover, as far as the dynamic withdrawal case is concerned, we also report the degree of the approximating polynomials employed by the Monte Carlo methods.

The initial values considered for the secant method are $\alpha_g = 0$ bp and $\alpha_g = 200$ bp. To reduce the run time we perform the secant iterations using an increasing number of time steps for all the methods. In particular, the values in Table 1 are those used for the last 3 iterations of the secant method.

Finally, the contract parameters employed for these tests are inspired by those employed by Chen and Forsyth (2008) and they are reported in Table 2.

5.1 Static withdrawal

The tests presented in this subsection assume the PH to withdraw at each event time an amount equal to G .

5.1.1 Test 1: Black–Scholes Hull–White model

In this test we price a GMWB product according to the Black–Scholes Hull–White model. In particular, for reasons of simplicity, we simply calibrate the Hull–White model considering a flat curve for the yield. Model parameters are shown in Table 3, while results are available in Table 4.

All the four methods behave well and in the configuration D they give results which are consistent with the benchmark. The HPDE method proves to be the best: all of its configurations give results very close to the benchmark. Then, APDE, SMC, and HMC give good results too. In particular, SMC performs a little better than HMC: the first method simulates the underlying value and the interest rate exactly and so it is enough to simulate the values at each event time t_i . On the contrary, HMC matches the first three moments of the Black–Scholes Hull–White r process, but it doesn't reproduce exactly its law and therefore we have to increase the number of steps per year to achieve convergence to the correct value of α_g . So, for a given run time, we can simulate less scenarios with HMC than with SMC and thus the confidence intervals

Table 1 Configuration parameters for the Black–Scholes Hull–White model and for the Heston model, assuming static and dynamic withdrawal, for a GMWB product with $T_2 = 10$ and $WF = 1$

	Black–Scholes Hull–White static				Heston static			
	HMC	SMC	HPDE	APDE	HMC	SMC	HPDE	APDE
<i>A</i>	$4 \times 9.2 \cdot 10^5$	$1 \times 1.7 \cdot 10^6$	260×250	$25 \times 250 \times 505$	$4 \times 5.8 \cdot 10^5$	$4 \times 5.2 \cdot 10^5$	270×250	$25 \times 250 \times 505$
<i>B</i>	$8 \times 1.8 \cdot 10^6$	$1 \times 5.7 \cdot 10^6$	420×500	$40 \times 400 \times 85$	$8 \times 1.2 \cdot 10^6$	$8 \times 1.2 \cdot 10^6$	520×500	$40 \times 400 \times 80$
<i>C</i>	$12 \times 6.3 \cdot 10^6$	$1 \times 2.9 \cdot 10^7$	780×1000	$60 \times 620 \times 125$	$12 \times 3.9 \cdot 10^6$	$12 \times 3.4 \cdot 10^6$	850×1000	$60 \times 620 \times 120$
<i>D</i>	$16 \times 1.9 \cdot 10^7$	$1 \times 1.2 \cdot 10^8$	1200×2000	$100 \times 1000 \times 215$	$16 \times 1.2 \cdot 10^7$	$16 \times 1.1 \cdot 10^7$	1400×2000	$100 \times 1000 \times 200$
Black–Scholes Hull–White dynamic								
	HMC	SMC	HPDE	APDE	HMC	SMC	HPDE	APDE
<i>A</i>	$4 \times 6.0 \cdot 10^4 \times 1$	$1 \times 6.5 \cdot 10^4 \times 1$	70×250	$8 \times 95 \times 30$	$4 \times 5.1 \cdot 10^4 \times 1$	$4 \times 5.5 \cdot 10^4 \times 1$	88×250	$10 \times 125 \times 25$
<i>B</i>	$8 \times 8.7 \cdot 10^4 \times 2$	$1 \times 9.5 \cdot 10^4 \times 2$	160×500	$14 \times 150 \times 48$	$8 \times 1.2 \cdot 10^4 \times 2$	$8 \times 1.3 \cdot 10^4 \times 2$	160×500	$15 \times 200 \times 40$
<i>C</i>	$12 \times 1.8 \cdot 10^5 \times 3$	$1 \times 1.9 \cdot 10^5 \times 3$	270×1000	$22 \times 250 \times 75$	$12 \times 2.3 \cdot 10^5 \times 3$	$12 \times 2.5 \cdot 10^5 \times 3$	266×1000	$25 \times 320 \times 60$
<i>D</i>	$16 \times 3.5 \cdot 10^5 \times 4$	$1 \times 3.5 \cdot 10^5 \times 4$	360×2000	$35 \times 400 \times 120$	$16 \times 4.2 \cdot 10^5 \times 4$	$16 \times 5.0 \cdot 10^5 \times 4$	350×2000	$40 \times 500 \times 90$

Table 2 Parameters used by Chen and Forsyth (2008)

Contract parameters		Model parameters	
Expiry time T	5, 10, 20 years	S_0	100.0
Withdrawal frequency WF	1 or 2 per year	r	0.05
Premium P	100.0	σ	0.20
Withdrawal penalty κ	0.10		
Management fees α_m	0		

Table 3 The model parameters for Test 1

Model parameters			
S_0	100.0	k	1.0
σ	0.20	ω	0.2
r_0	0.05	ρ	-0.5
<i>Curve</i>	<i>Flat</i>		

provided by HMC are generally larger than those yielded by SMC. On the contrary, the two PDE methods returns stable result and they often converge monotonically.

With regard to the numerical results, we observe that the values of α_g decrease when the maturity increases and they increase a little when the withdrawal frequency increases, just as in the Black–Scholes model (see Chen and Forsyth 2008).

5.1.2 Test 2: Heston model

In this test we price a GMWB product according to the Heston model. Model parameters are shown in Table 5, while results are available in Table 6.

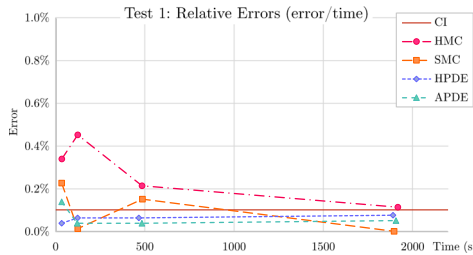
In this test, the Monte Carlo methods have more difficulties. In fact, all the values computed with the PDE methods are close to the benchmark, while some values from Monte Carlo methods are less accurate (but still consistent with the benchmark since the value of BM is inside the confidence interval). If we compare the two Monte Carlo approaches we can state that they are equivalent as they both rely on high order approximations of the Heston model. Specifically, HMC proves to be faster than SMC when using few time steps (we can exploit approximately 11% more simulations in configuration A), while SMC proves to be slightly faster when simulations with many time steps are considered, because of the more time needed to build the volatility tree (8% less simulations in configuration D). HPDE shows to be very stable (case $T = 10$, $WF = 2$, α_g do not change through configurations B–D) and APDE behaves well too (often monotone convergence).

With regard to the numerical results, we observe that the values of α_g decrease when the maturity increases and they increase a little when the withdrawal frequency increases, just as in the Black–Scholes model (see Chen and Forsyth 2008).

Table 4 Test 1: in the first table, the fair fee α_g (in basis points) for the Black–Scholes Hull–White model, considering annual or semi-annual withdrawals

T	$WF = 1$					$WF = 2$					
	HMC	SMC	HPDE	APDE	BM	HMC	SMC	HPDE	APDE	BM	
5	A	191.1 ± 0.8	190.8 ± 0.7	191.0	191.2	191.3 ± 0.1	196.9 ± 0.9	196.8 ± 0.9	196.4	196.5	196.8 ± 0.1
	B	191.8 ± 0.6	191.3 ± 0.3	191.2	191.6		196.9 ± 0.6	197.0 ± 0.5	196.6	196.6	
	C	191.3 ± 0.3	191.3 ± 0.2	191.3	191.5		196.7 ± 0.3	196.6 ± 0.2	196.6	196.9	
	D	191.2 ± 0.2	191.3 ± 0.1	191.3	191.4		196.6 ± 0.2	196.7 ± 0.1	196.7	196.7	
10	A	79.7 ± 0.8	79.3 ± 0.7	79.4	79.3	79.4 ± 0.1	81.4 ± 0.9	81.1 ± 0.95	81.0	81.3	81.0 ± 0.1
	B	79.8 ± 0.6	79.4 ± 0.3	79.4	79.4		80.9 ± 0.6	81.4 ± 0.5	81.0	80.6	
	C	79.6 ± 0.3	79.6 ± 0.2	79.4	79.4		81.1 ± 0.3	81.1 ± 0.2	81.0	81.0	
	D	79.4 ± 0.2	79.4 ± 0.1	79.4	79.4		80.9 ± 0.2	81.0 ± 0.1	81.0	81.0	
20	A	26.3 ± 1.0	25.0 ± 0.8	24.9	24.9	24.8 ± 0.1	25.0 ± 1.1	25.7 ± 1.1	25.3	25.2	25.2 ± 0.1
	B	25.9 ± 0.7	25.2 ± 0.4	24.9	24.7		25.9 ± 0.8	25.6 ± 0.5	25.2	25.2	
	C	25.2 ± 0.4	24.9 ± 0.2	24.8	24.8		25.2 ± 0.4	25.4 ± 0.2	25.2	25.2	
	D	25.0 ± 0.2	24.8 ± 0.1	24.8	24.8		25.4 ± 0.2	25.1 ± 0.1	25.2	25.2	

	HMC	SMC	HPDE	APDE
A	31 s	30 s	30 s	30 s
B	121 s	121 s	120 s	118 s
C	482 s	484 s	464 s	481 s
D	1920 s	1899 s	1893 s	1909 s



In the second table the run times for the case $WF = 1, T = 10$. Finally, the plot of the relative error (with reference to the benchmark value) for the four methods, again in the case $WF = 1, T = 10$

5.1.3 Test 3: delta computation

In this test we apply the different methods to calculate the value of the Greek Delta under the same assumptions of Test 1 and Test 2.

In this particular case, the value of α_g is assumed to be already settled (see Tables 7, 9). Specifically, the values considered are such as to cover the costs of the insurer and may be plausible on a real case. The results, which are available in Table 8 (all values in table must be multiplied by 10^{-4}) are very accurate for all the employed methods. Anyway, the values calculated by employing HPDE and APDE are the most accurate.

We remark that, despite the value of the fair fee changes a lot when changing the maturity of the policy, the value of Delta changes much less.

5.2 Dynamic withdrawal

The tests presented in this subsection assume the PH to withdraw at each event time the amount that maximizes her total wealth.

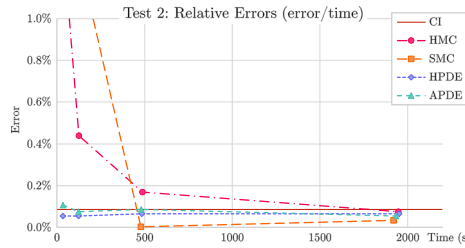
Table 5 The model parameters for Test 2

Model parameters			
S_0	100.0	k	1.0
r	0.05	ω	0.2
v_0	0.04	ρ	-0.5
θ	0.04		

Table 6 Test 2: in the first table, the fair fee α_g (in basis points) for the Heston model, considering annual or semi-annual withdrawals

T	$WF = 1$					$WF = 2$					
	HMC	SMC	HPDE	APDE	BM	HMC	SMC	HPDE	APDE	BM	
5	A	232.3 ± 1.0	232.8 ± 1.0	231.2	231.6	231.4 ± 0.1	239.8 ± 1.0	240.9 ± 1.6	239.3	239.5	239.5 ± 0.1
	B	231.7 ± 0.4	231.7 ± 0.6	231.4	231.6		240.4 ± 0.7	239.8 ± 0.8	239.5	239.4	
	C	231.4 ± 0.4	231.6 ± 0.4	231.4	231.5		239.8 ± 0.4	239.8 ± 0.4	239.6	239.7	
	D	231.4 ± 0.2	231.6 ± 0.2	231.5	231.5		239.5 ± 0.2	239.7 ± 0.2	239.6	239.6	
10	A	97.1 ± 1.1	97.6 ± 1.1	95.9	95.9	95.8 ± 0.1	98.3 ± 1.1	99.5 ± 1.1	98.0	98.3	98.0 ± 0.1
	B	96.2 ± 0.7	97.1 ± 0.8	95.9	95.9		98.6 ± 0.7	98.6 ± 0.8	98.0	98.8	
	C	95.7 ± 0.4	95.8 ± 0.4	95.9	95.9		98.1 ± 0.4	97.8 ± 0.4	98.0	98.0	
	D	95.9 ± 0.2	95.8 ± 0.2	95.9	95.9		97.9 ± 0.2	98.0 ± 0.2	98.0	98.0	
20	A	31.8 ± 1.2	31.8 ± 1.2	30.7	30.7	30.6 ± 0.1	31.4 ± 1.2	32.1 ± 1.3	31.2	31.3	31.1 ± 0.1
	B	31.4 ± 0.8	31.8 ± 0.9	30.6	30.6		31.7 ± 0.9	31.1 ± 0.9	31.1	31.1	
	C	30.5 ± 0.5	31.0 ± 0.5	30.6	30.6		31.0 ± 0.5	31.0 ± 0.5	31.1	31.1	
	D	30.6 ± 0.3	30.6 ± 0.3	30.6	30.6		31.5 ± 0.3	31.1 ± 0.3	31.1	31.1	

	HMC	SMC	HPDE	APDE
A	30 s	30 s	30 s	31 s
B	122 s	118 s	121 s	120 s
C	486 s	477 s	483 s	479 s
D	1951 s	1924 s	1956 s	1939 s



In the second table the run times for the case $WF = 1, T_2 = 10$. Finally, the plot of the relative error (with reference to the benchmark value) for the four methods, again in the case $WF = 1, T = 10$

Table 7 The α_g values used for Delta calculation in the static Black–Scholes Hull–White case (in basis points)

T	$WF = 1$	$WF = 2$
5	200	200
10	100	100
20	50	50

In this case, the SMC employed as benchmark is based on the regression by levels approach. Specifically, the degree of the employed polynomials is equal to 4 and we consider 10^6 scenarios (doubled by the antithetic variables technique), excluding the case $T_2 = 20, WF = 2$ where we used half scenarios (so as to contain the calculation time). In particular, the computational time required to perform the benchmark calcu-

Table 8 Test 3: delta estimates for the Black–Scholes Hull–White model

<i>T</i>	<i>WF</i> = 1					<i>WF</i> = 2				
	HMC	SMC	HPDE	APDE	BM	HMC	SMC	HPDE	APDE	BM
5	<i>A</i>	6212 ± 4	6214 ± 3	6213	6212	6213 ± 1	6178 ± 4	6180 ± 4	6181	6180 ± 1
	<i>B</i>	6213 ± 3	6213 ± 1	6213	6213	6213	6180 ± 3	6180 ± 2	6180	6180
	<i>C</i>	6211 ± 1	6213 ± 1	6213	6213	6213	6179 ± 1	6180 ± 1	6180	6180
	<i>D</i>	6213 ± 0	6213 ± 1	6213	6213	6213	6179 ± 1	6180 ± 1	6180	6180
10	<i>A</i>	7153 ± 7	7154 ± 6	7155	7153	7154 ± 1	7138 ± 7	7129 ± 8	7133	7127
	<i>B</i>	7155 ± 5	7152 ± 3	7154	7154	7154 ± 1	7134 ± 5	7132 ± 4	7132	7131
	<i>C</i>	7152 ± 3	7153 ± 1	7154	7154	7154	7132 ± 3	7131 ± 2	7132	7131
	<i>D</i>	7157 ± 2	7154 ± 1	7154	7154	7154	7133 ± 2	7131 ± 1	7132	7131
20	<i>A</i>	8018 ± 16	8010 ± 13	8017	8008	8016 ± 1	8010 ± 20	8005 ± 20	8005	8004 ± 1
	<i>B</i>	8023 ± 11	8016 ± 7	8017	8014	8016 ± 1	8014 ± 14	8005 ± 10	8005	8002
	<i>C</i>	8025 ± 6	8013 ± 3	8016	8015	8016 ± 1	8013 ± 7	8002 ± 4	8004	8003
	<i>D</i>	8020 ± 3	8015 ± 1	8016	8015	8016 ± 1	8007 ± 4	8001 ± 2	8004	8003

All results must be multiplied by 10^{-4} . The parameters used for this test are available in Tables 2, 3 and 7

Table 9 The α_g values used for Delta calculation in the static Heston case (in basis points)

T	$WF = 1$	$WF = 2$
5	250	250
10	100	100
20	50	50

lations varies from 30 min (case $T_2 = 5$, $WF = 1$) to 38 h (case $T_2 = 20$, $WF = 2$). On the contrary, when SMC and HMC are considered, we employ the full regression method which permits to obtain better results when a small computational time is required (Table 10).

5.2.1 Test 4: Black–Scholes Hull–White Model

In this test we price a GMWB product according to the Black–Scholes Hull–White model. Model parameters are shown in Table 3, while results are available in Table 11.

In this test PDE methods prove to be much more efficient than Monte Carlo ones. In fact, Monte Carlo methods use a least-squares regression approach to find the optimal withdrawal: this method needs many trajectories to approximate, through the polynomial regression, the value of the policy. Thus, working at fixed time, we can perform fewer scenarios than in the static withdrawal case, since much of the time is devoted to the least squares regression.

The two Monte Carlo methods give equivalent results: the differences in scenarios generation run-time are negligible because most of the time is spent in finding the best withdrawal. The HPDE method gives good and stable results, while APDE has more troubles, with estimated values which do not converge monotonically.

The case $(T, WF) = (20, 2)$ is the most challenging: the long maturity and the large number of withdrawal dates (40 event times) make the problem hard also for PDE methods. Moreover, the Monte Carlo methods in configuration A also give lower values than static approach (18.64 bp vs 25.20 bp), which is not possible: in this very particular case, due to the few scenarios considered, the least squares regression does not increase the gain of the PH.

5.2.2 Test 5: Heston Model

In this test we price a GMWB product according to the Heston model. Model parameters are shown in Table 5, while results are available in Table 12.

The results are similar to those of Test 4 but the accuracy of the results is higher: Monte Carlo methods provide good estimations of α_g , especially when using high level configurations. PDE methods behave good as usual. In particular, they both give good results except for the case $(T, WF) = (20, 2)$ where the initial approximations of α_g of APDE are too large. The case $(T, WF) = (20, 2)$ is still the most insidious, but this time we do not obtain any value lower than the value of α_g for the static withdrawal case.

Table 10 Test 3: delta calculation for the Heston model

<i>T</i>	<i>WF = 1</i>						<i>WF = 2</i>					
	HMC	SMC	HPDE	APDE	BM		HMC	SMC	HPDE	APDE	BM	
5	A	6132 ± 4	6141 ± 5	6132	6131	6131 ± 1	6101 ± 5	6107 ± 5	6099	6098	6098 ± 1	
	B	6134 ± 3	6136 ± 3	6131	6131		6101 ± 3	6104 ± 3	6098	6098	6098	
	C	6131 ± 2	6131 ± 2	6131	6131		6099 ± 2	6097 ± 2	6098	6098	6098	
	D	6131 ± 1	6131 ± 1	6131	6131		6098 ± 1	6098 ± 1	6098	6098	6098	
10	A	7287 ± 8	7297 ± 9	7286	7284	7285 ± 1	7277 ± 9	7273 ± 9	7263	7261	7262 ± 1	
	B	7289 ± 6	7287 ± 6	7285	7284		7266 ± 6	7269 ± 6	7262	7263		
	C	7287 ± 3	7287 ± 3	7284	7284		7264 ± 3	7262 ± 3	7262	7262		
	D	7285 ± 2	7287 ± 2	7284	7284		7263 ± 2	7264 ± 2	7262	7262		
20	A	8051 ± 19	8084 ± 19	8059	8058	8056 ± 1	8048 ± 19	8053 ± 19	8048	8045	8047 ± 1	
	B	8067 ± 13	8074 ± 14	8058	8056		8055 ± 13	8072 ± 14	8047	8045		
	C	8060 ± 7	8068 ± 7	8057	8056		8050 ± 7	8047 ± 8	8046	8045		
	D	8060 ± 4	8063 ± 4	8057	8056		8051 ± 4	8048 ± 4	8046	8045		

All results must be multiplied by 10^{-4} . The parameters used for this test are available in Tables 2, 5 and 9

Table 11 Test 4: in the first table, the fair fee α_g (in basis points) for the Black–Scholes Hull–White model, considering annual or semi-annual withdrawals

T_2	$WF = 1$					$WF = 2$					
	HMC	SMC	HPDE	APDE	BM	HMC	SMC	HPDE	APDE	BM	
5	A	226.2 ± 3.0	223.9 ± 2.7	282.2	278.3	282.0 ± 1.5	244.8 ± 5.9	242.1 ± 5.8	320.5	313.9	319.0 ± 1.7
	B	255.4 ± 2.4	256.3 ± 2.3	282.2	276.3		277.5 ± 5.2	275.5 ± 5.1	320.4	320.6	
	C	267.0 ± 1.7	265.1 ± 1.7	282.3	280.6		310.2 ± 3.6	308.8 ± 3.7	320.4	320.1	
	D	275.8 ± 1.3	272.6 ± 1.2	282.3	282.6		312.1 ± 2.8	311.2 ± 2.7	320.3	320.7	
10	A	128.4 ± 4.6	130.6 ± 4.0	163.5	160.4	162.5 ± 1.2	142.0 ± 11	141.7 ± 11	194.6	192.6	186.4 ± 1.4
	B	144.1 ± 3.8	145.4 ± 3.7	163.0	157.8		146.2 ± 5.1	149.6 ± 5.0	190.8	190.9	
	C	155.6 ± 2.7	155.5 ± 2.8	162.9	159.7		165.6 ± 4.0	169.6 ± 3.9	189.7	188.9	
	D	157.0 ± 2.0	155.3 ± 2.0	162.9	157.4		182.4 ± 3.3	180.4 ± 3.2	189.5	188.2	
20	A	65.9 ± 4.5	65.8 ± 5.5	90.7	62.1	84.0 ± 1.0	92.2 ± 24	18.6 ± 22	109.3	13.7	99.0 ± 1.6
	B	65.5 ± 3.1	67.7 ± 3.6	86.9	87.5		80.3 ± 12	79.7 ± 19	106.8	31.7	
	C	75.9 ± 2.7	76.3 ± 2.7	86.1	86.4		84.8 ± 6.2	83.3 ± 5.1	106.0	71.8	
	D	78.9 ± 2.1	81.2 ± 2.4	85.7	85.8		89.7 ± 4.3	92.4 ± 4.2	104.5	95.8	

	HMC	SMC	HPDE	APDE
A	29 s	31 s	31 s	30 s
B	121 s	124 s	121 s	123 s
C	484 s	484 s	489 s	474 s
D	1881 s	1927 s	1899 s	1901 s



In the second table the run times for the case $WF = 1$, $T = 10$. Finally, the plot of the relative error (with respect to the benchmark value) for the four methods, again in the case $WF = 1$, $T_2 = 10$

We note that the dynamic strategy increases the value of α_g more when the Black–Scholes Hull–White model is assumed than when the Heston is considered: probably, playing on interest rate lets the PH gain more than playing on volatility.

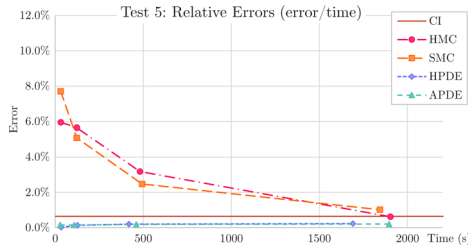
5.2.3 Test 6: Grid analysis

In Sect. 2.4 we observe that the optimal amount to be withdrawn is always a multiple of the guaranteed amount G . We did not prove this property, but we observed it through some numerical experiments. In this Test, we compute the fair price of the product of Tables 11 and 12 with maturity $T = 10$ and withdrawal frequency $WF = 1$, considering both the Black–Scholes Hull–White and the Heston model. We consider the uniform grid \mathcal{B} described in Sect. 4, which verifies $B_{i+1} - B_i = G$, and in addition two other uniform grids with $B_{i+1} - B_i = \frac{G}{2}$ and $B_{i+1} - B_i = \frac{G}{10}$, which successively refine the first grid. As shown in Table 13, the value of fair fee parameter α_{tot} does not change when the grid for B is refined, which confirms that the optimal withdrawals can be found in the set \mathcal{B} of all the multiples of G from 0 to P .

Table 12 Test 5: in the first table, the fair fee α_g (in basis points) for the Heston model, considering annual or semi-annual withdrawals

T_2	$WF = 1$					$WF = 2$					
	HMC	SMC	HPDE	APDE	BM	HMC	SMC	HPDE	APDE	BM	
5	A	238.4 ± 2.4	238.9 ± 2.4	246.3	246.3	246.5 ± 1.1	242.3 ± 4.4	245.9 ± 4.5	255.5	256.1	255.2 ± 1.1
	B	237.8 ± 1.5	240.3 ± 1.5	246.6	246.4		244.7 ± 2.9	246.5 ± 3.0	256.6	256.4	
	C	242.3 ± 1.1	243.0 ± 1.1	246.6	246.7		248.8 ± 2.2	250.9 ± 2.2	256.7	256.3	
	D	243.4 ± 0.8	243.1 ± 0.8	246.6	246.7		253.1 ± 1.6	251.8 ± 1.6	256.7	256.3	
10	A	125.8 ± 4.0	123.4 ± 4.1	133.7	133.9	133.7 ± 0.9	132.7 ± 12	119.8 ± 11	137.9	146.6	137.0 ± 0.9
	B	126.2 ± 2.5	126.9 ± 2.5	133.9	133.9		118.8 ± 4.6	122.6 ± 4.6	138.1	139.4	
	C	129.5 ± 1.8	130.4 ± 1.8	134.0	134.0		125.2 ± 3.1	124.1 ± 3.0	138.3	138.4	
	D	132.9 ± 1.2	132.4 ± 1.2	134.0	134.0		130.6 ± 2.3	130.3 ± 2.3	138.4	138.4	
20	A	37.7 ± 11.9	37.5 ± 9.7	72.3	74.1	69.4 ± 0.7	82.5 ± 26	61.2 ± 28	74.5	99.4	71.8 ± 1.1
	B	64.5 ± 3.8	69.2 ± 2.4	71.1	72.3		70.7 ± 7.8	67.1 ± 8.5	73.0	86.0	
	C	66.9 ± 3.2	66.1 ± 2.5	71.1	71.6		65.3 ± 3.4	66.5 ± 4.1	73.1	77.1	
	D	64.2 ± 2.6	66.9 ± 2.2	71.2	71.7		61.3 ± 2.6	62.2 ± 2.8	73.2	74.6	

	HMC	SMC	HPDE	APDE
A	32 s	32 s	29 s	30 s
B	123 s	124 s	122 s	118 s
C	483 s	475 s	474 s	495 s
D	1903 s	1882 s	1923 s	1947 s



In the second table the run times for the case $WF = 1, T = 10$. Finally, the plot of the relative error (with respect to the benchmark value) for the four methods, again in the case $WF = 1, T_2 = 10$

Table 13 Test 6: the fair fee α_g (in basis points), reducing the step ΔB of the grid \mathcal{G}

ΔB		BS-HW model		Heston model	
		HPDE	APDE	HPDE	APDE
G	A	163.5	160.4	133.7	133.9
	B	163.0	157.8	133.9	133.9
	C	162.9	159.7	134.0	134.0
	D	162.9	157.4	134.0	134.0
$\frac{G}{2}$	A	163.5	160.4	133.7	133.9
	B	163.0	157.8	133.9	133.9
	C	162.9	159.7	134.0	134.0
	D	162.9	157.4	134.0	134.0
$\frac{G}{10}$	A	163.5	160.4	133.7	133.9
	B	163.0	157.8	133.9	133.9
	C	162.9	159.7	134.0	134.0
	D	162.9	157.4	134.0	134.0

Table 14 The α_g values (in basis points) used for Delta calculation in the Dynamic Black–Scholes Hull–White case

T_2	$WF = 1$	$WF = 2$
5	350	350
10	200	200
20	150	150

Table 15 The α_g values (in basis points) used for Delta calculation in the Dynamic Heston case

T_2	$WF = 1$	$WF = 2$
5	300	300
10	150	150
20	100	100

5.2.4 Test 7: delta computation

In this test we apply the different numerical methods to calculate the value of the Greek Delta under the same assumptions of Test 4 and Test 5. The employed values for α_g are available in Tables 14 and 15, while results are presented in Tables 16 and 17.

The values obtained by HPDE are quite accurate and they are very regular despite the high dimension of the problem. Results from APDE are good but in this case they are a bit worse than HPDE, especially when the Black–Scholes Hull–White model is considered (see for example the case $(T, WF) = (20, 2)$). Monte Carlo methods suffer the few scenarios performed and sometimes the confidence interval is large.

5.2.5 Optimal Withdrawal Strategy Plots

In Figs. 3 and 4 we report the optimal withdrawals at the second event time t_2 for a GMWB with $(T, WF) = (10, 1)$ for both the Black–Scholes Hull–White model and the Heston model. In particular, these optimal withdrawals are computed by applying the HPDE method. We remark that these plots are very similar to those reported by Chen and Forsyth (2008). In particular, we observe the same structure around the bisector and the wide region of regular withdrawals.

6 Conclusions

In this article we propose four numerical methods to evaluate and to compute the Greeks of a GMWB contract. Regarding to the stochastic model, both stochastic interest rate and stochastic volatility effects are investigated. According to the policy holder's behavior, both static and dynamic strategies are considered.

All four methods give reliable results both for pricing and delta calculation, in particular when static withdrawal is considered. The PDE methods prove to be efficient and reliable in all conditions, while Monte Carlo methods prove to be less accurate and more computationally demanding especially when dynamic withdrawal is considered. The Hybrid PDE is the most effective method (good convergence speed and stability

Table 16 Test 7: delta calculation for the Black–Scholes Hull–White model

T_2	$WF = 1$						$WF = 2$					
	HMC	SMC	HPDE	APDE	BM		HMC	SMC	HPDE	APDE	BM	
5	A	4986 ± 311	4790 ± 310	4474	4465	4514 ± 62	4753 ± 537	4117 ± 429	4191	4187	4181 ± 68	
	B	4455 ± 171	4443 ± 179	4477	4469		4220 ± 182	4362 ± 222	4196	4191		
	C	4385 ± 130	4420 ± 122	4478	4473		4057 ± 196	3987 ± 170	4198	4195		
	D	4319 ± 103	4432 ± 94	4478	4476		4158 ± 167	4235 ± 170	4198	4198		
10	A	4734 ± 656	5152 ± 543	4630	4625	4593 ± 112	4612 ± 946	3881 ± 817	4270	4325	4291 ± 123	
	B	4577 ± 320	4367 ± 307	4636	4616		4628 ± 460	3846 ± 362	4296	4316		
	C	4665 ± 259	4548 ± 240	4639	4631		3898 ± 310	4122 ± 303	4304	4300		
	D	4517 ± 178	4537 ± 175	4639	4635		4492 ± 329	4201 ± 276	4306	4304		
20	A	4053 ± 302	4223 ± 112	4129	4149	4083 ± 143	4152 ± 150	4037 ± 144	3639	4062	3857 ± 211	
	B	4370 ± 105	4253 ± 108	4153	4118		4095 ± 74	4039 ± 68	3752	3924		
	C	4046 ± 332	4011 ± 326	4157	4150		4078 ± 72	4049 ± 60	3766	3803		
	D	3980 ± 268	3857 ± 242	4157	4145		3434 ± 294	3659 ± 332	3780	3798		

All results must be multiplied by 10^{-4} . The parameters used for this test are available in Tables 2, 14 and 5

Table 17 Test 7: delta calculation for the Heston model

<i>T</i>	<i>WF</i> = 1						<i>WF</i> = 2					
	HMC	SMC	HPDE	APDE	BM		HMC	SMC	HPDE	APDE	BM	
5	A	5732 ± 40	5727 ± 26	5629	5631	5637 ± 26	5878 ± 97	5803 ± 62	5571	5577	5599 ± 25	
	B	5715 ± 28	5699 ± 27	5628	5630		5678 ± 88	5603 ± 78	5570	5572		
	C	5614 ± 49	5668 ± 66	5628	5629		5674 ± 122	5635 ± 78	5570	5570		
	D	5607 ± 44	5653 ± 55	5628	5628		5695 ± 103	5618 ± 63	5569	5570		
10	A	6082 ± 179	6103 ± 157	6007	6009	5983 ± 58	6918 ± 712	6083 ± 392	5938	5915	5914 ± 55	
	B	5949 ± 133	5886 ± 125	6006	6007		6225 ± 263	6058 ± 142	5936	5942		
	C	5909 ± 117	6062 ± 136	6005	6006		5779 ± 151	6026 ± 116	5936	5939		
	D	6008 ± 113	6059 ± 97	6004	6005		5980 ± 121	5840 ± 114	5936	5937		
20	A	5604 ± 480	5658 ± 383	5636	5644	5635 ± 126	4162 ± 987	5886 ± 831	5540	5122	5343 ± 174	
	B	5428 ± 405	5855 ± 433	5635	5642		5056 ± 362	5543 ± 437	5540	5382		
	C	5410 ± 297	5571 ± 206	5635	5640		5421 ± 306	5297 ± 369	5542	5543		
	D	5734 ± 213	5571 ± 199	5635	5638		5379 ± 226	5416 ± 198	5543	5550		

All results must be multiplied by 10^{-4} . The parameters used for this test are available in Tables 2, 15 and 5

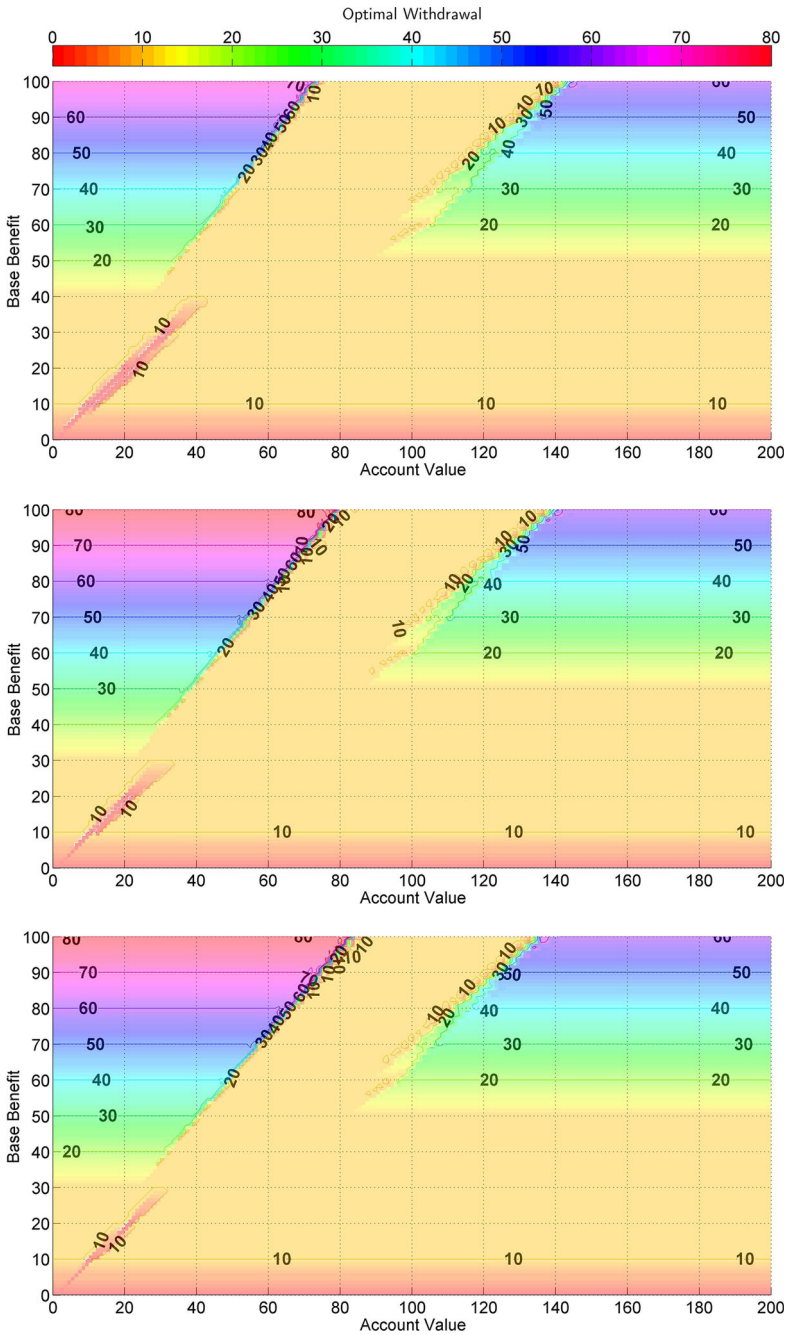


Fig. 3 Plots of the optimal withdrawals at time $t_2 = 2$ for the Black–Scholes Hull–White model according to different values of r_2 : from the top to the bottom $r_2 = 0.03$, $r_2 = 0.05$ and $r_2 = 0.07$. The parameters used to obtain these plots are the same as for Delta calculation for case $T = 10$, $WF = 1$: see Tables 2, 3 and 14

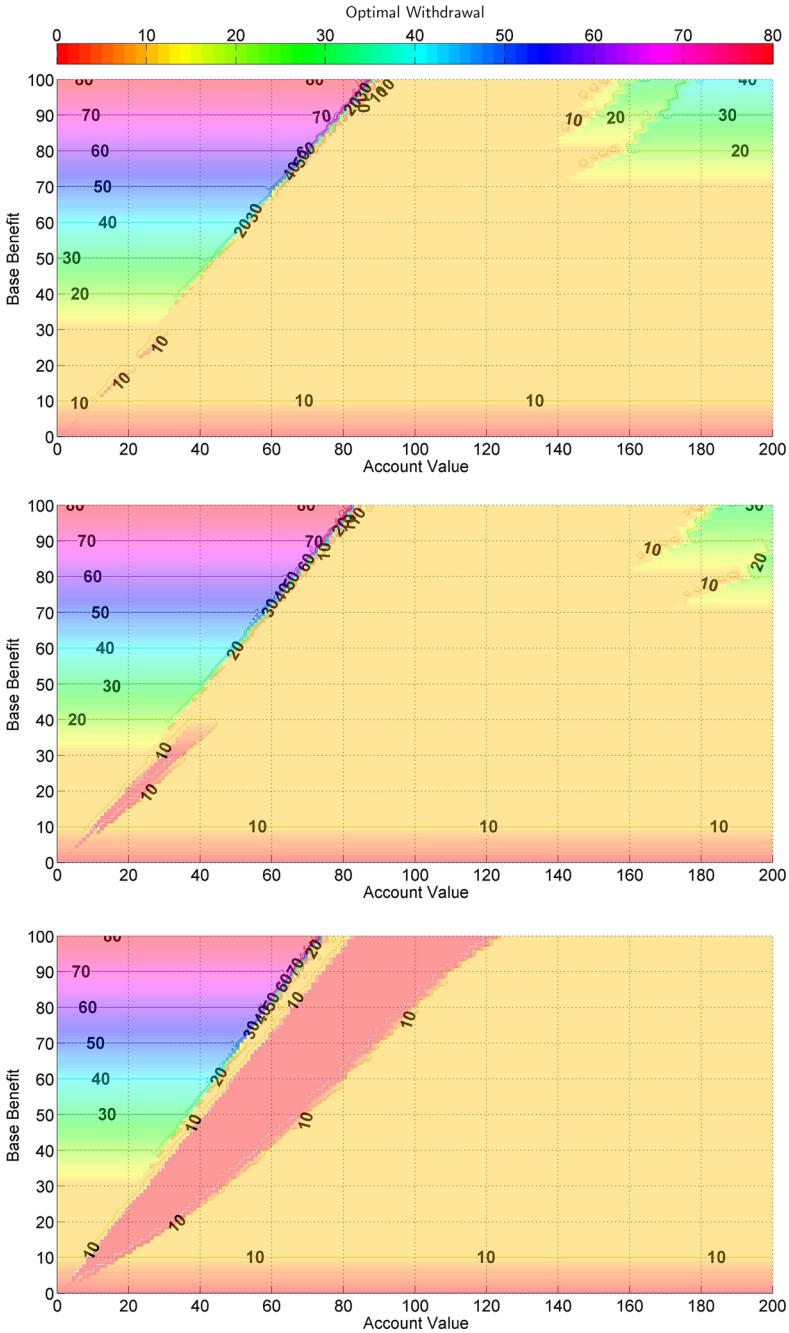


Fig. 4 Plots of the optimal withdrawals at time $t_2 = 2$ for the Heston model according to different values of the volatility v_2 : from the top to the bottom $v_2 = 0$, $v_2 = 0.04$ and $v_2 = 0.16$. The parameters used to obtain these plots are the same as for Delta calculation for case $T = 10$, $WF = 1$: see Tables 2, 5 and 15

of results). Also ADI PDE is accurate, but the implementation of this method is a little harder than Hybrid PDE one; moreover the choice of the good parameters for ADI PDE may be a source of issues. In the Black–Scholes Hull–White model case, the Standard Monte Carlo method, thanks to its exact simulation, outperforms the Hybrid method while, in the Heston model case, the two Monte Carlo methods prove to be roughly equivalent (even if the Hybrid Monte Carlo is easier to implement).

The gap among the different methods is clear cut when they are applied to evaluate a contract which allows the PH to dynamically withdraw. In this case the PDE methods are the most efficient. As far as the Monte Carlo methods are concerned, a good approximation of the correct value of the contract can be obtained by employing the regression by levels approach, but such a method is time demanding. Anyway, we have to remark that Monte Carlo methods offer confidence intervals for the numerical results, they are useful in risk measures calculation (for example VAR or ES), and they are preferred by insurance companies because of their attachment to the idea of financial scenario. Insurance companies may be very interested by these new Monte Carlo techniques that allow one to evaluate GMWB contracts under dynamic withdrawal without leaving Monte Carlo methods for PDE methods.

We conclude by pointing out that our methods are quite flexible in that they can accommodate a wide variety of policy holder withdrawal strategies such as ones derived from utility-based models.

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